Model reduction for controller design for infinite-dimensional systems

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Chapter 9

The gap metric

In this chapter we provide an alternative view towards ε-admissible right factor perturbations. This is done in terms of the gap metric.

We first consider the gap metric as a metric on the set of closed subspaces of a given Hilbert space. The relevant definition is as follows.

Definition 9.1. Let $\mathscr{K}_i$ $(i = 1, 2)$ be closed subspaces of the Hilbert space $\mathscr{H}$. Denote by P_i the orthogonal projection onto $\mathscr{K}_i$. Define

$$\delta(\mathscr{K}_1, \mathscr{K}_2) = \|P_1 - P_2\|.$$

The function δ is called the **gap metric**.

Lemmas 9.2 through 9.7 give some basic properties of the gap metric.

Lemma 9.2. *The gap metric is a metric on the set of closed subspaces of a given Hilbert space.*

Proof. Symmetry is obvious. It is also obvious that $\delta(\mathscr{K}_1, \mathscr{K}_2) = 0$ implies $\mathscr{K}_1 = \mathscr{K}_2$. The triangle inequality follows from the triangle inequality in $\mathcal{L}(\mathscr{H})$. □

Definition 9.3. Let $\mathscr{K}_i$ $(i = 1, 2)$ be closed subspaces of the Hilbert space $\mathscr{H}$. Denote by P_i the orthogonal projection onto $\mathscr{K}_i$. Define

$$\vec{\delta}(\mathscr{K}_1, \mathscr{K}_2) = \|(I - P_2)P_1\|.$$

The function $\vec{\delta}$ is called the **directed gap**.

Lemma 9.4. *We have*

$$\delta(\mathscr{K}_1, \mathscr{K}_2) = \max\{\vec{\delta}(\mathscr{K}_1, \mathscr{K}_2), \vec{\delta}(\mathscr{K}_2, \mathscr{K}_1)\}.$$

Proof. We have

$$(P_1 - P_2)x = P_1(I-P_2)x - (I-P_1)P_2x = P_1(I-P_2)^2x - (I-P_1)P_2^2x.$$

Using that $P_1(I-P_2)^2x$ and $(I-P_1)P_2^2x$ are orthogonal we obtain from this that

$$\begin{aligned}\|(P_1-P_2)x\|^2 &= \|P_1(I-P_2)^2x - (I-P_1)P_2^2x\|^2 \\ &= \|P_1(I-P_2)^2x\|^2 + \|(I-P_1)P_2^2x\|^2 \\ &\le \|P_1(I-P_2)\|^2\|(I-P_2)x\|^2 + \|(I-P_1)P_2\|^2\ \|P_2x\|^2.\end{aligned}$$

Since $\|(I-P_2)x\|^2 + \|P_2x\|^2 = \|x\|^2$ we obtain from this

$$\|(P_1-P_2)x\|^2 \le \max\{\|P_1(I-P_2)\|^2, \|(I-P_1)P_2\|^2\}\ \|x\|^2.$$

Since the adjoint of $(I-P_1)P_2$ equals $P_2(I-P_1)$ we have

$$\|(P_1-P_2)x\|^2 \le \max\{\|P_1(I-P_2)\|^2, \|P_2(I-P_1)\|^2\}\ \|x\|^2.$$

It follows that $\delta(\mathscr{K}_1,\mathscr{K}_2) \le \max\{\vec{\delta}(\mathscr{K}_1,\mathscr{K}_2), \vec{\delta}(\mathscr{K}_2,\mathscr{K}_1)\}$. The converse inequality follows from

$$\|(I-P_2)P_1\| = \|(P_1-P_2)P_1\| \le \|P_1-P_2\|$$

and the similar inequality with the roles of P_1 and P_2 reversed. □

Lemma 9.5. *We have $\delta(\mathscr{K}_1,\mathscr{K}_2) < 1$ if and only if P_1 restricts to a bijection from $\mathscr{K}_2$ onto $\mathscr{K}_1$. In this case we have $\delta(\mathscr{K}_1,\mathscr{K}_2) = \vec{\delta}(\mathscr{K}_1,\mathscr{K}_2) = \vec{\delta}(\mathscr{K}_2,\mathscr{K}_1)$.*

Proof. Assume that $\delta(\mathscr{K}_1,\mathscr{K}_2) < 1$. This implies that $\|P_1 - P_2\| < 1$, from which it follows that $T := I - P_1 + P_2$ has a bounded inverse. We have $P_1T = P_1P_2$. Since T maps $\mathscr{H}$ onto $\mathscr{H}$ and P_1 maps $\mathscr{H}$ onto $\mathscr{K}_1$, it follows that P_1P_2 maps $\mathscr{H}$ onto $\mathscr{K}_1$. Obviously it then follows that P_1 maps $\mathscr{K}_2$ onto $\mathscr{K}_1$. If $h \in \mathscr{K}_2$ is such that $P_1h = 0$, then $\|h\| = \|(I-P_1)h\|$, since $[P_1; I-P_1]$ is an isometry. Since $h \in \mathscr{K}_2$ we have $P_2h = h$ and so we obtain $\|h\| = \|(I-P_1)P_2h\| \le \|P_2 - P_1\|\ \|h\|$. Since by assumption $\|P_2 - P_1\| < 1$, this can only hold if $h = 0$. It follows that P_1 restricted to $\mathscr{K}_2$ is injective.

Now assume that P_1 restricts to a bijection from $\mathscr{K}_2$ onto $\mathscr{K}_1$. Denote this restriction by P_1^r. Define

$$\tau_1 := \inf_{h\in\mathscr{K}_2, \|h\|=1} \|P_1^r h\|.$$

Since P_1^r has a bounded inverse, we have $\tau_1 > 0$. Using that $\|(I-P_1)P_2h\|^2 = \|P_2h\|^2 - \|P_1P_2h\|^2$ (which follows from the Pythagorean Theorem), we have

$$\sup_{h\in\mathscr{K}_2,\|h\|=1} \|(I-P_1)P_2h\|^2 = 1 - \inf_{h\in\mathscr{K}_2,\|h\|=1} \|P_1h\|^2 = 1 - \tau_1^2.$$

From this we obtain $\|(I-P_1)P_2\|^2 = 1-\tau_1^2$. By interchanging the role of P_1 and P_2 we obtain $\|(I-P_2)P_1\|^2 = 1-\tau_2^2$, where $\tau_2 := \inf_{h\in\mathscr{K}_1,\|h\|=1}\|P_2^r h\|$ and P_2^r is the restriction of P_2 to $\mathscr{K}_1$. We now show that the adjoint of P_1^r equals P_2^r. Let $h_1 \in \mathscr{K}_1$ and $h_2 \in \mathscr{K}_2$, then

$$\langle P_1^r h_2, h_1\rangle_{\mathscr{K}_1} = \langle P_1 h_2, h_1\rangle_{\mathscr{H}} = \langle h_2, P_1 h_1\rangle_{\mathscr{H}} = \langle h_2, h_1\rangle_{\mathscr{H}}.$$

Similarly we obtain $\langle h_2, h_1\rangle_{\mathscr{H}} = \langle h_2, P_2^r h_1\rangle_{\mathscr{K}_2}$. We conclude that the adjoint of $P_1^r \in \mathcal{L}(\mathscr{K}_2,\mathscr{K}_1)$ is $P_2^r \in \mathcal{L}(\mathscr{K}_1,\mathscr{K}_2)$. From Lemma 7.28 we obtain $\tau_1 = \tau_2$. So we obtain $\vec{\delta}(\mathscr{K}_1,\mathscr{K}_2) = \vec{\delta}(\mathscr{K}_2,\mathscr{K}_1) < 1$. □

Lemma 9.6. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and $T \in \mathcal{L}(\mathscr{H}_1,\mathscr{H}_2)$. Assume that T is an isometry. Then the orthogonal projection onto the range of T is given by $P := TT^*$.*

Proof. For this we have to show three things:

1. P is a projection, i.e. $P = P^2$.
2. The projection is orthogonal, i.e. $P = P^*$.
3. P maps onto the image of T.

$P^2 = TT^*TT^* = TT^* = P$, where we have used that $T^*T = I$ since T is an isometry. That P is self-adjoint is obvious, so the projection is orthogonal. We have

$$\operatorname{Im}(P) = \operatorname{Im}(TT^*) \subset \operatorname{Im}(T) = \operatorname{Im}(TT^*T) \subset \operatorname{Im}(TT^*) = \operatorname{Im}(P),$$

where we have again used that $T^*T = I$. So $\operatorname{Im}(P) = \operatorname{Im}(T)$. □

Lemma 9.7. *Let $T_1 \in \mathcal{L}(\mathscr{H}_1,\mathscr{H}_2)$ be an isometry and $\mathscr{K}_2$ a closed subspace of $\mathscr{H}_2$. Define $\mathscr{K}_1 := \operatorname{Im}(T_1)$. Then $\mathscr{K}_1$ is a closed subspace of $\mathscr{H}_2$ and*

$$\vec{\delta}(\mathscr{K}_1,\mathscr{K}_2) = \|(I-P_2)T_1\|.$$

Proof. That the image of an isometry is closed is easily proven. Using Lemma 9.6 we obtain $P_1 = T_1T_1^*$, so

$$\vec{\delta}(\mathscr{K}_1,\mathscr{K}_2) = \|(I-P_2)T_1T_1^*\|.$$

So we need to show that $\|(I-P_2)T_1T_1^*\| = \|(I-P_2)T_1\|$. We show that in general for $S \in \mathcal{L}(\mathscr{H}_2)$ we have $\|ST_1T_1^*\| = \|ST_1\|$. We have

$$\|ST_1T_1^*\| \le \|ST_1\|\,\|T_1\| = \|ST_1\| = \|ST_1T_1^*T_1\| \le \|ST_1T_1^*\|\,\|T_1\| = \|ST_1T_1^*\|.$$

□

We use the space $\mathscr{V}(0)$ of stable input-output pairs for initial condition zero (see (6.2)) to define a distance between discrete-time systems.

Definition 9.8. Let Σ_i $(i = 1, 2)$ be discrete-time systems with the same input and output spaces. The **gap** $\delta(\Sigma_1, \Sigma_2)$ is defined as $\delta(\mathscr{V}_1(0), \mathscr{V}_2(0))$. The **directed gap** $\vec{\delta}(\Sigma_1, \Sigma_2)$ is defined as $\vec{\delta}(\mathscr{V}_1(0), \mathscr{V}_2(0))$.

Remark 9.9. Note that discrete-time systems with the same transfer function have gap zero. So the gap is not a metric on the set of discrete-time systems. Let $\mathscr{U} = \mathscr{Y} = \mathbb{C}$ and $\mathsf{G}(z) = \sqrt{z - \alpha}$. For $\alpha \neq 0$ this function is holomorphic at zero (with an appropriate choice of the branch cut). This implies that G can be realized as the transfer function of a discrete-time system. If $\hat{u} \in H^2$, then $\hat{y} := \mathsf{G}\hat{u}$ can never be in H^2 unless $\hat{u} = 0$ since otherwise $\mathsf{G} = \hat{y}/\hat{u}$ would be meromorphic in $\mathbb{D}$, which it clearly is not if $|\alpha| < 1$. This implies that for all realizations of G we have $\mathscr{V}(0) = \{0\}$. Since this is true for all $\alpha \neq 0$ with $|\alpha| < 1$ it is not even true that the gap is a distance on equivalence classes of discrete-time systems with the same transfer function.

Definition 9.10. Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a right factor $[\mathsf{M}; \mathsf{N}]$. Define the space $Z_{[\mathsf{M};\mathsf{N}]} \subset H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y})$ by $Z_{[\mathsf{M};\mathsf{N}]} = \{(\mathsf{M}_i v; \mathsf{N}_i v) : v \in H^2(\mathbb{D}, \mathscr{U})\}$.

Proposition 9.11. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a right factorization. The space $Z_{[\mathsf{M};\mathsf{N}]}$ equals $\hat{\mathscr{V}}(0)$ for all weakly right-coprime factors.*

Proof. This follows from Proposition 7.10. □

The following shows that the gap metric gives a metric on the space of holomorphic functions defined in a neighbourhood of zero that have a right factorization.

Proposition 9.12. *Let $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G}_i)$ $(i = 1, 2)$. Assume that G_i has a right factorization. Then $\delta(\mathsf{G}_1, \mathsf{G}_2) = 0$ implies that $\mathsf{G}_1 = \mathsf{G}_2$ in a neighbourhood of zero.*

Proof. It follows from Proposition 7.13 that G_i has a weakly right-coprime factor $[\mathsf{M}_i; \mathsf{N}_i]$. Since $\delta(\mathsf{G}_1, \mathsf{G}_2) = 0$ it follows that $\mathscr{V}_1(0) = \mathscr{V}_2(0)$. It follows from Proposition 9.11 that the spaces $Z_{[\mathsf{M}_i;\mathsf{N}_i]}$ for $i = 1, 2$ are equal. In particular $[\mathsf{M}_1(z); \mathsf{N}_1(z)]u = [\mathsf{M}_2(z); \mathsf{N}_2(z)]u$ for all $u \in \mathscr{U}$ and $z \in \mathbb{D}$. It follows that $[\mathsf{M}_1(z); \mathsf{N}_1(z)] = [\mathsf{M}_2(z); \mathsf{N}_2(z)]$ for all $z \in \mathbb{D}$. From this we obtain $\mathsf{G}_1(z) = \mathsf{N}_1(z)\mathsf{M}_1^{-1}(z) = \mathsf{N}_2(z)\mathsf{M}_2^{-1}(z) = \mathsf{G}_2(z)$ in a neighbourhood of zero. □

The next two propositions give alternative characterizations of the directed gap.

Proposition 9.13. *Let* $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G}_i)$ *(*$i = 1, 2$*). Assume that* G_1 *and* G_2 *have normalized weakly right-coprime factors* $[\mathsf{M}_1; \mathsf{N}_1]$ *and* $[\mathsf{M}_2; \mathsf{N}_2]$*, respectively. Denote by* $T_{[\mathsf{M}_1;\mathsf{N}_1]}$ *the multiplication operator and by* $P_{Z^{\perp}_{[\mathsf{M}_2;\mathsf{N}_2]}}$ *the orthogonal projection. The directed gap equals*

$$\vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) = \left\| P_{Z^{\perp}_{[\mathsf{M}_2;\mathsf{N}_2]}} T_{[\mathsf{M}_1;\mathsf{N}_1]} \right\|.$$

Proof. This follows from Lemma 9.7 and Proposition 9.11. □

Proposition 9.14. *Let* $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G}_i)$ *(*$i = 1, 2$*). Assume that* G_1 *and* G_2 *have normalized weakly right-coprime factors* $[\mathsf{M}_1; \mathsf{N}_1]$ *and* $[\mathsf{M}_2; \mathsf{N}_2]$*, respectively. Then*

$$\vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) = \inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))} \left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2 \\ \mathsf{N}_2 \end{bmatrix} \mathsf{V} \right\|.$$

Proof. This follows from Proposition 9.13 and Corollary A.22. □

Proposition 9.15. *Let* $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G}_i)$ *(*$i = 1, 2$*). Assume that* G_1 *has a normalized weakly right-coprime factor* $[\mathsf{M}_1; \mathsf{N}_1]$ *and that* G_2 *has a right factorization. Then*

$$\vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) \le \left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2 \\ \mathsf{N}_2 \end{bmatrix} \right\|,$$

where $[\mathsf{M}_2; \mathsf{N}_2]$ *is any right factor of* G_2*.*

Proof. This follows from Propositions 7.14 and 9.14. □

The following corollary deals with ε-right factor admissible perturbations (see Definition 8.12).

Corollary 9.16. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$*. Assume that* G *has a normalized weakly right-coprime factorization. Let* G_Δ *be a* ε*-right factor admissible perturbation of* G*. Then* $\vec{\delta}(\mathsf{G}, \mathsf{G}_\Delta) < \varepsilon$*.*

Proof. This follows immediately from Proposition 9.15. □

Corollary 9.17. *Let $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G}_i)$ $(i = 1, 2)$. Assume that G_1 has a normalized weakly right-coprime factor $[\mathsf{M}_1; \mathsf{N}_1]$ and G_2 has a right factorization. If $\mathscr{U}$ is finite-dimensional, then*

$$\vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) = \inf_{[\mathsf{M}_2;\mathsf{N}_2]} \left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2 \\ \mathsf{N}_2 \end{bmatrix} \right\|,$$

where $[\mathsf{M}_2; \mathsf{N}_2]$ is any right factor of G_2.

Proof. Let $[\mathsf{M}_2^0; \mathsf{N}_2^0]$ be a normalized weakly right-coprime factor of G_2. For each $\varepsilon > 0$ we will construct a $\tilde{\mathsf{V}} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$ which has a bounded inverse in zero and is such that

$$\left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2^0 \\ \mathsf{N}_2^0 \end{bmatrix} \tilde{\mathsf{V}} \right\| - \vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) < \varepsilon. \tag{9.1}$$

The desired equality then immediately follows using Proposition 7.14 (and Proposition 9.15). By Proposition 9.14 there exists a $\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$ such that

$$\left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2^0 \\ \mathsf{N}_2^0 \end{bmatrix} \mathsf{V} \right\| - \vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) < \varepsilon/2.$$

Define $\tilde{\mathsf{V}} := \mathsf{V} + \delta I$. It is then easily computed that

$$\left\| \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} - \begin{bmatrix} \mathsf{M}_2^0 \\ \mathsf{N}_2^0 \end{bmatrix} \tilde{\mathsf{V}} \right\| - \vec{\delta}_g(\mathsf{G}_1, \mathsf{G}_2) < \varepsilon/2 + \left\| \begin{bmatrix} \mathsf{M}_2^0 \\ \mathsf{N}_2^0 \end{bmatrix} \right\| \delta.$$

Define $\eta := \varepsilon/(2\|[\mathsf{M}_2^0; \mathsf{N}_2^0]\|)$. If we choose $\delta \in (0, \eta)$, then (9.1) is satisfied. Since $\mathscr{U}$ is finite-dimensional the interval $(-\eta, 0)$ must contain points that are in the resolvent set of $\mathsf{V}(0) \in \mathcal{L}(\mathscr{U})$. This implies that there exists a $\delta \in (0, \eta)$ such that $\tilde{\mathsf{V}}(0)$ has a bounded inverse. □

Proposition 9.18. *Let $\mathsf{G}_i : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G}_i)$ $(i = 1, 2)$. Assume that G_1 and G_2 have weakly right-coprime factors $[\mathsf{M}_1; \mathsf{N}_1]$ and $[\mathsf{M}_2; \mathsf{N}_2]$, respectively. Assume further that $[\mathsf{M}_1; \mathsf{N}_1]$ is normalized and that $\|\Delta\|_\infty < 1$, where $\Delta := [\mathsf{M}_2; \mathsf{N}_2] - [\mathsf{M}_1; \mathsf{N}_1]$. Then $\delta(\mathsf{G}_1, \mathsf{G}_2) = \vec{\delta}(\mathsf{G}_1, \mathsf{G}_2) = \vec{\delta}(\mathsf{G}_2, \mathsf{G}_1) < 1$.*

Proof. Let $h \in H^2(\mathscr{U})$, we consider the projection of $[\mathsf{M}_2; \mathsf{N}_2]h$ onto $Z_{[\mathsf{M}_1;\mathsf{N}_1]}$. Since $[\mathsf{M}_1; \mathsf{N}_1]$ is inner we have, using Lemma 9.6,

$$\begin{aligned} P_{Z_{[\mathsf{M}_1;\mathsf{N}_1]}} \begin{bmatrix} \mathsf{M}_2 \\ \mathsf{N}_2 \end{bmatrix} h &= \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} [\mathsf{M}_1^*, \mathsf{N}_1^*] \left(\begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} + \Delta \right) h \\ &= \begin{bmatrix} \mathsf{M}_1 \\ \mathsf{N}_1 \end{bmatrix} (I + [\mathsf{M}_1^*, \mathsf{N}_1^*]\Delta) h. \end{aligned}$$

We have $\|[\mathsf{M}_1^*, \mathsf{N}_1^*]\Delta\|_\infty < 1$, which implies that $I + [\mathsf{M}_1^*, \mathsf{N}_1^*]\Delta$ is invertible in $H^\infty(\mathcal{L}(\mathscr{U}))$. It follows that $P_{Z_{[\mathsf{M}_1;\mathsf{N}_1]}}$ maps $Z_{[\mathsf{M}_2;\mathsf{N}_2]}$ onto $Z_{[\mathsf{M}_1;\mathsf{N}_1]}$ and since $[\mathsf{M}_1; \mathsf{N}_1]$ is injective, this mapping is injective. Since both $[\mathsf{M}_1; \mathsf{N}_1]$ and $[\mathsf{M}_2; \mathsf{N}_2]$ are weakly right-coprime, we have $Z_{[\mathsf{M}_i;\mathsf{N}_i]} = \hat{\mathscr{V}}_i(0)$ $(i = 1, 2)$ by Proposition 9.11. Lemma 9.5 now gives the result. □

Definition 9.19. Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a right factorization. Let $\varepsilon > 0$. The **directed gap ball** with center G of radius ε is defined as

$$\vec{B}(\mathsf{G}, \varepsilon) := \{\mathsf{G}_\Delta : \mathsf{G}_\Delta \text{ has a right factorization, } \vec{\delta}_g(\mathsf{G}, \mathsf{G}_\Delta) < \varepsilon\}.$$

The **gap ball** with center G of radius ε is defined as

$$B(\mathsf{G}, \varepsilon) := \{\mathsf{G}_\Delta : \mathsf{G}_\Delta \text{ has a right factorization, } \delta_g(\mathsf{G}, \mathsf{G}_\Delta) < \varepsilon\}.$$

Proposition 9.20. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a normalized weakly right-coprime factorization and let $\mathsf{G}_\Delta \in \vec{B}(\mathsf{G}, \varepsilon)$. Assume that $\mathscr{U}$ is finite-dimensional. Then G_Δ is an ε-right factor admissible perturbation of G.*

Proof. This is immediate from Corollary 9.17. □

Combining Corollary 9.16 and Proposition 9.20 we obtain the following.

Corollary 9.21. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a normalized weakly right-coprime factorization and that $\mathscr{U}$ is finite-dimensional. Then $\mathsf{G}_\Delta \in \vec{B}(\mathsf{G}, \varepsilon)$ if and only if G_Δ is an ε-right factor admissible perturbation of G.*

Proof. This follows by combining Corollary 9.16 and Proposition 9.20. □

Proposition 9.22. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a normalized strongly right-coprime factorization. Then there exists an $\eta > 0$ such that for all $\varepsilon < \eta$ we have $\vec{B}(\mathsf{G}, \varepsilon) = B(\mathsf{G}, \varepsilon)$.*

Proof. This follows from Propositions 7.26 and 9.18. □

Corollary 9.23. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G has a normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$ and that $\mathscr{U}$ is finite-dimensional. Denote the Hankel operator of $[\mathsf{M}; \mathsf{N}]$ by $H_{[\mathsf{M};\mathsf{N}]}$. Then η in Proposition 9.22 can be taken equal to $\sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2}$.*

Proof. This follows as the proof of Proposition 9.22 but using Proposition 7.32 instead of Proposition 7.26. □

The following result relates ε-right factor admissible perturbations and the gap metric.

Corollary 9.24. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a normalized weakly right-coprime factorization, that* $\mathscr{U}$ *is finite-dimensional and that* $\varepsilon < \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2}$. *Then* $\mathsf{G}_\Delta \in B(\mathsf{G}, \varepsilon)$ *if and only if* G_Δ *is an* ε*-right factor admissible perturbation of* G.

Proof. This follows from Corollaries 9.21 and 9.23. □

Proposition 9.25. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Suppose that* G *has a normalized doubly coprime factorization and that* $\mathscr{U}$ *is finite-dimensional. Let* $H_{[\mathsf{M};\mathsf{N}]}$ *denote the Hankel operator of the normalized strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$. *Let* $\varepsilon \in (0, \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2})$. *Then there exists a* ε*-robust right factor stabilizing feedback function for* G *that stabilizes all* $\mathsf{G}_\Delta \in B(\mathsf{G}, \varepsilon)$.

Proof. This follows from Proposition 8.17 and Corollary 9.24. □

Corollary 9.26. *Suppose that* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *with* $0 \in D(\mathsf{G})$ *is a matrix-valued rational function. Let* $H_{[\mathsf{M};\mathsf{N}]}$ *denote the Hankel operator of a normalized strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$. *Let* $\varepsilon \in (0, \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2})$. *Then there exists a rational* ε*-robust right factor stabilizing admissible feedback function for* G *that stabilizes all* G_Δ *with* $\delta(\mathsf{G}, \mathsf{G}_\Delta) < \varepsilon$.

Proof. This follows from the proof of Proposition 9.25 using Corollary 8.19. □

Notes

The gap metric as a distance between closed subspaces of a Hilbert space was first introduced in Kreĭn and Krasnosel′skiĭ [46] under the name **aperture** (see Krasnosel′skiĭ et. al. [45]). Proposition 9.14 was first proven by Georgiou [33] for rational functions. The relation between the gap metric and right factor admissible perturbation was investigated by, among others, Georgiou and Smith [34] and Sefton and Ober [87].