Model reduction for controller design for infinite-dimensional systems

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Part II

Continuous-time systems

Chapter 11

Basic objects

In this chapter we provide a new framework for continuous-time systems.

11.1 Resolvent linear systems

A finite-dimensional linear system is usually described by specifying four matrices A, B, C, D and defining for a given initial state x_0 and an input function $u \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{\mathrm{u}})$ the state $x \in C(0,\infty;\mathbb{C}^{\mathrm{x}})$ and the output $y \in L^2_{\mathrm{loc}}(0,\infty;\mathbb{C}^{\mathrm{y}})$ as the unique solutions of

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad y(t) = Cx(t) + Du(t). \tag{11.1}$$

As is well-known, these unique solutions are given explicitly by

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\,ds, \tag{11.2}$$

$$y(t) = C\mathrm{e}^{At}x_0 + \int_0^t C\mathrm{e}^{A(t-s)}Bu(s)\,ds + Du(t).$$

If we Laplace transform the equations (11.1) and solve for x and y we obtain

$$\hat{x}(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}B\hat{u}(s) \tag{11.3}$$

$$\hat{y}(s) = C(sI-A)^{-1}x_0 + \big(C(sI-A)^{-1}B + D\big)\,\hat{u}(s).$$

Our approach to continuous-time infinite-dimensional systems will be to generalize the situation (11.3) rather than the situation (11.1) or (11.2).

In this section we study the generalizations of the matrix-valued functions $(sI-A)^{-1}$, $(sI-A)^{-1}B$, $C(sI-A)^{-1}$ and $C(sI-A)^{-1}B+D$. The generalization of the dynamical system (11.3) will be considered in Section 11.2. We first consider the generalization of the resolvent.

Definition 11.1. Let $\mathscr{X}$ be a Hilbert space and Λ a nonempty subset of the complex plane. A function $\mathfrak{a} : \Lambda \to \mathcal{L}(\mathscr{X})$ that satisfies the following **resolvent equation**

$$\mathfrak{a}(\beta) - \mathfrak{a}(\alpha) = (\alpha - \beta)\mathfrak{a}(\beta)\mathfrak{a}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda$$

is called a **pseudoresolvent**. A pseudoresolvent $\mathfrak{a}_{\max}$ is called a **maximal pseudoresolvent** if there is no pseudoresolvent that is a proper extension of $\mathfrak{a}_{\max}$.

Lemma 11.2. *Every pseudoresolvent has a unique extension to a maximal pseudoresolvent* $\mathfrak{a}_{\max} : \Lambda_{\max} \to \mathcal{L}(\mathscr{X})$. *The set* $\Lambda_{\max}$ *is open and* $\mathfrak{a}_{\max}$ *is holomorphic.*

Proof. This is contained in Hille and Phillips [38, Chapter 5.2]. □

We now consider the generalization of all the indicated matrix-valued functions.

Definition 11.3. A **resolvent linear system** on a triple of Hilbert spaces $(\mathscr{U}, \mathscr{X}, \mathscr{Y})$ consists of a nonempty subset Λ of the complex plane and four operator valued function $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}$ satisfying
$\mathfrak{a} : \Lambda \to \mathcal{L}(\mathscr{X})$ satisfies

$$\mathfrak{a}(\beta) - \mathfrak{a}(\alpha) = (\alpha - \beta)\mathfrak{a}(\beta)\mathfrak{a}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{11.4}$$

$\mathfrak{b} : \Lambda \to \mathcal{L}(\mathscr{U}, \mathscr{X})$ satisfies

$$\mathfrak{b}(\beta) - \mathfrak{b}(\alpha) = (\alpha - \beta)\mathfrak{a}(\beta)\mathfrak{b}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{11.5}$$

$\mathfrak{c} : \Lambda \to \mathcal{L}(\mathscr{X}, \mathscr{Y})$ satisfies

$$\mathfrak{c}(\beta) - \mathfrak{c}(\alpha) = (\alpha - \beta)\mathfrak{c}(\alpha)\mathfrak{a}(\beta) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{11.6}$$

$\mathfrak{d} : \Lambda \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ satisfies

$$\mathfrak{d}(\beta) - \mathfrak{d}(\alpha) = (\alpha - \beta)\mathfrak{c}(\beta)\mathfrak{b}(\alpha) \quad \text{for all } \alpha, \beta \in \Lambda. \tag{11.7}$$

The function $\mathfrak{a}$ is called the **pseudoresolvent**, $\mathfrak{b}$ the **incoming wave function**, $\mathfrak{c}$ the **outgoing wave function** and $\mathfrak{d}$ the **characteristic function** of the resolvent linear system. The pseudoresolvent is assumed to be maximal.

Proposition 11.4. *The pseudoresolvent, the wave functions and the characteristic function of a resolvent linear system are holomorphic.*

Proof. For the pseudoresolvent this was already stated in Lemma 11.2. For the other three functions it follows from the functional equations using that the pseudoresolvent is holomorphic. For example to prove that $\mathfrak{b}$ is holomorphic in a point β first fix a point α and note that the term on the right-hand side of (11.5) is holomorphic in β. It follows that the term on the left-hand side of the equation is and since $\mathfrak{b}(\alpha)$ is constant it follows that $\mathfrak{b}$ is holomorphic in β. □

A resolvent linear system is completely determined by the values of the pseudoresolvent, the wave functions and the characteristic function at one point in the following sense.

Proposition 11.5. *For $a \in \mathcal{L}(\mathscr{X})$, $b \in \mathcal{L}(\mathscr{U}, \mathscr{X})$, $c \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$, $d \in \mathcal{L}(\mathscr{U}, \mathscr{Y})$ and $\alpha \in \mathbb{C}$ there exists a unique resolvent linear system with $\alpha \in \Lambda$ and $\mathfrak{a}(\alpha) = a$, $\mathfrak{b}(\alpha) = b$, $\mathfrak{c}(\alpha) = c$, $\mathfrak{d}(\alpha) = d$.*

Proof. The function $\tilde{\mathfrak{a}} : \{\alpha\} \to \mathcal{L}(\mathscr{X})$ defined by $\tilde{\mathfrak{a}}(\alpha) = a$ defines a pseudoresolvent. By Lemma 11.2 it has a maximal extension which we denote by $\mathfrak{a}$ and whose domain we denote by Λ. Define the operator-valued functions $\mathfrak{b} : \Lambda \to \mathcal{L}(\mathscr{U}, \mathscr{X})$, $\mathfrak{c} : \Lambda \to \mathcal{L}(\mathscr{X}, \mathscr{Y})$, $\mathfrak{d} : \Lambda \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ by $\mathfrak{b}(s) := b + (\alpha - s)\mathfrak{a}(s)b$, $\mathfrak{c}(s) := c + (\alpha - s)c\mathfrak{a}(s)$, $\mathfrak{d}(s) := d + (\alpha - s)\mathfrak{c}(s)b$. It is easily seen that his gives a resolvent linear system. The desired uniqueness follows from the uniqueness of the maximal pseudoresolvent. □

We now show how unbounded operators A, B, C can be constructed that generalize the matrices considered earlier in this section. Assume that $\mathfrak{a}$ is the resolvent of a densely defined closed operator A with nonempty resolvent set. A necessary and sufficient condition for such an A to exist is that there exists an $\alpha \in \Lambda$ such that $\mathfrak{a}(\alpha)$ is one-to-one and has dense range. We now introduce two spaces. Let $\mathscr{X}_1$ be $D(A)$ with the norm $\|x\|_1 := \|(\alpha - A)x\|$. For every $\alpha \in \rho(A)$ this is a Hilbert space with norm equivalent to the graph norm. Let $\mathscr{X}_{-1}$ be the completion of $\mathscr{X}$ with respect to the norm $\|x\|_{-1} := \|\mathfrak{a}(\alpha)x\|$. The operator A has an extension $A_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}_{-1}$. Define $B : \mathscr{U} \to \mathscr{X}_{-1}$ by $B := (\alpha - A_{\mathscr{X}})\mathfrak{b}(\alpha)$, it follows from the functional equation (11.5) that B does not depend on α. Define the operator $C : \mathscr{X}_1 \to \mathscr{Y}$ by $C := \mathfrak{c}(\alpha)(\alpha - A)$, it follows from the functional equation (11.6) that C does not depend on α. A meaningful generalization of the matrix D is not always possible.

We make the following definition.

Definition 11.6. An **operator node** is a resolvent linear system for which the pseudoresolvent is the resolvent of a densely defined closed operator with nonempty resolvent set.

Remark 11.7. One can define an operator node through four generating operators. An operator A on the state space $\mathscr{X}$ which is densely defined and has nonempty resolvent set. An operator $B \in \mathcal{L}(\mathscr{U}, \mathscr{X}_{-1})$, an operator $C \in \mathcal{L}(\mathscr{X}_1, \mathscr{Y})$ and an operator $D \in \mathcal{L}(\mathscr{U}, \mathscr{Y})$. The corresponding resolvent linear system is then defined as follows. The pseudoresolvent is the resolvent of A. The incoming wave function is defined as $\mathfrak{b}(s) := (sI - A_{\mathscr{X}})^{-1}B$, the outgoing wave function by $\mathfrak{c}(s) := C(sI - A)^{-1}$ and the charcteristic function by fixing $\alpha \in \rho(A)$, defining $\mathfrak{d}(\alpha) = D$ and extending this to the whole of $\rho(A)$ by using (11.7).

11.2 Distributional resolvent linear systems

In this section we define a subclass of the set of resolvent linear systems for which the dynamical system (11.3) has a meaningful generalization.

Definition 11.8. A **distributional resolvent linear system** is a resolvent linear system with the additional propery that there exist constants $\alpha > 0, \beta \in \mathbb{R}$ and a polynomial p such that

$$\Lambda_E(\alpha, \beta) := \{s \in \mathbb{C} : \text{Re } s \geq \beta, \quad |\text{Im } s| \leq e^{\alpha \text{Re } s}\} \subset \Lambda \tag{11.8}$$

and

$$\|\mathfrak{a}(s)\| \leq p(|s|) \quad \forall\ s \in \Lambda_E. \tag{11.9}$$

A region Λ_E as above is called an exponential region (see Arendt, El-Mennaoui and Kéyantuo [2]). Note that the wave functions and characteristic function of a distributional resolvent linear system are also polynomially bounded on Λ_E (this follows from the functional equations in Definition 11.3).

Equivalently we could assume that the pseudoresolvent is polynomially bounded on a logarithmic region. A logarithmic region is a region of the form

$$\Lambda_L(a, b, c) := \{s \in \mathbb{C} : \text{Re } s \geq c, \text{Re } s \geq \frac{1}{a} \log |s| + b\} \tag{11.10}$$

with $a > 0$ and $b, c \in \mathbb{R}$. This is true since one can show that an exponential region is contained in a logarithmic region is contained in an exponential region (see Arendt, El-Mennaoui and Kéyantuo [2]).

We also define the following subclass of distributional resolvent linear systems where we do not work on an exponential region, but on a half-plane.

Definition 11.9. A distributional resolvent linear system is called **exponentially bounded** if there exists a $\gamma \in \mathbb{R}$ and a polynomial p such that

$$\Lambda_H(\gamma) := \{s \in \mathbb{C} : \text{Re } s \geq \gamma\} \subset \Lambda \tag{11.11}$$

and

$$\|\mathfrak{a}(s)\| \leq p(|s|) \quad \forall\ s \in \Lambda_H. \tag{11.12}$$

Remark 11.10. In the sequel we will need the following well-known characterization of Laplace transformable Banach space valued distributions by Schwartz. The image of the Schwartz-Laplace transformable Banach-space valued distributions is exactly the set of polynomially bounded holomorphic functions defined on some right half-plane. For details see Schwartz [86]. A generalization of this characterization is due to Kunstmann [49]. He defined a space of Banach space valued distributions that can be Laplace transformed and whose image under the Laplace transform is exactly the set of polynomially bounded holomorphic functions defined on some exponential region.

Using Remark 11.10 we are now in a position to generalize the dynamical system (11.3). Let u be a $\mathscr{U}$-valued Kunstmann-Laplace transformable distribution. For a distributional resolvent linear system $\mathfrak{a}(s)x_0 + \mathfrak{b}(s)\hat{u}(s)$ is holomorphic and polynomially bounded on some exponential region and therefore it is the Kunstmann-Laplace transform of some $\mathscr{X}$-valued Kunstmann-Laplace transformable distribution. Similar arguments apply to $\mathfrak{c}(s)x_0 + \mathfrak{d}(s)\hat{u}(s)$. This leads to the following definition.

Definition 11.11. The state x and output y of a distributional resolvent linear system corresponding to the initial state $x_0 \in \mathscr{X}$ and the input u (a $\mathscr{U}$-valued Kunstmann-Laplace transformable distribution) are defined through their Kunstmann-Laplace transforms by

$$\hat{x}(s) := \mathfrak{a}(s)x_0 + \mathfrak{b}(s)\hat{u}(s), \quad \hat{y}(s) := \mathfrak{c}(s)x_0 + \mathfrak{d}(s)\hat{u}(s), \tag{11.13}$$

where s is restricted to the intersection of Λ_E and the exponential region on which $\hat{u}$ is holomorphic and polynomially bounded.

Remark 11.12. If the distributional resolvent linear system in Definition 11.11 is assumed to be exponentially bounded and the input u is assumed to be a Schwartz-Laplace transformable distribution, then the state and output of the system are Schwartz-Laplace transformable distributions.

We recall the concept of a system node. See Staffans [89, Section 4.7].

Definition 11.13. A **system node** is an operator node for which A is the generator of a strongly continuous semigroup.

Remark 11.14. Since the resolvent of the generator of a strongly continuous semigroup is uniformly bounded on a right half-plane by the Hille-Yosida conditions, a system node defines an exponentially bounded distributional resolvent linear system.

The concept of a well-posed system as given below is equivalent to the usual one as can be found in Staffans [89].

Definition 11.15. A distributional resolvent linear system is called **well-posed** if there exists a $\sigma \in \mathbb{R}$ such that

- the pseudoresolvent is the resolvent of the generator of a strongly continuous semigroup,
- the restriction of $\mathfrak{b}(\cdot)^{\dagger} x_0$ to the right half-plane $\mathbb{C}_\sigma^+$ is an element of $H^2(\mathbb{C}_\sigma^+, \mathscr{U})$ for all $x_0 \in \mathscr{X}$,
- the restriction of $\mathfrak{c}(\cdot) x_0$ to the right half-plane $\mathbb{C}_\sigma^+$ is an element of $H^2(\mathbb{C}_\sigma^+, \mathscr{Y})$ for all $x_0 \in \mathscr{X}$,
- the restriction of the characteristic function to the right half-plane $\mathbb{C}_\sigma^+$ is an element of $H^\infty(\mathbb{C}_\sigma^+, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$.

Figure 11.1 gives a picture of the inclusion relationships between the different classes of systems we have encountered.

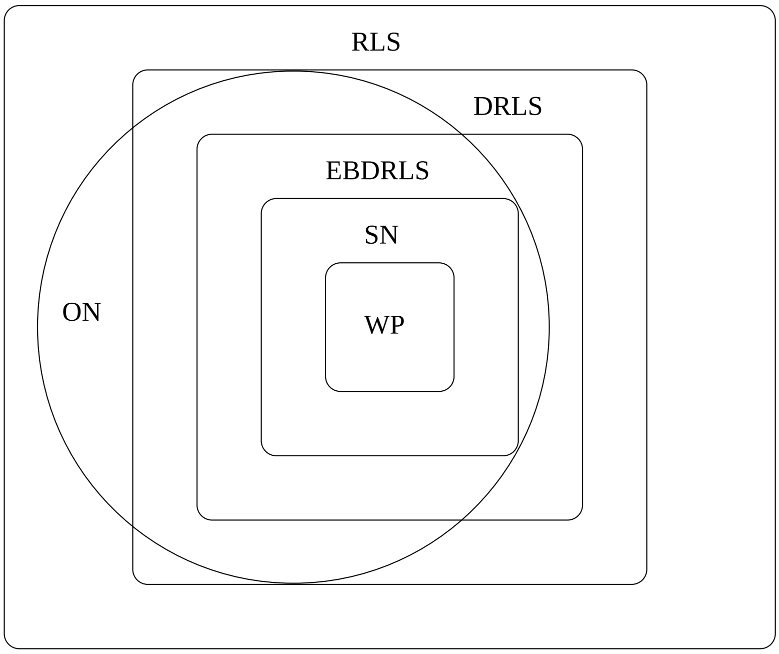

Figure 11.1: Classes of systems. WP=well-posed, SN=system nodes, EBDRLS=exponentially bounded distributional resolvent linear systems, DRLS=distributional resolvent linear systems, ON=operator nodes, RLS= resolvent linear systems.

Notes

The definition of resolvent linear system is taken from Opmeer [66], where also the subclasses of distributional resolvent linear systems and exponentially bounded distributional resolvent linear systems were introduced (the last class under the name integrated resolvent linear systems). See Opmeer [67] for the corresponding time-domain definitions.

The set of operator nodes is implicitly present in Salamon [85]. It is the set of systems that satisfy his assumption (S0) on page 385, but not necessarily his assumptions (S1) to (S4). We refer to Staffans [89, Section 4.7] for alternative characterizations of operator nodes and historical remarks.

Our assumption on the pseudoresolvent in the case of distributional resolvent linear systems (exponentially bounded or not) is much weaker than assumption (S1) of Salamon [85] (the system node assumption). Moreover, we drop assumptions (S2-S4) of Salamon. Hence we obtain a much larger class of systems than the well-posed linear systems introduced by Salamon in [85]. This class of well-posed linear systems has been the state-of-the-art for the last two decades (see Staffans [89]).

The concept of a distributional resolvent linear system is the natural generalization of the concept of distribution semigroup from systems with only a state to input/state/output systems. Distribution semigroups were introduced by Lions [52]. Important contributions were made by Chazarain [7]. The case of not necessarily densely defined generators A is treated in Kunstmann [48] and Wang [95]. The general case (including the degenerate case where the pseudoresolvent is not a resolvent) is treated in Kisyński [44]. See Fattorini [26] for further information on distribution semigroups.