Model reduction for controller design for infinite-dimensional systems

Opmeer, Mark Robertus

Chapter 6

The linear quadratic optimal control problem

In this chapter we consider the best-studied problem in systems theory, the **linear quadratic optimal control problem**. This problem is also known as the linear quadratic regulator or **LQR-problem**. We first review several well-known results. Many of these are available in the literature only under stronger assumptions then the ones we impose. Towards the end of this chapter some completely new results are presented that are of crucial importance in later chapters.

Problem 6.1. For a given discrete-time system, consider the cost function

$$J(x_0, u) = \|u\|^2_{l^2(\mathbb{Z}^+,\mathscr{U})} + \|y\|^2_{l^2(\mathbb{Z}^+,\mathscr{Y})}, \tag{6.1}$$

where y is the output for initial state x_0 and input u. The goal is the minimize this cost function over all inputs.

An obvious condition on the underlying system is that, for each initial state, there should exist an input that makes the cost finite. We formalize this in the following definition.

Definition 6.2. A discrete-time system satisfies the **finite cost condition** if the following holds. For each initial state $x_0 \in \mathscr{X}$ there exists an input $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ such that the corresponding output is in $l^2(\mathbb{Z}^+, \mathscr{Y})$.

The principal ingredient in the solution of the LQR-problem is the following well-known result, which is often referred to as the **orthogonal projection lemma**.

Proposition 6.3. *Let $\mathscr{H}$ be a Hilbert space and $\mathscr{K}$ a nonempty closed subspace of $\mathscr{H}$. Define, for $h_0 \in \mathscr{H}$, the affine set*

$$\mathscr{K}(h_0) := \{h \in \mathscr{H} : h = h_0 + k \text{ for some } k \in \mathscr{K}\}.$$

Then there exists a unique $h_{\min} \in \mathscr{K}(h_0)$ *such that*

$$\|h_{\min}\| = \min_{h \in \mathscr{K}(h_0)} \|h\|.$$

$h_{\min}$ *is characterized by the fact that it is the unique fixed point in* $\mathscr{K}(h_0)$ *of the orthogonal projection onto* $\mathscr{K}^{\perp}$.

Proof. See for example Kreyszig [47, Section 3.3]. □

We will first analyze a certain set associated with the system. For a discrete-time system consider the set of **stable input-output pairs**

$$\mathscr{V}(x_0) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \begin{bmatrix} l^2(\mathbb{Z}^+, \mathscr{U}) \\ l^2(\mathbb{Z}^+, \mathscr{Y}) \end{bmatrix} : y \text{ satisfies } (2.1) \right\}. \tag{6.2}$$

Note that $\mathscr{V}(x_0)$ is nonempty for every $x_0 \in \mathscr{X}$ if and only if the finite cost condition is satisfied. $\mathscr{V}(x_0)$ will play the role of $\mathscr{K}(h_0)$ in the orthogonal projection lemma.

Lemma 6.4. $\mathscr{V}(0)$ *is a closed linear subspace of* $l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y})$.

Proof. If $[u; y] \in \mathscr{V}(0)$, then

$$y_n = \sum_{k=0}^{n-1} CA^k B u_{n-k-1} + D u_n. \tag{6.3}$$

From this it is easily seen that $\mathscr{V}(0)$ is a linear space. We now prove that $\mathscr{V}(0)$ is closed. Let $[u^m; y^m] \in \mathscr{V}(0)$ and assume that there exist $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ and $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$ such that $u^m \to u$ in $l^2(\mathbb{Z}^+, \mathscr{U})$ and $y^m \to y$ in $l^2(\mathbb{Z}^+, \mathscr{Y})$. Then $u_n^m \to u_n$ in $\mathscr{U}$, from which we obtain

$$y_n^m = \sum_{k=0}^{n-1} CA^k B u_{n-k-1}^m + D u_n^m \to \sum_{k=0}^{n-1} CA^k B u_{n-k-1} + D u_n,$$

since we also have $y_n^m \to y_n$ in $\mathscr{Y}$ we obtain that y is the output corresponding to u. This shows that $\mathscr{V}(0)$ is closed. □

The next result establishes existence and uniqueness of the minimizing input.

Proposition 6.5. *If the finite cost condition is satisfied, then, for every* $x_0 \in \mathscr{X}$, *there exists a unique element in* $\mathscr{V}(x_0)$ *with minimal norm. This element is characterized by the fact that it is the unique fixed point in* $\mathscr{V}(x_0)$ *of the orthogonal projection onto* $\mathscr{V}(0)^{\perp}$.

Proof. We apply Proposition 6.3 with $\mathscr{H} = l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y})$ and $\mathscr{K} = \mathscr{V}(0)$.

Note that if $(u^1, y^1), (u^2, y^2) \in \mathscr{V}(x_0)$, then $(u^1 - u^2, y^1 - y^2) \in \mathscr{V}(0)$. So $\mathscr{V}(x_0)$ is a translation of the closed subspace $\mathscr{V}(0)$ just like $\mathscr{K}(h_0)$ is a translation of the closed set $\mathscr{K}$. $\mathscr{V}(0)$ is nonempty since it contains zero. That $\mathscr{V}(0)$ is a closed convex subset follows from Lemma 6.4. The above shows that all the conditions of Proposition 6.3 are fulfilled. This proposition now gives the desired result. □

Definition 6.6. Define for a system that satisfies the finite cost condition the operator

$$\mathcal{I}^+ : \mathscr{X} \to l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y}), \qquad \mathcal{I}^+ w := \begin{bmatrix} u_w^{\min} \\ y_w^{\min} \end{bmatrix},$$

that assigns to $w \in \mathscr{X}$ the element of $\mathscr{V}(w)$ with minimal norm. This operator is called the **minimizing operator** of the system.

Proposition 6.7. *The minimizing operator is linear.*

Proof. Let $w_1, w_2 \in \mathscr{X}$. We shall prove that $\mathcal{I}^+(w_1 + w_2) = \mathcal{I}^+ w_1 + \mathcal{I}^+ w_2$. Since the system is linear, we have that the output for initial state $w_1 + w_2$ and input $u_{w_1}^{\min} + u_{w_2}^{\min}$ is $y_{w_1}^{\min} + y_{w_2}^{\min}$. Hence $\mathcal{I}^+ w_1 + \mathcal{I}^+ w_2 \in \mathscr{V}(w_1 + w_2)$. Let P be the orthogonal projection onto $\mathscr{V}(0)^\perp$. Since $\mathcal{I}^+ w_1$ and $\mathcal{I}^+ w_2$ are both fixed points of P, it follows that $\mathcal{I}^+ w_1 + \mathcal{I}^+ w_2$ is. So $\mathcal{I}^+ w_1 + \mathcal{I}^+ w_2$ is a fixed point of P in $\mathscr{V}(w_1 + w_2)$. Since by Proposition 6.5 the element of $\mathscr{V}(\cdot)$ with minimal norm is the unique fixed point of P in this set, it follows that $\mathcal{I}^+ w_1 + \mathcal{I}^+ w_2$ is the element of minimal norm in $\mathscr{V}(w_1 + w_2)$. Hence $\mathcal{I}^+(w_1 + w_2) = \mathcal{I}^+ w_1 + \mathcal{I}^+ w_2$. □

Proposition 6.8. *The minimizing operator is bounded.*

Proof. We show that the minimizing operator is closed. It then follows from the closed graph theorem that it is bounded. Let $w^k \in \mathscr{X} \to w^\infty$ in $\mathscr{X}$, $\mathcal{I}^+ w^k = [u_{w^k}^{\min}; y_{w^k}^{\min}] \to [u^\infty; y^\infty]$ in $l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y})$. We need to show that $[u_{w^\infty}^{\min}; y_{w^\infty}^{\min}] = [u^\infty; y^\infty]$.

The output y for initial condition w and input u is given by

$$y_n = CA^n w + \sum_{i=0}^{n-1} CA^i B u_{n-1-i} + D u_n.$$

Applying this with $w = w^k$ and $u = u_{w^k}^{\min}$ we obtain

$$\left(y_{w^k}^{\min}\right)_n = CA^n w^k + \sum_{i=0}^{n-1} CA^i B \left(u_{w^k}^{\min}\right)_{n-1-i} + D \left(u_{w^k}^{\min}\right)_n.$$

Taking the limit for $k \to \infty$ we obtain

$$y_n^\infty = CA^n w^\infty + \sum_{i=0}^{n-1} CA^i B u_{n-1-i}^\infty + D u_n^\infty .$$

This shows that the output for initial state w^∞ and input u^∞ is y^∞. This shows that $[u^\infty; y^\infty] \in \mathscr{V}(w)$. We show that $[u_{w^\infty}^{\min}; y_{w^\infty}^{\min}] = [u^\infty; y^\infty]$ by proving the latter is a fixed point of the projection onto $\mathscr{V}(0)^\perp$. Since $[u_{w^k}^{\min}; y_{w^k}^{\min}]$ is the element with minimal norm in $\mathscr{V}(w^k)$, we have

$$P_{\mathscr{V}(0)^\perp} \begin{bmatrix} u_{w^k}^{\min} \\ y_{w^k}^{\min} \end{bmatrix} = \begin{bmatrix} u_{w^k}^{\min} \\ y_{w^k}^{\min} \end{bmatrix} .$$

Letting $k \to \infty$ we obtain

$$P_{\mathscr{V}(0)^\perp} \begin{bmatrix} u^\infty \\ y^\infty \end{bmatrix} = \begin{bmatrix} u^\infty \\ y^\infty \end{bmatrix} .$$

So $[u^\infty; y^\infty]$ is indeed a fixed point of the projection onto $\mathscr{V}(0)^\perp$. Since $[u^\infty; y^\infty] \in \mathscr{V}(w)$, and by the uniqueness of the fixed point, we have $[u_{w^\infty}^{\min}; y_{w^\infty}^{\min}] = [u^\infty; y^\infty]$. □

Definition 6.9. Define the following sesquilinear form for a system that satisfies the finite cost condition

$$q^{\min} : \mathscr{X} \times \mathscr{X} \to \mathbb{C}, \quad q^{\min}(w_1, w_2) = \langle \mathcal{I}^+ w_1, \mathcal{I}^+ w_2 \rangle .$$

This sesquilinear form is called the **optimal cost sesquilinear form** of the system.

Note that $q^{\min}(w, w)$ is the optimal cost for the initial condition w. We remind the reader that a sesquilinear form (linear in the first variable and anti-linear in the second) f is called hermitian if $f(x, y) = \overline{f(y, x)}$ for all x and y, nonnegative if $f(x, x) \geq 0$ for all x and positive if $f(x, x) > 0$ for all nonzero f.

Proposition 6.10. *The optimal cost sesquilinear form is continuous, hermitian and nonnegative.*

Proof. The optimal cost sesquilinear form is continuous, since the minimizing operator is continuous by Proposition 6.8. That it is hermitian, nonnegative follows immediately from the definition. □

Definition 6.11. For a system that satisfies the finite cost condition define the bounded self-adjoint nonnegative linear operator $Q^{\min} \in \mathcal{L}(\mathscr{X})$ by

$$q^{\min}(w_1, w_2) = \langle Q^{\min} w_1, w_2 \rangle.$$

This operator is called the **optimal cost operator** of the system.

Proposition 6.12. *Assume that Σ satisfies the finite cost condition. Σ is approximately observable if and only if the optimal cost sesquilinear form is positive (or equivalently, the optimal cost operator is positive).*

Proof. Assume that the optimal cost sesquilinear form is not positive. Then there exists a nonzero $w \in \mathscr{X}$ with zero optimal cost. It follows that the output for initial state w and zero input is zero. This contradicts approximate observability.

Assume that Σ is not approximately observable. Then there exists a nonzero $w \in \mathscr{X}$ such that with w as initial state and zero input the output is zero. It follows that the optimal cost with w as initial state is zero. Hence the optimal cost sesquilinear form is not positive. □

Definition 6.13. For a system that satisfies the finite cost condition define the operator

$$F^{\min} : \mathscr{X} \to \mathscr{U}, \quad F^{\min} w = \left(u_w^{\min}\right)_0.$$

This operator is called the **optimal cost feedback operator**.

Proposition 6.14. *The optimal cost feedback operator is linear and bounded.*

Proof. Denote by P the projection from $l^2(\mathbb{Z}^+, \mathscr{U} \times \mathscr{Y})$ onto the $\mathscr{U}$-component of the zero-th coordinate. Then $F^{\min} = P\mathcal{I}^+$. Since both P and $\mathcal{I}^+$ are linear and bounded it follows that $F^{\min}$ is. □

Proposition 6.15. *For every initial state $x_0 \in \mathscr{X}$ and input $u : \mathbb{Z}^+ \to \mathscr{U}$ we have*

$$q^{\min}(x_0) \leq \|u_0\|^2 + \|y_0\|^2 + q^{\min}(x_1), \tag{6.4}$$

where y is the output and x the state. Equality holds if and only if $u_0 = (u_{x_0}^{\min})_0$.

Proof. For notational simplicity we denote $u_{x_0}^{\min}$ by $u^{\min}$ in this proof. The input $[u_0, u_1^{\min}, u_2^{\min}, \ldots]$ is denoted by v. If (6.4) would not hold, then the input v would have a strictly lower cost than $u^{\min}$, which is impossible by definition of $u^{\min}$. So (6.4) must hold. If we have an equality in (6.4), then $u^{\min}$ and v give rise to the same cost. By the uniqueness of the optimal input we have $u^{\min} = v$ and so $u_0 = (u_{x_0}^{\min})_0$. □

Proposition 6.16. *For every initial state $x_0 \in \mathscr{X}$ and input $u : \mathbb{Z}^+ \to \mathscr{U}$ we have*

$$
\begin{aligned}
& q^{\min}(x_0) \leq \langle Cx_0, Cx_0 \rangle + q^{\min}(Ax_0) \\
& -\langle S^{-1}(B^* Q^{\min} A + D^* C)x_0, (B^* Q^{\min} A + D^* C)x_0 \rangle \qquad (6.5) \\
& +\langle u_0 + S^{-1}(B^* Q^{\min} A + D^* C)x_0, S\left(u_0 + S^{-1}(B^* Q^{\min} A + D^* C)x_0\right)\rangle,
\end{aligned}
$$

where $S := I + D^*D + B^* Q^{\min} B$. Equality holds if and only if $u_0 = u_{x_0}^{\min}$.

Proof. Some elementary algebraic manipulations show that the right-hand side of (6.5) is identical to the right-hand side of (6.4). The statement then follows from Proposition 6.15. □

Proposition 6.17. *Given $x_0 \in \mathscr{X}$ we have equality in* (6.5) *if and only if $u_0 = \left(I + D^*D + B^* Q^{\min} B\right)^{-1} (B^* Q^{\min} A + D^* C)x_0$.*

Proof. According to Proposition 6.16, for every $u_0 \in \mathscr{U}$ the inequality (6.5) holds and for exactly one we have equality. It follows that this u_0 is the one that minimizes

$$
\begin{aligned}
& \langle Cx_0, Cx_0 \rangle + q^{\min}(Ax_0) - \langle S^{-1}(B^* Q^{\min} A + D^* C)x_0, (B^* Q^{\min} A + D^* C)x_0 \rangle \\
& \quad +\langle u_0 + S^{-1}(B^* Q^{\min} A + D^* C)x_0, S\left(u_0 + S^{-1}(B^* Q^{\min} A + D^* C)x_0\right)\rangle.
\end{aligned}
$$

The first three terms do not depend on u_0. It follows that equality holds only for that u_0 that minimizes

$$
\langle u_0 + S^{-1}(B^* Q^{\min} A + D^* C)x_0, S\left(u_0 + S^{-1}(B^* Q^{\min} A + D^* C)x_0\right)\rangle.
$$

This function is nonnegative since S is nonnegative. It is zero if and only if $u_0 = -\left(I + D^*D + B^* Q^{\min} B\right)^{-1} (B^* Q^{\min} A + D^* C)x_0$. It follows that with this u_0 and only with this u_0, we have equality in (6.5). □

Proposition 6.18. *The optimal cost feedback operator can be written in terms of the optimal cost operator as follows:*

$$
F^{\min} = -\left(I + D^*D + B^* Q^{\min} B\right)^{-1} (B^* Q^{\min} A + D^* C).
$$

Proof. This follows from Proposition 6.16 which says that we have equality in (6.5) if and only if $u_0 = u_{x_0}^{\min}$ and Proposition 6.17 which says that we have equality in (6.5) if and only if $u_0 = -\left(I + D^*D + B^* Q^{\min} B\right)^{-1} (B^* Q^{\min} A + D^* C)x_0$. □

Proposition 6.19. *The optimal cost operator satisfies*

$$\begin{aligned}&A^*Q^{\min}A - Q^{\min} + C^*C\\&-(A^*Q^{\min}B + D^*C)(I + D^*D + B^*Q^{\min}B)^{-1}(AQ^{\min}B^* + DC^*) = 0.\end{aligned}$$

Proof. This follows from substituting u_0 from Proposition 6.17 into (6.5). □

Definition 6.20. The equation

$$\begin{aligned}&A^*QA - Q + C^*C\\&-(C^*D + A^*QB)(I + D^*D + B^*QB)^{-1}(D^*C + B^*QA) = 0.\end{aligned} \tag{6.6}$$

is called the **control algebraic Riccati equation**. We consider only bounded self-adjoint nonnegative solutions of this equation. With a nonnegative self-adjoint solution $Q \in \mathcal{L}(\mathscr{X})$ we associate the following operators:

$$S := I + D^*D + B^*QB, \quad F := -S^{-1}(D^*C + B^*QA). \tag{6.7}$$

For a bounded nonnegative seld-adjoint solution Q of the control algebraic Riccati equation and S as above, define the sesquilinear forms q and s by $q(x_1, x_2) := \langle Qx_1, x_2\rangle_{\mathscr{X}}$ and $s(u_1, u_2) := \langle Su_1, u_2\rangle_{\mathscr{U}}$, respectively. The triple (q, s, F) is called a **control Riccati triple**.

The next two propositions give alternative characterizations of control Riccati triples.

Proposition 6.21. *The triple (q, s, F) is a control Riccati triple if and only if*

- *$q : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ is a bounded nonnegative hermitian sesquilinear form.*
- *$s : \mathscr{U} \times \mathscr{U} \to \mathbb{C}$ is a bounded nonnegative hermitian sesquilinear form.*
- *$F : \mathscr{X} \to \mathscr{U}$ is a bounded linear operator.*
- *For all $w \in \mathscr{X}$, $u \in \mathscr{U}$ we have*

$$\begin{aligned}q(Aw) + \|Cw\|^2_{\mathscr{Y}} &= q(w) + s(Fw),\\ s(u) &= \|u\|^2_{\mathscr{U}} + \|Du\|^2_{\mathscr{Y}} + q(Bu),\\ -s(Fw, u) &= \langle Cw, Du\rangle_{\mathscr{Y}} + q(Aw, Bu).\end{aligned} \tag{6.8}$$

Proof. The second equation of (6.8) is easily seen to be equivalent to the definition of S in (6.7). The third equation of (6.8) is then seen to be equivalent to the definition of F in (6.7). Finally it follows that the first equation of (6.8) is equivalent to Q satisfying the control algebraic Riccati equation. □

Proposition 6.22. *The triple (q, s, F) is a control Riccati triple if and only if*

- *$q : \mathscr{X} \times \mathscr{X} \to \mathbb{C}$ is a bounded nonnegative hermitian sesquilinear form.*
- *$s : \mathscr{U} \times \mathscr{U} \to \mathbb{C}$ is a bounded nonnegative hermitian sesquilinear form.*
- *$F : \mathscr{X} \to \mathscr{U}$ is a bounded linear operator.*
- *For all $w \in \mathscr{X}$, $u \in \mathscr{U}$ we have*

$$q(Aw + Bu) + \|Cw + Du\|^2_{\mathscr{Y}} + \|u\|^2_{\mathscr{U}} = q(w) + s(Fw - u). \quad (6.9)$$

Proof. Writing out (6.9) shows that it is equivalent to

$$\begin{aligned} q(Aw) + q(Bu) + q(Aw, Bu) + q(Bu, Aw) + \|Cw\|^2 + \|Du\|^2 \\ + \langle Cw, Du\rangle + \langle Du, Cw\rangle + \|u\|^2 \\ = q(w) + s(Fw) + s(u) - s(Fw, u) - s(u, Fw). \end{aligned} \quad (6.10)$$

Using equations (6.8) we see that this holds. The first equation of (6.8) is (6.9) with $u = 0$, the second with $w = 0$. Using these first two equations we obtain that (6.10) reads

$$-[s(Fw, u) + s(u, Fw)] = q(Aw, Bu) + q(Bu, Aw) + \langle Cw, Du\rangle + \langle Du, Cw\rangle,$$

which is equivalent to

$$-\mathrm{Re}(s(Fw, u)) = \mathrm{Re}(\langle Cw, Du\rangle + q(Aw, Bu)).$$

Applying the above with iw instead of w gives equality of the imaginary parts of the third equation of (6.8). □

From Proposition 6.22 we obtain the following by induction.

Proposition 6.23. *If (q, s, F) is a control Riccati triple for the system Σ and $[u; x; y] \in \mathbb{B}$, then*

$$q(x_n) + \sum_{k=0}^{n-1} \|u_k\|^2 + \|y_k\|^2 = q(x_0) + \sum_{k=0}^{n-1} s(Fx_k - u_k).$$

Proof. This follows from (6.9) using induction. □

Proposition 6.24. *Let (q, s, F) be a control Riccati triple. Then for the input defined by $u_n := Fx_n$ we have*

$$J(x_0, u) \leq q(x_0),$$

where J is the cost function (6.1).

Proof. Proposition 6.23 with $u_k = Fx_k$ gives

$$q(x_n) + \sum_{k=0}^{n-1} \|u_k\|^2 + \|y_k\|^2 = q(x_0).$$

Since $q \geq 0$ we obtain from this

$$\sum_{i=0}^{n-1} \|u_k\|^2 + \|y_k\|^2 \leq q(x_0).$$

Letting $n \to \infty$ gives the desired result. □

Proposition 6.25. *If a discrete-time system has a bounded nonnegative self-adjoint solution to its control algebraic Riccati equation, then the discrete-time system satisfies the finite cost condition.*

Proof. Proposition 6.24 shows that, for given $x_0 \in \mathscr{X}$, the input defined by $u_n := Fx_n$ gives rise to a finite cost. □

Proposition 6.26. *Assume that the discrete-time system Σ satisfies the finite cost condition. Let (q, s, F) be a control Riccati triple of Σ. Then $q^{\min} \leq q$, where $q^{\min}$ is the optimal cost sesquilinear form of Σ.*

Proof. This follows from Proposition 6.24 since

$$q^{\min}(x_0) \leq J(x_0, u) \leq q(x_0),$$

where u is the input defined in Proposition 6.24. □

Corollary 6.27. *The optimal cost operator is the smallest bounded nonnegative self-adjoint solution of the control algebraic Riccati equation.*

Proof. This is a reformulation of Proposition 6.26. □

Proposition 6.28. *Let Σ satisfy the finite cost condition and let $[u; x; y] \in \mathbb{B}$ with $[u; y] \in \mathscr{V}(x_0)$. Then*

$$\lim_{n\to\infty} q^{\min}(x_n) = 0.$$

Proof. Since $q^{\min}(x_n)$ is the optimal cost when starting from state x_n we have

$$q^{\min}(x_n) \leq J(x_n, [u_n, u_{n+1}, \ldots]) = \sum_{k=n}^{\infty} \|u_k\|^2 + \|y_k\|^2.$$

The right hand side converges to zero since $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ and $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$. It follows that the left-hand side converges to zero as desired. □

Combining Propositions 6.23 and 6.28 we obtain the following.

Proposition 6.29. *Let Σ satisfy the finite cost condition and let $[u;x;y] \in \mathbb{B}$ with $[u;y] \in \mathscr{V}(x_0)$. Then*

$$\sum_{k=0}^{\infty} \|u_k\|^2 + \|y_k\|^2 = q^{\min}(x_0) + \sum_{k=0}^{\infty} s^{\min}(F^{\min} x_k - u_k).$$

Proof. This follows by letting $n \to \infty$ in Proposition 6.23 and using Proposition 6.28. □

Proposition 6.30. *Let Σ satisfy the finite cost condition and let $[u;x;y] \in \mathbb{B}$ with $[u;y] \in \mathscr{V}(x_0)$. Then $(F^{\min} x_k)_{k \geq 0}$ is in $l^2(\mathbb{Z}^+, \mathscr{U})$.*

Proof. It follows from Proposition 6.29 that

$$\sum_{k=0}^{\infty} s^{\min}(F^{\min} x_k - u_k) < \infty.$$

We also have

$$\sum_{k=0}^{\infty} \|F^{\min} x_k - u_k\|^2 \leq \sum_{k=0}^{\infty} \|(S^{\min})^{-1}\| \; s^{\min}(F^{\min} x_k - u_k)$$

and so $(F^{\min} x_k - u_k) \in l^2(\mathbb{Z}^+, \mathscr{U})$. Since $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ we have $(F^{\min} x_k) \in l^2(\mathbb{Z}^+, \mathscr{U})$. □

The following proposition gives another alternative characterization of control Riccati triples.

Proposition 6.31. *The equation* (6.9) *is equivalent to the following triple of equations.*

$$\begin{aligned} q(w) &= q((A+BF)w) + \|(C+DF)w\|^2_{\mathscr{Y}} + \|Fw\|^2_{\mathscr{U}}, \\ \|u\|^2_{\mathscr{U}} &= \|S^{-1/2}u\|^2_{\mathscr{U}} + \|DS^{-1/2}u\|^2_{\mathscr{Y}} + q(BS^{-1/2}u), \\ 0 &= \langle (C+DF)w, DS^{-1/2}u\rangle_{\mathscr{Y}} \\ &\quad + \langle Fw, S^{-1/2}u\rangle_{\mathscr{U}} + q((A+BF)w, BS^{-1/2}u). \end{aligned} \tag{6.11}$$

Proof. The second equation of (6.11) is easily seen to be equivalent to the formula for S in (6.7). The third equation of (6.11) is then seen to be equivalent to the formula for F in (6.7). Using this the first equation is seen to be equivalent to the control algebraic Riccati equation. □

To investigate the connection between the control algebraic Riccati equation and output stabilizability we introduce the following concept.

Definition 6.32. The **Riccati closed-loop system** associated with a control Riccati triple (q, s, F) is defined through its system operator

$$\left[\begin{array}{c|c} A+BF & BS^{-1/2} \\ \hline F & S^{-1/2} \\ C+DF & DS^{-1/2} \end{array}\right]. \tag{6.12}$$

In the case that $(q, s, F) = (q^{\min}, s^{\min}, F^{\min})$ the Riccati closed-loop system is called the **optimal closed loop system**.

Proposition 6.33. *Let (q, s, F) be a control Riccati triple. Then $[S^{1/2}F, I - S^{1/2}]$ is an admissible feedback pair and the corresponding closed-loop system is the Riccati closed-loop system.*

Proof. This is elementary. □

Proposition 6.34. *Let (q, s, F) be a control Riccati triple for the system Σ. Then the Riccati closed-loop system is energy preserving with storage operator Q. Hence the Riccati closed-loop system is output stable and input-output stable and Σ is output stabilizable.*

Proof. The necessary and sufficient conditions for energy preservation from Proposition 5.3 applied to the Riccati closed-loop system are exactly the equations (6.11). It follows from Proposition 5.2 that the Riccati closed-loop system is output stable and input-output stable. Since the Riccati closed-loop system is obtained from Σ by an admissible feedback pair, it follows that Σ is output stabilizable. □

In the case of the optimal closed-loop system we can say a bit more.

Proposition 6.35. *The observability gramian of the optimal closed loop system is $Q^{\min}$.*

Proof. Let $\mathcal{C}^{\min}$ be the output map of the optimal closed-loop system. We have

$$\langle L_C x_0, x_0\rangle = \|\mathcal{C}^{\min} x_0\|^2_{l^2(\mathbb{Z}^+, \mathscr{U}\times\mathscr{Y})} = \|u^{\min}\|^2_{l^2(\mathbb{Z}^+,\mathscr{U})} + \|y^{\min}\|^2_{l^2(\mathbb{Z}^+,\mathscr{Y})} = q^{\min}(x_0).$$

It follows that $L_C = Q^{\min}$. □

Proposition 6.36. *The following are equivalent statements about a discrete-time system Σ.*

1. *Σ satisfies the finite cost condition.*

2. *Σ is output stabilizable.*

3. *The control algebraic Riccati equation of Σ has a bounded nonnegative self-adjoint solution.*

Proof. (1) implies (3) follows from Proposition 6.19 which shows that the optimal cost operator is a solution of the control algebraic Riccati equation. (3) implies (2) is contained in Proposition 6.34. (2) implies (1) follows by choosing $u_n := (I - G)^{-1} F x_n$ where $[F, G]$ is the output stabilizing admissible feedback pair. Since the feedback pair is output stabilizing, it follows that, for each $x_0 \in \mathscr{X}$, $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ and the corresponding output $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$. Hence for each $x_0 \in \mathscr{X}$ the set of stable input-output pairs $\mathscr{V}(x_0)$ is nonempty. □

Definition 6.37. The triple (p, r, L) is called a **filter Riccati triple** of Σ if it is a control Riccati triple for the dual system of Σ.

All the results obtained for control Riccati triples have obvious counterparts for filter Riccati triples. In particular, the existence of a filter Riccati triple is equivalent to the following **filter algebraic Riccati equation** having a nonnegative self-adjoint solution $P \in \mathcal{L}(\mathscr{X})$

$$\begin{aligned} &APA^* - P + BB^* \\ &-(APC^* + BD^*)(I + DD^* + CPC^*)^{-1}(CPA^* + DB^*) = 0. \end{aligned} \tag{6.13}$$

In the proofs of the next few results (Proposition 6.38 up to Proposition 6.46) we need some algebraic calculations involving the control algebraic Riccati equation and the filter algebraic Riccati equation that can be found in Appendix B.

Proposition 6.38. *Let Σ be an input and output stabilizable discrete-time system. Assume there exists a control Riccati triple (q, s, F) such that the main operator of the corresponding Riccati closed-loop system is strongly stable. Then (q, s, F) is the unique control Riccati triple of Σ.*

Proof. For the proof we need the following algebraic relations, which are proven in Appendix B (Lemmas B.4 and B.5). Lemma B.4 gives the following relation between the main operator A_Q of the Riccati closed-loop system corresponding to an arbitrary control Riccati triple (q, s, F) and $A_P := A - (BD^* + APC^*)(I + DD^* + CPC^*)^{-1}C$, where P is a bounded nonnegative self-adjoint solution of the filter algebraic Riccati equation:

$$(I + PQ)A_Q = A_P(I + PQ). \tag{6.14}$$

The following algebraic relation is also proven in Appendix B (Lemma B.5). If Q_1 and Q_2 are bounded nonnegative self-adjoint solutions of the control algebraic Riccati equation and A_{Q_1} and A_{Q_2} denote the main operators of the corresponding Riccati closed-loop systems, then

$$Q_1 - Q_2 = A_{Q_2}^*(Q_1 - Q_2)A_{Q_1}. \tag{6.15}$$

By induction it follows that for all $n \in \mathbb{Z}^+$ we have

$$Q_1 - Q_2 = A_{Q_2}^{*n}(Q_1 - Q_2)A_{Q_1}^n. \tag{6.16}$$

Using these facts we now prove the proposition. Since Σ is input stabilizable, there exists a bounded nonnegative self-adjoint solution P of the filter algebraic Riccati equation. Since A_Q is assumed to be strongly stable and (6.14) shows that A_P is similar to A_Q, we have that A_P is also strongly stable. Now let $\tilde{Q}$ be an arbitrary bounded nonnegative self-adjoint solution of the control algebraic Riccati equation. According to (6.14), $A_{\tilde{Q}}$ is similar to the strongly stable operator A_P and hence is strongly stable. Since $A_{\tilde{Q}}$ is strongly stable there exists for every $x \in \mathscr{X}$ a real number c_x such that for every $n \in \mathbb{Z}^+$ we have $\|A_{\tilde{Q}}^n x\| \leq c_x$. By the uniform boundedness theorem this implies that there exists a real number c such that for every $n \in \mathbb{Z}^+$ we have $\|A_{\tilde{Q}}^n\| \leq c$. Using (6.16) with $Q_1 = Q$ and $Q_2 = \tilde{Q}$ we have for all $x \in \mathscr{X}$ and $n \in \mathbb{Z}^+$

$$\|(Q-\tilde{Q})x\| = \|A_{\tilde{Q}}^{*n}(Q-\tilde{Q})A_Q^n x\| \leq \|A_{\tilde{Q}}^{*n}\| \, \|Q-\tilde{Q}\| \, \|A_Q^n x\| \leq c \, \|Q-\tilde{Q}\| \, \|A_Q^n x\|.$$

Since A_Q is strongly stable, the right-hand side converges to zero as $n \to \infty$. This implies that the left-hand side is zero and so $\tilde{Q} = Q$. □

Proposition 6.39. *Let Σ be a discrete-time system. Assume that its control algebraic Riccati equation has a bounded nonnegative self-adjoint solution Q and that its filter algebraic Riccati equation has a bounded nonnegative self-adjoint solution P. Then the control Lyapunov equation of the Riccati closed-loop system corresponding to Q has a solution $L_b := (I + PQ)^{-1}P = P^{1/2}(I + P^{1/2}QP^{1/2})^{-1}P^{1/2} \geq 0$.*

Proof. This is proven in Appendix B on page 180. □

Corollary 6.40. *Let Σ be an input and output stabilizable discrete-time system. Then the Riccati closed-loop system associated with any solution of the control algebraic Riccati equation is input, output and input-output stable.*

Proof. That the Riccati closed-loop system is input and input-output stable follows from Proposition 6.34. From Proposition 6.39 we obtain that the control Lyapunov equation of the Riccati closed-loop system has a bounded nonnegative self-adjoint solution. It follows from Corollary 3.4 that the Riccati closed-loop system is input stable. □

Corollary 6.41. *Let Σ be an input and output stabilizable discrete-time system. Then the Hankel map of the Riccati closed-loop system associated with any solution of the control algebraic Riccati equation has norm strictly smaller than one.*

Proof. Using Propositions 6.34 and 6.39 we obtain solutions $L_c = Q$ and $L_b := (I + PQ)^{-1}P$ of the Lyapunov equations of the Riccati closed-loop system. So we have $L_c L_b = PQ(I + PQ)^{-1}$. We prove that the spectral radius of $L_c L_b$ is strictly smaller than one. Lemmas 3.18 and 3.19 then give the result. We have $r(L_c L_b) = r(Q^{1/2}PQ^{1/2}(I + Q^{1/2}PQ^{1/2})^{-1})$. It is easily seen that for any nonnegative self-adjoint operator T we have $T(I+T)^{-1} < I$. Denote $T := Q^{1/2}PQ^{1/2}$. Then from the above we obtain $r(L_c L_b) < 1$. $\square$

The following lemma on square roots of operators is needed in the proof of Proposition 6.43.

Lemma 6.42. *Let $P, Q \in \mathcal{L}(\mathscr{H})$ be nonnegative self-adjoint. Define $L := (I + PQ)^{-1}P$. Then, for all $h \in \mathscr{H}$,*

$$\|L^{1/2}h\| \leq \|P^{1/2}h\| + \frac{2}{\pi}\ (2 + \|L\|)\ \|Q\|\ \|Ph\|.$$

Proof. According to Kato [42, Lemma V.3.43 page 284] we have the following representation for the square root of a bounded nonnegative self-adjoint operator T:

$$T^{1/2}h = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(\lambda I + T)^{-1}Th\, d\lambda$$

and we have the following resolvent estimate [42, equation (V.3.38) page 279]

$$\|(\lambda I + T)^{-1}\| \leq \frac{1}{\lambda}$$

for $\lambda > 0$. Applying this with L and P we obtain

$$L^{1/2}h - P^{1/2}h = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\left[(\lambda I + L)^{-1}L - (\lambda I + P)^{-1}P\right] h\, d\lambda$$

and some rewriting of the integrand shows that this equals

$$\frac{1}{\pi}\int_0^\infty \lambda^{1/2}(\lambda I + L)^{-1}LQ(\lambda I + P)^{-1}Ph\, d\lambda.$$

Using the above resolvent estimate we obtain

$$\|\lambda^{1/2}(\lambda I + L)^{-1}LQ(\lambda I + P)^{-1}Ph\| \leq \lambda^{-3/2}\ \|L\|\ \|Q\|\ \|Ph\|$$

and so

$$\| \int_1^\infty \lambda^{1/2}(\lambda I + L)^{-1} LQ(\lambda I + P)^{-1} Ph \, d\lambda \| \leq 2\|L\| \; \|Q\| \; \|Ph\|.$$

Since $(\lambda I + L)^{-1}L = I - \lambda(\lambda I + L)^{-1}$ we obtain from the above resolvent estimate $\|(\lambda I + L)^{-1}L\| \leq 2$ and so

$$\|\lambda^{1/2}(\lambda I + L)^{-1} LQ(\lambda I + P)^{-1} Ph\| \leq 2\lambda^{-1/2} \; \|Q\| \; \|Ph\|,$$

which gives

$$\| \int_0^1 \lambda^{1/2}(\lambda I + L)^{-1} LQ(\lambda I + P)^{-1} Ph \, d\lambda \| \leq 4 \; \|Q\| \; \|Ph\|.$$

Combining the above two estimates we obtain

$$\| \int_0^\infty \lambda^{1/2}(\lambda I + L)^{-1} LQ(\lambda I + P)^{-1} Ph \, d\lambda \| \leq 2 \; (2 + \|L\|) \; \|Q\| \; \|Ph\|$$

and so

$$\|L^{1/2}h - P^{1/2}h\| \leq \frac{2}{\pi} \; (2 + \|L\|) \; \|Q\| \; \|Ph\|,$$

which gives

$$\|L^{1/2}h\| \leq \|P^{1/2}h\| + \frac{2}{\pi} \; (2 + \|L\|) \; \|Q\| \; \|Ph\|,$$

as desired. □

Proposition 6.43. *Let Σ be an input and output stabilizable discrete-time system. Let Q be a solution of the control algebraic Riccati equation of Σ and denote the optimal cost operator of the dual system of Σ by $P^{\min}$. Then the controllability gramian of the Riccati closed-loop system associated with Q is $L_B = (I + P^{\min} Q)^{-1} P^{\min}$.*

Proof. Proposition 6.35 applied to the dual system Σ_{dual} of Σ shows that $P^{\min}$ is the observability gramian of the optimal closed-loop system of Σ_{dual}. It follows from Lemma 3.7 that for all $h \in \mathscr{X}$ we have $(P^{\min})^{1/2} A_d^n h \to 0$ as $n \to \infty$, where A_d is the main operator of the optimal closed-loop system of Σ_{dual}. The operator A_d is given explicitely by

$$A_d = A^* - C^*(I + DD^* + C^* P^{\min} C)^{-1}(DB^* + CP^{\min} A^*).$$

Lemma B.4 gives

$$(I + P^{\min}Q)A_Q = A_{P^{\min}}(I + P^{\min}Q), \tag{6.17}$$

where $A_{P^{\min}} = A_d^*$ and A_Q is the main operator of the Riccati closed-loop system of Σ associated with Q. Since A_d is the adjoint of $A_{P^{\min}}$ it follows that $(P^{\min})^{1/2}A^{*n}_{P^{\min}}h \to 0$. Using (6.17) we obtain that

$$(P^{\min})^{1/2}(I + QP^{\min})^{-1}A_Q^{*n}(I + QP^{\min})h \to 0.$$

Using that $(I + (P^{\min})^{1/2}Q(P^{\min})^{1/2})^{-1}(P^{\min})^{1/2} = (P^{\min})^{1/2}(I + QP^{\min})^{-1}$ it follows that $(P^{\min})^{1/2}A_Q^{*n}w \to 0$ for all $w \in \mathscr{X}$.

Define $L := (I + P^{\min}Q)^{-1}P^{\min}$. It follows from Proposition 6.39 that L is a solution of the control Lyapunov equation of the Riccati closed-loop system. Lemma 6.42 gives

$$\|L^{1/2}h\| \le \|(P^{\min})^{1/2}h\| + \frac{2}{\pi}\,(2 + \|L\|)\,\|Q\|\,\|P^{\min}h\|.$$

With $h = A_Q^{*n}w$ we obtain from this that $L^{1/2}A_Q^{*n}w \to 0$ for all $w \in \mathscr{X}$. By Lemma 3.13 we obtain that L is the controllability gramian. □

Proposition 6.44. *Let Σ be an input and output stabilizable discrete-time system. Let $Q^{\min}$ be the optimal cost operator of Σ and denote the optimal cost operator of the dual system of Σ by $P^{\min}$. Denote the Hankel map of the optimal closed-loop system by $\mathcal{H}$. Then $\|\mathcal{H}\|^2 = r((I + P^{\min}Q^{\min})^{-1}P^{\min}Q^{\min})$.*

Proof. By Lemma 3.18 we have $\|\mathcal{H}\|^2 = r(L_B L_C)$, where L_B is the controllability gramian and L_C the observability gramian of the optimal closed-loop system. Proposition 6.35 shows that $L_B = Q^{\min}$ and Proposition 6.43 shows that $L_C = (I + P^{\min}Q^{\min})^{-1}P^{\min}$. The desired result follows. □

Proposition 6.45. *Let $\check{\Sigma}$ be an energy preserving discrete-time system with input space $\mathscr{U}$ and output space $\mathscr{U} \times \mathscr{Y}$. Assume that $\check{D}_1$ has a bounded inverse and that the storage operator L is nonnegative self-adjoint. Define the system Σ as in Proposition 2.23. Then L is a solution of the control algebraic Riccati equation of Σ.*

Proof. Define $Q := L$, $S := \check{D}_1^{-*}\check{D}_1^{-1}$, $F := \check{C}_1$. One easily checks the equations (6.11) using the equations from Proposition 5.3 applied to $\check{\Sigma}$. □

Proposition 6.46. *Let $\check{\Sigma}$ be an energy preserving discrete-time system with input space $\mathscr{U}$ and output space $\mathscr{U} \times \mathscr{Y}$. Assume that $\check{D}_1$ has a bounded inverse and that the storage operator L_c is nonnegative self-adjoint. Further* *assume that $\check{\Sigma}$ is input stable. Define the system Σ as in Proposition 2.23. Let L_b be a solution of the control Lyapunov equation of $\check{\Sigma}$ and assume that $1 \notin \sigma(L_b L_c)$. Then $P := (I - L_b L_c)^{-1} L_b$ is a solution of the filter algebraic Riccati equation of Σ.*

Proof. This is proven in Appendix B on page 183. □

In the following four Propositions 6.47-6.50, we compare the closed-loop systems associated with different control Riccati triples. These propositions are used in the chapter on coprime factorization (Chapter 7) to show that all Riccati closed-loop systems provide a strongly right-coprime factorization using the the optimal closed-loop system does.

The first of these propositions shows the relation between the transfer functions.

Proposition 6.47. *Let Σ be an output stabilizable discrete-time system. Let Σ_i $(i = 1, 2)$ be the Riccati closed-loop system associated with the control Riccati triple (q_i, s_i, F_i). Let S_i and Q_i be the operators corresponding to the sesquilinear forms s_i and q_i, respectively. Let Σ_s be the discrete-time system with system operator*

$$\left[\begin{array}{c|c} A + BF_1 & BS_1^{-1/2} \\ \hline S_2^{1/2}(F_1 - F_2) & S_2^{1/2} S_1^{-1/2} \end{array}\right].$$

Then $\mathsf{D}_1 = \mathsf{D}_2 \mathsf{D}_s$ in a neighbourhood of zero, where D_i is the transfer function of Σ_i and D_s is the transfer function of Σ_s.

Proof. Using Proposition 2.20 we see that once we prove that the transfer function of the series interconnection of Σ_s and Σ_2 equals the transfer function of Σ_1, then we are done.

We write down a realization of the transfer function of the series interconnection of Σ_s and Σ_2 using Lemma 2.21:

$$\left[\begin{array}{cc|c} A + BF_1 & 0 & BS_1^{-1/2} \\ 0 & A + BF_2 & 0 \\ \hline F_1 & F_2 & S_1^{-1/2} \\ C + DF_1 & C + DF_2 & DS_1^{-1/2} \end{array}\right].$$

Since the state operator is diagonal and the input operator has zero as its second component, the transfer function is equal to the transfer function of

$$\left[\begin{array}{c|c} A + BF_1 & BS_1^{-1/2} \\ \hline F_1 & S_1^{-1/2} \\ C + DF_1 & DS_1^{-1/2} \end{array}\right],$$

which is Σ_1. □

Proposition 6.48. *Let Σ be an output stabilizable discrete-time system. The system Σ_s from Proposition 6.47 is energy preserving with storage operator $\Delta := Q_1 - Q_2$.*

Proof. It is straightforward to check the necessary and sufficient conditions (5.2) using that Q_1 and Q_2 satisfy the control algebraic Riccati equation. □

Proposition 6.49. *Let Σ be an input and output stabilizable discrete-time system. Then the system Σ_s from Proposition 6.47 is input stable and input-output stable.*

Proof. It follows from Proposition 6.39 that any Riccati closed-loop system of Σ is input stable. Since the state operator and input operator of Σ_s are equal to those of the Riccati closed-loop system associated with the control Riccati triple (q_1, s_1, F_1) it follows that Σ_s is input stable. Propositions 5.8 and 6.48 now show that Σ_s is input-output stable. □

Proposition 6.50. *Let Σ be an input and output stabilizable discrete-time system. Then the transfer function of the system Σ_s from Proposition 6.47 has an inverse in $H^\infty(\mathbb{D}, \mathscr{U})$.*

Proof. Using Proposition 2.22 it is easily seen that a realization of the inverse of the transfer function of Σ_s is a system of the same form as Σ_s, but with the indices 1 and 2 interchanged. It follows from Proposition 6.49 that this realization is input-output stable. Hence D_s^{-1} is in $H^\infty(\mathbb{D}, \mathscr{U})$. □

Notes

The LQR problem for discrete-time systems was studied by Lee, Chow and Barr [51] and Zabczyk [100], [101], [102]. Our approach to this problem, based on the set of stable input-output pairs, follows Curtain and Zwart [18]. The properties of the Riccati closed-loop system given in this chapter are mainly taken from Opmeer and Curtain [71]. Proposition 6.43 is well-known in the case of exponentially stabilizable and detectable systems, see Curtain and Zwart [18, Lemma 9.4.10]. It was first proven in the generality considered here in Curtain and Opmeer [16]. Propositions 6.47-6.50 were also first proven in Curtain and Opmeer [16].