Model reduction for controller design for infinite-dimensional systems

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Appendix A

Hardy spaces

In this appendix we give some basic definitions and results on Hardy spaces that are needed in this thesis.

Definition A.1. Let $\mathscr{H}$ be a Hilbert space. The Hardy space $H^2(\mathbb{D};\mathscr{H})$ is defined as follows: $\mathsf{F} \in H^2(\mathbb{D};\mathscr{H})$ if $\mathsf{F} : \mathbb{D} \to \mathscr{H}$ is holomorphic and

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \|\mathsf{F}(re^{i\theta})\|^2_{\mathscr{H}}\, d\theta < \infty. \tag{A.1}$$

Lemma A.2. *1. The space $H^2(\mathbb{D};\mathscr{H})$ is a Hilbert space with as norm the square root of the expression on the left-hand side of* (A.1).

2. There is an isometric isomorphism between $l^2(\mathbb{Z}^+;\mathscr{H})$ and $H^2(\mathbb{D};\mathscr{H})$ given by $(a_n)_{n\in\mathbb{Z}^+} \mapsto \sum_{n=0}^{\infty} a_n z^n$.

Definition A.3. Let $\mathscr{B}$ be a Banach space. The Hardy space $H^\infty(\mathbb{D};\mathscr{B})$ is defined as follows: $\mathsf{F} \in H^\infty(\mathbb{D};\mathscr{B})$ if $\mathsf{F} : \mathbb{D} \to \mathscr{B}$ is holomorphic and

$$\sup_{|z|<1} \|\mathsf{F}(z)\|_{\mathscr{B}} < \infty. \tag{A.2}$$

Lemma A.4. *1. The space $H^\infty(\mathbb{D};\mathscr{B})$ is a Banach space with norm the expression on the left-hand side of* (A.2).

2. There is an isometric isomorphism between $H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U},\mathscr{Y}))$ and the set of bounded linear maps from $H^2(\mathbb{D};\mathscr{U})$ to $H^2(\mathbb{D};\mathscr{Y})$ that commute with multiplication by z. The latter are all of the form $h \mapsto \mathsf{F}h$ with $\mathsf{F} \in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U},\mathscr{Y}))$.

For $\delta \in (0,1)$ and $\zeta \in \mathbb{T}$ define Ω^δ_ζ as the convex hull of $\{z \in \mathbb{C} : |z| \leq \delta\} \cup \{\zeta\}$. The sets Ω^δ_ζ are used to define the following **nontangential limits**.

Lemma A.5. *If* $\mathsf{F} \in H^2(\mathbb{D}, \mathscr{H})$ *or* $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*, then for almost all* $\zeta \in \mathbb{T}$ *and all* $\delta \in (0,1)$

$$\lim_{z \to \zeta, z \in \Omega_\zeta^\delta} \mathsf{F}(z)$$

exists and is independent of δ*. The limit is taken in the strong topology in the case that* $\mathsf{F} \in H^2(\mathbb{D}, \mathscr{H})$ *and in the strong operator topology in the case that* $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*.*

Proof. This follows from Rosenblum and Rovnyak [84, Theorem 4.6.A]. □

In the cases considered in Lemma A.5 a **boundary function** F_b of F is defined almost everywhere on $\mathbb{T}$ by

$$\mathsf{F}_b(\zeta) := \lim_{z \to \zeta, z \in \Omega_\zeta^\delta} \mathsf{F}(z).$$

Definition A.6. Let $\mathscr{H}$ be a Hilbert space. The Hardy space $H^2(\mathbb{T}; \mathscr{H})$ is defined as follows: $\mathsf{F} \in H^2(\mathbb{T}; \mathscr{H})$ if $\mathsf{F} \in L^2(\mathbb{T}, \mathscr{H})$ and for all integers $n \geq 1$

$$\int_{\mathbb{T}} \mathsf{F}(\zeta)\zeta^n d\zeta = 0.$$

Definition A.7. The Hardy space $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is defined as follows: $\mathsf{F} \in H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ if $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ and for all integers $n \geq 1$

$$\int_{\mathbb{T}} \mathsf{F}(\zeta)\zeta^n d\zeta = 0.$$

Lemma A.8. *The Hardy space* $H^2(\mathbb{T}; \mathscr{H})$ *is a closed subspace of* $L^2(\mathbb{T}, \mathscr{H})$*. It follows that with the induced norm it is a Hilbert space.*

Lemma A.9. *The Hardy space* $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ *is a closed subspace of* $L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*. It follows that with the induced norm it is a Banach space.*

Lemma A.10. *The boundary function of a* $H^2(\mathbb{D}; \mathscr{H})$ *function is an element of* $H^2(\mathbb{T}; \mathscr{H})$*. This mapping is a unitary operator between these two Hardy spaces.*

Lemma A.11. *The boundary function of a* $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ *function is in* $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*. This mapping is an isometric operator onto the space* $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*.*

Due to Lemmas A.10 and A.11 we do not have to be careful about the distinction between Hardy spaces on the unit disc and on the unit circle.

Lemma A.12. *The space* $l^2(\mathbb{Z}, \mathscr{H})$ *is isometrically isomorphic to the space* $L^2(\mathbb{T}, \mathscr{H})$*. This isomorphism is given by*

$$(a_n)_{n\in\mathbb{Z}} \mapsto \sum_{n=-\infty}^{\infty} a_n \zeta^n.$$

The subspace $l^2(\mathbb{Z}^+, \mathscr{H})$ *is mapped onto* $H^2(\mathbb{T}, \mathscr{H})$*.*

The transformation from Lemma A.12 is called the **Z-transform**.

The space $H^\infty(\mathbb{D}^+; \mathscr{B})$ can be defined analogously to $H^\infty(\mathbb{D}; \mathscr{B})$. Here $\mathbb{D}^+$ is the (open) exterior of the closed unit disc. Define for $\mathsf{F} : \mathbb{D} \to \mathscr{B}$ the function $\mathsf{F}^- : \mathbb{D}^+ \to \mathscr{B}$ is by $\mathsf{F}^-(z) := \mathsf{F}(1/z)$. It is easily seen that $\mathsf{F} \mapsto \mathsf{F}^-$ is an isometric isomorphism from $H^\infty(\mathbb{D}; \mathscr{B})$ onto $H^\infty(\mathbb{D}^+; \mathscr{B})$. Functions in $H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ also have nontangential limits almost everywhere and we obtain a norm-preserving injection $H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y})) \to L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. Similar comments apply to the Hardy space $H^2(\mathbb{D}^+, \mathscr{H})$.

Lemma A.13. *If* $\mathsf{F} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y})) \cap H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*, then* F *is constant.*

Proof. Define for $u \in \mathscr{U}$ the function $\mathsf{F}_u : \mathbb{D} \to \mathscr{Y}$ by $\mathsf{F}_u(z) := \mathsf{F}(z)u$. It easily follows that $\mathsf{F}_u \in H^2(\mathbb{D}, \mathscr{Y}) \cap H^2(\mathbb{D}^+, \mathscr{Y})$. Using Lemma A.12 we can write

$$\mathsf{F}(z)u = \sum_{k=-\infty}^{\infty} a_k(u) z^k.$$

Since $\mathsf{F}_u \in H^2(\mathbb{D}, \mathscr{Y})$ it follows that $a_k(u) = 0$ for $k < 0$ for all $u \in \mathscr{U}$. From $\mathsf{F}_u \in H^2(\mathbb{D}^+, \mathscr{Y})$ we obtain that $a_k(u) = 0$ for $k > 0$ for all $u \in \mathscr{U}$. We conclude that

$$\mathsf{F}(z)u = a_0(u).$$

Hence F is a constant operator. □

Corollary A.14. *If* $\mathsf{H} \in L^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ *is the boundary function of both a function* H_1 *in* $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ *and a function* H_2 *in* $H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*, then* H *is constant.*

Proof. Define the function $\mathsf{F} : \mathbb{D} \cap \mathbb{D}^+ \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ by $\mathsf{F}(z) := \mathsf{H}_1(z)$ if $z \in \mathbb{D}$ and $\mathsf{F}(z) := \mathsf{H}_2(z)$ if $z \in \mathbb{D}^+$. Apply Lemma A.13 to this function. We conclude that F is constant. It follows that H_1 is constant. The boundary function of a constant function is obvously constant, so H is constant. □

Definition A.15. For $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ we define the operator $L_\mathsf{F} : L^2(\mathbb{T}, \mathscr{U}) \to L^2(\mathbb{T}, \mathscr{Y})$ by $L_\mathsf{F}\mathsf{H} = \mathsf{F}\mathsf{H}$. This operator is called the **Laurent operator** of F.

Definition A.16. For $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ we define the operator $T_\mathsf{F} : H^2(\mathbb{T}, \mathscr{U}) \to H^2(\mathbb{T}, \mathscr{Y})$ by $T_\mathsf{F} = P_{H^2(\mathbb{T},\mathscr{Y})} L_\mathsf{F}|_{H^2(\mathbb{T},\mathscr{U})}$. This operator is called the **Toeplitz operator** of F.

Note that we can identify the Toeplitz operator T_F with an operator from $H^2(\mathbb{D}, \mathscr{U})$ to $H^2(\mathbb{D}, \mathscr{Y})$ using the identification of functions on the unit disc with their boundary functions discussed earlier.

Lemma A.17. *We have $L_\mathsf{F}^* = L_{\mathsf{F}^*}$ and $T_\mathsf{F}^* = T_{\mathsf{F}^*}$, where the function $\mathsf{F}^* \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$ is defined by $\mathsf{F}^*(z) = \mathsf{F}(z)^*$.*

Lemma A.18. *Let $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. The Laurent operator L_F is isometric if and only if $\mathsf{F}(\zeta)$ is isometric for almost all $\zeta \in \mathbb{T}$.*

Definition A.19. A function $\mathsf{F} \in H^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is called **inner** if its Laurent operator L_F is an isometry.

Lemma A.20. *Let $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. The Toeplitz operator T_F is isometric if and only if the Laurent operator L_F is.*

The following is known as Sarason's theorem.

Lemma A.21. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces, $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ an inner function and $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2))$. Let $\operatorname{Im}(T_\mathsf{H})$ denote the image of the Toeplitz operator of H, $\operatorname{Im}(T_\mathsf{H})^\perp$ its orthogonal complement in $H^2(\mathbb{D}, \mathscr{H}_2)$ and $P_{\operatorname{Im}(T_\mathsf{H})^\perp} \in \mathcal{L}(H^2(\mathbb{D}, \mathscr{H}_2))$ the orthogonal projection onto this ortogonal complement. Then*

$$\|P_{\operatorname{Im}(T_\mathsf{H})^\perp} T_\mathsf{F}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2))}.$$

Proof. See for example Nikol′skiĭ [59, p191]. □

We need the following two corollaries of Sarason's theorem.

Corollary A.22. *Let $\mathscr{U}$ and $\mathscr{Y}$ be Hilbert spaces, $\mathsf{H}, \mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ with H an inner function. Then*

$$\|P_{\operatorname{Im}(T_\mathsf{H})^\perp} T_\mathsf{F}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \|\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))}.$$

Proof. We apply Lemma A.21 and denote the operators there with tildes to distinguish them from the operators used in the statement of this lemma. Apply Lemma A.21 with $\mathscr{H}_1 = \mathscr{U}$, $\mathscr{H}_2 = \mathscr{U} \times \mathscr{Y}$, $\tilde{\mathsf{H}} = \mathsf{H}$ and $\tilde{\mathsf{F}} = [\mathsf{F}, 0]$. It is easily seen from the form of $\tilde{\mathsf{F}}$ that

$$\|P_{\mathrm{Im}(T_{\mathsf{H}})^\perp} T_{\mathsf{F}}\| = \|P_{\mathrm{Im}(T_{\tilde{\mathsf{H}}})^\perp} T_{\tilde{\mathsf{F}}}\|.$$

The parameter $\tilde{\mathsf{V}}$ from Lemma A.21 can be decomposed as $\tilde{\mathsf{V}} = [\tilde{\mathsf{V}}_1, \tilde{\mathsf{V}}_2]$. We have

$$\begin{aligned}
&\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|\tilde{\mathsf{F}} - \tilde{\mathsf{H}}\tilde{\mathsf{V}}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\
= &\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|[\mathsf{F} - \mathsf{H}\tilde{\mathsf{V}}_1, -\mathsf{H}\tilde{\mathsf{V}}_2]\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\
= &\inf_{V \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \|[\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))},
\end{aligned}$$

where we have used that the infimum over $\tilde{\mathsf{V}}_2$ is reached for $\tilde{\mathsf{V}}_2 = 0$. From Lemma A.21 we now obtain the desired equality. □

Corollary A.23. *Let $\mathscr{U}$ and $\mathscr{Y}$ be Hilbert spaces, $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ an inner function and $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U} \times \mathscr{Y}))$. Then*

$$\|P_{\mathrm{Im}(T_{\mathsf{H}})^\perp} T_{\mathsf{F}}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))} \|\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U} \times \mathscr{Y}))}.$$

Proof. We apply Lemma A.21 and denote the operators there with tildes to distinguish them from the operators used in the statement of this lemma. Apply Lemma A.21 with $\mathscr{H}_1 = \mathscr{U}$, $\mathscr{H}_2 = \mathscr{U} \times \mathscr{Y}$, $\tilde{\mathsf{H}} = \mathsf{H}$ and $\tilde{\mathsf{F}} = [0, \mathsf{F}]$. It is easily seen from the form of $\tilde{\mathsf{F}}$ that

$$\|P_{\mathrm{Im}(T_{\mathsf{H}})^\perp} T_{\mathsf{F}}\| = \|P_{\mathrm{Im}(T_{\tilde{\mathsf{H}}})^\perp} T_{\tilde{\mathsf{F}}}\|.$$

The parameter $\tilde{\mathsf{V}}$ from Lemma A.21 can be decomposed as $\tilde{\mathsf{V}} = [\tilde{\mathsf{V}}_1, \tilde{\mathsf{V}}_2]$. We have

$$\begin{aligned}
&\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|\tilde{\mathsf{F}} - \tilde{\mathsf{H}}\tilde{\mathsf{V}}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\
= &\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|[-\mathsf{H}\tilde{\mathsf{V}}_1, \mathsf{F} - \mathsf{H}\tilde{\mathsf{V}}_2]\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\
= &\inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \|[\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U} \times \mathscr{Y}))},
\end{aligned}$$

where we have used that the infimum over $\tilde{\mathsf{V}}_1$ is reached for $\tilde{\mathsf{V}}_1 = 0$. From Lemma A.21 we now obtain the desired equality. □

Definition A.24. For $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ we define the operator $H_\mathsf{F} : L^2(\mathbb{T}, \mathscr{H}_1) \to L^2(\mathbb{T}, \mathscr{H}_2)$ by $H_\mathsf{F} := P_{H^2(\mathbb{D},\mathscr{H}_2)} L_\mathsf{F} P_{H^2(\mathbb{D},\mathscr{H}_1)^\perp}$. This operator is called the **Hankel operator** of F.

Remark A.25. We warn the reader that in the literature there are several different definitions of the concept of the Hankel operator of a function.

The next result relates the Hankel operator of a $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function with the Hankel map of an input-output stable discrete-time system (see the definition on page 9, Definition 3.1 and Proposition 3.22).

Lemma A.26. *Let Σ be an input-output stable discrete-time system with transfer function D and Hankel map $\mathcal{H}$. Denote the Hankel operator of $\mathsf{D} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ by H_D and the Z-transform from $l^2(\mathbb{Z}, \mathscr{H})$ to $L^2(\mathbb{T}, \mathscr{H})$ by $Z_\mathscr{H}$. Then $H_\mathsf{D} Z_\mathscr{U} = Z_\mathscr{Y} \mathcal{H}$. In particular, $\|H_\mathsf{D}\| = \|\mathcal{H}\|$.*

Proof. This follows easily from the definitions. □

The following result is known as Nehari's theorem (or sometimes Page's theorem).

Lemma A.27. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and $\mathsf{F} \in L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$. Then*

$$\|H_\mathsf{F}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{F}^* - \mathsf{V}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))}.$$

Proof. It is shown in e.g. Peller [75, page 68], Nikol′skiĭ [59, p191] that for $\mathsf{H} \in L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$

$$\|P_{H^2(\mathbb{D},\mathscr{H}_1)^\perp} L_\mathsf{H} P_{H^2(\mathbb{D},\mathscr{H}_2)}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{H} - \mathsf{V}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))}.$$

Applying this with $H = F^*$ gives

$$\|P_{H^2(\mathbb{D},\mathscr{H}_1)^\perp} L_\mathsf{F}^* P_{H^2(\mathbb{D},\mathscr{H}_2)}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{F}^* - \mathsf{V}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))}.$$

The desired result now follows from noting that the left-hand side of this last expression is the norm of the adjoint of the Hankel operator of F and that an operator and its adjoint have the same norm. □

Lemma A.28. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be finite-dimensional Hilbert spaces and $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ a rational function. Then, for each $\sigma > \|H_\mathsf{F}\|$ there exists a rational $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ such that*

$$\|\mathsf{F}^* - \mathsf{H}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))} \leq \sigma.$$

Proof. This follows from the explicit state space formulas given in McFarlane and Glover [54, Appendix B]. □

The second statement in the following result is known as the Corona theorem.

Lemma A.29. *Let* $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$. *If there exists a function* $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ *such that* $\mathsf{H}(s)\mathsf{F}(s) = I$ *for all* $s \in \mathbb{D}$, *then there exists a* $\varepsilon > 0$ *such that for all* $s \in \mathbb{D}$ *and all* $h \in \mathscr{H}_1$

$$\|\mathsf{F}(s)h\|_{\mathscr{H}_2} \geq \varepsilon \|h\|_{\mathscr{H}_1}. \tag{A.3}$$

If $\mathscr{H}_1$ *is finite-dimensional then the converse is also true, i.e., if there exists a* $\varepsilon > 0$ *such that for all* $s \in \mathbb{D}$ *and all* $h \in \mathscr{H}_1$ *we have* (A.3), *then there exists a* $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ *such that* $\mathsf{H}(s)\mathsf{F}(s) = I$ *for all* $s \in \mathbb{D}$.

Notes

General references on Hardy space theory are Duren [23], Garnett [32] and Hoffman [39]. The vector-valued case can be found in Nikol′skiĭ [61], [59], [60], Peller [75] and Rosenblum and Rovnyak [84].

Appendix B

Algebraic Riccati equations

In this appendix we prove some simple algebraic results concerning algebraic Riccati equations.

Lemma B.1. *Let P and Q be nonnegative self-adjoint operators. Define*

$$A_P := A - (BD^* + APC^*)(\underline{R} + CPC^*)^{-1}C, \tag{B.1}$$

$$A_Q := A - B(\underline{S} + B^*QB)^{-1}(D^*C + B^*QA), \tag{B.2}$$

$$\underline{A} := A - B\underline{S}^{-1}D^*C. \tag{B.3}$$

where $\underline{S} := I + D^*D$ and $\underline{R} := I + DD^*$. Then

$$A_P(I + PC^*\underline{R}^{-1}C) = \underline{A} = (I + B\underline{S}^{-1}B^*Q)A_Q, \tag{B.4}$$

$$A_Q = (I + B\underline{S}^{-1}B^*Q)^{-1}A_P(I + PC^*\underline{R}^{-1}C), \tag{B.5}$$

$$A_P = (I + B\underline{S}^{-1}B^*Q)A_Q(I + PC^*\underline{R}^{-1}C)^{-1}. \tag{B.6}$$

Proof. We prove that $A_P(I + PC^*\underline{R}^{-1}C) = \underline{A}$. The equality $\underline{A} = (I + B\underline{S}^{-1}B^*Q)A_Q$ is proved similarly. By writing out A_P in full we have

$$\begin{aligned}
&A_P(I + PC^*\underline{R}^{-1}C) \\
&\quad = A(I + PC^*\underline{R}^{-1}C) - (BD^* + APC^*)(\underline{R} + CPC^*)^{-1}C(I + PC^*\underline{R}^{-1}C) \\
&\quad = A(I + PC^*\underline{R}^{-1}C) - (BD^* + APC^*)(\underline{R} + CPC^*)^{-1}(\underline{R} + CPC^*)\underline{R}^{-1}C \\
&\quad = A + APC^*\underline{R}^{-1}C - (BD^* + APC^*)\underline{R}^{-1}C = A - BD^*\underline{R}^{-1}C \\
&\quad = A - B\underline{S}^{-1}D^*C,
\end{aligned}$$

since $D^*\underline{R}^{-1} = \underline{S}^{-1}D^*$. This completes the proof of (B.4). Equations (B.5) and (B.6) easily follow from (B.4). □

Remark B.2. Note that in the above lemma we have not assumed that P and Q are solutions of the Riccati equations.

We now prove that the Riccati equations can be written in several different but equivalent versions.

Lemma B.3. *Let P and Q be nonnegative self-adjoint operators.*

1. *P satisfies*

$$A_P P(I + C^* \underline{R}^{-1} CP)A_P^* - P + B\underline{S}^{-1}B^* = 0, \tag{B.7}$$

where A_P is defined by (B.1)*, if and only if it satisfies*

$$\underline{A}P(I + C^* \underline{R}^{-1} CP)^{-1}\underline{A}^* - P + B\underline{S}^{-1}B^* = 0, \tag{B.8}$$

where $\underline{A}$ is defined by (B.3).

2. *P satisfies* (B.7) *if and only if it satisifies the filter algebraic Riccati equation.*

3. *Q satisfies*

$$A_Q^*(I + QB\underline{S}^{-1}B^*)QA_Q - Q + C^*\underline{R}^{-1}C = 0, \tag{B.9}$$

where A_Q is defined by (B.2)*, if and only if it satisfies*

$$\underline{A}^*Q(I + B\underline{S}^{-1}B^*Q)^{-1}\underline{A} - Q + C^*\underline{R}^{-1}C = 0, \tag{B.10}$$

where $\underline{A}$ is defined by (B.3).

4. *Q satisfies* (B.9) *if and only if it satisifies the control algebraic Riccati equation.*

Proof. We shall prove the equivalence of the filter equations; the equivalence of the control equations is similar.

1. The equations (B.7) and (B.8) are equivalent if and only if the following holds:

$$\underline{A}P(I + C^*\underline{R}^{-1}CP)^{-1}\underline{A}^* = A_P P(I + C^*\underline{R}^{-1}CP)A_P^*. \tag{B.11}$$

We use Lemma B.1 (which tells us that $\underline{A} = A_P(I + PC^*\underline{R}^{-1}C)$) to write the left-hand side of (B.11) as

$$A_P(I + PC^*\underline{R}^{-1}C)P(I + C^*\underline{R}^{-1}CP)^{-1}(I + C^*\underline{R}^{-1}CP)A_P^*,$$

which is indeed equal to the right-hand side of (B.11).

2. To prove the equivalence of (B.7) and the filter algebraic Riccati equation we substitute in (B.7) for A_P from (B.1) and for $(I+C^*\underline{R}^{-1}CP)A_P^*$, we substitute $\underline{A}^*$ (using (B.4)) and then substitute (B.3) for $\underline{A}$. We then get

$$(A-(BD^*+APC^*)(\underline{R}+CPC^*)^{-1}C)P(A^*-C^*D\underline{S}^{-1}B^*)-P+B\underline{S}^{-1}B^*=0.$$

Rewriting this gives

$$\begin{aligned} APA^*-P+BB^* &= (BD^*+APC^*)(\underline{R}+CPC^*)^{-1}CPA^* \\ &\quad -(BD^*+APC^*)(\underline{R}+CPC^*)^{-1}CPC^*D\underline{S}^{-1}B^* \\ &\quad +APC^*D\underline{S}^{-1}B^*-B\underline{S}^{-1}B^*+BB^*. \end{aligned}$$

We now focus on the last two lines of this last equation. We note that $I-\underline{S}^{-1}=D^*D\underline{S}^{-1}$ and we can thus rewrite these last two lines as

$$-(BD^*+APC^*)(\underline{R}+CPC^*)^{-1}CPC^*D\underline{S}^{-1}B^*+APC^*D\underline{S}^{-1}B^*+BD^*D\underline{S}^{-1}B^*,$$

and this can be rewritten as

$$(BD^*+APC^*)(\underline{R}+CPC^*)^{-1}[-CPC^*+\underline{R}+CPC^*]D\underline{S}^{-1}B^*.$$

Noting that $\underline{R}D\underline{S}^{-1}=D$, we see that this is equal to

$$(BD^*+APC^*)(\underline{R}+CPC^*)^{-1}DB^*.$$

This completes the proof of the equivalence of (B.7) and the filter algebraic Riccati equation. □

Lemma B.4. *Assume the discrete-time system* Σ *has a solution* Q *of its control algebraic Riccati equation and* P *be of its filter algebraic Riccati equation. Define* A_P *and* A_Q *by* (B.1) *and* (B.2)*, respectively. Then*

$$(I+PQ)A_Q=A_P(I+PQ). \tag{B.12}$$

Proof. We use the equivalent version of the filter algebraic Riccati equation (B.7) to write

$$P=A_PP(I+C^*\underline{R}^{-1}CP)A_P^*+B\underline{S}^{-1}B^*,$$

which leads to

$$I+PQ=I+A_PP(I+C^*\underline{R}^{-1}CP)A_P^*Q+B\underline{S}^{-1}B^*Q$$

and so

$$(I+PQ)A_Q=(I+B\underline{S}^{-1}B^*Q)A_Q+A_PP(I+C^*\underline{R}^{-1}CP)A_P^*QA_Q.$$

We use (B.5) to write the right-hand side as

$$A_P(I + PC^*\underline{R}^{-1}C) + A_P P(I + C^*\underline{R}^{-1}CP)A_P^*QA_Q.$$

Rearranging gives

$$A_P + A_P P[C^*\underline{R}^{-1}C + (I + C^*\underline{R}^{-1}CP)A_P^*QA_Q],$$

and using (B.5) again we obtain

$$A_P + A_P P[C^*\underline{R}^{-1}C + A_Q^*(I + QB\underline{S}^{-1}B^*)QA_Q].$$

According to the version (B.9) of the control algebraic Riccati equation, the term in square brackets equals Q. So the above is equal to $A_P(I + PQ)$. □

We now prove a relation concerning the difference of two solutions of a Riccati equation.

Lemma B.5. *Assume the discrete-time system Σ has solutions Q_1 and Q_2 of its control algebraic Riccati equation. Define A_{Q_1} and A_{Q_2} similarly to* (B.2)*. Then*

$$Q_1 - Q_2 = A_{Q_2}^*(Q_1 - Q_2)A_{Q_1}.$$

Proof. Subtract the form (B.9) of the control algebraic Riccati equation for Q_1 and Q_2 to obtain

$$Q_1 - Q_2 = A_{Q_1}^*(I + Q_1B\underline{S}^{-1}B^*)Q_1A_{Q_1} - A_{Q_2}^*(I + Q_2B\underline{S}^{-1}B^*)Q_2A_{Q_2}. \quad \text{(B.13)}$$

According to Lemma B.1 (say with $P = I$) we have

$$\begin{aligned} A_{Q_2} &= (I + B\underline{S}^{-1}B^*Q_2)^{-1}A_P(I + PC^*\underline{R}^{-1}C) \\ &= (I + B\underline{S}^{-1}B^*Q_2)^{-1}(I + B\underline{S}^{-1}B^*Q_1)A_{Q_1}. \end{aligned} \quad \text{(B.14)}$$

Combining (B.13) and (B.14) we obtain

$$\begin{aligned} Q_1 - Q_2 &= A_{Q_2}^*(I + Q_2B\underline{S}^{-1}B^*)Q_1A_{Q_1} - A_{Q_2}^*Q_2(I + B\underline{S}^{-1}B^*Q_1)A_{Q_1} \\ &= A_{Q_2}^*(Q_1 - Q_2)A_{Q_1}. \end{aligned}$$

□

Proof of Proposition 6.39. Denote the Riccati closed-loop system associated with the solution Q by Σ_Q. We need to show that

$$A_Q P(I + PQ)^{-1} - P(I + PQ)^{-1} + B_Q B_Q^* = 0.$$

From (B.5) and (B.12) we see that this is equivalent to

$$(I+B\underline{S}^{-1}B^*Q)^{-1}A_P(I+PC^*\underline{R}^{-1}C)PA_P^*(I+QP)^{-1}-P(I+QP)^{-1}+B_QB_Q^*=0. \tag{B.15}$$

It is easily proven that

$$(I+B\underline{S}^{-1}B^*Q)B_QB_Q^*=B\underline{S}^{-1}B^*.$$

Using this we see that (B.15) is equivalent to

$$\begin{aligned} &(I+B\underline{S}^{-1}B^*Q)^{-1} \times \\ &\big[A_P(I+PC^*\underline{R}^{-1}C)PA_P^*-(I+B\underline{S}^{-1}B^*Q)P+B\underline{S}^{-1}B^*(I+QP)\big] \times \\ &(I+QP)^{-1} = 0. \end{aligned}$$

The term in square brackets is zero since it is the equivalent version (B.7) of the filter Riccati equation. □

Lemma B.6. *Suppose that the discrete-time system with system operator* $[\check{A},\check{B};[\check{C}_1;\check{C}_2],[\check{D}_1;\check{D}_2]]$ *is such that* $\check{D}_1$ *has a bounded inverse and that there exists a nonnegative self-adjoint operator* V *such that*

$$\check{B}^*V\check{A}+\check{D}^*\check{C}=0. \tag{B.16}$$

Define A,B,C,D *as in Proposition 2.23,* $\underline{S}:=I+D^*D$ *and* $\underline{R}:=I+DD^*$. *Then*

1. $\check{A}=A-B(\underline{S}+B^*VB)^{-1}(D^*C+B^*VA)$ *and*
2. $\check{A}^*V\check{A}-V+\check{C}^*\check{C}=\check{A}^*(I+VB\underline{S}^{-1}B^*)V\check{A}-V+C^*\underline{R}^{-1}C.$

Proof. We first prove the equality

$$\underline{S}\check{C}_1=-(B^*V\check{A}+D^*C). \tag{B.17}$$

From (B.16) and (2.5) we obtain

$$\check{D}_1^*B^*V\check{A}+\check{D}_1^*\check{C}_1+\check{D}_2^*\check{C}_2=0.$$

Thus

$$\check{C}_1=-B^*V\check{A}-D^*\check{C}_2=-B^*V\check{A}-D^*(C+D\check{C}_1)$$

and this yields (B.17):

$$\underline{S}\check{C}_1=(I+D^*D)\check{C}_1=-B^*V\check{A}-D^*C.$$

We now prove the first equality stated in the lemma. We take the equality just proved (B.17) and substitute $\check{A} = A + B\check{C}_1$ to obtain

$$\underline{S}\check{C}_1 = -(B^*V(A + B\check{C}_1) + D^*C).$$

Thus

$$(\underline{S} + B^*VB)\check{C}_1 = -(B^*VA + D^*C).$$

We now solve for $\check{C}_1$ and substitute to obtain

$$\check{A} = A + B\check{C}_1 = A - B(\underline{S} + B^*VB)^{-1}(B^*VA + D^*C).$$

We now prove the equality

$$\check{C}^*\check{C} = \check{A}^*VB\underline{S}^{-1}B^*V\check{A} + C^*\underline{R}^{-1}C. \tag{B.18}$$

We have

$$\check{C}^*\check{C} = \check{C}_1^*\check{C}_1 + \check{C}_2^*\check{C}_2$$

and substituting for $\check{C}_2$ from (2.5) gives

$$\check{C}^*\check{C} = \check{C}_1^*\check{C}_1 + (C + D\check{C}_1)^*(C + D\check{C}_1).$$

Finally, substituting for $\check{C}_1$ from (B.17) and simplifying gives the result. The second equality stated in the lemma follows easily from (B.18). □

Lemma B.7. *Suppose that the discrete-time system with system operator* $[\check{A}, \check{B}; [\check{C}_1; \check{C}_2], [\check{D}_1; \check{D}_2]]$ *is such that* $\check{D}_1$ *has a bounded inverse and assume that a nonnegative self-adjoint operator* V *exists such that*

$$\check{B}^*V\check{B} + \check{D}^*\check{D} = I.$$

Define A, B, C, D *as in Proposition 2.23. Then we have*

1. $B^*VB + \underline{S} = \check{D}_1^{-*}\check{D}_1^{-1}$ *and*
2. $\check{B}\check{B}^*(I + VB\underline{S}^{-1}B^*) = B\underline{S}^{-1}B^*$.

Proof. 1. The given equation for V translates to

$$\check{D}_1^*B^*VB\check{D}_1 + \check{D}_1^*\check{D}_1 + \check{D}_1^*D^*D\check{D}_1 = I,$$

and multiplying from the left with $\check{D}_1^{-*}$ and from the right with $\check{D}_1^{-1}$ gives the result.

2. The first equality implies that $(\underline{S} + B^*VB)^{-1} = \check{D}_1\check{D}_1^*$ and so $B(\underline{S} + B^*VB)^{-1}B^*$
$= \check{B}\check{B}^*$. Hence

$$\begin{aligned} \check{B}\check{B}^*(I + VB\underline{S}^{-1}B^*) &= B(\underline{S} + B^*VB)^{-1}B^*(I + VB\underline{S}^{-1}B^*) \\ &= B(\underline{S} + B^*VB)^{-1}(\underline{S} + B^*VB)\underline{S}^{-1}B^* = B\underline{S}^{-1}B^*, \end{aligned}$$

which proves the second equality. □

Proof of Lemma 6.46. It easily follows from Proposition 6.45 that $\check{\Sigma}$ is the Riccati closed-loop system of Σ corresponding to the solution of the control Riccati equation $Q := L_c$. This in particular implies that $\check{A} = A_Q$, where A_Q is defined by (B.2). Define $P := L(I - QL)^{-1}$ and define A_P by (B.1). We establish the identity

$$(I - LQ)A_P = \check{A}(I - LQ) \tag{B.19}$$

or by (B.6) the equivalent identity

$$(I - LQ)(I + B\underline{S}^{-1}B^*Q)\check{A} = \check{A}(I - LQ)(I + PC^*\underline{R}^{-1}C). \tag{B.20}$$

Since $P = L(I - QL)^{-1} = (I - LQ)^{-1}L$ the right-hand side of (B.20) is equal to

$$\check{A} - \check{A}LQ + \check{A}LC^*\underline{R}^{-1}C.$$

We substitute $Q - C^*\underline{R}^{-1}C = \check{A}^*(I + QB\underline{S}^{-1}B^*)Q\check{A}$ (this identity holds because Q is a solution of the filter Riccati equation; see (B.9)) to obtain for the right-hand side of (B.20)

$$\check{A} - \check{A}L\check{A}^*(I + QB\underline{S}^{-1}B^*)Q\check{A}.$$

The control Lyapunov equation tells us that $\check{A}L\check{A}^* = L - \check{B}\check{B}^*$ and so the right-hand side of (B.20) is equal to

$$\check{A} - L(I + QB\underline{S}^{-1}B^*)Q\check{A} + \check{B}\check{B}^*(I + QB\underline{S}^{-1}B^*)Q\check{A}.$$

Substituting $\check{B}\check{B}^*(I + QB\underline{S}^{-1}B^*) = B\underline{S}^{-1}B^*$ from Lemma B.7 with $V = Q$ we obtain for the right-hand side of (B.20)

$$\check{A} - L(I + QB\underline{S}^{-1}B^*)Q\check{A} + B\underline{S}^{-1}B^*Q\check{A},$$

which is equal to the left-hand side of (B.20). This proves (B.19). We show that P is a solution of the filter algebraic Riccati equation. We start with the control Lyapunov equation

$$\check{A}L\check{A}^* - L + \check{B}\check{B}^* = 0$$

and substitute $\check{A} = (I - LQ)A_P(I - LQ)^{-1}$ from (B.19) and $\check{A}^* = (I + C^*\underline{R}^{-1}CP)A_P^*(I + QB\underline{S}^{-1}B^*)^{-1}$ from (B.6) to obtain

$$(I-LQ)A_P(I-LQ)^{-1}L(I+C^*\underline{R}^{-1}CP)A_P^*(I+QB\underline{S}^{-1}B^*)^{-1}-L+\check{B}\check{B}^* = 0.$$

We multiply by $(I - LQ)^{-1}$ from the left and by $(I + QB\underline{S}^{-1}B^*)$ from the right to obtain

$$A_PP(I+C^*\underline{R}^{-1}CP)A_P^*-P(I+QB\underline{S}^{-1}B^*)+(I-LQ)^{-1}\check{B}\check{B}^*(I+QB\underline{S}^{-1}B^*)=0.$$

We again use the fact that $\check{B}\check{B}^*(I + QB\underline{S}^{-1}B^*) = B\underline{S}^{-1}B^*$ to obtain

$$A_PP(I+C^*\underline{R}^{-1}CP)A_P^*-P-PQB\underline{S}^{-1}B^*+(I-LQ)^{-1}B\underline{S}^{-1}B^* = 0. \tag{B.21}$$

Using that $P = (I - LQ)^{-1}L$ we see that the sum of the two last terms of the left-hand side of (B.21) equals $B\underline{S}^{-1}B^*$. This proves that P is a solution of the equivalent version (B.7) of the filter algebraic Riccati equation. □

Lemma B.8. *Let Q be a solution of the control algebraic Riccati equation of Σ. Define A_Q by* (B.2)*. Denote by $\overline{\mathbb{D}}$ the closed unit disc. Then the component of $1/\rho(A)\cap\overline{\mathbb{D}}$ that contains zero is contained in the component of $1/\rho(A_Q)\cap\overline{\mathbb{D}}$ that contains zero.*

Proof. We first show that if $\lambda \in \overline{\mathbb{D}}$ is in the approximate point spectrum of A_Q, then it is in the appoximate point spectrum of A.

We first note that for all $x \in \mathscr{X}$ and $\lambda \in \mathbb{C}$

$$\begin{aligned}&||(\lambda I - A)x|| \leq ||(A_Q - A)x|| + ||(\lambda I - A_Q)x|| \leq \\ &||B(\underline{S} + B^*QB)^{-1}||\ (||D^*Cx|| + ||B^*QAx||) + ||(\lambda I - A_Q)x||.\end{aligned} \tag{B.22}$$

Second we note that the control algebraic Riccati equation (it follows most easily from the equivalent version (B.9)) implies that for every $x \in \mathscr{X}$

$$\|\underline{S}^{-1/2}B^*QA_Qx\|^2 + \|\underline{R}^{-1/2}Cx\|^2 = \|Q^{1/2}x\|^2 - \|Q^{1/2}A_Qx\|^2. \tag{B.23}$$

For every λ in the exterior of the open unit disc we have that the right-hand side of (B.23) is smaller than or equal to $|\lambda|^2\ \|Q^{1/2}x\|^2 - \|Q^{1/2}A_Qx\|^2$, which equals

$$-\langle Q(\lambda I - A_Q)x, (\lambda I - A_Q)x\rangle + \lambda\langle Qx, (\lambda I - A_Q)x\rangle + \bar{\lambda}\langle(\lambda I - A_Q)x, Qx\rangle.$$

It is easily computed that $B^*QA_Q = \underline{S}(\underline{S} + B^*QB)^{-1}B^*QA$. It follows that the left-hand side of (B.23) is equal to

$$||\underline{S}^{1/2}(\underline{S} + B^*QB)^{-1}B^*QAx||^2 + ||R^{1/2}Cx||^2.$$

Hence we obtain

$$
\begin{aligned}
&||\underline{S}^{1/2}(\underline{S}+B^*QB)^{-1}B^*QAx\|^2+\|\underline{R}^{-1/2}Cx\|^2 \\
&\le -\|Q^{1/2}(\lambda I-A_Q)x\|^2+\lambda\langle Qx,(\lambda I-A_Q)x\rangle+\bar{\lambda}\langle(\lambda I-A_Q)x,Qx\rangle.
\end{aligned}
\tag{B.24}
$$

Assume that $\lambda \in \overline{\mathbb{D}}$ is in the approximate point spectrum of A_Q. Then there exists an sequence $x_n \in \mathscr{X}$ with $\|x_n\| = 1$ such that $\|(\lambda I - A_Q)x_n\| \to 0$. It follows from (B.24) that $\|B^*QAx_n\| \to 0$ and $\|Cx_n\| \to 0$. It then follows using (B.22) that $\|(\lambda I-A)x_n\| \to 0$. This means that λ is in the approximate point spectrum of A. So we have $\sigma_a(A_Q) \cap \overline{\mathbb{D}^+} \subset \sigma_a(A) \cap \overline{\mathbb{D}^+}$.

Let μ be an element of the component of $1/\rho(A) \cap \overline{\mathbb{D}}$ that contains zero. Then there exists a path l in $1/\rho(A) \cap \overline{\mathbb{D}}$ that has zero and μ as its endpoints. Assume that μ is not an element of the component of $1/\rho(A_Q) \cap \overline{\mathbb{D}}$ that contains zero. Consider the sets

$$
V_\sigma := \{z \in l_\mu : 1/z \in \sigma(A_Q) \cap \overline{\mathbb{D}^+}\}, \quad V_\rho := \{z \in l_\mu : 1/z \in \rho(A_Q) \cap \overline{\mathbb{D}^+}\}.
$$

Since l is contained in the closed unit disc it follows that $l = V_\sigma \cup V_\rho$. It is easily seen that V_σ is closed. Let $p : [0,1] \to l$ be a parametrization of l. We have $p(0) = 0 \in V_\rho$ and $p(1) = \mu \in V_\sigma$. We have $[0,1] = p^{-1}(V_\sigma) \cup p^{-1}(V_\rho)$. Since $p^{-1}(V_\sigma)$ is closed it has a smallest element. Denote this smallest element by $t_{\min}$. Since $0 \in p^{-1}(V_\rho)$ we have $t_{\min} > 0$. Denote $\lambda_{\min} = p(t_{\min})$. It follows by construction that $\lambda_{\min}$ is an element of the boundary of $\sigma(A_Q)$. Since the boundary of the spectrum consists of approximate eigenvalues we have $\lambda_{\min} \in \sigma_a(A_Q) \cap \overline{\mathbb{D}^+}$. It follows, using the above established relation between the approximate eigenvalues, that $\lambda_{\min} \in \sigma_a(A) \cap \overline{\mathbb{D}^+}$. But this contradicts the fact that l is contained in $1/\rho(A) \cap \overline{\mathbb{D}}$. The desired result follows. $\square$