Model reduction for controller design for infinite-dimensional systems

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Chapter 14

Basic results

In this chapter we translate the main results obtained in part I of this thesis to continuous-time.

We first define the dual system of a resolvent linear system

Definition 14.1. The **dual system** of a resolvent linear system is the resolvent linear system on the set $\overline{\Lambda}$ of complex conjugates with pseudoresolvent $\mathfrak{a}_{\text{dual}}(s) := \mathfrak{a}(\bar{s})^*$, incoming wave function $\mathfrak{b}_{\text{dual}}(s) := \mathfrak{c}(\bar{s})^*$, outgoing wave function $\mathfrak{c}_{\text{dual}}(s) := \mathfrak{b}(\bar{s})^*$ and characteristic function $\mathfrak{d}_{\text{dual}}(s) := \mathfrak{d}(\bar{s})^*$.

It is easily seen that the Cayley transform with parameter α of the dual system is the dual system of the Cayley transform with parameter $\bar{\alpha}$.

Definition 14.2. A distributional resolvent linear system is called **approximately observable** if for input zero the output is only zero if the initial state is zero. It is called **approximately controllable** if its dual system is approximately observable and it is called **minimal** if it is both approximately controllable and approximately observable.

It is easily seen that approximate observability translates under the Cayley transform. It follows that approximate controllability and minimality do also.

14.1 Stability

Definition 14.3. A distributional resolvent linear system is called

- **exponentially stable** if for all $x_0 \in \mathscr{X}$ the state x defined by (11.13) with $u = 0$ is in $L^2(\mathbb{R}^+, \mathscr{X})$.

- **output stable** if for all $x_0 \in \mathscr{X}$ the output y defined by (11.13) with $u = 0$ is in $L^2(\mathbb{R}^+, \mathscr{Y})$.
- **input stable** if the dual system is output stable.
- **input-output stable** if for all $u \in L^2(\mathbb{R}^+, \mathscr{U})$ the output y defined by (11.13) with $x_0 = 0$ is in $L^2(\mathbb{R}^+, \mathscr{Y})$.

Proposition 14.4. *Let Σ be a distributional resolvent linear system and $\alpha \in \Lambda_E$ and $\alpha > 0$. Let Σ_d be its Cayley transform with parameter α. Then Σ is output stable if and only if Σ_d is, Σ is input stable if and only if Σ_d is, Σ is input-output stable if and only if Σ_d is.*

Proof. This follows easily using Proposition 13.6. □

Remark 14.5. Note that exponential stability almost never translates under the Cayley transform. It is easily seen that -1 is in the resolvent set of the state operator of the Cayley transform if and only if $\mathfrak{a}(\alpha)$ has a bounded inverse. It follows that if $\mathfrak{a}(\alpha)$ does not have a bounded inverse, then the Cayley transform is not exponentially stable.

Definition 14.6. Let Σ be an output stable distributional resolvent linear system. Let $y_w \in L^2(\mathbb{R}^+, \mathscr{Y})$ be the output for initial state $w \in \mathscr{X}$ and zero input. The **observability gramian** $L_C \in \mathcal{L}(\mathscr{X})$ is defined by $\langle L_C w, w \rangle = \|y_w\|^2_{L^2(\mathbb{R}^+, \mathscr{Y})}$.

Proposition 14.7. *Let Σ be an output stable distributional resolvent linear system and $\alpha \in \Lambda_E$ with $\alpha > 0$. The observability gramian of Σ equals the observability gramian of the Cayley transform with parameter α of Σ.*

Proof. This is easily seen using that the Möbius operator, which relates the output of the system and its Cayley transform, is unitary. □

Definition 14.8. Let Σ be an input stable distributional resolvent linear system. The **controllability gramian** $L_B \in \mathcal{L}(\mathscr{X})$ is defined as the observability gramian of the dual system.

Remark 14.9. In the special case that Σ is a system node one can show that the observability gramian is the minimal nonnegative self-adjoint solution of the observation Lyapunov equation

$$\langle Lw, Aw \rangle + \langle Aw, Lw \rangle = \|Cw\|^2, \quad w \in D(A).$$

A dual statement holds for the controllability gramian.

14.2 Stabilizability

Definition 14.10. An **admissible feedback pair** for a resolvent linear system is a pair $[\mathfrak{f},\mathfrak{g}] : \Lambda \to \mathcal{L}(\mathscr{X} \times \mathscr{U}, \mathscr{U})$ that satisfies

$$\begin{aligned}\mathfrak{f}(\beta) - \mathfrak{f}(\alpha) &= (\alpha-\beta)\mathfrak{f}(\alpha)\mathfrak{a}(\beta), \\ \mathfrak{g}(\beta) - \mathfrak{g}(\alpha) &= (\alpha-\beta)\mathfrak{f}(\alpha)\mathfrak{b}(\beta),\end{aligned} \tag{14.1}$$

and such that $I - \mathfrak{g}(s)$ has a bounded inverse for some $s \in \Lambda$.

The **closed-loop system** of a resolvent linear system with an admissible feedback pair is the resolvent linear system

$$\mathfrak{a}^{\mathrm{cl}} := \mathfrak{a} + \mathfrak{b}(I-\mathfrak{g})^{-1}\mathfrak{f}, \quad \mathfrak{b}^{\mathrm{cl}} := \mathfrak{b}(I-\mathfrak{g})^{-1},$$

$$\mathfrak{c}^{\mathrm{cl}} := \begin{bmatrix} (I-\mathfrak{g})^{-1}\mathfrak{f} \\ \mathfrak{c} + \mathfrak{d}(I-\mathfrak{g})^{-1}\mathfrak{f} \end{bmatrix}, \quad \mathfrak{d}^{\mathrm{cl}} := \begin{bmatrix} (I-\mathfrak{g})^{-1} \\ \mathfrak{d}(I-\mathfrak{g})^{-1} \end{bmatrix}.$$

Remark 14.11. Note that Λ^{cl}, the domain of definition of the closed-loop system, consists of those $s \in \Lambda$ for which $I - \mathfrak{g}(s)$ has a bounded inverse.

Lemma 14.12. *Let Σ be a resolvent linear system and $[\mathfrak{f},\mathfrak{g}]$ and admissible feedback pair. Assume that there exists an $\alpha \in \Lambda$ with $\alpha > 0$ such that $I - \mathfrak{g}(\alpha)$ has a bounded inverse. Denote the Cayley transform with parameter α of Σ by Σ_d. Then $[\sqrt{2\alpha}\mathfrak{f}(\alpha), \mathfrak{g}(\alpha)]$ is an admissible feedback pair for Σ_d. Moreover, the corresponding closed-loop system equals the Cayley transform with parameter α of the closed-loop system of Σ with the admissible feedback pair $[\mathfrak{f},\mathfrak{g}]$.*

Proof. That $[\sqrt{2\alpha}\mathfrak{f}(\alpha), \mathfrak{g}(\alpha)]$ is an admissible feedback pair for Σ_d is immediate. The indicated equality of systems also follows immediately from the definitions. □

Definition 14.13. An **exponential region admissible feedback pair** for a distributional resolvent linear system is an admissible feedback pair $[\mathfrak{f},\mathfrak{g}]$ such that $I - \mathfrak{g}(s)$ has a bounded inverse for all s in some exponential region and $(I - \mathfrak{g}(s))^{-1}$ is polynomially bounded in this exponential region.

A **half-plane admissible feedback pair** for an exponentially bounded distributional resolvent linear system is an admissible feedback pair $[\mathfrak{f},\mathfrak{g}]$ such that $I - \mathfrak{g}(s)$ has a bounded inverse for all s in some right half-plane and $(I - \mathfrak{g}(s))^{-1}$ is polynomially bounded in this right half-plane.

Proposition 14.14. *The closed-loop system of a distributional resolvent linear system with an exponential region admissible feedback pair is a distributional resolvent linear system.*

The closed-loop system of an exponentially bounded distributional resolvent linear system with a half-plane admissible feedback pair is an exponentially bounded distributional resolvent linear system.

Proof. This is easily checked. □

Remark 14.15. If Σ is a system node and $[\mathfrak{f}, \mathfrak{g}]$ is a half-plane admissible feedback pair, then the closed-loop system will in general not be a system node. A concept of feedback under which the closed-loop system is again a system node is given in Staffans [89, Section 7.4]. This concept is very complicated and seems to be impossible to check.

Remark 14.16. If Σ is well-posed and $[\mathfrak{f}, \mathfrak{g}]$ is a half-plane admissible feedback pair that is uniformly bounded on the indicated right half-plane, then the closed-loop system is again a well-posed system.

Definition 14.17. A distributional resolvent linear system is called **output stabilizable** if there exists an exponential region admissible feedback pair such that the corresponding closed-loop system is output stable.

Remark 14.18. The other stabilizability notions introduced earlier for discrete-time systems also have (now hopefully obvious) counterparts in continuous-time.

Remark 14.19. It follows using Lemma 14.12 and Proposition 14.4 that, with the right choice of α, output stabilizability translates under the Cayley transform.

14.3 The LQ optimal control problem

Definition 14.20. We say that a distributional resolvent linear system satisfies the **finite cost condition** if for every $x_0 \in \mathscr{X}$ the set $\mathscr{V}(x_0)$ is nonempty.

Proposition 14.21. *Let Σ be a distributional resolvent linear system and $\alpha \in \Lambda_E$ and $\alpha > 0$. Let Σ_d be its Cayley transform with parameter α. Σ satisfies the finite cost condition if and only if Σ_d does. In this case the optimal cost operators are equal.*

Proof. This follows immediately from Proposition 13.6. □

Proposition 14.22. *Let Σ be a distributional resolvent linear system that satisfies the finite cost condition and let $\alpha > 0$ be such that $\alpha \in \Lambda_E$. Let Σ_d be the Cayley transform of Σ with parameter α. Denote the admissible feedback pair from Proposition 6.33 that gives the optimal closed-loop system of Σ_d*

by $[F_d, G_d]$. *Define* $[\mathfrak{f}, \mathfrak{g}]$ *by* $[\mathfrak{f}(\alpha), \mathfrak{g}(\alpha)] = [F_d/\sqrt{2\alpha}, G_d]$ *and extending to* Λ *using the functional equations* (14.1). *Then* $[\mathfrak{f}, \mathfrak{g}]$ *is an exponential region admissible feedback pair for* Σ. *Denote the closed-loop system of* Σ *and this admissible feedback pair by* $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$. *Then the optimal closed-loop system of* Σ_d *is the Cayley transform with parameter* α *of* $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$. *The set* Λ *on which* $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$ *is defined and polynomially bounded contains* $\Lambda_E \cap \mathbb{C}_0^+$.

Proof. The equations (14.1) are satisfied by definition. Since $I - \mathfrak{g}(\alpha) = I - G_d$, it follows that $[\mathfrak{f}, \mathfrak{g}]$ is an admissible feedback pair. We will show that $I - \mathfrak{g}(s)$ has a bounded inverse on the exponential region $\Lambda_E^0 := \Lambda_E \cap \mathbb{C}_0^+$. Note that under the map $z = (\alpha - s)/(\alpha + s)$ the region Λ_E^0 is mapped into the connected component of $1/\rho(A_d) \cap \overline{\mathbb{D}}$ that contains zero. Define the function $\mathfrak{G}(z) := \mathfrak{g}(s)$, where $z = (\alpha - s)/(\alpha + s)$. It follows that $\mathfrak{G}(z) = G_d + F_d z(I - zA_d)^{-1}B_d$. We have that $I - \mathfrak{G}(z)$ is invertible on $\rho(A_Q)$, where A_Q is the state operator of the optimal closed-loop system of Σ_d. Lemma B.8 shows that the connected component of $1/\rho(A_Q) \cap \overline{\mathbb{D}}$ that contains zero contains the connected component of $1/\rho(A_d) \cap \overline{\mathbb{D}}$ that contains zero. It follows that it contains the image of Λ_E^0. From this we see that indeed $I - \mathfrak{g}(s)$ has a bounded inverse on the exponential region Λ_E^0. Lemma 14.12 shows that the optimal closed-loop system of Σ_d is the Cayley transform with parameter α of $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$. Since the optimal closed-loop system of Σ_d is input-output stable it follows that $I - \mathfrak{g}$ has an inverse that extends to a function in $H^\infty(\mathbb{C}_0^+, \mathscr{U})$. So $(I - \mathfrak{g})^{-1}$ is uniformly bounded on Λ_E^0. It follows that $[\mathfrak{f}, \mathfrak{g}]$ is an exponential region admissible feedback pair. The other statements follow easily from the above. □

Definition 14.23. Let Σ be a distributional resolvent linear system that satisfies the finite cost condition and let $\alpha > 0$ be such that $\alpha \in \Lambda_E$. The exponential region admissible feedback pair $[\mathfrak{f}, \mathfrak{g}]$ from Lemma 14.22 is called the α-**optimal feedback pair**. The corresponding closed-loop system is called the α-**optimal closed-loop system**.

Note that it easily follows from the proof of Proposition 14.22 that an α-optimal feedback pair for an exponentially bounded distributional resolvent linear system is a half-plane admissible feedback pair.

Proposition 14.24. *Let* Σ *be a distributional resolvent linear system that satisfies the finite cost condition and let* $\alpha > 0$ *be such that* $\alpha \in \Lambda_E$. *Then the* α*-optimal closed-loop system is output stable and input-output stable.*

Proof. This follows from the corresponsing discrete-time results and Proposition 14.4 which shows that output stability and input-output stability translate under the Cayley transform. □

Proposition 14.25. *Let Σ be a distributional resolvent linear that satisifies the finite cost condition. Denote the element of minimal norm in $\mathscr{V}(x_0)$ by $[u_{x_0}^{\min}; y_{x_0}^{\min}]$. Let $\alpha \in \Lambda_E$ with $\alpha > 0$ and let $[\mathfrak{f}, \mathfrak{g}]$ be an α-optimal feedback pair. Then $\hat{u}_{x_0}^{\min}(s) = (I - \mathfrak{g}(s))^{-1}\mathfrak{f}(s)x_0$ for $s \in \Lambda_E \cap \mathbb{C}_0^+$.*

Proof. Denote the Cayley transform of Σ with parameter α by Σ_d. Note that $\mathscr{V}(x_0)$ has a unique element of minimal norm by Proposition 13.6 and the fact that $\mathscr{V}_d(x_0)$ has a unique element of minimal norm $[u^d; y^d]$. Proposition 14.22 shows that the optimal closed-loop system of Σ_d is the Cayley transform with parameter α of $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$. The first component of the output for initial state x_0 and input zero of the optimal closed-loop system of Σ_d is easily seen to be the optimal input u^d. Since this optimal closed-loop system of Σ_d is the Cayley transform with parameter α of $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$, it follows that the first component of the output for initial state x_0 and input zero of $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$ equals $M^{-1}u_d$. But $M^{-1}u_d$ also equals $\hat{u}_{x_0}^{\min}$, the first component of the element of minimal norm in $\mathscr{V}(x_0)$. It follows that $\hat{u}_{x_0}^{\min}$ is the first component of the output for initial state x_0 and input zero of $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$. This gives the desired formula $\hat{u}_{x_0}^{\min}(s) = (I - \mathfrak{g}(s))^{-1}\mathfrak{f}(s)x_0$ for s in all exponential regions on which $\Sigma_{[\mathfrak{f},\mathfrak{g}]}$ has the polynomial boundedness property. That $\Lambda_E \cap \mathbb{C}_0^+$ is such a region follows from Proposition 14.22. □

14.4 Coprime factorization

We will focus exclusively on strongly right-coprime factorizations. The other cases treated in Chapter 7 can be treated analogously.

Definition 14.26. Let $\mathsf{M} \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$, $\mathsf{N} \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_3))$.

The functions M and N are called **strongly right-coprime** if $[\mathsf{M}; \mathsf{N}]$ has a left-inverse in $H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{H}_2 \times \mathscr{H}_3, \mathscr{H}_1))$, i.e. if there exist $\tilde{\mathsf{X}} \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ and $\tilde{\mathsf{Y}} \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{H}_3, \mathscr{H}_1))$ such that

$$\tilde{\mathsf{X}}(s)\mathsf{M}(s) - \tilde{\mathsf{Y}}(s)\mathsf{N}(s) = I_{\mathscr{H}_1} \quad \forall s \in \mathbb{C}_0^+. \tag{14.2}$$

The functions $\tilde{\mathsf{X}}$ and $\tilde{\mathsf{Y}}$ are called **right Bezout factors** for the pair (M, N).

Definition 14.27. Let $\mathsf{G} : \Lambda_E \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$, with Λ_E an exponential region, be holomorphic and polynomially bounded.

G has a **right factorization** if there exist $\mathsf{M} \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U}))$ and $\mathsf{N} \in H^\infty(\mathbb{C}_0^+; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ such that $\mathsf{M}(s)$ is invertible for s in some exponential region, M^{-1} is polynomially bounded on an exponential region, and $\mathsf{G}(s) = \mathsf{N}(s)\mathsf{M}(s)^{-1}$ for s in some exponential region. The factor $[\mathsf{M}; \mathsf{N}]$ provides a **strongly right-coprime factorization** if M and N are strongly right-coprime. The right factor $[\mathsf{M};\mathsf{N}]$ is called **normalized** when multiplication with $[\mathsf{M};\mathsf{N}]$ is an isometry from $H^2(\mathbb{C}_0^+,\mathscr{U})$ into $H^2(\mathbb{C}_0^+,\mathscr{U}\times\mathscr{Y})$.

Note that characteristic functions of distributional resolvent linear systems belong to the class of functions to which the above definitions of factorization apply. The following result shows under which conditions the characteristic function of a distributional resolvent linear system has a strongly right-coprime factorization.

Proposition 14.28. *Assume that the distributional resolvent linear system Σ and its dual system both satisfy the finite cost condition and let $\alpha > 0$ be such that $\alpha \in \Lambda_E$. Then the characteristic function of the α-optimal closed-loop system has a holomorphic extension to $\mathbb{C}_0^+$ and this extension provides a normalized strongly right-coprime factorization of the characteristic function of Σ.*

Proof. Let $\alpha \in \Lambda_E$ and $\alpha > 0$. Denote the Cayley transform of Σ with parameter α by Σ_d. It follows from Proposition 7.20 that the transfer function of Σ_d has a normalized strongly right-coprime factor $[\mathsf{M}_d;\mathsf{N}_d]$ with Bezout pair $[\tilde{\mathsf{X}}_d,\tilde{\mathsf{Y}}_d]$. The function $[\mathsf{M}_d;\mathsf{N}_d]$ is the transfer function of the optimal closed-loop system of Σ_d. Define $[\mathsf{M}(s);\mathsf{N}(s)] := [\mathsf{M}_d(z);\mathsf{N}_d(z)]$, $[\tilde{\mathsf{X}}(s),\tilde{\mathsf{Y}}(s)] := [\tilde{\mathsf{X}}_d(z),\tilde{\mathsf{Y}}_d(z)]$, where z and s are related by $z = (\alpha - s)/(\alpha + s)$. It easily follows that the indicated functions are in H^∞ and that (14.2) holds. It is also easily seen that M^{-1} equals $(I-\mathfrak{g}(s))^{-1}\mathfrak{f}(s)$ on $\Lambda_E\cap\mathbb{C}_0^+$, where $[\mathfrak{f},\mathfrak{g}]$ is the α-optimal feedback pair. It follows that M is invertible on $\Lambda_E\cap\mathbb{C}_0^+$ and that M^{-1} is polynomially bounded on $\Lambda_E\cap\mathbb{C}_0^+$. The equality $\mathfrak{d}(s) = \mathsf{N}(s)\mathsf{M}(s)^{-1}$ on $\Lambda_E\cap\mathbb{C}_0^+$ also follows. □

Remark 14.29. In Section 14.7 we will see that any function that has a strongly right-coprime factorization coincides on some exponential region with the characteristic function of some distributional resolvent linear system that satisfies the finite cost condition and whose dual system satisfies the finite cost condition.

Remark 14.30. It is easily shown that, as in the discrete-time case, the existence of a strongly left-coprime factorization, the existence of a normalized strongly left-coprime factorization, the existence of a strongly right-coprime factorization, the existence of a normalized strongly right-coprime factorization, the existence of a doubly coprime factorization and the existence of a normalized doubly coprime factorization are all equivalent.

Remark 14.31. We consider the special case that G is holomorphic and polynomially bounded on a right half-plane and not just on an exponential region.

The Bezout equation gives $\tilde{\mathsf{X}} - \tilde{\mathsf{Y}}(s)\mathsf{G}(s) = \mathsf{M}(s)^{-1}$ for s in an exponential region. Using that the left-hand side is a holomorphic and polynomially bounded function on a right half-plane it is not difficult to see that $\mathsf{M}(s)$ is invertible for s in a right half-plane and that the inverse function is polynomially bounded on a right half-plane.

It follows similarly that if G is holomorphic and uniformly bounded on a right half-plane, then so is M^{-1}.

14.5 The gap metric

The gap between distributional resolvent linear systems is defined similarly as for discrete-time systems using the gap metric on subspaces of a given Hilbert space (see Definition 9.1).

Definition 14.32. Let Σ_i $(i = 1, 2)$ be distributional resolvent linear systems with the same input and output spaces. The **gap** $\delta(\Sigma_1, \Sigma_2)$ is defined to be $\delta(\mathscr{V}_1(0), \mathscr{V}_2(0))$.

Proposition 14.33. *Let Σ_i $(i = 1, 2)$ be distributional resolvent linear systems with the same input and output spaces, $\alpha \in \Lambda_E$ for both systems, and $\alpha > 0$. Let Σ_d^i be the respective Cayley transforms with parameter α. Then $\delta(\Sigma_1, \Sigma_2) = \delta(\Sigma_d^1, \Sigma_d^2)$.*

Proof. This follows easily using that $\mathscr{V}_i(0)$ is isometrically isomorphic to $\mathscr{V}_d^i(0)$ $(i = 1, 2)$ under the Möbius operator by Proposition 13.6. □

14.6 Stabilization

Definition 14.34. Let $\mathsf{G} : \Lambda_E \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ with Λ_E an exponential region be holomorphic and polynomially bounded. We say that K is an **admissible feedback function** for G if $\mathsf{K} : \Lambda_E \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ is holomorphic and polynomially bounded and $I - \mathsf{KG}$ has a bounded inverse on an exponential region.

Definition 14.35. An admissible feedback function K for G is called **stabilizing** if

$$\begin{bmatrix} (I - \mathsf{KG})^{-1} & \mathsf{K}(I - \mathsf{GK})^{-1} \\ \mathsf{G}(I - \mathsf{KG})^{-1} & (I - \mathsf{GK})^{-1} \end{bmatrix}$$

extends to a function in $H^\infty(\mathbb{C}_0^+, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U} \times \mathscr{Y}))$.

We discuss the continuous-time analogue of the discrete-time robust right factor stabilizing feedback function from Definition 8.13. We only do this for the finite-dimensional case as this is all we need in the sequel and a full discussion of the general case would take us too far afield (see Curtain [11] for this case). Assume that $\mathscr{U}, \mathscr{X}$ and $\mathscr{Y}$ are finite-dimensional and the system Σ is described by

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) + Du(t). \tag{14.3}$$

Further assume that this system has solutions Q and P to its (continuous-time) control and filter algebraic Riccati equation, respectively. Here the (continuous-time) control algebraic Riccati equation is

$$A^*Q + QA + C^*C = (QB + C^*D)(I + D^*D)^{-1}(B^*Q + D^*C),$$

and the (continuous-time) filter algebraic Riccati equation is

$$AP + PA^* + BB^* = (PC^* + BD^*)(I + DD^*)^{-1}(CP + DB^*).$$

Let $\varepsilon < 1/\sqrt{1+\mu_1^2}$, where μ_1 is the largest LQG-characteristic value of Σ. Define the controller by its system operator

$$\left[\begin{array}{c|c} A + BF + WPC^*(C + DF) & WPC^* \\ \hline B^*Q & -D^* \end{array}\right],$$

where $F := -(I + D^*D)^{-1}(D^*C + B^*Q)$ and $W := ((1-\varepsilon^2)I + \varepsilon^2 PQ)^{-1}$. Denote the transfer function of the controller by K and the transfer function of Σ by G. Then K is an admissible feedback function and it is stabilizing for all G_Δ with $\delta_g(\mathsf{G}, \mathsf{G}_\Delta) \le \varepsilon$. The above follows from McFarlane and Glover [54].

14.7 Balanced realizations

Definition 14.36. An input and output stable distributional resolvent linear system is called **Lyapunov-balanced** if its controllability and observability gramian are equal.

Proposition 14.37. *Any function in $H^\infty(\mathbb{C}_0^+, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ has a minimal Lyapunov-balanced realization. Minimal Lyapunov-balanced realizations are unique up to a unitary similarity transformation in the state space. The pseudoresolvent is the resolvent of the generator of a strongly continuous contraction semigroup. Both the minimal Lyapunov-balanced realization and its dual system are strongly stable.*

Proof. See Theorem 9.5.6 of Staffans [89]. □

LQG-characteristic values are defined as in discrete-time.

Definition 14.38. Let Σ be a distributional resolvent linear system that satisfies the finite cost condition and whose dual system satisfies the finite cost condition. Denote the optimal cost operator by $Q^{\min}$ and the optimal cost operator of the dual system by $P^{\min}$. The square roots of the points in the spectrum of $P^{\min}Q^{\min}$, with the exception of zero, are called the **LQG-characteristic values** of Σ.

Note that, using the Cayley transform and the corresponding discrete-time result, it is easily seen that two distributional resolvent linear systems whose characteristic functions coincide on an exponential region have the same LQG-characteristic values.

Definition 14.39. A distributional resolvent linear system is called **LQG-balanced** if it and its dual system satisfy the finite cost condition and the optimal cost operator of the system and of its dual system are equal. It is called **compact LQG-balanced** if it is LQG-balanced and the optimal cost operator is compact.

Proposition 14.40. *Let Σ be a distributional resolvent linear system and $\alpha \in \Lambda_E$ with $\alpha > 0$. Σ is LQG-balanced if and only if its Cayley transform with parameter α is. It is compact LQG-balanced if and only if its Cayley transform with parameter α is.*

Proof. This easily follows using Proposition 14.21. □

Proposition 14.41. *Let $\mathsf{G} : \Lambda_E \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$, with Λ_E an exponential region, be holomorphic and polynomially bounded. Assume that G has a normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$. Then there exists a minimal LQG-balanced distributional resolvent linear system whose characteristic function coincides with G on some exponential region. Such a system is unique up to a unitary transformation in the state space.*

Proof. By Proposition 14.37 $[\mathsf{M}; \mathsf{N}]$ has a minimal Lyapunov-balanced realization $\check{\Sigma}_{\text{LYAP}}$. Note that since a minimal Lyapunov-balanced realization is strongly stable the set $\check{\Lambda}$ of $\check{\Sigma}_{\text{LYAP}}$ contains the whole right half-plane. Since $\check{\mathfrak{d}}_1$, the first component of the characteristic function of $\check{\Sigma}_{\text{LYAP}}$, coincides with M on the right half-plane we have that $\check{\mathfrak{d}}_1(s)$ has a bounded inverse for s in some exponential region and that $\check{\mathfrak{d}}_1^{-1}$ is polynomially bounded on some exponential region. Define the distributional resolvent linear system Σ by

$$\left[\begin{array}{c|c} \mathfrak{a} & \mathfrak{b} \\ \hline \mathfrak{c} & \mathfrak{d} \end{array}\right] := \left[\begin{array}{c|c} \check{\mathfrak{a}} - \check{\mathfrak{b}}\check{\mathfrak{d}}_1^{-1}\check{\mathfrak{c}}_1 & \check{\mathfrak{b}}\check{\mathfrak{d}}_1^{-1} \\ \hline \check{\mathfrak{c}}_2 - \check{\mathfrak{d}}_2\check{\mathfrak{d}}_1^{-1}\check{\mathfrak{c}}_1 & \check{\mathfrak{d}}_2\check{\mathfrak{d}}_1^{-1} \end{array}\right].$$

It follows from the above that this is indeed a distributional resolvent linear system. Apply the similarity transformation $(I - L^2)^{1/4}$ to Σ to obtain a distributional resolvent linear system Σ_{LQG}. Here L is the gramian of the minimal Lyapunov-balanced realization $\check{\Sigma}_{\mathrm{LYAP}}$. Now let $\alpha > 0$ be in the exponential region of all the above constructed systems. Applying the Cayley transform with parameter α to the above systems and comparing with the proof of Proposition 10.34 shows that the Cayley transform of Σ_{LQG} is minimal and LQG-balanced. It follows using Proposition 14.40 that Σ_{LQG} is. Its characteristic function $\check{\mathfrak{d}}_2\check{\mathfrak{d}}_1^{-1}$ equals $\mathsf{N}\mathsf{M}^{-1}$ on an exponential region and this equals G. The desired result follows. □

Remark 14.42. It is easily seen from the proof of Proposition 14.41 and remark 14.31 that if G is holomorphic and polynomially bounded on a right half-plane instead of only on an exponential region, then a minimal LQG-balanced realization is an exponentially bounded distributional resolvent linear system. Similarly it follows that a minimal LQG-balanced realization is well-posed if G is holomorphic and uniformly bounded on a right half-plane.

Definition 14.43. Given a compact LQG-balanced distributional resolvent linear system Σ, let (w_i) be an ordered sequence of eigenvectors of the optimal cost operator $Q^{\min}$ (the ordering is such that the corresponding eigenvalues μ_i form a nonincreasing sequence). Let $\alpha \in \Lambda_E$ with $\alpha > 0$. Let $n \in \mathbb{Z}^+$ be such that $\mu_n > \mu_{n+1}$. The α**-truncated LQG-balanced realization** of dimension n with respect to the sequence of eigenvectors (w_i) is defined as the restriction/projection of the operator

$$\begin{bmatrix} \mathfrak{a}(\alpha) & \mathfrak{b}(\alpha) \\ \mathfrak{c}(\alpha) & \mathfrak{d}(\alpha) \end{bmatrix}$$

onto $\mathscr{X}_n := \{w_i : i = 1, \ldots, n\}$.

The following result shows that the Cayley transform and α-LQG-balanced truncation commute.

Proposition 14.44. *Let Σ be a compact LQG-balanced distributional resolvent linear system and $\alpha \in \Lambda_E$ with $\alpha > 0$. The Cayley transform with parameter α of the α-truncated LQG-balanced realization of dimension n equals the truncated LQG-balanced realization of the Cayley transform with parameter α of Σ.*

Proof. This follows trivially from the definitions. □

From the above we immediately obtain the following analogues of Propositions 10.45, 10.46 and 10.47.

Proposition 14.45. *Let Σ be a compact LQG-balanced distributional resolvent linear system with $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional. Assume that the LQG-characteristic values (μ_i) form a summable sequence. Let $\alpha \in \Lambda_E$ with $\alpha > 0$. Denote by Σ_n the α-truncated LQG-balanced system of dimension n of Σ. Then we have*

$$\vec{\delta}_g(\Sigma, \Sigma_n) \leq 2 \sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}.$$

Proposition 14.46. *Let Σ be a compact LQG-balanced distributional resolvent linear system with $\mathscr{U}$ and $\mathscr{Y}$ finite-dimensional. Assume that the LQG-characteristic values (μ_i) form a summable sequence. Let $\alpha \in \Lambda_E$ with $\alpha > 0$. Denote by Σ_n the α-truncated LQG-balanced system of dimension n of Σ. Then there exists a $N \in \mathbb{Z}^+$ such that*

$$2 \sum_{i=N+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}} < \frac{1}{\sqrt{1+\mu_1^2}}. \tag{14.4}$$

For $n \geq N$ we have

$$\delta_g(\Sigma, \Sigma_n) \leq 2 \sum_{i=n+1}^{\infty} \frac{\mu_i}{\sqrt{1+\mu_i^2}}. \tag{14.5}$$

Remark 14.47. The condition that the LQG-characteristic values are summable is satisfied by many systems, but there are also many systems for which it is not satisfied. In Chapter 15 we consider a typical case in which the condition is satisfied. In that example it is crucial that the damping parameter β is positive, for $\beta = 0$ the LQG-characteristic values are not summable.

Remark 14.48. It follows from Proposition 14.46 and the results collected in Section 14.6 that the robust right factor stabilizing feedback function reviewed in Section 14.6 designed for the α-truncated LQG-balanced system of dimension n for n large enough stabilizes Σ. In fact, it also stabilizes all systems close to Σ in the gap metric and by choosing n large enough the robustness radius converges to the optimal robustness radius $1/\sqrt{1+\mu_1^2}$.

Notes

The definition of admissible feedback pair as given in this chapter are taken from Opmeer [66]. The solution of the linear quadratic optimal control problem as given here were also presented earlier in [66]. The results on coprime factorization are slight modifications of those in Curtain and Opmeer [16]. The results on LQG-balanced realizations are from Opmeer [68]. Earlier results on Lyapunov-balanced realizations in continuous-time for infinite-dimensional systems are among other Glover, Curtain and Partington [35] and Ober and Montgomery-Smith [62]. To obtain a complete theory of admissible feedback functions one needs to consider **controllers with internal loop** as in Weiss and Curtain [97] and Curtain, Weiss and Weiss [17].