Model reduction for controller design for infinite-dimensional systems

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Chapter 16

Concluding remarks

By collecting various results from the previous chapters we obtain the following theorem.

Theorem 16.1. *Let H_i $(i = 1, 2, 3, 4)$ be*

1. *The set of functions* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *that are holomorphic and with* $0 \in D(\mathsf{G})$.

2. *The set of functions* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *defined on an exponential region that are holomorphic and bounded in norm by a polynomial.*

3. *The set of functions* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *defined on a right half-plane that are holomorphic and bounded in norm by a polynomial.*

4. *The set of functions* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *defined on a right half-plane that are holomorphic and uniformly bounded in norm.*

Let S_i $(i = 1, 2, 3, 4)$ be

1. *The class of discrete-time systems.*

2. *The class of distributional resolvent linear systems.*

3. *The class of exponentially bounded distributional resolvent linear systems.*

4. *The class of well-posed linear systems.*

Let $\mathsf{G} \in H_i$. *Then the following are equivalent.*

- G *has a strongly right-coprime factorization.*

- G *has normalized strongly right-coprime factorization.*

- G *has a strongly left-coprime factorization.*
- G *has normalized strongly left-coprime factorization.*
- G *has a doubly coprime factorization.*
- G *has a normalized doubly coprime factorization.*
- G *has a input and output stabilizable realization in* S_i.
- G *has a minimal input and output stabilizable realization in* S_i.
- G *has a realization in* S_i *that satisfies the finite cost condition and whose dual system also satisfies the finite cost condition.*
- G *has a minimal realization in* S_i *that satisfies the finite cost condition and whose dual system also satisfies the finite cost condition.*
- G *has a LQG-balanced realization in* S_i.
- G *has a minimal LQG-balanced realization in* S_i.

In the case that $i = 1$ *(the discrete-time case) the above is also equivalent with*

- *There exists a stabilizing admissible feedback function for* G.
- G *has a realization in* S_i *that has bounded nonnegative self-adjoint solutions to both its control and its filter algebraic Riccati equation.*
- G *has a minimal realization in* S_i *that has bounded nonnegative self-adjoint solutions to both its control and its filter algebraic Riccati equation.*

In the other cases $(i = 2, 3, 4)$ *this is also true, but one should use controllers with internal loop and a more general form of the algebraic Riccati equations than the usual continuous-time ones.*

In the case that G has a compact LQG-balanced realization, the input and output space are finite-dimensional, and the LQG characteristic values are summable we obtained that LQG-balanced truncations converge in the gap metric, or equivalently, we have convergence of normalized strongly right-coprime factors in H^∞. Using a controller design that is robust with respect to right factor perturbations in this case the plant with transfer function G can be stabilized by a finite-dimensional controller. The performance of this controller approaches the performance of the corresponding infinite-dimensional controller as the state space dimension of the approximation goes to infinity.

Appendix A

Hardy spaces

In this appendix we give some basic definitions and results on Hardy spaces that are needed in this thesis.

Definition A.1. Let $\mathscr{H}$ be a Hilbert space. The Hardy space $H^2(\mathbb{D};\mathscr{H})$ is defined as follows: $\mathsf{F} \in H^2(\mathbb{D};\mathscr{H})$ if $\mathsf{F} : \mathbb{D} \to \mathscr{H}$ is holomorphic and

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \|\mathsf{F}(re^{i\theta})\|^2_{\mathscr{H}}\, d\theta < \infty. \tag{A.1}$$

Lemma A.2. *1. The space $H^2(\mathbb{D};\mathscr{H})$ is a Hilbert space with as norm the square root of the expression on the left-hand side of* (A.1)*.*

2. There is an isometric isomorphism between $l^2(\mathbb{Z}^+;\mathscr{H})$ and $H^2(\mathbb{D};\mathscr{H})$ given by $(a_n)_{n\in\mathbb{Z}^+} \mapsto \sum_{n=0}^{\infty} a_n z^n$.

Definition A.3. Let $\mathscr{B}$ be a Banach space. The Hardy space $H^\infty(\mathbb{D};\mathscr{B})$ is defined as follows: $\mathsf{F} \in H^\infty(\mathbb{D};\mathscr{B})$ if $\mathsf{F} : \mathbb{D} \to \mathscr{B}$ is holomorphic and

$$\sup_{|z|<1} \|\mathsf{F}(z)\|_{\mathscr{B}} < \infty. \tag{A.2}$$

Lemma A.4. *1. The space $H^\infty(\mathbb{D};\mathscr{B})$ is a Banach space with norm the expression on the left-hand side of* (A.2)*.*

2. There is an isometric isomorphism between $H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U},\mathscr{Y}))$ and the set of bounded linear maps from $H^2(\mathbb{D};\mathscr{U})$ to $H^2(\mathbb{D};\mathscr{Y})$ that commute with multiplication by z. The latter are all of the form $h \mapsto \mathsf{F}h$ with $\mathsf{F} \in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U},\mathscr{Y}))$.

For $\delta \in (0,1)$ and $\zeta \in \mathbb{T}$ define Ω_ζ^δ as the convex hull of $\{z \in \mathbb{C} : |z| \leq \delta\} \cup \{\zeta\}$. The sets Ω_ζ^δ are used to define the following **nontangential limits**.

Lemma A.5. *If* $\mathsf{F} \in H^2(\mathbb{D}, \mathscr{H})$ *or* $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*, then for almost all* $\zeta \in \mathbb{T}$ *and all* $\delta \in (0,1)$

$$\lim_{z \to \zeta, z \in \Omega_\zeta^\delta} \mathsf{F}(z)$$

exists and is independent of δ*. The limit is taken in the strong topology in the case that* $\mathsf{F} \in H^2(\mathbb{D}, \mathscr{H})$ *and in the strong operator topology in the case that* $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*.*

Proof. This follows from Rosenblum and Rovnyak [84, Theorem 4.6.A]. □

In the cases considered in Lemma A.5 a **boundary function** F_b of F is defined almost everywhere on $\mathbb{T}$ by

$$\mathsf{F}_b(\zeta) := \lim_{z \to \zeta, z \in \Omega_\zeta^\delta} \mathsf{F}(z).$$

Definition A.6. Let $\mathscr{H}$ be a Hilbert space. The Hardy space $H^2(\mathbb{T}; \mathscr{H})$ is defined as follows: $\mathsf{F} \in H^2(\mathbb{T}; \mathscr{H})$ if $\mathsf{F} \in L^2(\mathbb{T}, \mathscr{H})$ and for all integers $n \geq 1$

$$\int_{\mathbb{T}} \mathsf{F}(\zeta) \zeta^n d\zeta = 0.$$

Definition A.7. The Hardy space $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is defined as follows: $\mathsf{F} \in H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ if $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ and for all integers $n \geq 1$

$$\int_{\mathbb{T}} \mathsf{F}(\zeta) \zeta^n d\zeta = 0.$$

Lemma A.8. *The Hardy space* $H^2(\mathbb{T}; \mathscr{H})$ *is a closed subspace of* $L^2(\mathbb{T}, \mathscr{H})$*. It follows that with the induced norm it is a Hilbert space.*

Lemma A.9. *The Hardy space* $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ *is a closed subspace of* $L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*. It follows that with the induced norm it is a Banach space.*

Lemma A.10. *The boundary function of a* $H^2(\mathbb{D}; \mathscr{H})$ *function is an element of* $H^2(\mathbb{T}; \mathscr{H})$*. This mapping is a unitary operator between these two Hardy spaces.*

Lemma A.11. *The boundary function of a* $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ *function is in* $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*. This mapping is an isometric operator onto the space* $H^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$*.*

Due to Lemmas A.10 and A.11 we do not have to be careful about the distinction between Hardy spaces on the unit disc and on the unit circle.

Lemma A.12. *The space $l^2(\mathbb{Z}, \mathscr{H})$ is isometrically isomorphic to the space $L^2(\mathbb{T}, \mathscr{H})$. This isomorphism is given by*

$$(a_n)_{n\in\mathbb{Z}} \mapsto \sum_{n=-\infty}^{\infty} a_n \zeta^n.$$

The subspace $l^2(\mathbb{Z}^+, \mathscr{H})$ is mapped onto $H^2(\mathbb{T}, \mathscr{H})$.

The transformation from Lemma A.12 is called the **Z-transform**.

The space $H^\infty(\mathbb{D}^+; \mathscr{B})$ can be defined analogously to $H^\infty(\mathbb{D}; \mathscr{B})$. Here $\mathbb{D}^+$ is the (open) exterior of the closed unit disc. Define for $\mathsf{F} : \mathbb{D} \to \mathscr{B}$ the function $\mathsf{F}^- : \mathbb{D}^+ \to \mathscr{B}$ is by $\mathsf{F}^-(z) := \mathsf{F}(1/z)$. It is easily seen that $\mathsf{F} \mapsto \mathsf{F}^-$ is an isometric isomorphism from $H^\infty(\mathbb{D}; \mathscr{B})$ onto $H^\infty(\mathbb{D}^+; \mathscr{B})$. Functions in $H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ also have nontangential limits almost everywhere and we obtain a norm-preserving injection $H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y})) \to L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. Similar comments apply to the Hardy space $H^2(\mathbb{D}^+, \mathscr{H})$.

Lemma A.13. *If $\mathsf{F} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y})) \cap H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$, then F is constant.*

Proof. Define for $u \in \mathscr{U}$ the function $\mathsf{F}_u : \mathbb{D} \to \mathscr{Y}$ by $\mathsf{F}_u(z) := \mathsf{F}(z)u$. It easily follows that $\mathsf{F}_u \in H^2(\mathbb{D}, \mathscr{Y}) \cap H^2(\mathbb{D}^+, \mathscr{Y})$. Using Lemma A.12 we can write

$$\mathsf{F}(z)u = \sum_{k=-\infty}^{\infty} a_k(u) z^k.$$

Since $\mathsf{F}_u \in H^2(\mathbb{D}, \mathscr{Y})$ it follows that $a_k(u) = 0$ for $k < 0$ for all $u \in \mathscr{U}$. From $\mathsf{F}_u \in H^2(\mathbb{D}^+, \mathscr{Y})$ we obtain that $a_k(u) = 0$ for $k > 0$ for all $u \in \mathscr{U}$. We conclude that

$$\mathsf{F}(z)u = a_0(u).$$

Hence F is a constant operator. □

Corollary A.14. *If $\mathsf{H} \in L^\infty(\mathbb{T}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is the boundary function of both a function H_1 in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ and a function H_2 in $H^\infty(\mathbb{D}^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$, then H is constant.*

Proof. Define the function $\mathsf{F} : \mathbb{D} \cap \mathbb{D}^+ \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ by $\mathsf{F}(z) := \mathsf{H}_1(z)$ if $z \in \mathbb{D}$ and $\mathsf{F}(z) := \mathsf{H}_2(z)$ if $z \in \mathbb{D}^+$. Apply Lemma A.13 to this function. We conclude that F is constant. It follows that H_1 is constant. The boundary function of a constant function is obvously constant, so H is constant. □

Definition A.15. For $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ we define the operator $L_\mathsf{F} : L^2(\mathbb{T}, \mathscr{U}) \to L^2(\mathbb{T}, \mathscr{Y})$ by $L_\mathsf{F}\mathsf{H} = \mathsf{F}\mathsf{H}$. This operator is called the **Laurent operator** of F.

Definition A.16. For $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ we define the operator $T_\mathsf{F} : H^2(\mathbb{T}, \mathscr{U}) \to H^2(\mathbb{T}, \mathscr{Y})$ by $T_\mathsf{F} = P_{H^2(\mathbb{T},\mathscr{Y})} L_\mathsf{F}|_{H^2(\mathbb{T},\mathscr{U})}$. This operator is called the **Toeplitz operator** of F.

Note that we can identify the Toeplitz operator T_F with an operator from $H^2(\mathbb{D}, \mathscr{U})$ to $H^2(\mathbb{D}, \mathscr{Y})$ using the identification of functions on the unit disc with their boundary functions discussed earlier.

Lemma A.17. *We have $L_\mathsf{F}^* = L_{\mathsf{F}^*}$ and $T_\mathsf{F}^* = T_{\mathsf{F}^*}$, where the function $\mathsf{F}^* \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$ is defined by $\mathsf{F}^*(z) = \mathsf{F}(z)^*$.*

Lemma A.18. *Let $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. The Laurent operator L_F is isometric if and only if $\mathsf{F}(\zeta)$ is isometric for almost all $\zeta \in \mathbb{T}$.*

Definition A.19. A function $\mathsf{F} \in H^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is called **inner** if its Laurent operator L_F is an isometry.

Lemma A.20. *Let $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. The Toeplitz operator T_F is isometric if and only if the Laurent operator L_F is.*

The following is known as Sarason's theorem.

Lemma A.21. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces, $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ an inner function and $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2))$. Let $\mathrm{Im}(T_\mathsf{H})$ denote the image of the Toeplitz operator of H, $\mathrm{Im}(T_\mathsf{H})^\perp$ its orthogonal complement in $H^2(\mathbb{D}, \mathscr{H}_2)$ and $P_{\mathrm{Im}(T_\mathsf{H})^\perp} \in \mathcal{L}(H^2(\mathbb{D}, \mathscr{H}_2))$ the orthogonal projection onto this ortogonal complement. Then*

$$\|P_{\mathrm{Im}(T_\mathsf{H})^\perp} T_\mathsf{F}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2))}.$$

Proof. See for example Nikol′skiĭ [59, p191]. □

We need the following two corollaries of Sarason's theorem.

Corollary A.22. *Let $\mathscr{U}$ and $\mathscr{Y}$ be Hilbert spaces, $\mathsf{H}, \mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ with H an inner function. Then*

$$\|P_{\mathrm{Im}(T_\mathsf{H})^\perp} T_\mathsf{F}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \|\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))}.$$

Proof. We apply Lemma A.21 and denote the operators there with tildes to distinguish them from the operators used in the statement of this lemma. Apply Lemma A.21 with $\mathscr{H}_1 = \mathscr{U}$, $\mathscr{H}_2 = \mathscr{U} \times \mathscr{Y}$, $\tilde{\mathsf{H}} = \mathsf{H}$ and $\tilde{\mathsf{F}} = [\mathsf{F}, 0]$. It is easily seen from the form of $\tilde{\mathsf{F}}$ that

$$\|P_{\mathrm{Im}(T_{\mathsf{H}})^{\perp}} T_{\mathsf{F}}\| = \|P_{\mathrm{Im}(T_{\tilde{\mathsf{H}}})^{\perp}} T_{\tilde{\mathsf{F}}}\|.$$

The parameter $\tilde{\mathsf{V}}$ from Lemma A.21 can be decomposed as $\tilde{\mathsf{V}} = [\tilde{\mathsf{V}}_1, \tilde{\mathsf{V}}_2]$. We have

$$\begin{aligned} &\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|\tilde{\mathsf{F}} - \tilde{\mathsf{H}}\tilde{\mathsf{V}}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\ = &\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|[\mathsf{F} - \mathsf{H}\tilde{\mathsf{V}}_1, -\mathsf{H}\tilde{\mathsf{V}}_2]\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\ = &\inf_{V \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \|[\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))}, \end{aligned}$$

where we have used that the infimum over $\tilde{\mathsf{V}}_2$ is reached for $\tilde{\mathsf{V}}_2 = 0$. From Lemma A.21 we now obtain the desired equality. □

Corollary A.23. *Let $\mathscr{U}$ and $\mathscr{Y}$ be Hilbert spaces, $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ an inner function and $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U} \times \mathscr{Y}))$. Then*

$$\|P_{\mathrm{Im}(T_{\mathsf{H}})^{\perp}} T_{\mathsf{F}}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))} \|\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U} \times \mathscr{Y}))}.$$

Proof. We apply Lemma A.21 and denote the operators there with tildes to distinguish them from the operators used in the statement of this lemma. Apply Lemma A.21 with $\mathscr{H}_1 = \mathscr{U}$, $\mathscr{H}_2 = \mathscr{U} \times \mathscr{Y}$, $\tilde{\mathsf{H}} = \mathsf{H}$ and $\tilde{\mathsf{F}} = [0, \mathsf{F}]$. It is easily seen from the form of $\tilde{\mathsf{F}}$ that

$$\|P_{\mathrm{Im}(T_{\mathsf{H}})^{\perp}} T_{\mathsf{F}}\| = \|P_{\mathrm{Im}(T_{\tilde{\mathsf{H}}})^{\perp}} T_{\tilde{\mathsf{F}}}\|.$$

The parameter $\tilde{\mathsf{V}}$ from Lemma A.21 can be decomposed as $\tilde{\mathsf{V}} = [\tilde{\mathsf{V}}_1, \tilde{\mathsf{V}}_2]$. We have

$$\begin{aligned} &\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|\tilde{\mathsf{F}} - \tilde{\mathsf{H}}\tilde{\mathsf{V}}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\ = &\inf_{\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))} \|[-\mathsf{H}\tilde{\mathsf{V}}_1, \mathsf{F} - \mathsf{H}\tilde{\mathsf{V}}_2]\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))} \\ = &\inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \|[\mathsf{F} - \mathsf{H}\mathsf{V}\|_{H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U} \times \mathscr{Y}))}, \end{aligned}$$

where we have used that the infimum over $\tilde{\mathsf{V}}_1$ is reached for $\tilde{\mathsf{V}}_1 = 0$. From Lemma A.21 we now obtain the desired equality. □

Definition A.24. For $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ we define the operator $H_\mathsf{F} : L^2(\mathbb{T}, \mathscr{H}_1) \to L^2(\mathbb{T}, \mathscr{H}_2)$ by $H_\mathsf{F} := P_{H^2(\mathbb{D},\mathscr{H}_2)} L_\mathsf{F} P_{H^2(\mathbb{D},\mathscr{H}_1)^\perp}$. This operator is called the **Hankel operator** of F.

Remark A.25. We warn the reader that in the literature there are several different definitions of the concept of the Hankel operator of a function.

The next result relates the Hankel operator of a $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function with the Hankel map of an input-output stable discrete-time system (see the definition on page 9, Definition 3.1 and Proposition 3.22).

Lemma A.26. *Let Σ be an input-output stable discrete-time system with transfer function D and Hankel map $\mathcal{H}$. Denote the Hankel operator of $\mathsf{D} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ by H_D and the Z-transform from $l^2(\mathbb{Z}, \mathscr{H})$ to $L^2(\mathbb{T}, \mathscr{H})$ by $Z_\mathscr{H}$. Then $H_\mathsf{D} Z_\mathscr{U} = Z_\mathscr{Y} \mathcal{H}$. In particular, $\|H_\mathsf{D}\| = \|\mathcal{H}\|$.*

Proof. This follows easily from the definitions. □

The following result is known as Nehari's theorem (or sometimes Page's theorem).

Lemma A.27. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and $\mathsf{F} \in L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$. Then*

$$\|H_\mathsf{F}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{F}^* - \mathsf{V}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))}.$$

Proof. It is shown in e.g. Peller [75, page 68], Nikol′skiĭ [59, p191] that for $\mathsf{H} \in L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$

$$\|P_{H^2(\mathbb{D},\mathscr{H}_1)^\perp} L_\mathsf{H} P_{H^2(\mathbb{D},\mathscr{H}_2)}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{H} - \mathsf{V}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))}.$$

Applying this with $H = F^*$ gives

$$\|P_{H^2(\mathbb{D},\mathscr{H}_1)^\perp} L_\mathsf{F}^* P_{H^2(\mathbb{D},\mathscr{H}_2)}\| = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \|\mathsf{F}^* - \mathsf{V}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))}.$$

The desired result now follows from noting that the left-hand side of this last expression is the norm of the adjoint of the Hankel operator of F and that an operator and its adjoint have the same norm. □

Lemma A.28. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be finite-dimensional Hilbert spaces and $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ a rational function. Then, for each $\sigma > \|H_\mathsf{F}\|$ there exists a rational $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ such that*

$$\|\mathsf{F}^* - \mathsf{H}\|_{L^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))} \leq \sigma.$$

Proof. This follows from the explicit state space formulas given in McFarlane and Glover [54, Appendix B]. □

The second statement in the following result is known as the Corona theorem.

Lemma A.29. *Let* $\mathsf{F} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$*. If there exists a function* $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ *such that* $\mathsf{H}(s)\mathsf{F}(s) = I$ *for all* $s \in \mathbb{D}$*, then there exists a* $\varepsilon > 0$ *such that for all* $s \in \mathbb{D}$ *and all* $h \in \mathscr{H}_1$

$$\|\mathsf{F}(s)h\|_{\mathscr{H}_2} \geq \varepsilon \|h\|_{\mathscr{H}_1}. \tag{A.3}$$

If $\mathscr{H}_1$ *is finite-dimensional then the converse is also true, i.e., if there exists a* $\varepsilon > 0$ *such that for all* $s \in \mathbb{D}$ *and all* $h \in \mathscr{H}_1$ *we have* (A.3)*, then there exists a* $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ *such that* $\mathsf{H}(s)\mathsf{F}(s) = I$ *for all* $s \in \mathbb{D}$.

Notes

General references on Hardy space theory are Duren [23], Garnett [32] and Hoffman [39]. The vector-valued case can be found in Nikol′skiĭ [61], [59], [60], Peller [75] and Rosenblum and Rovnyak [84].