Model reduction for controller design for infinite-dimensional systems

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Chapter 13

The Cayley transform

In this section we investigate the relationship between the class of resolvent linear systems and the class of discrete-time systems. This is the tool we shall use to deduce many properties of resolvent linear systems from the corresponding ones for discrete-time systems.

We first define the Cayley transforms of a resolvent linear system. Note that in the literature usually the Cayley transform with parameter $\alpha = 1$ is used.

Definition 13.1. Let $\alpha > 0$. The Cayley transform with parameter α of a resolvent linear system with $\alpha \in \Lambda$ is the discrete-time system with generating operators

$$\begin{aligned} A_d &:= -I + 2\alpha\ \mathfrak{a}(\alpha), & B_d &:= \sqrt{2\alpha}\ \mathfrak{b}(\alpha), \\ C_d &:= \sqrt{2\alpha}\ \mathfrak{c}(\alpha), & D_d &:= \mathfrak{d}(\alpha). \end{aligned} \tag{13.1}$$

Proposition 13.2. *The Cayley transform with parameter α gives a one-to-one correspondence between the set of resolvent linear systems with $\alpha \in \Lambda$ and the set of discrete-time systems.*

Proof. This follows from Proposition 11.5. □

Remark 13.3. The pseudoresolvent of a resolvent linear system is a resolvent if and only if -1 is not in the point spectrum of the state operator of its Cayley transform. A resolvent linear system is an operator node if and only if -1 is not in the point spectrum and not in the residual spectrum of the state operator of its Cayley transform. These conditions are very often hard, if not impossible to check. This was one of the reasons for introducing the class of resolvent linear systems instead of working with the class of operator nodes.

In the following remark we recall some facts about linear fractional transformations.

Remark 13.4. Let $\alpha \in \mathbb{C}$ be nonzero. The map $s \mapsto z = (\alpha - s)/(\alpha + s)$, with inverse $z \mapsto s = \alpha(1-z)/(1+z)$, maps $\mathbb{C} - \{-\alpha\}$ one-to-one onto $\mathbb{C} - \{-1\}$. The unit circle in the z-plane is the image of the line $\{s \in \mathbb{C} : \mathrm{Im}\alpha \ \mathrm{Im}s + \mathrm{Re}\alpha \ \mathrm{Re}s = 0\}$ in the s-plane. In particular, whenever α is real, the unit circle is the image of the imaginary axis. If $\alpha > 0$ the unit disc is the image of the right half-plane. For $\alpha > 0$ the map $M : H^2(\mathbb{C}_0^+, \mathscr{H}) \to H^2(\mathbb{D}, \mathscr{H})$ given by

$$(Mg)(z) = \frac{\sqrt{2\alpha}}{1+z} \ g\left(\alpha \ \frac{1-z}{1+z}\right), \tag{13.2}$$

is unitary with inverse

$$(M^{-1}f)(s) = \frac{\sqrt{2\alpha}}{\alpha+s} \ f\left(\frac{\alpha-s}{\alpha+s}\right). \tag{13.3}$$

The operator M is called the **Möbius operator**. With some abuse of notation we will denote the unitary operator $L^2(\mathbb{R}^+, \mathscr{H}) \to l^2(\mathbb{Z}^+, \mathscr{H})$ induced by M using the Z-transform and the Laplace transform by the same letter.

The above indicates that the above linear fractional transformation has nice mapping properties between the unit disc and the right half-plane. The situation is however drastically different when we look at arbitrary right half-planes, exponential regions and arbitrary discs centered at zero.

The line $\mathrm{Re}s = x$ in the s-plane is mapped onto the circle with center $-x/(\alpha + x)$ and radius $\alpha/(\alpha + x)$. Note that this circle contains the point -1, which is the image of the point at infinity. If α is chosen in the right half-plane $\mathrm{Re}s > x$, then the circle has zero in its interior.

An exponential region $\Lambda_E(a, b) := \{s \in \mathbb{C} : \mathrm{Re}\ s \geq b, \quad |\mathrm{Im}\ s| \leq \mathrm{e}^{a\mathrm{Re}\ s}\}$ that contains α in its interior is mapped onto a subset of the above indicated disc (where x is replaced by b is the formulas) since it is contained in the right half-plane $\mathrm{Re}s \geq b$. Also here -1 is on the boundary of the image since it is the image of the point at infinity.

A disc in the z-plane centered at zero with radius strictly smaller than one can never be mapped onto an exponential region or a right half-plane. This follows since the indicated disc does not have -1 on its boundary whereas the images of exponential regions and right half-planes do. Actually, the image of the indicated disc is a bounded region in the s-plane.

We have the following relation between a resolvent linear system and the resolvent, the wave functions and the characteristic function of its Cayley transform.

Proposition 13.5. *Let Σ be a resolvent linear system with $\alpha \in \Lambda$ where $\alpha > 0$. Let Σ_d be its Cayley transform with parameter α as defined in Definition 13.1. Denote the resolvent of Σ_d by $\mathfrak{A}$, its incoming wave function by $\mathfrak{B}$, its outgoing wave function by $\mathfrak{C}$ and its characteristic function by $\mathfrak{D}$. Let $s \in \Lambda$ and define $z := (\alpha - s)/\alpha + s)$. If $z \in 1/\rho(A_d)$, then*

$$\mathfrak{a}(s) = (1+z)\mathfrak{A}(z)\mathfrak{a}(\alpha), \qquad \mathfrak{b}(s) = \frac{1+z}{z\sqrt{2\alpha}}\,\mathfrak{B}(z), \tag{13.4}$$
$$\mathfrak{c}(s) = \frac{1+z}{\sqrt{2\alpha}}\,\mathfrak{C}(z), \qquad \mathfrak{d}(s) = \mathfrak{D}(z).$$

Proof. We first show that the equation

$$(I - zA_d)\mathfrak{a}(s) = (1+z)\mathfrak{a}(\alpha) \tag{13.5}$$

is equivalent to the functional equation (11.4). Substituting for A_d from (13.1) we see that (13.5) is equivalent to

$$(I - z\,[-I + 2\alpha\ \mathfrak{a}(\alpha)])\mathfrak{a}(s) = (1+z)\mathfrak{a}(\alpha).$$

Simplyfying the left-hand side shows that this is equivalent to

$$(1+z)\left[I - \frac{2\alpha}{1+z}\mathfrak{a}(\alpha)\right]\mathfrak{a}(s) = (1+z)\mathfrak{a}(\alpha).$$

Noting that $2\alpha z/(1+z) = \alpha - s$ and cancelling $1+z$ on both sides shows that this is equivalent to

$$[I - (\alpha - s)\mathfrak{a}(\alpha)]\,\mathfrak{a}(s) = \mathfrak{a}(\alpha),$$

and this is obviously equivalent to (11.4). Since $s \in \Lambda$ we have that (11.4) and therefore (13.5) holds. Since by assumption $z \in 1/\rho(A_d)$ it follows from (13.5) that $\mathfrak{a}(s) = (1+z)\mathfrak{A}(z)\mathfrak{a}(\alpha)$.

We now turn to the equation relating the incoming wave functions. We first note that the equation

$$(I - zA_d)\mathfrak{b}(s) = (1+z)\mathfrak{b}(\alpha)$$

is equivalent to the functional equation (11.5). The proof is almost exactly the same as the equivalence of (13.5) and (11.4) proven above and is left to the reader. The given equation for the incoming wave function follows easily. The argument for the equation relating the outgoing wave functions is entirely similar.

We prove the equation relating the characteristic functions. We have

$$\mathfrak{D}(z) = D_d + C_d \mathfrak{B}(z).$$

Using the relation between the incoming wave functions and substituting for D_d and C_d from (13.1) we obtain

$$\mathfrak{D}(z) = \mathfrak{d}(\alpha) + \frac{2\alpha z}{1+z} \mathfrak{c}(\alpha)\mathfrak{b}(s).$$

Since $2\alpha z/(1+z) = \alpha - s$ this is equivalent to

$$\mathfrak{D}(z) = \mathfrak{d}(\alpha) + (\alpha - s)\mathfrak{c}(\alpha)\mathfrak{b}(s).$$

Using the functional equation (11.7) it follows that $\mathfrak{D}(z) = \mathfrak{d}(s)$ as desired. □

Analogous to the discrete-time case for a distributional resolvent linear system we define the set of **stable input-output pairs**

$$\mathscr{V}(x_0) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \begin{bmatrix} L^2(\mathbb{R}^+; \mathscr{U}) \\ L^2(\mathbb{R}^+; \mathscr{Y}) \end{bmatrix} : y \text{ satisfies } (11.13) \right\}.$$

The following theorem shows that, for a suitably chosen parameter α, there is a one-to-one relationship between the stable input-output pairs of a distributional resolvent linear system and those of its Cayley transform.

Proposition 13.6. *Let Σ be a distributional resolvent linear system and $\alpha \in \Lambda_E$ and $\alpha > 0$. Let Σ_d be its Cayley transform with parameter α. Then $[u; y] \in \mathscr{V}(x_0)$ if and only if $[Mu; My] \in \mathscr{V}_d(x_0)$.*

Proof. Let $[u; y] \in \mathscr{V}(x_0)$. From (11.13) we obtain

$$\hat{y}(s) := \mathfrak{c}(s)x_0 + \mathfrak{d}(s)\hat{u}(s). \tag{13.6}$$

Since $\hat{u} \in H^2(\mathbb{C}_0^+, \mathscr{U})$ the above holds for $s \in \Lambda_E \cap \mathbb{C}_0^+$. Define $\Lambda_\alpha := \Lambda_E \cap \mathbb{C}_0^+$. Then Λ_α is an exponential region and it contains α. With $z := (\alpha - s)/\alpha + s)$ and using Proposition 13.5 we obtain from (13.6) that for $s \in \Lambda_\alpha$

$$\hat{y}(s) = \frac{1+z}{\sqrt{2\alpha}}\, \mathfrak{C}(z)x_0 + \mathfrak{D}(z)\hat{u}(s).$$

It follows that for z in a neighbourhood of zero (we use here that $\alpha \in \Lambda_\alpha$ is mapped to zero)

$$\frac{\sqrt{2\alpha}}{1+z}\hat{y}\left(\alpha\, \frac{1-z}{1+z}\right) = \mathfrak{C}(z)x_0 + \mathfrak{D}(z)\frac{\sqrt{2\alpha}}{1+z}\hat{u}\left(\alpha\, \frac{1-z}{1+z}\right). \tag{13.7}$$

But the right-hand side of (13.7) is the Z-transform of the output of Σ_d for initial state x_0 and input $M\hat{u}$ and the left-hand side equals $M\hat{y}$. Using the identity theorem for holomorphic functions we obtain that the output of Σ_d for initial state x_0 and input $M\hat{u}$ is $M\hat{y}$.

That $[u; y] \in \mathscr{V}_d(x_0)$ implies $[M^{-1}u; M^{-1}y] \in \mathscr{V}(x_0)$ follows in the same way. □

Proposition 13.6 is the key connection between continuous-time systems and their Cayley transforms. It is this result that will allow us to translate many results from discrete-time to continuous-time.

Notes

The definition of the Cayley transform presented here is inspired by and generalizes the one in Staffans [89, Section 12.3]. The main idea presented in this chapter, Proposition 13.6, was first put forward in Opmeer and Curtain [70] for well-posed linear systems.