Model reduction for controller design for infinite-dimensional systems

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Chapter 12

Partial differential equations

In this chapter we illustrate how partial differential equations with boundary control and observation fit into the framework presented in Chapter 11. We emphasize that the examples given in this chapter are certainly not the only ones that can be formulated in that framework.

In Section 12.1 we recall the concept of an abstract boundary control systems as studied in Salamon [85, Section 2.2] and show that in this setting our wave functions and characteristic function are solution operators of certain elliptic problems. In Section 12.2 we review some results on elliptic differential operators. In Section 12.3 we study partial differential equations which are first order in time (in particular the heat equation) and in Section 12.4 partial differential equations which are second order in time (in particular the wave equation).

12.1 Abstract boundary control systems

We review the concept of an abstract boundary control system.

Definition 12.1. An abstract boundary control system on a quadruple of Hilbert spaces $(\mathscr{U}, \mathscr{K}, \mathscr{X}, \mathscr{Y})$ where $\mathscr{K} \subset \mathscr{X}$ with a continuous and dense injection consists of three operators: $\Delta \in \mathcal{L}(\mathscr{K}, \mathscr{X})$, $\Gamma \in \mathcal{L}(\mathscr{K}, \mathscr{U})$, $K \in \mathcal{L}(\mathscr{K}, \mathscr{Y})$ that satisfy: Γ is onto, $\ker \Gamma$ is dense in $\mathscr{X}$, there exists a $\mu \in \mathbb{R}$ such that $\ker \mu I - \Delta \cap \ker \Gamma = \{0\}$ and $\mu I - \Delta$ is onto.

Let A be the restriction of Δ to $\ker \Gamma$, let C be the restriction of K to $\ker \Gamma$, and given $u \in U$, choose $x \in \mathscr{K}$ such that $\Gamma x = u$ and define

$$Bu = \Delta x - Ax, \quad \mathfrak{d}(\mu) = Kx - C(\mu I - A)^{-1}(\mu x - \Delta x).$$

(note that the A in the definition of B and $\mathfrak{d}$ above is the extension to an operator in $\mathcal{L}(\mathscr{X}, \mathscr{X}_{-1})$ as studied in Section 11.1 and that the definitions are independent of the particular x that is chosen). Then it follows as in Salamon [85, Proposition 2.8] that $A, B, C, \mathfrak{d}(\mu)$ determine an operator node (and hence a resolvent linear system).

It is interesting to note (see Salamon [85, p 391]) that for $\mu \in \rho(A)$ the operator $\mathfrak{b}(\mu)$ is the solution operator for the abstract elliptic problem

$$(\mu - \Delta)x = 0, \quad \Gamma x = u, \tag{12.1}$$

in the sense that for $u \in \mathscr{U}$ the solution is given by $x = \mathfrak{b}(\mu)u$. Similarly, $\mathfrak{a}(\mu)$ is the solution operator of the abstract elliptic problem

$$(\mu - \Delta)x = x_0, \quad \Gamma x = 0, \tag{12.2}$$

$\mathfrak{c}(\mu)$ is the solution operator of the abstract elliptic problem

$$(\mu - \Delta)x = x_0, \quad \Gamma x = 0, \quad Kx = y, \tag{12.3}$$

and $\mathfrak{d}(\mu)$ is the solution operator of the abstract elliptic problem

$$(\mu - \Delta)x = 0, \quad \Gamma x = u, \quad Kx = y. \tag{12.4}$$

Since it is not always easy to see what the space $\mathscr{K}$ should be, we will work with the abstract elliptic problems (12.1-12.4) and not directly with abstract boundary control systems.

With an abstract boundary control system the following dynamical system is associated

$$\begin{aligned} \dot{x}(t) = \Delta x(t), \quad x(0) = x_0, \\ \Gamma x(t) = u(t), \\ y(t) = Kx(t). \end{aligned}$$

We refer to Salamon [85, Section 2.2] and Staffans [89, Section 5.2] for more on abstract boundary control systems.

12.2 An elliptic differential operator

In this section we review some results from the literature on elliptic differential operators. In this section $\Omega \subset \mathbb{R}^n$ is a bounded open domain whose boundary $\partial\Omega$ is a compact orientable C^∞-manifold. We denote the standard Sobolev spaces by $H^s(\Omega)$. The space of infinitely differentiable functions with compact support in Ω is denoted by $C_0^\infty(\Omega)$. The space $H_0^s(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the $H^s(\Omega)$ norm.

An n-tuple of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$ is called a multi-index. We define

$$\zeta^\alpha = \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}, \qquad |\alpha| = \sum_{i=1}^n \alpha_i, \qquad D^\alpha = \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x^{\alpha_n}}.$$

We consider the differential operator L from $H^{2m}(\Omega)$ to $L^2(\Omega)$ defined by

$$L\varphi := \sum_{|\alpha| \leq 2m} a_\alpha D^\alpha \varphi,$$

with complex-valued coefficients a_α in $C^\infty(\overline{\Omega})$. The operator L is called **strongly elliptic** if there exists a constant $c > 0$ such that

$$\text{Re } (-1)^m \sum_{|\alpha|=2m} a_\alpha(\xi)\zeta^\alpha \geq c|\zeta|^{2m} \quad \xi \in \overline{\Omega}, \zeta \in \mathbb{R}^n.$$

The formal adjoint of L is the differential operator

$$L^*\psi := \sum_{|\alpha| \leq 2m} (-1)^{|\alpha|} D^\alpha \left(\overline{a_\alpha} \psi\right),$$

which is strongly elliptic if and only if L is.

A **Dirichlet form** is a sesquilinear form d on $H^m(\Omega)$ defined by

$$d(\varphi, \psi) := \sum_{|\rho|, |\sigma| \leq m} \langle D^\rho \varphi, a_{\rho\sigma} D^\sigma \psi \rangle_{L^2(\Omega)},$$

here $a_{\rho\sigma}$ are complex-valued functions in $C^\infty(\overline{\Omega})$. A Dirichlet form is called strongly elliptic if

$$\sum_{|\rho|, |\sigma| = m} a_{\rho\sigma}(\xi) \zeta^\rho \zeta^\sigma \geq c|\zeta|^{2m} \quad \xi \in \overline{\Omega}, \zeta \in \mathbb{R}^n,$$

for some constant $c > 0$. The adjoint of the Dirichlet form d is the Dirichlet form d^* defined by $d^*(\psi, \varphi) = \overline{d(\varphi, \psi)}$. A Dirichlet form d is a Dirichlet form for the operator L if

$$d(\varphi, \psi) = \langle \varphi, L\psi \rangle_{L^2(\Omega)} \quad \text{for all } \varphi, \psi \in C_0^\infty(\Omega).$$

Every differential operator as above has an associated Dirichlet form (this follows from integration by parts), however different Dirichlet forms can correspond to the same operator. This nonuniqueness will not be a problem for us. The differential operator L is strongly elliptic if and only if every Dirichlet form for L is strongly elliptic. If $d = d^*$, then $L = L^*$ and if $L = L^*$ then we can choose an associated Dirichlet form such that $d = d^*$.

The above can be found in Folland [27]. See also Agmon [1], Friedman [29] and Bers at al. [4].

12.3 First order equations

We consider the first order (in time) PDE with Dirichlet boundary control described by the equations

$$\frac{\partial x}{\partial t}(\xi,t)+Lx(\xi,t)=0,\quad \xi\in\Omega, t>0, \tag{12.5}$$

$$D_\nu^j x(\xi,t)=u_j(\xi,t),\quad \xi\in\partial\Omega, t>0, j=0,\dots m-1, \tag{12.6}$$

where $\Omega\subset\mathbb{R}^n$ is a bounded open domain whose boundary $\partial\Omega$ is a compact orientable C^∞-manifold, L is a strongly elliptic differential operator of order $2m$ (as defined in Section 12.2) and D_ν the normal derivative at $\partial\Omega$ directed towards the exterior of Ω.

We define the observation

$$y_j(\xi,t)=D_\nu^j x(\xi,t),\quad \xi\in\partial\Omega, t>0, j=m,\dots 2m-1. \tag{12.7}$$

This system can be written as an abstract boundary control system with the formal operators

$$\Delta=-L$$

$$\Gamma x=\begin{bmatrix} D_\nu^0 x|_{\partial\Omega}\\ \vdots \\ D_\nu^{m-1}x|_{\partial\Omega}\end{bmatrix}, Kx=\begin{bmatrix} D_\nu^m x|_{\partial\Omega}\\ \vdots \\ D_\nu^{2m-1}x|_{\partial\Omega}\end{bmatrix}.$$

However, the spaces $\mathscr{U}$, $\mathscr{K}$, $\mathscr{X}$, $\mathscr{Y}$ on which these formal operators have the desired properties are not obvious. To obtain these spaces we study the elliptic problems (12.1)-(12.4) with the operators Δ, Γ, K as above.

The pseudoresolvent

We first study the partial differential equation (12.5) with zero Dirichlet boundary conditions. This is a well-studied problem and we recall its solution. Define $A\varphi=-L\varphi$ on $D(A):=H^{2m}(\Omega)\cap H_0^m(\Omega)$. It follows as in Pazy [74, Section 7.2] that A generates an analytic semigroup on $L^2(\Omega)$.

Some spaces

We introduce some spaces needed in the sequel. The Hilbert space $\Xi^r(\Omega)$ for $r\in\mathbb{R}$ is defined as in Lions and Magenes [53, Section 2.6.3 p170]. We need these spaces for $r\in[-2m,0]$. The only properties of these spaces that we need are

$$\Xi^0(\Omega)=L^2(\Omega),\quad L^2(\Omega)\subset\Xi^r(\Omega),$$

with a continuous injection for $r \leq 0$. Fix $\mu \in \rho(A) \cap \mathbb{R}$ and define the space $D^r_{L+\mu}(\Omega)$ for $r \in [0, 2m]$ as in [53, Section 2.7.2 p 186]

$$D^r_{L+\mu}(\Omega) := \{x \in H^r(\Omega) : (L + \mu)x \in \Xi^{r-2m}(\Omega)\},$$

provided with the graph norm

$$\|x\|_{D^r_{L+\mu}(\Omega)} := \sqrt{\|x\|^2_{H^r(\Omega)} + \|(L + \mu)x\|^2_{\Xi^{r-2m}(\Omega)}},$$

which makes $D^r_{L+\mu}(\Omega)$ a Hilbert space. Note that for $r \in [0, 2m]$ we have $D^r_{L+\mu}(\Omega) \subset L^2(\Omega)$ with a continuous injection.

The incoming wave function

We study the incoming wave function. That is, we study the solution operator of the elliptic problem

$$\begin{aligned} (L + \mu)x &= 0 \quad \text{on } \Omega, \\ \Gamma x &= u \quad \text{on } \partial\Omega, \end{aligned}$$

where $\mu \in \rho(A)$ and L, Γ as above.

Define for $r \in [0, 2m]$ the space

$$\mathscr{U}^r := \Pi_{j=0}^{m-1} H^{r-j-1/2}(\partial\Omega).$$

It follows from [53, Theorem 7.4 p 188] that for all $r \in [0, 2m]$ the map $u \mapsto x$ from $\mathscr{U}^r$ to $D^r_{L+\mu}(\Omega)$ is bounded. It follows that the map $u \mapsto x$ from $\mathscr{U}^r$ to $L^2(\Omega)$ is bounded for all $r \in [0, 2m]$. Hence $\mathfrak{b}(\mu) \in \mathcal{L}(\mathscr{U}^r, L^2(\Omega))$.

The outgoing wave function

We study the outgoing wave function. We consider the problem

$$\begin{aligned} (L + \mu)x &= x_0 \quad \text{on } \Omega, \\ \Gamma x &= 0 \quad \text{on } \partial\Omega, \\ y &= Kx \quad \text{on } \partial\Omega, \end{aligned} \tag{12.8}$$

where $\mu \in \rho(A)$ and L, Γ, K are as above.

Define for $r \in [0, 2m]$ the space

$$\mathscr{Y}^r := \Pi_{j=0}^{m-1} H^{r-m-j-1/2}.$$

It follows from [53, Theorem 7.4 p 188] that for all $r \in [0, 2m]$ the map $x_0 \mapsto x$, defined by the first two equations of (12.8), from $\Xi^{r-2m}(\Omega)$ to $D^r_{L+\mu}(\Omega)$ is bounded. It follows from [53, Theorem 7.3 p 187] that for all $r \in [0, 2m]$ the operator $K : D^r_{L+\mu} \to \mathscr{Y}^r$ is bounded. It follows that the map $x_0 \mapsto y$ from $L^2(\Omega)$ to $\mathscr{Y}^r$ is bounded for all $r \in [0, 2m]$. Hence $\mathfrak{c}(\mu) \in \mathcal{L}(L^2(\Omega), \mathscr{Y}^r)$.

The characteristic function

We study the characteristic function. In order to do so we consider the elliptic problem

$$\begin{aligned} (L+\mu)x &= 0 \quad \text{on } \Omega, \\ \Gamma x &= u \quad \text{on } \partial\Omega, \\ y &= Kx \quad \text{on } \partial\Omega, \end{aligned} \tag{12.9}$$

where $\mu \in \rho(A)$ and L, Γ, K are as above.

It follows as in the case of the incoming wave function that for all $r \in [0, 2m]$ the map $u \mapsto x$, defined by the first two equations of (12.9), from $\mathscr{U}^r$ to $D^r_{L+\mu}(\Omega)$ is bounded. Combined with the result mentioned above on the operator K we obtain that for all $r \in [0, 2m]$ the map $u \mapsto y$ from $\mathscr{U}^r$ to $\mathscr{Y}^r$ is bounded.

First order equations: conclusion

The results obtained show that the PDE (12.5-12.7) can be formulated as a distributional resolvent linear system (even as a system node) on the state space $\mathscr{X} = L^2(\Omega)$ with possible choices of input and output spaces

$$\mathscr{U}^r := \Pi_{j=0}^{m-1} H^{r-j-1/2}(\partial\Omega), \quad \mathscr{Y}^r := \Pi_{j=0}^{m-1} H^{r-m-j-1/2},$$

for $r \in [0, 2m]$.

12.4 Second order equations

We consider the following second order (in time) PDE with Dirichlet boundary control and boundary observation

$$\frac{\partial^2 x}{\partial t^2}(\xi, t) + Lx(\xi, t) = 0 \quad \xi \in \Omega, t > 0, \tag{12.10}$$

$$D^j_\nu x(\xi, t) = u_j(\xi, t), \quad \xi \in \partial\Omega, t > 0, j = 0, \dots m-1, \tag{12.11}$$

$$y_j(\xi, t) = D^j_\nu x(\xi, t), \quad \xi \in \partial\Omega, t > 0, j = m, \dots 2m-1. \tag{12.12}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded open domain whose boundary $\partial\Omega$ is a compact orientable C^∞-manifold and $L = L^*$ is a self-adjoint strongly elliptic differential operator (see Section 12.2).

As in section 12.3 the formal differential operator, formal boundary control operator and formal boundary observation operator are obvious:

$$\tilde{\Delta} = \begin{bmatrix} 0 & I \\ -L & 0 \end{bmatrix}, \quad \tilde{\Gamma} := [\Gamma \;\; 0], \quad \tilde{K} := [K \;\; 0],$$

where Γ and K are as in Section 12.3. We use the theory of cosine functions and that of elliptic problems to determine the spaces $\mathscr{U}$, $\mathscr{K}$, $\mathscr{X}$, $\mathscr{Y}$ on which these formal operators have the desired properties.

The pseudoresolvent

We first study the operator A as defined in Section 12.3 further for the case $L = L^*$ as considered here. It follows as in Fattorini [25, Section IV.8] that A generates a cosine function on $L^2(\Omega)$ (note that the arguments in [25] only make use of the fact that $d = d^*$). This implies that

$$\tilde{A} := \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

with domain $H^{2m}(\Omega) \cap H_0^m(\Omega) \times L^2(\Omega)$ generates an exponentially bounded integrated semigroup on $L^2(\Omega) \times L^2(\Omega)$ (see Arendt et al. [3, Theorem 3.14.7]).

The incoming wave function

We see that the elliptic problem (12.1) is equivalent to

$$(L + \mu^2)x_1 = 0, \quad \Gamma x_1 = u, \quad x_2 = \mu x_1,$$

so it follows as in the case of the incoming wave function for first order equations that the map $u \mapsto x = [x_1; x_2]$ is bounded from $\mathscr{U}^r$ to $D^r_{L+\mu^2}(\Omega) \times \mathscr{H}$ for any Hilbert space $\mathscr{H}$ such that $D^r_{L+\mu^2}(\Omega) \subset \mathscr{H}$ continuously for all $r \in [0, 2m]$ for $\mu^2 \in \rho(A)$. it follows that the map $u \mapsto x = [x_1; x_2]$ is bounded from $\mathscr{U}^r$ to $L^2(\Omega) \times L^2(\Omega)$ for all $r \in [0, 2m]$ for $\mu^2 \in \rho(A)$. Hence $\mathfrak{b}(\mu) \in \mathcal{L}(\mathscr{U}^r, L^2(\Omega) \times L^2(\Omega))$.

The outgoing wave function

We see that the elliptic problem (12.3) is equivalent to

$$(L + \mu^2)x_1 = x_2^0 + \mu x_1^0, \quad x_2 = \mu x_1 - x_1^0, \quad \Gamma x_1 = 0, \quad y = K x_1,$$

so it follows as in the case of the incoming wave function for first order equations that the map $x^0 = [x_1^0; x_2^0] \mapsto y$ is bounded from $\Xi^{r-2m}(\Omega) \times \Xi^{r-2m}(\Omega)$ to $\mathscr{Y}^r$ for all $r \in [0, 2m]$ for $\mu^2 \in \rho(A)$. Hence we obtain that the map $x^0 = [x_1^0; x_2^0] \mapsto y$ is bounded from $L^2(\Omega) \times L^2(\Omega)$ to $\mathscr{Y}^r$ for all $r \in [0, 2m]$. Hence $\mathfrak{c}(\mu) \in \mathcal{L}(L^2(\Omega) \times L^2(\Omega), \mathscr{Y}^r)$.

The characteristic function

We see that the elliptic problem (12.4) is equivalent to

$$(L+\mu^2)x_1 = 0, \quad x_2 = \mu x_1 - x_1^0, \quad \Gamma x_1 = u, \quad y = Kx_1,$$

so it follows as in the case of the characteristic function for first order equations that the map $u \mapsto y$ is bounded from $\mathscr{U}^r$ to $\mathscr{Y}^r$ for all $r \in [0, 2m]$ for $\mu^2 \in \rho(A)$. Hence $\mathfrak{d}(\mu) \in \mathcal{L}(\mathscr{U}^r, \mathscr{Y}^r)$.

Second order equations: conclusion

The results obtained in this section show that the PDE (12.10-12.12) can be formulated as a distributional resolvent linear system on the state space $\mathscr{X} = L^2(\Omega) \times L^2(\Omega)$ with possible choices of input and output spaces

$$\mathscr{U}^r := \Pi_{j=0}^{m-1} H^{r-j-1/2}(\partial\Omega), \quad \mathscr{Y}^r := \Pi_{j=0}^{m-1} H^{r-m-j-1/2},$$

for $r \in [0, 2m]$. Note that since $\tilde{A}$ does not generate a strongly continuous semigroup on $L^2(\Omega) \times L^2(\Omega)$, this distributional resolvent linear system is not a system node.

Notes

The content of this chapter appeared before in Opmeer [67]. Virtually all results depend on the study of non-homogeneous boundary value problems performed in Lions and Magenes [53].