Model reduction for controller design for infinite-dimensional systems

Opmeer, Mark Robertus

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Chapter 7

Coprime factorization

In this chapter and the following two chapters we consider the following set of holomorphic functions.

Definition 7.1. Let $\mathscr{U}$ and $\mathscr{Y}$ be Hilbert spaces. The set $H_0(\mathscr{U},\mathscr{Y})$ consists of functions $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U},\mathscr{Y})$ that are holomorphic with $0 \in D(\mathsf{G})$.

Remark 7.2. Note the transfer function of a discrete-time system is always in our set of holomorphic functions. Moreover, it follows from Proposition 2.12 that any function in this set is the transfer function of some discrete-time system.

In this chapter we study coprime factorization over H^∞. We study both a strong and a weak form of coprimeness. Since we are dealing with operator-valued functions, we have to distinguish between right coprimeness and left coprimeness.

Definition 7.3. Let $\mathsf{M} \in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_1,\mathscr{H}_2))$ and $\mathsf{N} \in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_1,\mathscr{H}_3))$.

The functions M and N are called **weakly right-coprime** if for every Z-transformable sequence $h : \mathbb{Z}^+ \to \mathscr{H}_1$ with $[\mathsf{M}\hat{h};\mathsf{N}\hat{h}] \in H^2(\mathbb{D},\mathscr{H}_2 \times \mathscr{H}_3)$ we have $h \in l^2(\mathbb{Z}^+,\mathscr{H}_1)$.

The functions M and N are called **strongly right-coprime** if $[\mathsf{M};\mathsf{N}]$ has a left-inverse in $H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_2 \times \mathscr{H}_3,\mathscr{H}_1))$, meaning if there exist $\tilde{\mathsf{X}} \in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_2,\mathscr{H}_1))$ and $\tilde{\mathsf{Y}} \in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_3,\mathscr{H}_1))$ such that

$$\tilde{\mathsf{X}}(z)\mathsf{M}(z) - \tilde{\mathsf{Y}}(z)\mathsf{N}(z) = I_{\mathscr{H}_1} \quad \forall z \in \mathbb{D}. \tag{7.1}$$

The functions $\tilde{\mathsf{X}}$ and $\tilde{\mathsf{Y}}$ are called **right Bezout factors** for the pair (M,N).

Let $\tilde{\mathsf{M}} \in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_1,\mathscr{H}_2))$ and $\tilde{\mathsf{N}} \in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_3,\mathscr{H}_2))$.

The functions $\tilde{\mathsf{M}}$ and $\tilde{\mathsf{N}}$ are called **strongly left-coprime** if $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$ has a right-inverse in $H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_2,\mathscr{H}_1 \times \mathscr{H}_3))$, that is to say, if there exist $\mathsf{X} \in$

$H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_2,\mathscr{H}_1))$ and $\mathsf{Y}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{H}_2,\mathscr{H}_3))$ such that

$$\tilde{\mathsf{M}}(z)\mathsf{X}(z)-\tilde{\mathsf{N}}(z)\mathsf{Y}(z)=I_{\mathscr{H}_2}\quad\forall z\in\mathbb{D}. \tag{7.2}$$

The functions X and Y are called **left Bezout factors** for the pair $(\tilde{\mathsf{M}},\tilde{\mathsf{N}})$.

Definition 7.4. Let $\mathsf{G}:D(\mathsf{G})\subset\mathbb{C}\to\mathcal{L}(\mathscr{U},\mathscr{Y})$ be holomorphic with $0\in D(\mathsf{G})$.

G has a **right factorization** if there exist $\mathsf{M}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U}))$ and $\mathsf{N}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U},\mathscr{Y}))$ such that $\mathsf{M}(z)$ is invertible for z in a neighbourhood of zero and $\mathsf{G}(z)=\mathsf{N}(z)\mathsf{M}(z)^{-1}$ for z in a neighbourhood of zero. The factor $[\mathsf{M};\mathsf{N}]$ provides a **weakly right-coprime factorization** if M and N are weakly right-coprime and a **strongly right-coprime factorization** if M and N are strongly right-coprime. The right factor $[\mathsf{M};\mathsf{N}]$ is called **normalized** when multiplication with $[\mathsf{M};\mathsf{N}]$ is an isometry from $H^2(\mathbb{D},\mathscr{U})$ into $H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})$.

G has a **left factorization** if there exist $\tilde{\mathsf{M}}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{Y}))$ and $\tilde{\mathsf{N}}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U},\mathscr{Y}))$ such that $\tilde{\mathsf{M}}(z)$ is invertible for z in a neighbourhood of zero and $\mathsf{G}(z)=\tilde{\mathsf{M}}(z)^{-1}\tilde{\mathsf{N}}(z)$ for z in a neighbourhood of zero. $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$ is a **strongly left-coprime factor** if $\tilde{\mathsf{M}}$ and $\tilde{\mathsf{N}}$ are strongly left-coprime. The left factor $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$ is called **normalized** when multiplication with $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$ is a co-isometry from $H^2(\mathbb{D},\mathscr{Y}\times\mathscr{U})$ into $H^2(\mathbb{D},\mathscr{Y})$.

G has a **doubly coprime factorization** if it has a left factorization and a right factorization and there exist $\tilde{\mathsf{X}}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{U}))$, $\tilde{\mathsf{Y}}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{Y},\mathscr{U}))$, $\mathsf{X}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{Y}))$ and $\mathsf{Y}\in H^\infty(\mathbb{D};\mathcal{L}(\mathscr{Y},\mathscr{U}))$ such that

$$\begin{bmatrix}\tilde{\mathsf{X}} & -\tilde{\mathsf{Y}}\\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}}\end{bmatrix}\begin{bmatrix}\mathsf{M} & \mathsf{Y}\\ \mathsf{N} & \mathsf{X}\end{bmatrix}=I=\begin{bmatrix}\mathsf{M} & \mathsf{Y}\\ \mathsf{N} & \mathsf{X}\end{bmatrix}\begin{bmatrix}\tilde{\mathsf{X}} & -\tilde{\mathsf{Y}}\\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}}\end{bmatrix}. \tag{7.3}$$

The doubly coprime factorization is called **normalized** when both the right factor $[\mathsf{M};\mathsf{N}]$ and the left factor $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$ are normalized.

Remark 7.5. In this chapter we will prove results for right factorizations. However, all results translate to left factorizations by considering $\mathsf{G}^\dagger$.

The next proposition is a first step towards relating state space closed-loop systems (see Definition 4.1) and factorizations.

Proposition 7.6. *Let $\mathsf{G}:D(\mathsf{G})\subset\mathbb{C}\to\mathcal{L}(\mathscr{U},\mathscr{Y})$ be holomorphic with $0\in D(\mathsf{G})$. Let Σ be a realization of G and let $[F,G]$ be an admissible feedback pair for Σ. Denote the transfer function of the closed-loop system by $[\mathsf{M};\mathsf{N}]$. Then $\mathsf{M}(z)$ is invertible for z in a neighbourhood of zero and $\mathsf{G}(z)=\mathsf{N}(z)\mathsf{M}(z)^{-1}$ in a neighbourhood of zero.*

Proof. This follows from Proposition 2.23. □

The following proposition provides a fundamental property of weakly right-coprime functions.

Proposition 7.7. *Assume that the functions* $\mathsf{M} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ *and* $\mathsf{N} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{H}_1, \mathscr{H}_3))$ *are weakly right-coprime. If for a holomorphic* $\mathsf{R} : D(\mathsf{R}) \to \mathcal{L}(\mathscr{H}_4, \mathscr{H}_1)$ *with* $0 \in D(\mathsf{R})$ *we have* $[\mathsf{MR}; \mathsf{NR}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_4, \mathscr{H}_2 \times \mathscr{H}_3))$*, then* $\mathsf{R} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_4, \mathscr{H}_1))$*.*

Proof. Let $h \in H^2(\mathbb{D}, \mathscr{H}_4)$. Then since $[\mathsf{MR}; \mathsf{NR}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_4, \mathscr{H}_2 \times \mathscr{H}_3))$ we have $[\mathsf{MR}h; \mathsf{NR}h] \in H^2(\mathbb{D}, \mathscr{H}_2 \times \mathscr{H}_3)$. Since $[\mathsf{M}; \mathsf{N}]$ is weakly right-coprime it follows that $\mathsf{R}h \in H^2(\mathbb{D}, \mathscr{H}_1)$. So multiplication by R maps $H^2(\mathbb{D}, \mathscr{H}_4)$ into $H^2(\mathbb{D}, \mathscr{H}_1)$. It is easily shown that a multiplication operator is closed from H^2 to H^2. By the closed graph theorem it follows that multiplication with R is a continuous operator from $H^2(\mathbb{D}, \mathscr{H}_4)$ to $H^2(\mathbb{D}, \mathscr{H}_1)$. By Lemma A.4 it follows that $\mathsf{R} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_4, \mathscr{H}_1))$. □

The following lemma gives additional conditions under which a weakly right-coprime factor is strongly right coprime (see Corollary 7.9). This will be useful in Chapter 8.

Lemma 7.8. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U})$ *be holomorphic with* $0 \in D(\mathsf{G})$*. If* $[\mathsf{M}; \mathsf{N}]$ *is a weakly right-coprime factor of* G*,* $I - \mathsf{G}(0)$ *has a bounded inverse and* $(I - \mathsf{G})^{-1} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$*, then* $\mathsf{M}(0) - \mathsf{N}(0)$ *has a bounded inverse and* $(\mathsf{M} - \mathsf{N})^{-1} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$*.*

Proof. We have $\mathsf{M} - \mathsf{N} = (I - \mathsf{G})\mathsf{M}$, which shows that $\mathsf{M}(0) - \mathsf{N}(0)$ has a bounded inverse. We have $\mathsf{M}(\mathsf{M} - \mathsf{N})^{-1} = (I - \mathsf{G})^{-1}$ and $\mathsf{N}(\mathsf{M} - \mathsf{N})^{-1} = \mathsf{G}(I - \mathsf{G})^{-1} = (I - \mathsf{G})^{-1} - I$. Proposition 7.7 now shows that $(\mathsf{M} - \mathsf{N})^{-1} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$. □

Corollary 7.9. *Under the assumptions of Lemma 7.8 we have that* $[\mathsf{M}; \mathsf{N}]$ *is strongly right-coprime.*

Proof. We can choose the Bezout factors $\tilde{\mathsf{X}} = \tilde{\mathsf{Y}} = (\mathsf{M} - \mathsf{N})^{-1}$. □

Weak right-coprimeness is connected to the linear quadratic optimal control problem as the following proposition shows. The set $\hat{\mathscr{V}}(x_0)$ is defined as the set of Z-transforms of sequences in $\mathscr{V}(x_0)$, which was defined in (6.2).

Proposition 7.10. *Let* D *be the transfer function of the discrete-time system* Σ *and let* $[\mathsf{M}; \mathsf{N}]$ *be a right factor. Then multiplication by* $[\mathsf{M}; \mathsf{N}]$ *is an injection from* $H^2(\mathbb{D}, \mathscr{U})$ *into* $\hat{\mathscr{V}}(0)$*. The factorization is weakly right-coprime if and only if multiplication by* $[\mathsf{M}; \mathsf{N}]$ *is a bijection from* $H^2(\mathbb{D}, \mathscr{U})$ *onto* $\hat{\mathscr{V}}(0)$*.*

Proof. That multiplication with $[\mathsf{M};\mathsf{N}]$ maps $H^2(\mathbb{D},\mathscr{U})$ into $H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})$ follows from the fact that $[\mathsf{M};\mathsf{N}]\in H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U},\mathscr{U}\times\mathscr{Y}))$. Let $r\in l^2(\mathbb{Z}^+,\mathscr{U})$, we show that $[\mathsf{M};\mathsf{N}]\hat{r}\in\hat{\mathscr{V}}(0)$. Define $\hat{u}:=\mathsf{M}\hat{r}$. We have to show that $\mathsf{N}\hat{r}=\mathsf{D}\mathsf{M}\hat{r}$. This follows since $\mathsf{D}\mathsf{M}\hat{r}=\mathsf{N}\mathsf{M}^{-1}\mathsf{M}\hat{r}=\mathsf{N}\hat{r}$. It follows that multiplication by $[\mathsf{M};\mathsf{N}]$ maps $H^2(\mathbb{D},\mathscr{U})$ into $\hat{\mathscr{V}}(0)$.

We show that multiplication with $[\mathsf{M};\mathsf{N}]$ is injective. Suppose that there are two Z-transformable sequences $r_i:\mathbb{Z}^+\to\mathscr{U}$ $(i=1,2)$ with $[\mathsf{M};\mathsf{N}]\hat{r}_1=[\mathsf{M};\mathsf{N}]\hat{r}_2$. Then $\mathsf{M}(z)\hat{r}_1(z)=\mathsf{M}(z)\hat{r}_2(z)$ in a neighbourhood of zero and since $\mathsf{M}(z)$ is invertible for z in a neighbourhood of zero we have $\hat{r}_1(z)=\hat{r}_2(z)$ in a neighbourhood of zero. This shows that $r_1=r_2$. Hence multiplication with $[\mathsf{M};\mathsf{N}]$ is injective.

Multiplication with $[\mathsf{M};\mathsf{N}]$ is onto if and only if for every $[u;y]\in\mathscr{V}(0)$ there exists an $r\in l^2(\mathbb{Z}^+,\mathscr{U})$ such that $[\hat{u};\hat{y}]=[\mathsf{M};\mathsf{N}]\hat{r}$. Suppose that multiplication with $[\mathsf{M};\mathsf{N}]$ is onto, and let $h:\mathbb{Z}^+\to\mathscr{U}$ be a Z-transformable sequence with $[\mathsf{M};\mathsf{N}]\hat{h}\in H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})$. Then $[\mathsf{M};\mathsf{N}]\hat{h}\in\hat{\mathscr{V}}(0)$ and since multiplication by $[\mathsf{M};\mathsf{N}]$ maps $H^2(\mathbb{D},\mathscr{U})$ onto $\hat{\mathscr{V}}(0)$ there exists an $r\in l^2(\mathbb{Z}^+,\mathscr{U})$ such that $[\mathsf{M};\mathsf{N}]\hat{h}=[\mathsf{M};\mathsf{N}]\hat{r}$. Since multiplication by $[\mathsf{M};\mathsf{N}]$ is injective as proven above it follows that $h=r$. Hence $h\in l^2(\mathbb{Z}^+,\mathscr{U})$ and so $[\mathsf{M};\mathsf{N}]$ is weakly right-coprime. Suppose that $[\mathsf{M};\mathsf{N}]$ is weakly right-coprime. Let $[u;y]\in\mathscr{V}(0)$. Define $r:\mathbb{Z}^+\to\mathscr{U}$ through its Z-transform: $\hat{r}(z):=\mathsf{M}(z)^{-1}\hat{u}(z)$ for z in a neighbourhood of zero. We then have $[\mathsf{M};\mathsf{N}]\hat{r}=[\hat{u};\hat{y}]$. So $[\mathsf{M};\mathsf{N}]\hat{r}\in H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})$. By weak right-coprimeness we have $r\in l^2(\mathbb{Z}^+,\mathscr{U})$. This shows that muliplication by $[\mathsf{M};\mathsf{N}]$ maps $H^2(\mathbb{D},\mathscr{U})$ onto $\hat{\mathscr{V}}(0)$. □

The following result connects the existence of normalized weakly right-coprime factorizations to the linear quadratic optimal control problem.

Proposition 7.11. *Let Σ be an output stabilizable discrete-time system. Then the transfer function of its optimal closed-loop system provides a normalized weakly right-coprime factorization of the transfer function of Σ.*

Proof. That the transfer function $[\mathsf{M};\mathsf{N}]$ of the optimal closed-loop system satisfies $\mathsf{D}(z)=\mathsf{N}(z)\mathsf{M}(z)^{-1}$ in a neighbourhood of zero follows from Proposition 7.6. Combining Propositions 6.34 and 6.35 we see that optimal closed-loop system is energy-preserving with as storage operator the observability gramian. By Proposition 5.2 the optimal closed-loop system is input-output stable, so $[\mathsf{M};\mathsf{N}]\in H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U},\mathscr{U}\times\mathscr{Y}))$. Proposition 5.4, and the fact that $l^2(\mathbb{Z}^+,\mathscr{U})$ and $H^2(\mathbb{D},\mathscr{U})$ are isometrically isomorphic under the Z-transform, shows that the factorization is normalized. We show that it is weakly right-coprime. Let $[u;y]\in\mathscr{V}(0)$. Let x be the corresponding state for initial state zero and define $r_k:=-(S^{\min})^{1/2}F^{\min}x_k+(S^{\min})^{1/2}u_k$. Then by Proposition

6.30 we have $r \in l^2(\mathbb{Z}^+, \mathscr{U})$. The sequence r is the output for input u of the system Σ_- defined by its system operator $[A, B; -(S^{\min})^{1/2}F^{\min}, (S^{\min})^{1/2}]$. Using Proposition 2.22 we see that the inverse of the transfer function of Σ_- has a realization $[A + BF^{\min}, B(S^{\min})^{-1/2}; F, (S^{\min})^{-1/2}]$. But this is a realization of M and so we conclude that Σ_- has M^{-1} as its transfer function. So $\hat{r}(z) = \mathsf{M}(z)^{-1}\hat{u}(z)$. It follows that $[\hat{u}(z); \hat{y}(z)] = [\mathsf{M}(z); \mathsf{N}(z)]\hat{r}(z)$. Hence each element of $\mathscr{V}(0)$ is in the range of the operator of multiplication by $[\mathsf{M}; \mathsf{N}]$. By Proposition 7.10 the pair (M, N) is weakly right-coprime. □

The existence of a right factorization and of an output stabilizable realization are equivalent as the following proposition shows.

Proposition 7.12. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$*. Then the following are equivalent:*

1. G *has a right factorization.*
2. G *has an output stabilizable realization.*

Proof. If G has an output stabilizable realization, then by Proposition 7.11 it has a right factorization. Assume that G has a right factor $[\mathsf{M}; \mathsf{N}]$. This right factor has a realization $\check{\Sigma}$ that is output stable (for example the backward shift realization from Remark 2.13 which is output stable by Example 3.3). Since $\mathsf{M}(0)$ has a bounded inverse, we can use Proposition 2.23 to obtain a realization Σ of G. It follows from Corollary 4.15 that Σ is output stabilizable. □

The following proposition shows that existence of a right factorization implies the existence of a normalized weakly right-coprime factorization.

Proposition 7.13. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$*. If* G *has a right factorization, then it has a normalized weakly right-coprime factorization.*

Proof. From Proposition 7.12 we see that G has an output stabilizable realization Σ. Proposition 7.10 shows that the optimal closed-loop system of Σ provides a normalized weakly right-coprime factorization of G. □

The following proposition gives a parametrization of all right factorizations.

Proposition 7.14. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$ *and assume that* G *has a right factorization. Let* $[\mathsf{M}_0, \mathsf{N}_0]$ *be a weakly right-coprime factor. Then all right factors are parametrized as follows:*

$$\mathsf{M} = \mathsf{M}_0\mathsf{V}, \qquad \mathsf{N} = \mathsf{N}_0\mathsf{V},$$

where V *runs through the set of* $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$ *functions that have a bounded inverse in zero. The weakly right-coprime factors are exactly those for which* V^{-1} *is in* $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$ *as well.*

Proof. The above M and N obviously provide a factorization. Assume that $[\mathsf{M}_1;\mathsf{N}_1]$ is a right factor. Define $\mathsf{V} := \mathsf{M}_0^{-1}\mathsf{M}_1$. Then $\mathsf{M}_1 = \mathsf{M}_0\mathsf{V}$ and $\mathsf{N}_1 = \mathsf{G}\mathsf{M}_1 = \mathsf{G}\mathsf{M}_0\mathsf{V} = \mathsf{N}_0\mathsf{V}$. By Proposition 7.7 we have that $\mathsf{V} \in H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$.

If V has an inverse in $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$, then from $[\mathsf{M};\mathsf{N}]h = [\mathsf{M}_0;\mathsf{N}_0]\mathsf{V}h \in H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})$ we obtain $h = \mathsf{V}^{-1}\mathsf{V}h \in H^2(\mathbb{D},\mathscr{U})$. This shows that in this case $[\mathsf{M};\mathsf{N}]$ is weakly right-coprime. If $[\mathsf{M};\mathsf{N}]$ is weakly right-coprime, then it follows from symmetry considerations that V must have an inverse in $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$. □

Proposition 7.15. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U},\mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$ *and assume that* G *has a right factorization. Let* $[\mathsf{M}_0,\mathsf{N}_0]$ *be a normalized weakly right-coprime factor. Then all normalized weakly right-coprime factors are parametrized as follows:*

$$\mathsf{M} = \mathsf{M}_0\mathsf{V}, \qquad \mathsf{N} = \mathsf{N}_0\mathsf{V},$$

where $\mathsf{V} \in \mathcal{L}(\mathscr{U})$ *is unitary.*

Proof. That the above M and N provide a normalized weakly right-coprime factorization is obvious. Assume that the pair $[\mathsf{M};\mathsf{N}]$ is a normalized weakly right-coprime factor. From Proposition 7.14 we obtain that a normalized weakly right-coprime factor must be of the indicated form, but we only know that V and its inverse are in $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$. So we still need to show that this function is constant and that this constant is a unitary operator. Since the factorizations are normalized we have $\mathsf{M}^*\mathsf{M} + \mathsf{N}^*\mathsf{N} = I$ and $\mathsf{M}_0^*\mathsf{M}_0 + \mathsf{N}_0^*\mathsf{N}_0 = I$ almost everywhere on the unit circle by Lemmas A.18 and A.20. Since $\mathsf{M} = \mathsf{M}_0\mathsf{V}$ (on the open unit disc, but this extends to almost everywhere on the unit circle) it follows that $\mathsf{V}^*\mathsf{V} = I$ almost everywhere on the unit circle. Since V has an inverse in $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$ its boundary function has an inverse in $L^\infty(\mathbb{T},\mathcal{L}(\mathscr{U}))$ and since $\mathsf{V}^*\mathsf{V} = I$, this inverse must equal V^*. Hence V^* is the boundary function of a function in $H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U}))$, namely of V^{-1}. Define $\mathsf{V}_- : \mathbb{D}^+ \to \mathcal{L}(\mathscr{U})$ by $\mathsf{V}_-(z) = \mathsf{V}(1/\bar{z})^*$. Then $\mathsf{V}^- \in H^\infty(\mathbb{D}^+,\mathcal{L}(\mathscr{U}))$ since it is obviously holomorphic and

$$\sup_{z\in\mathbb{D}^+} \|\mathsf{V}_-(z)\| = \sup_{z\in\mathbb{D}^+} \|\mathsf{V}(1/\bar{z})^*\| = \sup_{z\in\mathbb{D}^+} \|\mathsf{V}(1/\bar{z})\| = \sup_{s\in\mathbb{D}} \|\mathsf{V}(s)\| = \|\mathsf{V}\|_\infty.$$

The boundary function of V_- equals V^*. Hence V^* is the boundary function of a function in $H^\infty(\mathbb{D}^+,\mathcal{L}(\mathscr{U}))$, namely of V_-. So V^* is the boundary function of both a $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$ function and a $H^\infty(\mathbb{D}^+, \mathcal{L}(\mathscr{U}))$ function. It follows from Corollary A.14 that V^* is constant. Hence V is constant. It follows from the earlier established $\mathsf{V}^*\mathsf{V} = I$ almost everywhere on the unit circle and the fact that V has an inverse that V is unitary. □

Proposition 7.16. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *If* G *has a strongly right-coprime factorization, then all weakly right-coprime factorizations are strongly right-coprime.*

Proof. Assume that $[\mathsf{M}_0; \mathsf{N}_0]$ is a strongly right-coprime factor. Let $\tilde{\mathsf{X}}$ and $\tilde{\mathsf{Y}}$ be right Bezout factors. According to Proposition 7.14 all weakly right-coprime factors are of the form $[\mathsf{M}_0; \mathsf{N}_0]\mathsf{V}$ with both V and its inverse in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}))$. It is easily seen that $\mathsf{V}^{-1}\tilde{\mathsf{X}}$ and $\mathsf{V}^{-1}\tilde{\mathsf{Y}}$ are right Bezout factors for $[\mathsf{M}_0; \mathsf{N}_0]\mathsf{V}$. It follows that $\mathsf{M}_0\mathsf{V}$ and $\mathsf{N}_0\mathsf{V}$ are strongly right-coprime. □

In the following proposition we need the **Hankel operator** which is defined in Definition A.24. We further note that a H^∞ function is called inner if the corresponding multiplication operator is an isometry (Definition A.19, see also Lemma A.20).

Proposition 7.17. *Let* $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$. *Assume that* G *is inner and that it has a left inverse in* $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$. *Then the norm of the associated Hankel operator is strictly less than one.*

Proof. We have that the Hankel operator has norm less than or equal to one, since it is the composition of an isometric operator with two projections, each of which have norm smaller than or equal to one.

We show that the norm of the Hankel operator cannot be one. Suppose it is. Then there exists a sequence $h^n \in L^2(\mathbb{T}, \mathscr{H}_1)$ with norm one such that $\|P_+ L_\mathsf{G} P_- h_n\| \to 1$. Here P_- is the projection from $L^2(\mathbb{T}, \mathscr{H}_1)$ onto the subspace of functions whose nonnegative Fourier coefficients are zero, P_+ is the projection from $L^2(\mathbb{T}, \mathscr{H}_2)$ onto the subspace of functions whose negative Fourier coefficients are zero and L_G is the operator multiplication with G (see Definition A.15). We can assume without loss of generality that the h^n have zero nonnegative Fourier coefficients. Define $f^n := L_\mathsf{G} P_- h_n$, $f^n_+ := P_+ f^n$, $f^n_- := P_- f^n$. Then since G is inner we have $\|f^n\| = 1$ and we have $\|f^n\|^2 = \|f^n_+\|^2 + \|f^n_-\|^2$. Since by assumption $\|f^n_+\| \to 1$ it follows that $\|f^n_-\| \to 0$. By assumption there exists a $\mathsf{H} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ such that $\mathsf{H}\mathsf{G} = I$. We then have $h^n = L_\mathsf{H} L_\mathsf{G} h^n = L_\mathsf{H} f^n = L_\mathsf{H} f^n_+ + L_\mathsf{H} f^n_-$. Since $L_\mathsf{H} f^n_+$ has zero negative Fourier coefficients and h^n has zero nonnegative Fourier coefficients we have $\langle h^n, L_\mathsf{H} f^n_+ \rangle = 0$. So

$$0 = \langle h^n, L_\mathsf{H} f^n_+ \rangle = \langle h^n, h^n \rangle - \langle h^n, L_\mathsf{H} f^n_- \rangle \to 1.$$

This contradiction shows that the Hankel operator must have norm strictly smaller than one. □

The following proposition complements the previous one.

Proposition 7.18. *Let* $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$. *Assume that* G *is inner and that the norm of the associated Hankel operator is strictly less than one. Then* G *has a left inverse in* $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$.

Proof. We apply Proposition A.27 (the Nehari theorem) to G^*. Since $\|H_\mathsf{G}\| < 1$, this gives the existence of a $\mathsf{K} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ such that

$$\|\mathsf{G}^* + \mathsf{K}\|_{L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} < 1.$$

Since G is inner we have $\mathsf{G}^*\mathsf{G} = I$ almost everywhere on the unit circle from Proposition A.18. From this we obtain $I + \mathsf{KG} = \mathsf{G}^*\mathsf{G} + \mathsf{KG} = (\mathsf{G}^* + \mathsf{K})\mathsf{G}$ almost everywhere on the unit circle, which gives

$$\|I + \mathsf{KG}\|_{L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_1))} \leq \|\mathsf{G}^* + \mathsf{K}\|_{L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))} \, \|\mathsf{G}\|_{L^\infty(\mathbb{T}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))} < 1.$$

Since $\mathsf{KG} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1))$, which is a Banach algebra, we obtain that KG has an inverse R in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1))$ from the geometric series theorem. In particular $\mathsf{RKG} = I$, which implies that RK is a left inverse of G. □

Combining Propositions 7.17 and 7.18 we obtain the following.

Corollary 7.19. *Assume* $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2))$ *is inner. Then* G *has a left inverse in* $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1))$ *if and only if the norm of the Hankel operator of* G *is strictly less than one.*

The following result connects the existence of normalized strongly right-coprime factorizations to the linear quadratic optimal control problem.

Proposition 7.20. *Let* Σ *be an input and output stabilizable discrete-time system. Then the transfer function of the optimal closed-loop system of* Σ *is a normalized strongly right-coprime factor of the transfer function of* Σ.

Proof. From Proposition 7.11 we obtain that the transfer function of the optimal closed-loop system of Σ is a normalized right factor. Corollary 6.41 shows that the Hankel map of this system has norm strictly smaller than one. Since the Hankel map and the Hankel operator have the same norm by Lemma A.26, Proposition 7.18 then gives the result. □

The following proposition shows that not only the optimal closed-loop system provides a strongly right-coprime factorization, but that every Riccati closed-loop system does. Note that we may not obtain a normalized factorization in this case.

Proposition 7.21. *Let Σ be an input and output stabilizable discrete-time system. Then the transfer function of any Riccati closed-loop system of Σ is a strongly right-coprime factor of the transfer function of Σ.*

Proof. That we obtain a factorization follows from Propositions 6.34 and 7.6. Application of Propositions 6.47, 6.49 and 6.50 shows that the transfer function of an arbitrary Riccati closed-loop system of Σ can be obtained by multiplying the transfer function of the optimal closed-loop system from the right with a function that is in H^∞ and whose inverse is H^∞. Using that by Proposition 7.20 the transfer function of the optimal closed-loop system is strongly right-coprime it then easily follows that the transfer function of any Riccati closed-loop system of Σ is a strongly right-coprime. □

The following proposition shows that existence of a strongly right-coprime factorization and the existence of an input and output stabilizable realization are equivalent.

Proposition 7.22. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Then the following are equivalent:*

1. G *has an input and output stabilizable realization.*
2. G *has a normalized strongly right-coprime factorization.*
3. G *has a strongly right-coprime factorization.*

Proof. If G has an input and output stabilizable realization, then by Proposition 7.20 it has a normalized strongly right-coprime factorization. Assume that G has a strongly right-coprime factorization. It follows from Propositions 7.13 and 7.16 that G has a normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$. By Proposition 7.17 the norm of the Hankel operator associated to $[\mathsf{M}; \mathsf{N}]$ is strictly smaller than one. The function $[\mathsf{M}; \mathsf{N}]$ has an approximately controllable input and output stable realization $\check{\Sigma}$ (for example the restricted backward shift realization from Remark 2.13 which is output stable by Example 3.3 and input stable by Example 3.25). From Proposition 5.7 we obtain that $\check{\Sigma}$ is energy preserving with the observability gramian as storage operator (note that the condition on the equality of the norm of the input and output in Proposition 5.7 is satisfied since the factorization is normalized). We use Proposition 2.23 to obtain the corresponding realization Σ of G. It follows from Corollary 4.15 that Σ is output stabilizable. It follows from Lemma 3.18 combined with Lemma A.26 that the spectral radius of $L_B L_C$, the product of the controllability and the observability gramian of $\check{\Sigma}$, is strictly smaller than one. This implies that the operator $I - L_B L_C$ has a bounded inverse. Proposition 6.46 now shows that $P := (I - L_B L_C)^{-1} L_B$ provides a solution of the filter algebraic Riccati equation of Σ. The dual version of Proposition 6.36 now shows that Σ is input stabilizable. □

The following lemma shows that we can always pick right Bezout factors with a nice property.

Lemma 7.23. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$ *and assume that* G *has a strongly right-coprime factorization. For every strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$ *there exists a pair of right Bezout factors with* $\tilde{\mathsf{Y}}(0) = 0$ *and* $\tilde{\mathsf{X}}(0) = \mathsf{M}(0)^{-1}$.

Proof. Let $[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]$ be an arbitrary pair of Bezout factors. Define $\tilde{\mathsf{Y}}(z) := (I - \mathsf{M}(0)^{-1}\mathsf{M}(z))\tilde{\mathsf{Y}}_1(z)$ and $\tilde{\mathsf{X}}(z) = \mathsf{M}(0)^{-1} + (I - \mathsf{M}(0)^{-1}\mathsf{M}(z))\tilde{\mathsf{X}}_1(z)$. Then obviously $\tilde{\mathsf{Y}}(0) = 0$ and $\tilde{\mathsf{X}}(0) = \mathsf{M}(0)^{-1}$ and it is not hard to see that $\tilde{\mathsf{X}}$, $\tilde{\mathsf{Y}}$ is a right Bezout pair. □

The following proposition shows that the existence of a doubly coprime factorization follows from the existence of a strongly right-coprime factorization.

Proposition 7.24. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$*. The following are equivalent:*

1. G *has a normalized strongly right-coprime factorization.*
2. G *has a normalized strongly left-coprime factorization.*
3. G *has a normalized doubly coprime factorization.*

Moreover, any given normalized strongly right-coprime factorization and normalized strongly left-coprime factorization can be embedded in a normalized doubly coprime factorization.

Proof. That (1) and (2) are equivalent follows from Proposition 7.22 noting that the second condition in that proposition holds for G if and only if it holds for $\mathsf{G}^\dagger$. It is clear that (3) implies (1) and (2). We show that (1) implies (3).

Now assume that G has the normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$ with corresponding right Bezout factors $[\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}]$ and the normalized strongly left-coprime factor $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$ with the corresponding left Bezout factors $[\mathsf{X}_1; \mathsf{Y}_1]$. By Propostition 7.23 we can assume that $\tilde{\mathsf{Y}}(0) = 0$ and $\tilde{\mathsf{X}}(0) = \mathsf{M}(0)^{-1}$. Define $\Delta := \tilde{\mathsf{X}}\mathsf{Y}_1 - \tilde{\mathsf{Y}}\mathsf{X}_1$ and $\mathsf{Y} := -\mathsf{M}\Delta + \mathsf{Y}_1$, $\mathsf{X} := -\mathsf{N}\Delta + \mathsf{X}_1$. It is now easily verified that with this X and Y we have

$$\begin{bmatrix} \tilde{\mathsf{X}} & -\tilde{\mathsf{Y}} \\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}} \end{bmatrix} \begin{bmatrix} \mathsf{M} & \mathsf{Y} \\ \mathsf{N} & \mathsf{X} \end{bmatrix} = I. \tag{7.4}$$

Next we show that

$$\begin{bmatrix} \mathsf{M} & \mathsf{Y} \\ \mathsf{N} & \mathsf{X} \end{bmatrix} \begin{bmatrix} \tilde{\mathsf{X}} & -\tilde{\mathsf{Y}} \\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}} \end{bmatrix} = I.$$

Since $\tilde{\mathsf{Y}}(0) = 0$ and $\tilde{\mathsf{X}}(0) = \mathsf{M}(0)^{-1}$ we have that

$$\begin{bmatrix} \tilde{\mathsf{X}}(0) & -\tilde{\mathsf{Y}}(0) \\ -\tilde{\mathsf{N}}(0) & \tilde{\mathsf{M}}(0) \end{bmatrix}$$

has the bounded inverse

$$\begin{bmatrix} \tilde{\mathsf{M}}(0) & 0 \\ \tilde{\mathsf{M}}(0)^{-1}\tilde{\mathsf{N}}(0)\tilde{\mathsf{M}}(0) & \tilde{\mathsf{M}}(0)^{-1} \end{bmatrix}.$$

The function $[\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}; -\tilde{\mathsf{N}}, \tilde{\mathsf{M}}]$ is holomorphic at zero which implies that it has a realization Σ. Since the function value at zero has a bounded inverse operator, it follows from Proposition 2.22 that $[\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}; -\tilde{\mathsf{N}}, \tilde{\mathsf{M}}]$ is invertible in a neighbourhood of zero. It follows from (7.4) that $[\mathsf{M}, \mathsf{Y}; \mathsf{N}, \mathsf{X}]$ equals this inverse. By the identity theorem for holomorphic functions we have that (7.3) holds on $\mathbb{D}$. Hence G has a doubly coprime factorization. This doubly coprime factorization is obviously normalized. By construction both the given normalized strongly right-coprime factor $[\mathsf{M}; \mathsf{N}]$ and the given normalized strongly left-coprime factor $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$ are embedded in the doubly coprime factor. □

The following result gives a parametrization of all right Bezout factors.

Proposition 7.25. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$. *Then it has a strongly left-coprime factor* $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$. *Let* $\tilde{\mathsf{X}}_0$, $\tilde{\mathsf{Y}}_0$ *be right Bezout factors for* $[\mathsf{M}; \mathsf{N}]$ *and let* $\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$. *Then* $\tilde{\mathsf{X}} := \tilde{\mathsf{X}}_0 + \mathsf{V}\tilde{\mathsf{N}}$, $\tilde{\mathsf{Y}} := \tilde{\mathsf{Y}}_0 + \mathsf{V}\tilde{\mathsf{M}}$ *are right Bezout factors for* $[\mathsf{M}; \mathsf{N}]$. *Moreover, all right Bezout factors for* $[\mathsf{M}; \mathsf{N}]$ *are of this form.*

Proof. That G has a strongly left-coprime factorization follows from (the proof of) Proposition 7.24. That the indicated functions are right Bezout factors is easily checked. We show that all right Bezout factors are of this form. Let $\tilde{\mathsf{X}}_0$, $\tilde{\mathsf{Y}}_0$ be arbitrarty right Bezout factors for $[\mathsf{M}; \mathsf{N}]$. Define V in a neighbourhood of zero by $\mathsf{V} = (\tilde{\mathsf{Y}} - \tilde{\mathsf{Y}}_0)\tilde{\mathsf{M}}^{-1}$. It follows that $\tilde{\mathsf{Y}} = \tilde{\mathsf{Y}}_0 + \mathsf{V}\tilde{\mathsf{M}}$ in a neighbourhood of zero. Using the Bezout equation (7.1) we have $(\tilde{\mathsf{X}} - \tilde{\mathsf{X}}_0)\mathsf{M} = (\tilde{\mathsf{Y}} - \tilde{\mathsf{Y}}_0)\mathsf{N}$. Using the above equation for $\tilde{\mathsf{Y}}$ we see that this equals $\mathsf{V}\tilde{\mathsf{M}}\mathsf{N}$ in a neighbourhood of zero. Since $\tilde{\mathsf{M}}\mathsf{N} = \tilde{\mathsf{N}}\mathsf{M}$ we obtain $(\tilde{\mathsf{X}} - \tilde{\mathsf{X}}_0)\mathsf{M} = \mathsf{V}\tilde{\mathsf{N}}\mathsf{M}$ in a neighbourhood of zero. It follows that $\tilde{\mathsf{X}} = \tilde{\mathsf{X}}_0 + \mathsf{V}\tilde{\mathsf{N}}$ in a neighbourhood of zero. The only thing left to show is that $\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$. This follows since

$$\mathsf{V} = \mathsf{V}(\tilde{\mathsf{M}}\mathsf{X} - \tilde{\mathsf{N}}\mathsf{Y}) = (\tilde{\mathsf{Y}} - \tilde{\mathsf{Y}}_0)\mathsf{X} - (\tilde{\mathsf{X}}_0 - \tilde{\mathsf{X}})\mathsf{Y},$$

where X, Y are left Bezout factors for $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$. □

The set of all strongly right-coprime pairs is open as the following proposition shows.

Proposition 7.26. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$*. Assume that* G *has a strongly right-coprime factor* $[\mathsf{M}; \mathsf{N}]$*. Then there exists a* $\varepsilon > 0$ *such that for all* $\Delta = [\Delta_{\mathsf{M}}; \Delta_{\mathsf{N}}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ *with* $\|\Delta\|_\infty < \varepsilon$ *the functions* $\mathsf{M} + \Delta_{\mathsf{M}}$ *and* $\mathsf{N} + \Delta_{\mathsf{N}}$ *are strongly right-coprime.*

Proof. From Proposition 7.24 we obtain the existence of $\mathsf{X} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}))$ and $\mathsf{Y} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$ such that $[\mathsf{M}, \mathsf{Y}; \mathsf{N}, \mathsf{X}]$ is invertible in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))$. The result follows using that the invertible elements in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U} \times \mathscr{Y}))$ form an open set. □

In Proposition 7.32 we give an explicit ε under which the result of Proposition 7.26 holds under the assumption that $\mathscr{U}$ is finite-dimensional. The following results (Lemma 7.27 up to Proposition 7.31) are used in the proof of Proposition 7.32.

Lemma 7.27. *Let* $\mathscr{H}_1, \mathscr{H}_2$ *be Hilbert spaces,* $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ *and* $S \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$*. Assume that* $TS = I_{\mathscr{H}_2}$ *and* $ST = I_{\mathscr{H}_1}$*. Then*

$$\inf_{h \in \mathscr{H}_2 : \|h\| = 1} \|Sh\| = \frac{1}{\|T\|}.$$

Proof. We have for each $h \in \mathscr{H}_2$ that $\|h\| = \|TSh\| \leq \|T\| \, \|Sh\|$. This implies

$$\inf_{h \in \mathscr{H}_2 : \|h\| = 1} \|Sh\| \geq \frac{1}{\|T\|}.$$

There exist $f_n \in \mathscr{H}_1$ with norm one such that $\|Tf_n\| \to \|T\|$. Define $h_n := Tf_n / \|Tf_n\|$. Then $\|h_n\| = 1$ and $Sh_n = f_n / \|Tf_n\|$. So $\|Sh_n\| = 1/\|Tf_n\|$. For $n \to \infty$ we have $\|Sh_n\| \to 1/\|T\|$. This implies that $1/\|T\|$ is not only a lower bound, but the largest lower bound, i.e. it is the desired infimum. □

Lemma 7.28. *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and $T \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$. Then*

$$\inf_{\|x\|=1} \|Tx\| = \inf_{\|y\|=1} \|T^* y\|,$$

provided that both are positive.

Proof. We have

$$\inf_{\|x\|=1} \|Tx\|^2 = \inf_{\|x\|=1} \langle T^*Tx, x\rangle.$$

It is well-known (see for example Kreyzsig [47, p467]) that the number on the right-hand side is the smallest spectral value of T^*T. Similarly, $\inf_{\|y\|=1} \|T^*y\|^2$ is the smallest spectral value of TT^*. It follows from Lemma 3.16 that the spectra of T^*T and TT^* are equal, with the possible exception of zero. The result follows. □

Lemma 7.29. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be holomorphic with $0 \in D(\mathsf{G})$. Assume that G that has a normalized doubly coprime factorization. Denote the normalized strongly left-coprime factor by $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$, the normalized strongly right-coprime factor by $[\mathsf{M}; \mathsf{N}]$ and the left Bezout factor by $[\mathsf{X}; \mathsf{Y}]$. Denote the Hankel operator of $[\tilde{\mathsf{M}}, \tilde{\mathsf{N}}]$ by $H_{[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]}$. Then*

$$\inf_{\mathsf{V} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))} \left\| \begin{bmatrix} \mathsf{Y} \\ \mathsf{X} \end{bmatrix} - \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} \mathsf{V} \right\| = \frac{1}{\sqrt{1 - \|H_{[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]}\|^2}}. \tag{7.5}$$

Proof. Let $T_{[\mathsf{M};\mathsf{N}]} : H^2(\mathbb{D}, \mathscr{U}) \to H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y})$ be the operator of multiplication by $[\mathsf{M}; \mathsf{N}]$. Since $T_{[\mathsf{M};\mathsf{N}]}$ is an isometry its range is closed and we have the orthogonal decomposition

$$H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y}) = \operatorname{Im}(T_{[\mathsf{M};\mathsf{N}]}) \oplus \operatorname{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp. \tag{7.6}$$

Denote by $P_{\operatorname{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp}$ the orthogonal projection onto the second component in this decomposition. Define $T_{[\mathsf{Y};\mathsf{X}]}$ similarly to $T_{[\mathsf{M};\mathsf{N}]}$. Define $T : H^2(\mathbb{D}, \mathscr{Y}) \to H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y})$ by

$$T := P_{\operatorname{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp} T_{[\mathsf{Y};\mathsf{X}]}. \tag{7.7}$$

We obtain from Corollary A.23 that the infimum on the left-hand side of (7.5) equals $\|T\|$. Define $S : \operatorname{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp \subset H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y}) \to H^2(\mathbb{D}, \mathscr{Y})$ as the restriction to $\operatorname{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp$ of multiplication by $[-\tilde{\mathsf{N}}, \tilde{\mathsf{M}}]$, i.e.

$$S = T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}|_{\operatorname{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp}.$$

We show that S is the inverse of T. First note that for any $y \in H^2(\mathscr{Y})$ there exists a $u \in H^2(\mathscr{U})$ such that

$$Ty = \begin{bmatrix} \mathsf{Y} \\ \mathsf{X} \end{bmatrix} y + \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} u.$$

It follows using (7.3) that $STy = y$ for all $y \in H^2(\mathscr{Y})$. From (7.3) we also obtain

$$\begin{bmatrix} \mathsf{Y} \\ \mathsf{X} \end{bmatrix} [-\tilde{\mathsf{N}}, \tilde{\mathsf{M}}] + \begin{bmatrix} \mathsf{M} \\ \mathsf{N} \end{bmatrix} [\tilde{\mathsf{X}}, -\tilde{\mathsf{Y}}] = I.$$

Restricting to $\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp$ and projecting onto $\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp$ shows that TS equals the identity operator on $\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp$.

Using Lemma 7.27 we obtain

$$\inf_{w \in \mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp : \|w\|=1} \|Sw\|_{H^2(\mathbb{D},\mathscr{Y})} = \frac{1}{\|T\|}.$$

Let $T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} : H^2(\mathbb{D}, \mathscr{U} \times \mathscr{Y}) \to H^2(\mathbb{D}, \mathscr{Y})$ be the Toeplitz operator of $[-\tilde{\mathsf{N}}, \tilde{\mathsf{M}}]$. From (7.3) we obtain that $T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} T_{[\mathsf{M};\mathsf{N}]} = 0$. So $T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}$ is zero on $\mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})$. It follows that $T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}$ splits with respect to the decomposition (7.6) as

$$T_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} = [0, S].$$

Since $T^*_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} = T_{[-\tilde{\mathsf{N}}^*;\tilde{\mathsf{M}}^*]}$ we have, with respect to the decomposition (7.6),

$$T_{[-\tilde{\mathsf{N}}^*;\tilde{\mathsf{M}}^*]} = \begin{bmatrix} 0 \\ S^* \end{bmatrix}.$$

It follows that

$$\inf_{y \in H^2(\mathbb{D},\mathscr{Y}) : \|y\|=1} \|T_{[-\tilde{\mathsf{N}}^*;\tilde{\mathsf{M}}^*]} y\|_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})} = \inf_{y \in H^2(\mathbb{D},\mathscr{Y}) : \|y\|=1} \|S^* y\|_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})}. \quad (7.8)$$

Let $y \in H^2(\mathscr{Y})$. Since $[-\tilde{\mathsf{N}}^*; \tilde{\mathsf{M}}^*]$ is inner we have

$$\begin{aligned}
\|y\|^2_{H^2(\mathbb{D},\mathscr{U})} &= \left\| \begin{bmatrix} -\tilde{\mathsf{N}}^* \\ \tilde{\mathsf{M}}^* \end{bmatrix} y \right\|^2_{L^2(\mathbb{T},\mathscr{U}\times\mathscr{Y})} \\
&= \left\| P_{H^2(\mathscr{U}\times\mathscr{Y})} \begin{bmatrix} -\tilde{\mathsf{N}}^* \\ \tilde{\mathsf{M}}^* \end{bmatrix} y \right\|^2_{L^2(\mathbb{T},\mathscr{U}\times\mathscr{Y})} \\
&\quad + \left\| P_{H^2(\mathscr{U}\times\mathscr{Y})^\perp} \begin{bmatrix} -\tilde{\mathsf{N}}^* \\ \tilde{\mathsf{M}}^* \end{bmatrix} y \right\|^2_{L^2(\mathbb{T},\mathscr{U}\times\mathscr{Y})} \\
&= \|T_{[-\tilde{\mathsf{N}}^*;\tilde{\mathsf{M}}^*]} y\|^2 + \|H^*_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} y\|^2,
\end{aligned}$$

where

$$H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} := P_{H^2(\mathbb{D},\mathscr{Y})} L_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]} P_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})^\perp} : L^2(\mathbb{T},\mathscr{U}\times\mathscr{Y}) \to L^2(\mathbb{T},\mathscr{Y})$$

is the Hankel operator of $[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]$. It follows that

$$\inf_{y\in H^2(\mathbb{D},\mathscr{Y}):\|y\|=1} \|T_{[-\tilde{\mathsf{N}}^*;\tilde{\mathsf{M}}^*]}y\|^2_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})} = 1 - \|H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}\|^2. \tag{7.9}$$

Combining (7.8) and (7.9) we obtain

$$\inf_{y\in H^2(\mathbb{D},\mathscr{Y}):\|y\|=1} \|S^*y\|^2_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})} = 1 - \|H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}\|^2. \tag{7.10}$$

Using the dual version of Proposition 7.17 we conclude from the fact that $\tilde{\mathsf{N}}$ and $\tilde{\mathsf{M}}$ are strongly left-coprime that $\|H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}\| < 1$, so that the number in (7.10) is positive. We use Lemma 7.28 to conclude that

$$\inf_{w\in \mathrm{Im}(T_{[\mathsf{M};\mathsf{N}]})^\perp:\|w\|=1} \|Sw\|_{H^2(\mathbb{D},\mathscr{Y})} = \inf_{y\in H^2(\mathbb{D},\mathscr{Y}):\|y\|=1} \|S^*y\|_{H^2(\mathbb{D},\mathscr{U}\times\mathscr{Y})} \tag{7.11}$$

Note that we have already established that both sides of (7.11) are positive so that Lemma 7.28 is indeed applicable. We earlier established that the left-hand side of (7.11) equals $1/\|T\|$ and that this equals one over the infimum in the statement of the lemma. The right-hand side of (7.11) we have shown to be equal to $\sqrt{1-\|H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}\|^2}$. Noting that $\|H_{[-\tilde{\mathsf{N}},\tilde{\mathsf{M}}]}\| = \|H_{[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]}\|$ gives the desired result. □

Applying Lemma 7.29 to $\mathsf{G}^\dagger$ we obtain the following.

Corollary 7.30. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U},\mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a normalized doubly coprime factorization. Denote the normalized strongly left-coprime factor by* $[\tilde{\mathsf{M}},\tilde{\mathsf{N}}]$, *the normalized strongly right-coprime factor by* $[\mathsf{M};\mathsf{N}]$ *and the right Bezout factor by* $[\tilde{\mathsf{X}};\tilde{\mathsf{Y}}]$. *Denote the Hankel operator of* $[\mathsf{M},\mathsf{N}]$ *by* $H_{[\mathsf{M},\mathsf{N}]}$. *Then*

$$\inf_{\mathsf{V}\in H^\infty(\mathbb{D},\mathcal{L}(\mathscr{Y},\mathscr{U}))} \left\| \begin{bmatrix} \tilde{\mathsf{Y}} \\ \tilde{\mathsf{X}} \end{bmatrix} - \mathsf{V} \begin{bmatrix} \tilde{\mathsf{M}} \\ \tilde{\mathsf{N}} \end{bmatrix} \right\|_{H^\infty(\mathbb{D},\mathcal{L}(\mathscr{Y}\times\mathscr{U},\mathscr{U}))} = \frac{1}{\sqrt{1-\|H_{[\mathsf{M},\mathsf{N}]}\|^2}}.$$

Proof. This follows from applying Lemma 7.29 to $\mathsf{G}^\dagger$. □

Proposition 7.31. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U},\mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a normalized doubly coprime factorization. Denote the Hankel operator of a normalized strongly right-coprime factor* $[\mathsf{M};\mathsf{N}]$ *by* $H_{[\mathsf{M};\mathsf{N}]}$. *Then for all* $z \in \mathbb{D}$ *and* $u \in \mathscr{U}$

$$\left\| \begin{bmatrix} \mathsf{M}(z) \\ \mathsf{N}(z) \end{bmatrix} u \right\|^2 \geq (1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2)\ \|u\|^2.$$

Proof. Denote $\eta := \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2}$. We have $\eta \in (0,1]$ by Proposition 7.17. Let $\delta \in (0, \eta^2)$. Define

$$\varepsilon := \frac{1}{\sqrt{\eta^2 - \delta}} - \frac{1}{\eta}.$$

It easily follows that $\varepsilon > 0$.

Denote a Bezout factor of $[\mathsf{M};\mathsf{N}]$ by $[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]$. We have for $z \in \mathbb{D}$ and $u \in \mathscr{U}$

$$\|u\| = \|[\tilde{\mathsf{X}}_1(z), \tilde{\mathsf{Y}}_1(z)][\mathsf{M}(z);\mathsf{N}(z)]u\| \leq \|[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]\|_\infty \, \|[\mathsf{M}(z);\mathsf{N}(z)]u\|.$$

From this we obtain

$$\left\| \begin{bmatrix} \mathsf{M}(z) \\ \mathsf{N}(z) \end{bmatrix} u \right\|^2 \geq \frac{1}{\|[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]\|_\infty^2} \|u\|^2. \tag{7.12}$$

It is easily computed that if $[\tilde{\mathsf{X}}, \tilde{\mathsf{Y}}]$ is a Bezout factor, then so is $[\tilde{\mathsf{X}} - \mathsf{V}\tilde{\mathsf{N}}, \tilde{\mathsf{Y}} - \mathsf{V}\tilde{\mathsf{M}}]$ for any $\mathsf{V} \in H^\infty$. Using this we obtain from Corollary 7.30 that for each $\tilde{\varepsilon} > 0$ there exists a right Bezout factor $[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]$ with

$$\|[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]\|_\infty \leq \frac{1}{\eta} + \tilde{\varepsilon}. \tag{7.13}$$

In particular we can choose $\tilde{\varepsilon} = \varepsilon$, where ε is as above. With that choice the right-hand side of (7.13) equals $1/\sqrt{\eta^2 - \delta}$. It follows that

$$\frac{1}{\|[\tilde{\mathsf{X}}_1, \tilde{\mathsf{Y}}_1]\|_\infty^2} \geq \eta^2 - \delta. \tag{7.14}$$

Combining (7.12) and (7.14) we obtain

$$\left\| \begin{bmatrix} \mathsf{M}(z) \\ \mathsf{N}(z) \end{bmatrix} u \right\|^2 \geq \left(1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2 - \delta\right) \; \|u\|^2.$$

Since this holds for every $\delta \in (0, \eta^2)$ we obtain the desired result. □

The following proposition provides an explicit ball around a strongly right-coprime factor that only contains strongly right-coprime pairs.

Proposition 7.32. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a strongly right-coprime factor* $[\mathsf{M};\mathsf{N}]$ *and that* $\mathscr{U}$ *is finite-dimensional. Denote the Hankel operator of* $[\mathsf{M};\mathsf{N}]$ *by* $H_{[\mathsf{M};\mathsf{N}]}$. *If* $\Delta = [\Delta_\mathsf{M}; \Delta_\mathsf{N}] \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{U} \times \mathscr{Y}))$ *is such that* $\|\Delta\|_\infty < \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2}$, *then the functions* $\mathsf{M} + \Delta_\mathsf{M}$ *and* $\mathsf{N} + \Delta_\mathsf{N}$ *are strongly right-coprime.*

Proof. Define $\varepsilon := \sqrt{1 - \|H_{[\mathsf{M};\mathsf{N}]}\|^2} - \|\Delta\|_\infty > 0$. Using that

$$\begin{aligned} \left\| \begin{bmatrix} \mathsf{M}(z) \\ \mathsf{N}(z) \end{bmatrix} u \right\| &\leq \left\| \begin{bmatrix} \mathsf{M}(z) + \Delta_{\mathsf{M}}(z) \\ \mathsf{N}(z) + \Delta_{\mathsf{N}}(z) \end{bmatrix} u \right\| + \|\Delta(z)u\| \\ &\leq \left\| \begin{bmatrix} \mathsf{M}(z) + \Delta_{\mathsf{M}}(z) \\ \mathsf{N}(z) + \Delta_{\mathsf{N}}(z) \end{bmatrix} u \right\| + \|\Delta\|_\infty \|u\|, \end{aligned}$$

we have

$$\left\| \begin{bmatrix} \mathsf{M}(z) + \Delta_{\mathsf{M}}(z) \\ \mathsf{N}(z) + \Delta_{\mathsf{N}}(z) \end{bmatrix} u \right\| \geq \left\| \begin{bmatrix} \mathsf{M}(z) \\ \mathsf{N}(z) \end{bmatrix} u \right\| - \|\Delta\|_\infty \, \|u\| \geq \varepsilon \, \|u\|,$$

where we have also used Proposition 7.31. The Corona Theorem (Proposition A.29) then shows that $\mathsf{M} + \Delta_{\mathsf{M}}$ and $\mathsf{N} + \Delta_{\mathsf{N}}$ are strongly right-coprime. □

The following proposition will be useful in the next two chapters.

Proposition 7.33. *Let* $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ *be holomorphic with* $0 \in D(\mathsf{G})$. *Assume that* G *has a normalized doubly coprime factorization. Define* $\mathsf{W} : \mathbb{T} \to \mathcal{L}(\mathscr{U} \times \mathscr{Y})$ *(almost everywhere) by*

$$\mathsf{W}(z) = \begin{bmatrix} \mathsf{M}(z) & -\tilde{\mathsf{N}}(z)^* \\ \mathsf{N}(z) & \tilde{\mathsf{M}}(z)^* \end{bmatrix}.$$

Then $\mathsf{W}(z)$ *is unitary for almost all* $z \in \mathbb{T}$.

Proof. We first show that $\mathsf{W}(z)$ is an isometry, i.e. that $\mathsf{W}(z)^*\mathsf{W}(z) = I$ for almost all $z \in \mathbb{T}$. We have

$$\begin{aligned} \mathsf{W}(z)^*\mathsf{W}(z) &= \begin{bmatrix} \mathsf{M}(z)^* & \mathsf{N}(z)^* \\ -\tilde{\mathsf{N}}(z) & \tilde{\mathsf{M}}(z) \end{bmatrix} \begin{bmatrix} \mathsf{M}(z) & -\tilde{\mathsf{N}}(z)^* \\ \mathsf{N}(z) & \tilde{\mathsf{M}}(z)^* \end{bmatrix} \\ &= \begin{bmatrix} \mathsf{M}(z)^*\mathsf{M}(z) + \mathsf{N}(z)^*\mathsf{N}(z) & \tilde{\mathsf{N}}(z)^*\tilde{\mathsf{M}}(z)^* - \mathsf{M}(z)^*\tilde{\mathsf{N}}(z)^* \\ \tilde{\mathsf{M}}(z)\mathsf{N}(z) - \tilde{\mathsf{N}}(z)\mathsf{M}(z) & \tilde{\mathsf{M}}(z)\tilde{\mathsf{M}}(z)^* + \tilde{\mathsf{N}}(z)\tilde{\mathsf{N}}(z)^* \end{bmatrix}. \end{aligned}$$

The diagonal entries equal the identity since both the right and the left factorization is normalized. The off-diagonal entries are zero by (7.3). We show that $\mathsf{W}(z)$ is surjective. Since a surjective isometry is unitary this proves the proposition. We use that $\mathsf{W}(z)$ is surjective if and only if its range is closed and $\mathsf{W}(z)^*$ is injective. We first show that the range of any isometry T is closed. Let $y_n \in \mathrm{Im}(T)$ and assume that y_n converges to y. Let x_n be such that $y_n = Tx_n$ and define $x = T^*y$. Then

$$y \leftarrow Tx_n = TT^*Tx_n \to TT^*y = Tx.$$

So $y \in \mathrm{Im}(T)$ from which it follows that the range of T is closed. We now show that $\mathsf{W}(z)^*$ is injective. We use (7.3) and the normalization property to obtain

$$\begin{aligned}
[\mathsf{M}^*, \mathsf{N}^*] &= [\mathsf{M}^*, \mathsf{N}^*] \begin{bmatrix} \mathsf{M} & \mathsf{Y} \\ \mathsf{N} & \mathsf{X} \end{bmatrix} \begin{bmatrix} \tilde{\mathsf{X}} & -\tilde{\mathsf{Y}} \\ -\tilde{\mathsf{N}} & \tilde{\mathsf{M}} \end{bmatrix} \\
&= [\tilde{\mathsf{X}} - \mathsf{M}^*\mathsf{Y}\tilde{\mathsf{N}} - \mathsf{N}^*\mathsf{X}\tilde{\mathsf{N}},\ -\tilde{\mathsf{Y}} + \mathsf{M}^*\mathsf{Y}\tilde{\mathsf{M}} + \mathsf{N}^*\mathsf{X}\tilde{\mathsf{M}}],
\end{aligned}$$

on the unit circle. Assume $[u; y] \in \ker \mathsf{W}(z)^*$. Then $\mathsf{M}^*u + \mathsf{N}^*y = 0$ and $-\tilde{\mathsf{N}}u + \tilde{\mathsf{M}}y = 0$. We obtain from the above $0 = \tilde{\mathsf{X}}u - \tilde{\mathsf{Y}}y$. Using (7.3) we obtain from $-\tilde{\mathsf{N}}u + \tilde{\mathsf{M}}y = 0$ and $\tilde{\mathsf{X}}u - \tilde{\mathsf{Y}}y = 0$ that $[u; y] = 0$. It follows that $\mathsf{W}(z)^*$ is injective. This completes the proof. □

Notes

An excellent account of the use of coprime factorizations in systems and control theory is Vidyasagar [94]. The relation with state space systems was made by Khargonekar and Sontag [43] and Nett, Jacobson and Balas [58] in the case of rational functions. The relation between state space systems and normalized coprime factorizations of rational functions was established in Meyer and Franklin [55].

The concept of weak coprimeness as used here is due to Mikkola [56]. The results presented here on weakly coprime factorizations are also due to Mikkola [56]. Our proofs differ only slightly from his. Proposition 7.17 is due to Glover and McFarlane [36] in the rational case. Earlier generalizations to the general, not necessarily rational, case can be found in Curtain and Zwart [18, Lemma 9.4.7] and Oostveen [64, Lemma 7.2.4].

Propositions 7.18 to 7.22 were first given by Curtain and Opmeer [16] for continuous-time systems. This sequence of propositions constitutes our main original contribution on coprime factorizations. The sequence of propositions establishes a long sought after necessary and sufficient state space condition for existence of strongly coprime factorizations over H^∞. Partial result in this direction were obtained in, among others, Curtain and Zwart [19], Curtain, Weiss and Weiss [10], Curtain and Oostveen [12] and Staffans [90]. We note that in [16] also state space formulas for the Bezout factors are given for the continuous-time case. These are based on state space formulas for the continuous-time suboptimal Nehari problem obtained in Curtain and Opmeer [15]. Similar state space formulas can be obtained in discrete-time using the same approach.

Lemma 7.29 is due to Glover and McFarlane [36] for rational functions. The nonrational case was proven by Georgiou and Smith [34] for $\mathscr{U}$ and $\mathscr{Y}$

finite-dimensional. Our proof, also valid for $\mathscr{U}$ and $\mathscr{Y}$ infinite-dimensional, does not significantly differ from the one given by Georgiou and Smith.

Proposition 7.33 is due to Glover and McFarlane [36] for the rational case and to Curtain [11] for the general case considered here.

For a different viewpoint on coprime factorizations for not necessarily rational functions we refer to Quadrat [78], [79], [80], [81], [82], [83].