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Optimum maintenance strategy under uncertainty in the lifetime distribution

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ABSTRACT

The problem of determining the optimal maintenance strategy for a machine given its lifetime distribution has been studied extensively. Solutions to this problem are outlined in the academic literature, prescribed in professional handbooks, implemented in reliability engineering software systems and widely used in practice. These solutions typically assume that the lifetime distribution and its parameter values are known with certainty, although this is usually not the case in practice. In this paper we study the effect of parameter uncertainty on the optimum age-based maintenance strategy. The effect of uncertainty is evaluated by considering both a theoretical uniform lifetime distribution and a more realistic Weibull lifetime distribution. The results show that admitting to the uncertainty does influence the optimal maintenance age and also provides a quantifiable cost benefit. The results can help maintenance managers in making maintenance decisions under uncertainty, and also in deciding when it is worthwhile to invest in advanced data improvement procedures.

1. Introduction

It is widely accepted that preventive maintenance is important for achieving Operational Excellence [10,34], since it aids in reducing system downtime. Preventive maintenance policies can roughly be subdivided into two categories, namely condition-based and time-based. Recent advances in sensor technology have lead to increased popularity of condition-based maintenance. However, condition monitoring may be technically impossible for some assets, the benefits of condition-based maintenance may not outweigh the investment costs required to enable condition monitoring, and condition-based maintenance activities are more difficult to plan than activities that are fixed in time. Due to these limitations of condition-based maintenance, much of the preventive maintenance in practice is still time-directed.

An important type of time-directed maintenance is age-based maintenance [16]. The effectiveness of this maintenance strategy is determined by the age at which preventive maintenance takes place. Early (and therefore frequent) maintenance actions result in a high maintenance cost per unit time. Late (or infrequent) maintenance actions result in a higher probability of failure (and costs associated with failure). There are widely used handbooks [12,16,1,32] and software systems [16,8] that prescribe how to determine the optimal replacement age given the component lifetime distribution. These systems generally require the specification of the lifetime distribution and its parameter values, and do not allow for potential uncertainty in these inputs.

There are several reasons why estimates of equipment lifetime distributions may not be accurate. Firstly, vendor guidelines may not be (fully) compatible due to lack of knowledge of the actual use and maintenance of the equipment [27,39]. Furthermore, maintenance and reliability engineers have bemoaned the lack of credibility in collected data for years [25,11,6]. Maintenance records and historic failure data are often inaccurate or incomplete. A third source of uncertainty is the fact that historic failure data are likely to be (heavily) right-censored because of preventive maintenance in the past [7]. Finally, there is often an insufficient amount of data to determine accurate estimates for the model parameters.

The consequences of uncertainty in the lifetime distribution in terms of the optimum maintenance policy and the cost reduction that can be achieved by including the uncertainty did not receive much attention yet, and is the focus of the current paper. The approach that we follow is to accept the current modus operandi of many maintenance engineers by assuming a pre-defined distribution (e.g., uniform or Weibull), and to include uncertainty in its parameters, rather than taking point estimates. The results will show that admitting to the uncertainty does influence the optimal maintenance age. The significance of this influence ranges from very little to quite substantial, depending on the circumstances.

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This means that, in some cases, it is essential to take uncertainty into account.

This paper is organized as follows. In Section 2, we discuss the existing literature. In Section 3, we formally describe the problem we consider as well as our approach. In Section 4, we evaluate the uncertainty by considering a simple setting with a uniform lifetime distribution, in which the optimum maintenance age can be obtained explicitly. In Section 5, we evaluate a more realistic setting, with uncertainty in the scale parameter of the Weibull distribution, which we will evaluate numerically. Section 6 provides conclusions and extensions for future research.

2. Literature

Time-based preventive maintenance was first studied by Barlow and Hunter [3]. One of the two introduced policies, age-based maintenance, is further studied by Glasser [13], Tadikamalla [38], and Nakagawa and Yasui [28]. These papers all assume that the component lifetime distribution is known with certainty. Examples of other studies that make this assumption are Kijima et al. [19], Makis and Jardine [24] and Jiang et al. [17], who report on the periodical replacement problem with repairs at failures that bring the system to a certain better state; and Yeh and Lo [40], who determine the optimal maintenance strategy during a warranty period with a given length.

2.1. Uncertainty

The fact that the lifetime distribution of a component is typically uncertain was highlighted by several authors. Uncertainty is divided into model uncertainty and parameter uncertainty; in both categories, several solutions were proposed. Zhang and Mahadevan [41] propose a Bayesian procedure that includes both model uncertainty and parameter uncertainty. Hoeting et al. [15] use Bayesian model averaging to take account of model uncertainty in a rather general setting.

A common approach of many reliability engineers is to assume a certain distribution and to consider the parameters of this distribution as unknown (parameter uncertainty). This assumption is not too restrictive if a flexible distribution is chosen that provides a good description of many types of failure data. An example of such a distribution is the Weibull distribution [32,1]. Gertsbakh [12] describes an approach to determine the optimal preventive maintenance age if the parameters of the lifetime distribution take a specific value from a small set of values, each with a specific probability. However, this line of reasoning is not progressed in detail.

The distribution that is used to model parameter uncertainty can be based on expert judgment and/or on data. In a Bayesian approach, the opinions of experts can be used to determine a so-called prior distribution [5,29]. Kraan and Bedford [20] and Zuashkiani et al. [42] discuss methods to translate expert judgment on model outputs into uncertainties on model parameter values. The prior distribution could be updated based on data that becomes available, resulting in a posterior distribution. If no (expert) knowledge is available, commonly a reference prior [31] or a non-informative prior [14] is used.

There are also other methods used to model/estimate parameter uncertainty. Laggoune et al. [22] consider a setting with few failure data and use the Bootstrap technique to model the uncertainties in the parameter values. This technique draws a large amount of subsamples from the data, and base the random distribution that is used to model the uncertainty in the parameters on the maximum likelihood estimations for the parameters of the various subsamples. The optimal maintenance age turns out to change if this uncertainty is taken into account. Rocco et al. [33] use a two steps evolutionary approach and apply this to a maintenance optimization problem in a multi-component system. The results mainly focus on the computational complexity of this approach.

2.2. Consequences of uncertainty

Some studies consider the consequences of uncertainty in the lifetime distribution on the optimum maintenance plan. Baker and Scarf [2] consider the excess cost of making suboptimal decisions using point estimates for the parameters based on data, instead of using the true but unknown optimal decisions. Their focus is mainly on the optimal inspection interval. Although large sample sizes are required to accurately estimate the optimal value of maintenance decision variables, it turns out that the excess cost is already relatively small for modest sample sizes. Our study shows that this is not true in general; the additional cost of not taking into account uncertainty can already be significant for small levels of uncertainty, depending on the situation.

Bunea and Bedford [7] consider age-based maintenance planning under the presence of censored data. They distinguish three levels of dependence between the censoring times and the failure times, and show that the suboptimal replacement age and suboptimal replacement costs can be dramatically nonoptimal when the wrong level of dependence is considered. The study focuses on modeling dependence between censoring times and failure times, and does not consider the effect of parameter uncertainty on the optimal maintenance age as we do. Furthermore, they find that the effects are most significant when the failure rate increases slowly, which is the exact opposite of our results.

Mazzuchi and Soyer [26] present a Bayesian theoretic approach that updates random distributions, modeling uncertainty in the parameters of lifetime distributions, as more data becomes available. Under the uncertainty in the paremeters, the maintenance age that minimizes the expected cost is minimized. A single example indicates a maintenance age that decreases whenever a new duration is observed. General insights on the behavior of the maintenance age, as we present, are lacking.

Silver and Fiechter [35] use Bayesian updating in the simple case with only two possible lifetimes with unknown corresponding probabilities. In this case the optimal maintenance age is always either one of two values. The emphasis is mainly on the complexity of the required calculations and on two heuristics. A numerical test shows that if the level of uncertainty is high enough, the optimal maintenance age might change if the uncertainty is taken into account. Silver and Fiechter [36] generalize this to a discretely distributed lifetime and show that in some circumstances the optimal maintenance age decreases if the uncertainty is taken into account. Again, general insights are lacking, and the results based on nonparametric discrete lifetime distributions do not carry over to more realistic parametric continuous distributions.

2.3. Contribution

As outlined, most papers that include lifetime distribution uncertainty mainly focus on how to model this uncertainty, and on how to update the model when more data becomes available. Authors have focused on specific effects and considered few examples to illustrate those, not leading to general insights. This paper is devoted to the effects of uncertainty and does present general insights for the age-based maintenance strategy.

3. Approach

The age-based maintenance strategy considers a single unit with lifetime distribution F. When the unit fails, an emergency repair is performed at normalized cost 1. Furthermore, when the unit reaches a specified age T, a preventive maintenance action is performed at cost $c<1$. Both after an emergency repair and after a preventive maintenance action, the unit is assumed to be as-good-as-new. The cost rate of the age-based maintenance strategy is

$$\eta_{\text{age}}(T)=\frac{F(T)+(1-F(T))c}{\int_0^T(1-F(x))\,dx}. \quad (1)$$

This formula was first presented by Barlow and Hunter [3] and is also included in many textbooks [4,12]. Minimizing the cost rate $\eta_{\text{age}}(T)$ provides us with the optimal maintenance age T_{opt}. However, analysis of the age-based maintenance strategy requires the exact lifetime distribution of the unit, which is rarely known in practice.

As motivated in the introduction, we assume a specific lifetime distribution and allow for uncertainty in its parameters. This uncertainty will be modeled using a random distribution. We remark that the true parameter values are fixed, but unknown. Their estimated values depend on expert knowledge and random data that becomes available, and are therefore random. The used random distribution represents the likelihood that the true parameters have certain values. This paper will not focus on determining a proper distribution to model the uncertainty in the parameters, but on the effect of parameter (mis)specification on the optimal maintenance age.

Let us represent the vector of parameters of the lifetime distribution by s. We will denote the joint density function that models the uncertainty in s by $g(s)$, which is defined on $\mathbb{R}^n$. It then follows that the expected cost rate (as a function of the maintenance age T) equals

$$\eta^E_{\text{age}}(T)=\int_{s\in\mathbb{R}^n} g(s)\frac{F(T;s)+(1-F(T;s))c}{\int_0^T(1-F(x;s))\,dx}\,ds_1\ldots ds_n. \quad (2)$$

Minimizing this function provides us with the optimal preventive maintenance age T^E_{opt}, i.e., the maintenance age that minimizes the expected cost rate.

4. Uniformly distributed lifetime

We start our study with a uniformly distributed lifetime. Although unrealistic in many cases, an important advantage of this distribution is that it is relatively simple to get the relevant input from a maintenance engineer as only the minimum and maximum lifetime is needed. As maintenance will clearly not take place before the minimum lifetime is reached, we set it to zero in our model. The distribution function of the uniform distribution on an interval $[0,s]$ is

$$F(t;s)=\begin{cases}0, & t<0,\\ \frac{t}{s}, & 0\le t\le s,\\ 1, & t>s.\end{cases}$$

It follows that

$$\int_0^T(1-F(x;s))\,dx=\begin{cases}T\left(1-\dfrac{T}{2s}\right), & T\le s,\\ \dfrac{1}{2}s, & T\ge s.\end{cases} \quad (3)$$

We will first consider the case that the value of the parameter s is known with certainty. Without loss of generality, we will rescale the uniform distribution such that $s=1$. The cost rate (1) is then equal to

$$\eta_{\text{age}}(T;\hat{s})=\frac{T+(1-T)c}{T\left(1-\dfrac{T}{2}\right)}=\frac{T(1-c)+c}{T-\dfrac{1}{2}T^2}, \quad 0<T\le 1.$$

Restricted to $T>0$, this function has a unique minimum that is attained at

$$T_{\text{opt}}=\frac{\sqrt{c(2-c)}-c}{1-c}. \quad (4)$$

For any value of $c\in(0,1)$ this optimal maintenance age is contained in the interval $(0,1)$. Furthermore, T_{opt} is a strictly increasing function in c. This confirms the intuition that a higher preventive maintenance cost, as a fraction of the cost of an emergency repair, results in a higher preventive maintenance age.

As motivated before, it is not realistic to assume that the true value of the parameter s is known with certainty. This uncertainty should be taken into account while determining the optimal maintenance age. We will model the uncertainty in the parameters using a uniform distribution on the interval $[1-\alpha,1+\alpha]$, with $0\le\alpha\le 1$. Thus, the function $g(s)$ in (2) equals

$$g(s)=\begin{cases}(2\alpha)^{-1}, & s\in[1-\alpha,1+\alpha],\\ 0, & \text{elsewhere}.\end{cases}$$

The value of α can be interpreted as a measure of the uncertainty for our initial point estimate $s=1$.

If the parameter of the lifetime distribution is uncertain, the distribution of the lifetime can be seen as a composite distribution. In our current setting, where the parameter s follows a uniform distribution on the interval $[1-\alpha,1+\alpha]$, this composite distribution can be stated explicitly. Fig. 1 shows the density function of this composite distribution for various values of α. This figure provides us with a first idea of the effect of uncertainty in the parameter s on the optimal maintenance age.

If the uncertainty increases, i.e. if α increases, the composite distribution becomes more dense on the interval $[0,1-\alpha]$. As long as the optimal maintenance age is also contained in this interval, the probability of a failure prior to the maintenance action increases during this entire period. This, in turn, implies a decreasing optimal maintenance age. However, if the uncertainty increases further, the probability of a failure at the start of the lifetime of the unit increases significantly. Furthermore, the tail of the distribution becomes fatter. The early failures cannot be avoided, unless a very low maintenance age is chosen. On the other hand, as we will show, the fatter tail results in an increasing optimal maintenance age. Thus, the optimal maintenance age first decreases if the uncertainty in the parameter s increases, but increases if the uncertainty increases further.

We will continue with an explicit determination of the expected cost rate and the optimal maintenance age. The main results will be stated here, detailed calculations can be found in Appendix A. The expression for the expected cost rate $\eta^E_{\text{age}}(T)$

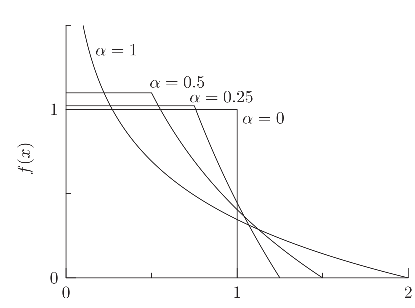

Fig. 1. Density function of the uniform lifetime distribution for $\hat{s}=1$ and for various values of α.

depends on the value of T and is equal to

$$\eta^E_{\mathrm{age}}(T)=\begin{cases}\frac{1}{2\alpha}\left(1-\frac{1}{2}c\right)\ln\left(\frac{2+2\alpha-T}{2-2\alpha-T}\right)+\frac{c}{T}, & \text{if } T\le 1-\alpha,\\ \frac{1}{\alpha}(\ln T-\ln(1-\alpha))+\frac{1}{2\alpha}\left(1-\frac{1}{2}c\right)\ln\left(\frac{2+2\alpha-T}{T}\right) & \\ \quad +\frac{c}{2\alpha}\left(\frac{1+\alpha}{T}-1\right), & \text{if } 1-\alpha\le T\le 1+\alpha.\end{cases}$$

On the interval $(0,1+\alpha)$, the function $\eta^E_{\mathrm{age}}(T)$ has a unique minimum. If $\alpha\le(1-c)(1+c)^{-1}$, this minimum is attained on the interval $(0,1-\alpha]$ at

$$\frac{\sqrt{c(2-c)-2\alpha^2c(1-c)}-c}{1-c}.$$

If, on the other hand, $\alpha\ge(1-c)(1+c)^{-1}$, this minimum is attained on the interval $[1-\alpha,1+\alpha)$ at $c(1+\alpha)$. Thus, the optimal maintenance age T^E_{opt} that minimizes the expected cost rate equals

$$T^E_{\mathrm{opt}}=\begin{cases}\frac{\sqrt{c(2-c)-2\alpha^2c(1-c)}-c}{1-c}, & \alpha\le\frac{1-c}{1+c},\\ c(1+\alpha), & \alpha\ge\frac{1-c}{1+c}.\end{cases}\tag{5}$$

The first part of this expression decreases in α. Thus, the optimal maintenance age T^E_{opt} first decreases if the uncertainty in the parameter s increases. If α increases further, above $(1-c)(1+c)^{-1}$, the optimal maintenance age increases linearly in α. Furthermore, (5) is equivalent to (4) if $\alpha=0$. Fig. 2(a) shows the optimal maintenance age T^E_{opt} as a function of the level of uncertainty α in the parameter s, for various values of the relative cost c of preventive maintenance.

Another notable observation is that all knots in the graphs in Fig. 2(a) are on the line $T^E_{\mathrm{opt}}=1-\alpha$. A knot is attained at the point $\alpha=(1-c)(1+c)$, which is equivalent to $c=(1-\alpha)(1+\alpha)$. The optimal maintenance age equals $c(1+\alpha)$, which reduces to $1-\alpha$ for $c=(1-\alpha)(1+\alpha)$.

We are not only interested in the effect of the uncertainty in the parameter s on the optimal maintenance age, but also to what extent the expected costs increase if we do not take this uncertainty into account. Denoting again the optimal preventive maintenance age based on the point estimate $\hat{s}$ of the parameter s by T_{opt}, and the optimal preventive maintenance age that takes the uncertainty in s into account by T^E_{opt}, the percentage increase in cost if we do not take the uncertainty in the parameter s into account equals

$$\left(\frac{\eta^E_{\mathrm{age}}(T_{\mathrm{opt}})}{\eta^E_{\mathrm{age}}(T^E_{\mathrm{opt}})}-1\right)\times 100\%.$$

Fig. 2(b) shows this percentage increase in expected costs for different values of the relative cost c of preventive maintenance, and as a function of the level of uncertainty α in the estimate of the parameter s. Especially for a low relative cost c of performing preventive maintenance and a high level of uncertainty α in the parameter s, the expected percentage increase in costs is high if we do not take the uncertainty into account. This percentage rises to 9.5% for $c=0.05$. The percentage increase in cost first increases if α increases and starts to decrease if α increases further. This is consistent with the results in Fig. 2(a). First, the optimal maintenance age decreases if the uncertainty increases. This results in a greater difference between the optimal maintenance age if we do not take the uncertainty into account and the optimal maintenance if we do so. As the optimal maintenance age starts to increase, this gap becomes smaller and taking into account the uncertainty becomes less profitable. The second peak in the graphs in Fig. 2(b) for high values of c is caused by an optimal maintenance age that increases even further than the optimal maintenance age without taking into account the uncertainty.

5. Weibull distributed lifetime

We continue our study with a more realistic case, namely that of a unit with a Weibull distributed lifetime, and show that the insights obtained in the previous section carry over. The Weibull distribution is the most commonly used distribution to model lifetimes and has been found to provide a good description of many types of lifetime data. For systems with multiple critical units, the lifetime approximately follows a Weibull distribution [23]. Other physical phenomena for which the Weibull distribution is a suitable distribution to model lifetimes are given by Rinne [32]. Furthermore, Rinne [32] also states that for various degradation processes the time until failure, i.e., the time at which a certain degradation level will be reached, closely corresponds to a Weibull distribution.

We consider the Weibull distribution with two parameters, a shape parameter k and a scale parameter λ. We note that in some situations the use of a three-parameter Weibull distribution with

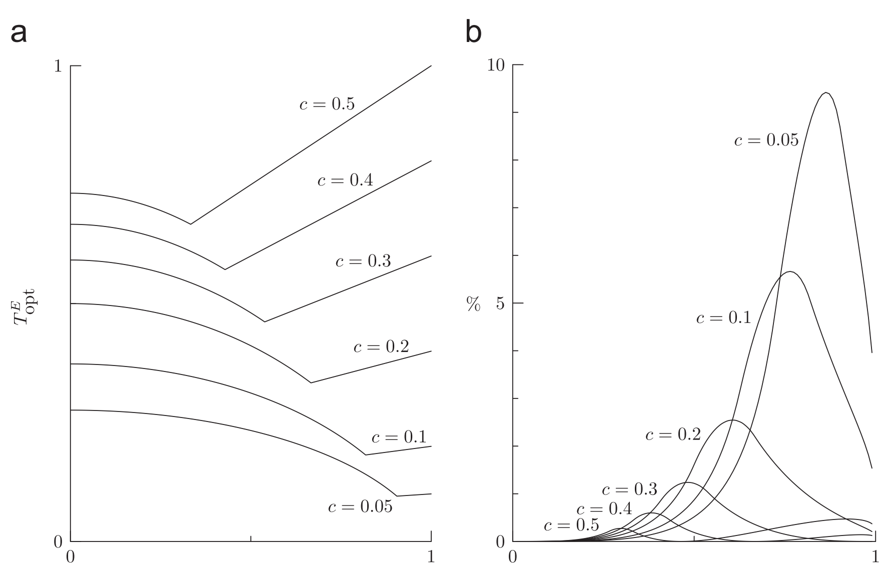

Fig. 2. The optimal preventive maintenance age T^E_{opt} under uncertainty in the right end point s of the uniform distribution (a) and the percentage increase in expected cost if this uncertainty is ignored (b).

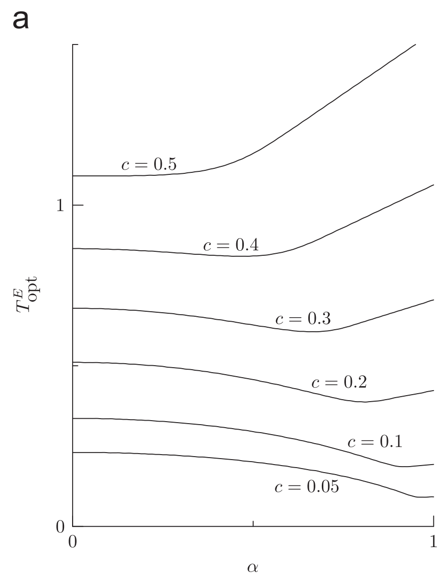

Fig. 3. The optimal preventive maintenance age T^E_{opt} under uncertainty in the scale parameter λ of the Weibull distribution. (a) $k=2$, (b) $k=5$.

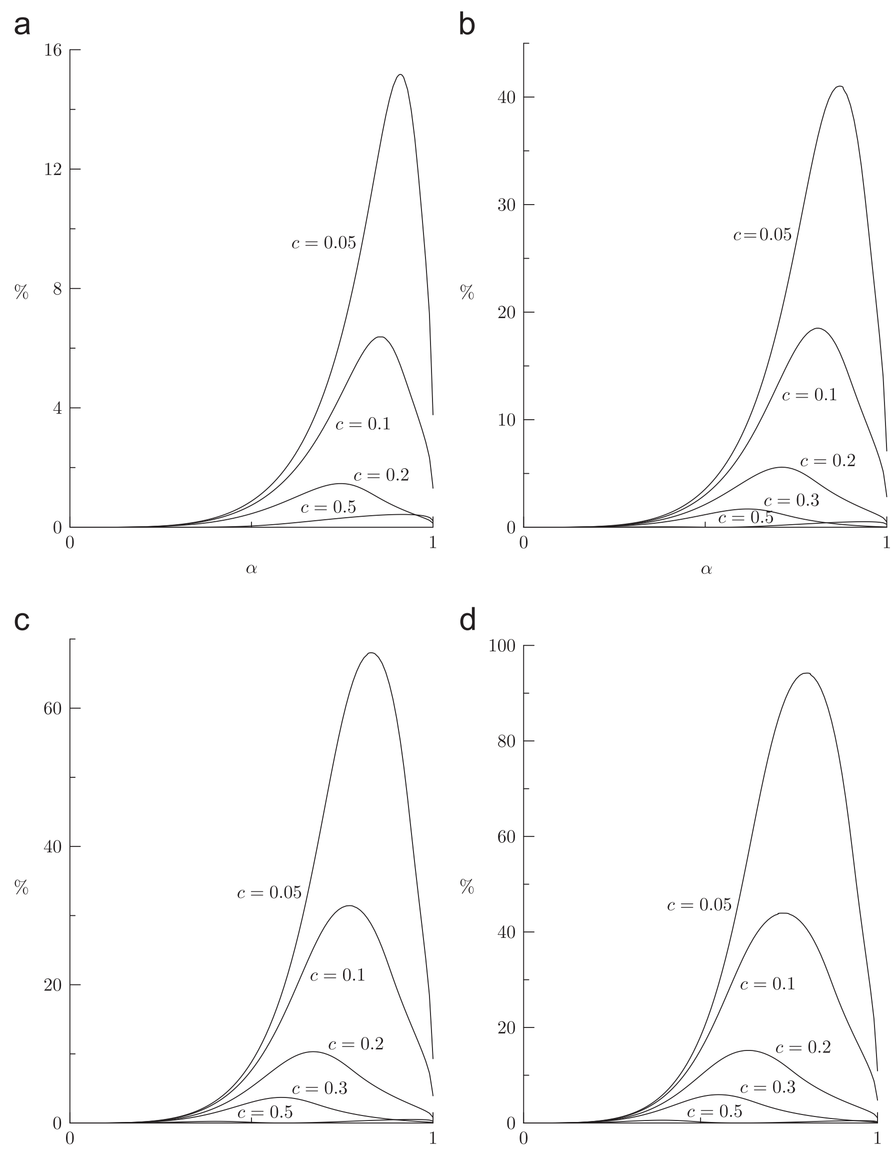

Fig. 4. The percentage increase in expected cost if the uncertainty in the scale parameter λ of the Weibull distribution is ignored. (a) $k=2$, (b) $k=3$, (c) $k=4$, (d) $k=5$.

an additional location parameter, corresponding to a failure-free period, is more appropriate. Because preventive maintenance will obviously not be performed during this failure-free period, we set the location parameter to zero as we also did with the minimum lifetime of the uniform distribution in the previous section.

Weibull experts generally agree that the value of the shape parameter k tends to be reasonably constant for specific or generic failure modes [1], and that it is much more complicated to give a reasonable estimate for the value of the scale parameter λ [42]. Examples of failure modes for which the value of k is predictable given by Abernethy [1] include failures of solid state electronics ($k=0.75$), random failures ($k=1$), roller bearing failures ($k=1.5$), ball bearing failures ($k=2$), V-belts ($k=2.5$), and stress corrosion ($k=5$). Furthermore, for some classes of products, such as vacuum tubes [18], an appropriate value of k may be known from previous test experience [37]. Therefore, in Weibull reliability analysis, it is frequently the case that the value of the shape parameter is known, but that there is uncertainty in the scale parameter [32]. Other studies that assume a fixed shape and random scale parameter include Kwon [21], Papadopoulos and Tsokos [30], and Canavos and Tsokos [9]. In line with these studies, we also assume a known shape parameter and an uncertain scale parameter. We remark that this case has similarities with the uniformly distributed lifetime with uncertainty in its right end point studied in the previous section. In fact, changing the right end point of a uniform distribution is equivalent to changing the scale of this distribution. Furthermore, we will only consider values for the shape parameter k that are greater than 1. A Weibull distribution with a shape parameter smaller than 1 has a decreasing failure rate, implying that preventive maintenance will never be beneficial.

The distribution function of the Weibull distribution is given by

$$F(t;\lambda,k)=1-\exp\left(-(t/\lambda)^k\right),\quad t\geq 0.$$

We will model the uncertainty in λ using a uniform distribution. We will again rescale this distribution such that it is centered around 1. The function $g(s)$ in (2) is therefore again equal to

$$g(s)=\begin{cases}(2\alpha)^{-1}, & s\in[1-\alpha,1+\alpha],\\ 0, & \text{elsewhere},\end{cases}$$

with the level of uncertainty α in the interval $[0,1]$. In this setting the expected cost rate (2), as a function of the maintenance age T, equals

$$\eta^E_{\text{age}}(T)=(2\alpha)^{-1}\int_{1-\alpha}^{1+\alpha}\frac{1-(1-c)\exp(-(T/\lambda)^k)}{\int_0^T\exp(-(x/\lambda)^k)\,dx}\,d\lambda.$$

The value of T that minimizes this function is the optimal preventive maintenance age T^E_{opt}. Because no explicit solution of the integral of the distribution function of the Weibull distribution exists, we have to rely on a numerical analysis of the expected mean cost per unit time. An immediate and inevitable consequence is that all results in this section are approximations.

Fig. 3 shows the optimal maintenance age as a function of the level of uncertainty α for $k=2$ (a) and for $k=5$ (b). We see a similar pattern as in the case with a uniformly distributed lifetime. The optimal maintenance age is first decreasing if the uncertainty increases. However, if the level of uncertainty exceeds some threshold, the optimal maintenance age starts to increase. Both effects become stronger if the value of the shape parameter k increases.

The percentage increase in expected cost if the uncertainty is not taken into account for $k=2$, 3, 4, and 5 is shown in Fig. 4. For all considered values of k, this cost increase is in general small for low levels of uncertainty, but substantial for higher levels of uncertainty. The biggest losses from ignoring uncertainty are related to a too high failure risk, and this is most pronounced if the failure cost is relatively large, i.e., if the cost c of preventive maintenance is small. Fig. 4 only includes the cost increases for values of c of at least 0.05. In practice, the relative cost of preventive maintenance might even be lower, resulting in even higher cost increases. See for example the case study presented by Laggoune et al. [22] where c ranges approximately from 0.005 to 0.025.

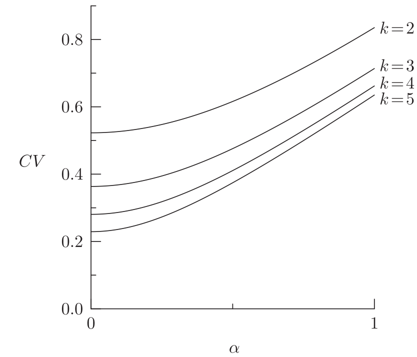

Fig. 5. Coefficient of variation CV of the composite distribution of the Weibull time until failure for various values of the shape parameter k and the level of uncertainty α.

An increase in the shape parameter k implies less variance in the lifetime for a given scale parameter λ and therefore a relatively larger impact of uncertainty in the scale parameter. Fig. 5 shows the coefficient of variation CV of the Weibull distribution for various values of the fixed shape parameter k and for the scale parameter λ uniformly distributed on the interval $[1-\alpha,1+\alpha]$. A derivation is given in Appendix B. Higher levels of uncertainty α obviously result in a higher CV. However, this effect is much stronger for higher values of k. A Weibull distribution with a low shape parameter k already has a relatively high CV without uncertainty in λ.

6. Conclusions and future extensions

We have studied the optimal age-based maintenance strategy under uncertainty in the lifetime distribution of a unit. Although this uncertainty is usually present in practice, it is ignored by most existing research and software. The lifetime distribution is often assumed to be known, or an estimate based on available data is considered as the true lifetime distribution. We considered certain lifetime distributions and studied the effect of parameter uncertainty on the optimal preventive maintenance age. Furthermore, we investigated the expected cost saving from taking the uncertainty into account.

We started our analysis with a setting that can be evaluated algebraically, namely that of a uniform lifetime distribution with uncertainty in its right end point. After that, we considered the commonly used Weibull lifetime distribution, which is appropriate for modeling lifetimes of a wide variety of units and systems. The value of the shape parameter of the Weibull distribution can often be determined accurately based on the failure mode of the unit. It is, on the other hand, very common that there is considerable uncertainty in the value of the scale parameter. The level of this uncertainty depends on the amount of failure data that is available and on the perceived quality of the information provided by the original equipment manufacturer, and so we considered uncertainty in the value of the scale parameter.

The effect of parameter uncertainty on the optimal maintenance age reveals a similar pattern for both the uniform and the Weibull lifetime distribution, and for various relative costs of performing preventive maintenance (as a fraction of the cost of a breakdown). The optimal maintenance age first decreases as the level of uncertainty in the considered parameter increases. So, more uncertainty in the lifetime distribution implies a higher maintenance frequency. However, less intuitive, if the level of uncertainty exceeds a certain threshold, the optimal maintenance age starts to increase as the uncertainty increases. At some point, the increased probability of a large remaining lifetime starts to outweigh the risk of an imminent failure.

We have outlined in what cases the uncertainty in the lifetime distribution is most relevant when determining a maintenance strategy. Firstly, the additional cost of ignoring the uncertainty is highest for low preventive maintenance costs. In such situations where failures are relatively costly, ignoring uncertainty leads to a maintenance frequency below optimal, and therefore yields too many costly failures.

Secondly, the additional cost that results from neglecting uncertainty is highest for lifetime distributions with a low variance. In this case, without parameter uncertainty, the distribution of the time until failure is concentrated around its mean time between failures (MTBF), implying that maintenance can be planned very effectively (i.e., just before failure). However, parameter uncertainty increases the variance of the lifetime distribution, requiring a more conservative strategy. This effect also occurs if the lifetime distribution itself already has a high variance, but the additional variance caused by parameter uncertainty then has a less significant impact on the variance and costs.

So, parameter uncertainty often implies that maintenance should be performed more frequently, especially if the parameter uncertainty constitutes a large fraction of the total uncertainty and if preventive maintenance costs are low. The more frequent maintenance actions ensure that costly failures are unlikely to occur. An immediate consequence, however, is that hardly any failure data will be collected, sustaining the uncertainty. If one aims to reduce this uncertainty, maintenance actions should be postponed or, if this is undesirable, tests should be performed in a controlled setting. The latter is, for example, done in full-scale tests on aircraft structures, which are performed before a new aircraft type enters service. Future research could study more dynamic policies that first 'allow' some failures to occur in order to obtain more information on the lifetime distribution, and then increase the maintenance frequency to ensure the required reliability levels.

Another research avenue is to study situations with uncertainty in both the scale and the shape parameter of the lifetime distribution. Although uncertainty in the scale parameter, studied in this paper, is more common, there may also be uncertainty in the shape parameter if, e.g., the failure mode is not known or if a system with multiple competing failure modes is considered.

One step further is to assume that the distribution family itself is also not known with certainty, and to study the effect of this uncertainty. Potential difficulties for this type of research are the selection of candidates for the true distribution family and the corresponding prior probabilities. Finally, we considered the expected cost rate as the optimality criterion. This leads to the best decisions averaged over all possible outcomes, but these decisions might be perceived as unacceptable for certain values of the unknown parameter. If this is the case, one could consider other optimality criteria related to the robustness as well.

Appendix A. Calculations for the uniform distribution with uniformly distributed right end parameter

For a uniform lifetime distribution on the interval $[0, s]$ with s uniformly distributed on the interval $[1-\alpha, 1+\alpha]$, the expected mean cost per unit time equals

$$\eta^E_{\text{age}}(T) = \frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha} \frac{F(T;s)+c(1-F(T;s))}{\int_0^T (1-F(x;s))\,dx}\,ds.$$

For $T \le 1-\alpha$, this is equal to

$$\begin{aligned}
\eta^E_{\text{age}}(T) &= \frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha} \frac{T(1-c)+cs}{Ts-\frac{1}{2}T^2}\,ds \\
&= \frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha} \frac{c}{T}\frac{T^2c^{-1}-\frac{1}{2}T^2+Ts-\frac{1}{2}T^2}{Ts-\frac{1}{2}T^2}\,ds \\
&= \frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha} \left(\frac{c}{T}\frac{T^2c^{-1}-\frac{1}{2}T^2}{Ts-\frac{1}{2}T^2}+\frac{c}{T}\right) ds \\
&= \frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha} \left(\frac{1-\frac{1}{2}c}{s-\frac{1}{2}T}+\frac{c}{T}\right) ds \\
&= \frac{1}{2\alpha}\Big|_{1-\alpha}^{1+\alpha}\left(\left(1-\frac{1}{2}c\right)\ln\left(s-\frac{1}{2}T\right)+\frac{c}{T}s\right) \\
&= \frac{1}{2\alpha}\left(1-\frac{1}{2}c\right)\ln\left(\frac{2+2\alpha-T}{2-2\alpha-T}\right)+\frac{c}{T}.
\end{aligned}$$

Likewise, for $1-\alpha \le T \le 1+\alpha$, it is equal to

$$\begin{aligned}
\eta^E_{\text{age}}(T) &= \frac{1}{2\alpha}\int_{1-\alpha}^{T} \frac{1}{\frac{1}{2}s}\,ds + \frac{1}{2\alpha}\int_{T}^{1+\alpha} \frac{T(1-c)+cs}{Ts-\frac{1}{2}T^2}\,ds \\
&= \frac{1}{2\alpha}\Big|_{1-\alpha}^{T} 2\ln s + \frac{1}{2\alpha}\Big|_{T}^{1+\alpha}\left((1-\tfrac{1}{2}c)\ln\left(s-\tfrac{1}{2}T\right)+\frac{c}{T}s\right) \\
&= \frac{1}{\alpha}\ln\frac{T}{1-\alpha}+\frac{1}{2\alpha}\left(1-\frac{1}{2}c\right)\ln\left(\frac{2+2\alpha-T}{T}\right)+\frac{c}{2\alpha}\left(\frac{1+\alpha}{T}-1\right).
\end{aligned}$$

We will first analyze the behavior of $\eta^E_{\text{age}}(T)$ for $T \le 1-\alpha$. In this case, the derivative of the mean cost per unit time equals

$$\begin{aligned}
\frac{d\eta^E_{\text{age}}(T)}{dT} &= \frac{1}{2\alpha}\left(1-\frac{1}{2}c\right)\left(\frac{1}{2-2\alpha-T}-\frac{1}{2+2\alpha-T}\right)-\frac{c}{T^2} \\
&= \frac{(2-c)T^2-c(2-2\alpha-T)(2+2\alpha-T)}{(2-2\alpha-T)(2+2\alpha-T)T^2} \\
&= \frac{2\left[(1-c)T^2+2cT+2c(\alpha^2-1)\right]}{(2-2\alpha-T)(2+2\alpha-T)T^2}.
\end{aligned}$$

The denominator of this derivative is always positive, and its sign is therefore determined by the sign of the numerator, which is a polynomial in T. Because the coefficient of T^2 is positive, the value of this polynomial is negative in between its roots, and positive elsewhere. Using the quadratic formula, it turns out that its roots are

$$\begin{aligned}
T_{1,2} &= \frac{-2c \pm \sqrt{4c^2-4(1-c)\cdot 2c(\alpha^2-1)}}{2(1-c)} \\
&= \frac{-c \pm \sqrt{c(2-c)-2\alpha^2c(1-c)}}{1-c}.
\end{aligned}$$

The first root (with the $-$ sign) is always negative. Only the second root (with the $+$ sign) could possibly be contained in the considered interval $[0, 1-\alpha]$. This root is nonnegative if

$$c(2-c)-2\alpha^2c(1-c) > c^2,$$

which is equivalent to $\alpha^2 \le 1$. As this is always satisfied, the second root is always nonnegative. The second root is smaller than or equal to $1-\alpha$ if

$$\sqrt{c(2-c)-2\alpha^2c(1-c)} \le (1-\alpha)(1-c)+c,$$

i.e., if

$$c(2-c)-2\alpha^2c(1-c)\le(1-\alpha(1-c))^2.$$

This can be rewritten as

$$(c^2-1)\alpha^2-2(c-1)\alpha-(c^2-2c+1)\le 0.$$

By again applying the quadratic formula, this inequality turns out to be satisfied if

$$\alpha\le\frac{1-c}{1+c}.$$

Thus, if $\alpha\le(1-c)(1+c)^{-1}$, the expected mean cost per unit time $\eta_{\text{age}}^E(T)$ is decreasing for

$$T\in(0,\frac{-c+\sqrt{c(2-c)-2\alpha^2c(1-c)}}{1-c}]$$

and is increasing for

$$T\in[\frac{-c+\sqrt{c(2-c)-2\alpha^2c(1-c)}}{1-c},1-\alpha];$$

and otherwise $\eta_{\text{age}}^E(T)$ is strictly decreasing for $T\in(0,1-\alpha]$.

We will continue with the behavior of $\eta_{\text{age}}^E(T)$ for $1-\alpha\le T\le 1+\alpha$. The derivative of the mean cost per unit time is now equal to

$$\begin{aligned}\frac{\mathrm{d}\eta_{\text{age}}^E(T)}{\mathrm{d}T}&=\frac{1}{\alpha T}-\frac{1-\frac{1}{2}c}{2\alpha(2+2\alpha-T)}-\frac{1-\frac{1}{2}c}{2\alpha T}-\frac{c(1+\alpha)}{2\alpha T^2}\\&=\frac{2\,T(2+2\alpha-T)-(1-\frac{1}{2}c)T^2-(1-\frac{1}{2}c)T(2+2\alpha-T)-c(1+\alpha)(2+2\alpha-T)}{2\alpha T^2(2+2\alpha-T)}\\&=\frac{-2\,T^2+2(\alpha+1)(c+1)T-2c(\alpha+1)^2}{2\alpha T^2(2+2\alpha-T)}.\end{aligned}$$

The denominator of this derivative is also always positive, and its sign is therefore also determined by its numerator, which is also a polynomial in T. The coefficient of T^2 is again negative, implying that its value is negative in between its roots, and positive elsewhere. The two roots of this polynomial are

$$\begin{aligned}T_{1,2}&=\frac{-2(\alpha+1)(c+1)\pm\sqrt{4(\alpha+1)^2(c+1)^2-16c(\alpha+1)^2}}{-4}\\&=c(1+\alpha)\quad\text{and}\quad 1+\alpha.\end{aligned}$$

The first root is contained in the considered interval $[1-\alpha,1+\alpha]$ if

$$\alpha\ge\frac{1-c}{1+c},$$

the second root always coincides with the end of this interval. Thus, if $\alpha\ge(1-c)/(1+c)$, the expected mean cost per unit time $\eta_{\text{age}}^E(T)$ is decreasing for $T\in[1-\alpha,c(1+\alpha)]$ and is increasing for $T\in[c(1+\alpha),1+\alpha]$; and otherwise $\eta_{\text{age}}^E(T)$ is strictly increasing for $T\in[1-\alpha,1+\alpha]$.

Summarizing the above, the expected mean cost per unit time $\eta_{\text{age}}^E(T)$ always has a unique minimum. The value of T at which this minimum is attained is the optimal maintenance age T_{opt}^E. Two cases can be distinguished. If

$$\alpha\le\frac{1-c}{1+c},$$

the optimal maintenance age equals

$$T_{\text{opt}}^E=\frac{-c+\sqrt{c(2-c)-2\alpha^2c(1-c)}}{1-c},$$

and if

$$\alpha\ge\frac{1-c}{1+c},$$

the optimal maintenance age equals

$$T_{\text{opt}}^E=c(1+\alpha).$$

Appendix B. The coefficient of variation of the Weibull distribution with uniformly distributed scale parameter

If the lifetime X follows a Weibull distribution with known shape parameter k and a scale parameter λ that is uniformly distributed on the interval $[1-\alpha,1+\alpha]$, $\alpha\in[0,1]$, then the density function f_X of the composite distribution of X equals

$$f_X(t)=\frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha}f(t;\lambda,k)\,\mathrm{d}\lambda,$$

where $f(t;\lambda,k)$ denotes the density function of the Weibull distribution:

$$f(t;\lambda,k)=\frac{k}{\lambda}\left(\frac{t}{\lambda}\right)^{k-1}\mathrm{e}^{-(t/\lambda)^k}.$$

The variance of X equals

$$\begin{aligned}\sigma^2&=\int_0^\infty(x-\mu)^2\frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha}f(x;\lambda,k)\,\mathrm{d}\lambda\,\mathrm{d}x\\&=\frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha}\int_0^\infty(x-\mu)^2f(x;\lambda,k)\,\mathrm{d}x\,\mathrm{d}\lambda\\&=\frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha}\left[\int_0^\infty x^2f(x;\lambda,k)\,\mathrm{d}x-2\mu\int_0^\infty xf(x;\lambda,k)\,\mathrm{d}x\right.\\&\quad\left.+\mu^2\int_0^\infty f(x;\lambda,k)\,\mathrm{d}x\right]\mathrm{d}\lambda.\end{aligned}$$

Since λ has mean 1, the mean μ of X equals the mean of the Weibull distribution with shape parameter 1, i.e., $\mu=\Gamma(1+1/k)$. The first moment of the Weibull distribution equals $\int_0^\infty xf(x;\lambda,k)\,\mathrm{d}x=\lambda\Gamma(1+1/k)$, and the second moment equals $\int_0^\infty x^2f(x;\lambda,k)\,\mathrm{d}x=\lambda^2\Gamma(1+2/k)$. It follows that

$$\begin{aligned}\sigma^2&=\frac{1}{2\alpha}\int_{1-\alpha}^{1+\alpha}\left[\lambda^2\Gamma(1+2/k)-2\lambda(\Gamma(1+1/k))^2+(\Gamma(1+1/k))^2\right]\mathrm{d}\lambda\\&=\frac{1}{2\alpha}\Big|_{1-\alpha}^{1+\alpha}\left[\frac{1}{3}\lambda^3\Gamma(1+2/k)-\lambda^2(\Gamma(1+1/k))^2+\lambda(\Gamma(1+1/k))^2\right]\\&=\left(1+\frac{1}{3}\alpha^2\right)\Gamma(1+2/k)-(\Gamma(1+1/k))^2.\end{aligned}$$

The coefficient of variation CV of X equals

$$CV=\sqrt{\frac{\sigma^2}{\mu^2}}=\sqrt{\left(1+\frac{1}{3}\alpha^2\right)\frac{\Gamma(1+2/k)}{(\Gamma(1+1/k))^2}-1}.$$

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