Model reduction for controller design for infinite-dimensional systems

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Chapter 8

Robust stabilization

In this chapter we consider so-called frequency domain feedback controller design. We are mainly interested in feedback controllers that provide a certain type of robustness. The following definition is basic for this chapter.

Definition 8.1. Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. We say that K is an **admissible feedback function** for G if $\mathsf{K} : D(\mathsf{K}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{Y}, \mathscr{U})$ is holomorphic with $0 \in D(\mathsf{K})$ and $I - \mathsf{KG}$ has a bounded inverse in zero.

Lemma 8.2. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$ and let K be an admissible feedback function for G. Then $I - \mathsf{GK}$ has a bounded inverse in zero.*

Proof. This follows from Lemma 3.16 with $Z = \mathsf{G}(0)$, $T = \mathsf{K}(0)$ and $\lambda = 1$. □

Remark 8.3. Note that it follows from Lemma 8.2 that K is an admissible feedback function for G if and only if G is an admissible feedback function for K.

Definition 8.4. An admissible feedback function K for G is called **stabilizing** if

$$\begin{bmatrix} (I - \mathsf{KG})^{-1} & \mathsf{K}(I - \mathsf{GK})^{-1} \\ \mathsf{G}(I - \mathsf{KG})^{-1} & (I - \mathsf{GK})^{-1} \end{bmatrix} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U} \times \mathscr{Y})). \tag{8.1}$$

Remark 8.5. Note that the function in (8.1) is the inverse of

$$\begin{bmatrix} I & -\mathsf{K} \\ -\mathsf{G} & I \end{bmatrix}.$$

The intuition behind the above definitions is that G is the transfer function of the plant and that K is the transfer function of the controller. The interconnection shown in figure 8.1 of the two systems is well-defined when K is an admissible feedback function for G. The condition (8.1) is equivalent to the transfer function from $[u_1; u_2]$ to $[e_1; e_2]$ being in H^∞.

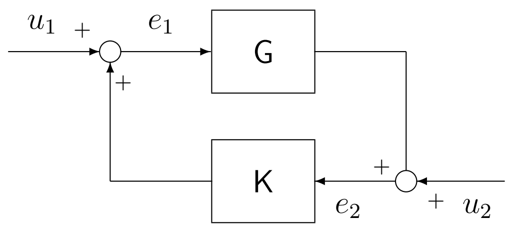

Figure 8.1: Feedback interconnection of G and K.

The following proposition, and most of the other propositions in this chapter, are formulated for right factorizations. By applying them to $\mathsf{G}^\dagger$ one obtains the obvious left versions of the results.

Proposition 8.6. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$. *Let* $[\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}]$ *be a right Bezout pair for this factorization and assume that* $\tilde{\mathsf{X}}$ *has a bounded inverse in zero. Define* $\mathsf{K} := \tilde{\mathsf{X}}^{-1}\tilde{\mathsf{Y}}$. *Then* K *is a stabilizing admissible feedback function for* G.

Proof. We have

$$I - \mathsf{KG} = I - \tilde{\mathsf{X}}^{-1}\tilde{\mathsf{Y}}\mathsf{N}\mathsf{M}^{-1} = \tilde{\mathsf{X}}^{-1}\left(\tilde{\mathsf{X}}\mathsf{M} - \tilde{\mathsf{Y}}\mathsf{N}\right)\mathsf{M}^{-1} = \tilde{\mathsf{X}}^{-1}\mathsf{M}^{-1},$$

which shows that $I - \mathsf{KG}$ has a bounded inverse in zero. Furthermore, we have

$$\begin{bmatrix} (I-\mathsf{KG})^{-1} & \mathsf{K}(I-\mathsf{GK})^{-1} \\ \mathsf{G}(I-\mathsf{KG})^{-1} & (I-\mathsf{GK})^{-1} \end{bmatrix} = \begin{bmatrix} \mathsf{M}\tilde{\mathsf{X}} & \mathsf{M}\tilde{\mathsf{Y}} \\ \mathsf{N}\tilde{\mathsf{X}} & I + \mathsf{N}\tilde{\mathsf{Y}} \end{bmatrix},$$

which is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U} \times \mathscr{Y}))$. To obtain the formula for $(I - \mathsf{GK})^{-1}$ we have used that

$$(I - \mathsf{GK})^{-1} = I + \mathsf{G}(I - \mathsf{KG})^{-1}\mathsf{K},$$

which is easily checked. □

Proposition 8.7. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that there exists a stabilizing admissible feedback function* K *for* G. *Then* G *has a strongly right-coprime factorization.*

Proof. Define $\mathsf{M}_1 := (I - \mathsf{KG})^{-1}$ and $\mathsf{N}_1 := \mathsf{G}(I - \mathsf{KG})^{-1}$. Since the feedback function is stabilizing we have $\mathsf{M}_1, \mathsf{N}_1 \in H^\infty$ and obviously $\mathsf{G} = \mathsf{N}_1\mathsf{M}_1^{-1}$. So G has a right factorization. By Proposition 7.13 G has a weakly right-coprime factorization $[\mathsf{M};\mathsf{N}]$. Similarly K has a weakly right-coprime factorization: $\mathsf{K} = \mathsf{YX}^{-1}$. Clearly,

$$\mathsf{N}_2 := \begin{bmatrix} 0 & \mathsf{Y} \\ \mathsf{N} & 0 \end{bmatrix}, \quad \mathsf{M}_2 := \begin{bmatrix} \mathsf{M} & 0 \\ 0 & \mathsf{X} \end{bmatrix}$$

provides a weakly right-coprime factorization of $\mathsf{G}_2 := [0, \mathsf{K}; \mathsf{G}, 0]$. Since the feedback function is stabilizing we have $(I - \mathsf{G}_2)^{-1} \in H^\infty$. It follows from Lemma 7.8 that $(\mathsf{M}_2 - \mathsf{N}_2)^{-1} \in H^\infty$. Denote the upper row of $(\mathsf{M}_2 - \mathsf{N}_2)^{-1}$ by $[\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}]$, then $\tilde{\mathsf{X}}\mathsf{M} - \tilde{\mathsf{Y}}\mathsf{N} = I$. Hence $[\mathsf{M};\mathsf{N}]$ is a strongly right-coprime factor. □

Corollary 8.8. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Any stabilizing feedback function for* G *has a strongly right-coprime factorization.*

Proof. The symmetry mentioned in remark 8.3 and Proposition 8.7 give the result. □

Proposition 8.9. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Then the following are equivalent:*

1. G *has a strongly right-coprime factorization.*
2. *There exists a stabilizing admissible feedback function for* G.

Proof. (2) implies (1) is Proposition 8.7. (1) implies (2) follows from Proposition 8.6 using Lemma 7.23. □

Proposition 8.10. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a doubly coprime factorization. Then all stabilizing admissible feedback functions are given by*

$$\mathsf{K} = (\mathsf{Y} + \mathsf{MV})(\mathsf{X} + \mathsf{NV})^{-1}, \tag{8.2}$$

where $\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$ *is such that* $\mathsf{X} + \mathsf{NV}$ *has a bounded inverse in zero, but is otherwise arbitrary.*

Proof. We first show that the function defined by (8.2) is a stabilizing admissible feedback function. Define $\underline{\mathsf{Y}} := \mathsf{Y} + \mathsf{MV}$, $\underline{\mathsf{X}} := \mathsf{X} + \mathsf{NV}$. Then

$$\begin{bmatrix} \mathsf{M} & \underline{\mathsf{Y}} \\ \mathsf{N} & \underline{\mathsf{X}} \end{bmatrix} = \begin{bmatrix} \mathsf{M} & \mathsf{Y} \\ \mathsf{N} & \mathsf{X} \end{bmatrix} \begin{bmatrix} I & \mathsf{V} \\ 0 & I \end{bmatrix},$$

from which it easily follows that with $\underline{\tilde{\mathsf{X}}} := \tilde{\mathsf{X}} + \mathsf{V}\tilde{\mathsf{N}}$ and $\underline{\tilde{\mathsf{Y}}} := \tilde{\mathsf{Y}} + \mathsf{V}\tilde{\mathsf{M}}$ we obtain (7.3) with the Bezout factors replaced by the underlined versions. It follows from the left version of Proposition 8.6 that if $\underline{\mathsf{X}}$ is invertible at zero, then $\underline{\mathsf{Y}}\underline{\mathsf{X}}^{-1}$ is a stabilizing admissible feedback function.

Assume that K is a stabilizing admissible feedback function. We will show that it is of the form (8.2). Since there exists a stabilizing admissible feedback function, G has a doubly coprime factorization by Propositions 7.24 and 8.7. From Proposition 7.24 and Corollary 8.8 we obtain that K has a strongly left-coprime factorization: $\mathsf{K} = \tilde{\mathsf{W}}^{-1}\tilde{\mathsf{Z}}$ and $\tilde{\mathsf{W}}\mathsf{S} - \tilde{\mathsf{Z}}\mathsf{R} = I$. Define $\Delta := \tilde{\mathsf{Z}}\mathsf{N} - \tilde{\mathsf{W}}\mathsf{M}$. Since $\Delta = \tilde{\mathsf{W}}(\mathsf{KG} - I)\mathsf{M}$ we have that Δ is invertible in zero. It is easily calculated that the matrix in (8.1) can be written as

$$\begin{bmatrix} (I - \mathsf{KG})^{-1} & \mathsf{K}(I - \mathsf{GK})^{-1} \\ \mathsf{G}(I - \mathsf{KG})^{-1} & (I - \mathsf{GK})^{-1} \end{bmatrix} = \begin{bmatrix} -\mathsf{M}\Delta^{-1}\tilde{\mathsf{W}} & -\mathsf{M}\Delta^{-1}\tilde{\mathsf{Z}} \\ -\mathsf{N}\Delta^{-1}\tilde{\mathsf{W}} & I - \mathsf{N}\Delta^{-1}\tilde{\mathsf{Z}} \end{bmatrix}. \quad (8.3)$$

Using the above Bezout equation for the strongly left-coprime factorization of K we obtain $\mathsf{M}\Delta^{-1} = \mathsf{M}\Delta^{-1}\tilde{\mathsf{W}}\mathsf{S} - \mathsf{M}\Delta^{-1}\tilde{\mathsf{Z}}\mathsf{R}$, which is in H^∞. Similarly we obtain $\mathsf{N}\Delta^{-1} \in H^\infty$. Using Proposition 7.7 we see that $\Delta^{-1} \in H^\infty$. Define $\mathsf{V} := \Delta^{-1}(\tilde{\mathsf{W}}\mathsf{Y} - \tilde{\mathsf{Z}}\mathsf{X}) \in H^\infty$. We show that $\mathsf{K}(\mathsf{X} + \mathsf{NV}) = \mathsf{Y} + \mathsf{MV}$. Using (8.3) we obtain

$$\begin{aligned} \mathsf{Y} + \mathsf{MV} &= \mathsf{Y} + \mathsf{M}\Delta^{-1}(\tilde{\mathsf{W}}\mathsf{Y} - \tilde{\mathsf{Z}}\mathsf{X}) = (I + \mathsf{M}\Delta^{-1}\tilde{\mathsf{W}})\mathsf{Y} - \mathsf{M}\Delta^{-1}\tilde{\mathsf{Z}}\mathsf{X} \\ &= (I - (I - \mathsf{KG})^{-1}))\mathsf{Y} + \mathsf{K}(I - \mathsf{GK})^{-1}\mathsf{X}, \end{aligned}$$

and

$$\begin{aligned} \mathsf{X} + \mathsf{NV} &= \mathsf{X} + \mathsf{N}\Delta^{-1}(\tilde{\mathsf{W}}\mathsf{Y} - \tilde{\mathsf{Z}}\mathsf{X}) = \mathsf{N}\Delta^{-1}\tilde{\mathsf{W}}\mathsf{Y} + (I - \mathsf{N}\Delta^{-1}\tilde{\mathsf{Z}})\mathsf{X} \\ &= -\mathsf{G}(I - \mathsf{KG})^{-1}\mathsf{Y} + (I - \mathsf{GK})^{-1}\mathsf{X}. \end{aligned} \quad (8.4)$$

From this it easily follows that $\mathsf{K}(\mathsf{X} + \mathsf{NV}) = \mathsf{Y} + \mathsf{MV}$. So the only thing left to show is that $\mathsf{X} + \mathsf{NV}$ has a bounded inverse in zero. Using (8.4) we obtain

$$\tilde{\mathsf{M}}(I - \mathsf{GK})(\mathsf{X} + \mathsf{NV}) = -\tilde{\mathsf{N}}\mathsf{Y} + \tilde{\mathsf{M}}\mathsf{X} = I,$$

where the last identity follows from (7.3). Since $\tilde{\mathsf{M}}(I - \mathsf{GK})$ has a bounded inverse in zero it follows that $\mathsf{X} + \mathsf{NV}$ does and that this inverse equals $\tilde{\mathsf{M}}(I - \mathsf{GK})$. □

Lemma 8.11. *If in a Banach algebra we have* $\|x\|^2 + \|y\|^2 \leq \alpha^2 < 1$*, then* $I - y$ *is invertible and* $\|(I-y)^{-1}x\|^2 \leq \alpha^2/(1-\alpha^2)$.

Proof. That $I - y$ has a bounded inverse follows from the Neumann series theorem. From this theorem we also obtain $\|(I-y)^{-1}\| \leq 1/(1-\|y\|)$. It follows that $\|(I-y)^{-1}x\|^2 \leq \|x\|^2/(1-\|y\|)^2$. Denote $x_1 := \|x\|$ and $y_1 := \|y\|$. Using elementary vector calculus one sees that the function $x_1^2/(1-y_1)^2$ under the constraint $x_1^2 + y_1^2 \leq \alpha^2 < 1$ has the maximum $\alpha^2/(1-\alpha^2)$. The desired result follows. □

We now focus our attention on feedback functions that provide a certain robustness.

Definition 8.12. Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Suppose that G has a normalized weakly right-coprime factor $[\mathsf{M}; \mathsf{N}]$. Let $\varepsilon > 0$. The function G_Δ is called a ε-**right factor admissible perturbation** of G if G_Δ has a right factor $[\mathsf{M}_\Delta; \mathsf{N}_\Delta]$ with $\|[\mathsf{M}_\Delta; \mathsf{N}_\Delta] - [\mathsf{M}; \mathsf{N}]\|_\infty < \varepsilon$.

Note that, due to Proposition 7.15, the set of ε-right factor admissible perturbations does not depend on the specific normalized weakly right-coprime factor chosen to define it.

Definition 8.13. Suppose that G has a normalized weakly right-coprime factor $[\mathsf{M}; \mathsf{N}]$. Let $\varepsilon > 0$. An admissible feedback function K for G is called ε-**robust right factor stabilizing** if it is a stabilizing admissible feedback function for all ε-right factor admissible perturbations of G.

Proposition 8.14. *Suppose that* G *has a normalized doubly coprime factorization. Let* $\varepsilon \in (0,1)$*. Suppose that* $[\tilde{\mathsf{V}}, \tilde{\mathsf{U}}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))$ *satisfies*

$$\left\| [\mathsf{M}^*, \mathsf{N}^*] - [\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}] \right\| \leq \sqrt{1-\varepsilon^2},$$

and that $\tilde{\mathsf{V}}$ *has a bounded inverse in zero. Then* $\mathsf{K} := \tilde{\mathsf{V}}^{-1}\tilde{\mathsf{U}}$ *is a* ε*-robust right factor stabilizing admissible feedback function for* G.

Proof. Let $\mathsf{W} : \mathbb{T} \to \mathcal{L}(\mathscr{U} \times \mathscr{Y})$ be the function from Proposition 7.33, i.e.

$$\mathsf{W}(z) = \begin{bmatrix} \mathsf{M}(z) & -\tilde{\mathsf{N}}(z)^* \\ \mathsf{N}(z) & \tilde{\mathsf{M}}(z)^* \end{bmatrix}.$$

Define $\mathsf{F} \in L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))$ by

$$\mathsf{F} := \Big([\mathsf{M}^*, \mathsf{N}^*] - [\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}] \Big) \mathsf{W} = [I - \tilde{\mathsf{V}}\mathsf{M} + \tilde{\mathsf{U}}\mathsf{N}, \tilde{\mathsf{V}}\tilde{\mathsf{N}}^* + \tilde{\mathsf{U}}\tilde{\mathsf{M}}^*], \tag{8.5}$$

where we have used (7.3). Since $\mathsf{W}(z)$ is unitary we have

$$\|\mathsf{F}\|_\infty \leq \sqrt{1-\varepsilon^2}.$$

It follows that $\|I - \tilde{\mathsf{V}}\mathsf{M} + \tilde{\mathsf{U}}\mathsf{N}\|_\infty < 1$. Since $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$ is a Banach algebra it follows that $\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N}$ has an inverse in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$. Denote an arbitrary ε-right factor admissible perturbation of G by G_Δ. Denote a right factor of G_Δ as in Definition 8.12 by $[\mathsf{M}_\Delta; \mathsf{N}_\Delta]$. Further denote $\Delta = [\Delta_\mathsf{M}; \Delta_\mathsf{N}] = [\mathsf{M}_\Delta; \mathsf{N}_\Delta] - [\mathsf{M}; \mathsf{N}]$. We have

$$\begin{aligned} I - \mathsf{K}\mathsf{G}_\Delta &= I - \tilde{\mathsf{V}}^{-1}\tilde{\mathsf{U}}(\mathsf{N} + \Delta_\mathsf{N})(\mathsf{M} + \Delta_\mathsf{M})^{-1} \\ &= \tilde{\mathsf{V}}^{-1}\left(\tilde{\mathsf{V}}\mathsf{M} + \tilde{\mathsf{V}}\Delta_\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N} - \tilde{\mathsf{U}}\Delta_\mathsf{N}\right)(\mathsf{M} + \Delta_\mathsf{M})^{-1} \\ &= \tilde{\mathsf{V}}^{-1}\left(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N} + [\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}]\begin{bmatrix} \Delta_\mathsf{M} \\ \Delta_\mathsf{N} \end{bmatrix}\right)(\mathsf{M} + \Delta_\mathsf{M})^{-1} \\ &= \tilde{\mathsf{V}}^{-1}(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})\,(I + \mathsf{S}\Delta)\,(\mathsf{M} + \Delta_\mathsf{M})^{-1}, \end{aligned}$$

where

$$\mathsf{S} := (\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}[\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}].$$

It follows that $I - \mathsf{K}\mathsf{G}_\Delta$ has an inverse in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$ if and only if $I + \mathsf{S}\Delta$ does. The latter is true if $\|\mathsf{S}\|_\infty < 1/\varepsilon$. We have

$$\begin{aligned} \|\mathsf{S}\|_\infty^2 &= \|\mathsf{S}\mathsf{W}\|_\infty^2 = \|[I, -(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}(\tilde{\mathsf{V}}\tilde{\mathsf{N}}^* + \tilde{\mathsf{U}}\tilde{\mathsf{M}}^*)]\|^2 \\ &= 1 + \|(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}(\tilde{\mathsf{V}}\tilde{\mathsf{N}}^* + \tilde{\mathsf{U}}\tilde{\mathsf{M}}^*)]\|^2 = 1 + \|(I - \mathsf{F}_1)^{-1}\mathsf{F}_2\|^2, \end{aligned}$$

where $\mathsf{F} = [\mathsf{F}_1, \mathsf{F}_2]$ is the function from (8.5). From Lemma 8.11 we obtain

$$\|\mathsf{S}\|_\infty^2 \leq \frac{1}{\varepsilon^2},$$

as desired.

To check that K stabilizes G_Δ we have to show that

$$\begin{bmatrix} (I - \mathsf{K}\mathsf{G}_\Delta)^{-1} & \mathsf{K}(I - \mathsf{G}_\Delta\mathsf{K})^{-1} \\ \mathsf{G}_\Delta(I - \mathsf{K}\mathsf{G}_\Delta)^{-1} & (I - \mathsf{G}_\Delta\mathsf{K})^{-1} \end{bmatrix} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y})).$$

We already saw that $(I - \mathsf{K}\mathsf{G}_\Delta)^{-1} \in H^\infty$. We compute

$$\mathsf{G}_\Delta(I - \mathsf{K}\mathsf{G}_\Delta)^{-1} = (\mathsf{N} + \Delta_\mathsf{N})(I + \mathsf{S}\Delta)^{-1}(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}\tilde{\mathsf{V}},$$

which is clearly in H^∞. Similarly we have

$$\mathsf{K}(I - \mathsf{G}_\Delta\mathsf{K})^{-1} = (I - \mathsf{K}\mathsf{G}_\Delta)^{-1}\mathsf{K} = (\mathsf{M} + \Delta_\mathsf{M})(I + \mathsf{S}\Delta)^{-1}(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}\tilde{\mathsf{U}},$$

and

$$(I - \mathsf{G}_\Delta\mathsf{K})^{-1} = I + \mathsf{G}_\Delta(I - \mathsf{K}\mathsf{G}_\Delta)^{-1}\mathsf{K} = I + (\mathsf{N} + \Delta_\mathsf{N})(I + \mathsf{S}\Delta)^{-1}(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})^{-1}\tilde{\mathsf{U}},$$

which are both in H^∞. □

Proposition 8.15. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Suppose that* G *has a normalized doubly coprime factorization. Let* $H_{[\mathsf{M};\mathsf{N}]}$ *denote the Hankel operator of the normalized strongly right-coprime factor* $[\mathsf{M};\mathsf{N}]$. *Let* $\varepsilon < \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2}$. *Then there exists a* $[\tilde{\mathsf{V}}, \tilde{\mathsf{U}}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}, \mathscr{U}))$ *such that*

$$\left\| [\mathsf{M}^*, \mathsf{N}^*] - [\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}] \right\| \leq \sqrt{1 - \varepsilon^2}.$$

If the input space $\mathscr{U}$ *is finite-dimensional, then* $[\tilde{\mathsf{V}}, \tilde{\mathsf{U}}]$ *can be chosen such that* $\tilde{\mathsf{V}}$ *has a bounded inverse in zero.*

Proof. The existence of $[\tilde{\mathsf{V}}, \tilde{\mathsf{U}}]$ such that the desired inequality is satisfied follows easily from Nehari's theorem (Lemma A.27). We now show that if $\mathscr{U}$ is finite-dimensional we can choose $\tilde{\mathsf{V}}$ such that it has a bounded inverse in zero. First choose a $\tilde{\varepsilon} \in (\varepsilon, \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2})$ and find a $[\tilde{\mathsf{V}}_1, \tilde{\mathsf{U}}_1]$ such that the desired inequality is satisfied with $\tilde{\varepsilon}$ instead of ε. Define $[\tilde{\mathsf{V}}_\delta, \tilde{\mathsf{U}}_\delta] := [\tilde{\mathsf{V}}_1, \tilde{\mathsf{U}}_1] - \delta[I, 0]$. Then this satisfies

$$\left\| [\mathsf{M}^*, \mathsf{N}^*] - [\tilde{\mathsf{V}}_\delta, -\tilde{\mathsf{U}}_\delta] \right\| \leq \sqrt{1 - \tilde{\varepsilon}^2} + \delta.$$

It follows that for $\delta \in (0, \sqrt{1 - \varepsilon^2} - \sqrt{1 - \tilde{\varepsilon}^2})$ the desired inequality is satisfied. Since $\mathscr{U}$ is finite-dimensional, this interval must contain a point in the resolvent set of $\tilde{\mathsf{V}}_1(0)$. For such a δ we have that $\tilde{\mathsf{V}}_\delta$ has a bounded inverse in zero. □

Remark 8.16. If $\mathsf{G}(0) = 0$, then the conclusion of Proposition 8.15 is also true without the assumption that the input space $\mathscr{U}$ is finite-dimensional. We indicate why this is true. From the proof of Proposition 8.14 we obtain that $\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N}$ has an inverse in H^∞. It follows that $(\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})(0)$ has a bounded inverse. We have $\mathsf{N}(0) = \mathsf{G}(0)\mathsf{M}(0) = 0$ and so $\tilde{\mathsf{V}}(0)\mathsf{M}(0) = (\tilde{\mathsf{V}}\mathsf{M} - \tilde{\mathsf{U}}\mathsf{N})(0)$ has a bounded inverse. Since $\mathsf{M}(0)$ has a bounded inverse it follows that $\tilde{\mathsf{V}}$ has a bounded inverse in zero as desired.

Proposition 8.17. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Suppose that* G *has a normalized doubly coprime factorization and that* $\mathscr{U}$ *is finite-dimensional. Let* $H_{[\mathsf{M};\mathsf{N}]}$ *denote the Hankel operator of the normalized strongly right-coprime factor* $[\mathsf{M};\mathsf{N}]$. *Let* $\varepsilon \in (0, \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2})$. *Then there exists an* ε*-robust right factor stabilizing admissible feedback function for* G.

Proof. This follows from combining Propositions 8.14 and 8.15. □

Remark 8.18. The assumption in Proposition 8.17 that $\mathscr{U}$ is finite-dimensional is made because that assumption had to be made in Proposition 8.15 to obtain invertiblity of $\tilde{\mathsf{V}}$ in zero. We can avoid making this assumption in Proposition 8.17 by considering controllers with internal loop as in Curtain, Weiss and Weiss [17].

Corollary 8.19. *Suppose that* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *with* $0 \in D(\mathsf{G})$ *is a matrix-valued rational function. Let* $H_{[\mathsf{M};\mathsf{N}]}$ *denote the Hankel operator of a normalized strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$*. Let* $\varepsilon \in (0, \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2})$*. Then there exists a rational* ε*-robust right factor stabilizing admissible feedback function for* G*.*

Proof. This follows from the proof of Proposition 8.17, using that in the proof of Proposition 8.15 we can choose $[\tilde{\mathsf{V}}, -\tilde{\mathsf{U}}]$ rational by Lemma A.28. □

Notes

We refer to Vidyagagar [94], Zhou, Doyle and Glover [103], Doyle, Francis and Tannenbaum [22] and Francis [28] for general information on stabilizing feedback functions for rational functions.

For finite-dimensional input and output spaces Proposition 8.9 is originally due to Inouye [40]. An independent proof was given by Smith [88]. The general case, also valid for infinite-dimensional input and output spaces, was first proven by Mikkola [56], whose proof we followed. Proposition 8.10 is due to Youla, Jabr and Bongiorno [98] in the case of rational functions. Using controllers with internal loop this result in the general case is due to Curtain, Weiss and Weiss [17]. The results presented here on robust right factor stabilizing feedback functions are due to McFarlane and Glover [36], [54] in the rational case. Continuous-time versions of the general case are given in Curtain and Zwart [18, Section 9.4], Oostveen [64, Section 7.2] and Curtain [11]. All of these references closely follow the arguments in McFarlane and Glover as do we.