Model reduction for controller design for infinite-dimensional systems

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Chapter 5

Energy preserving systems

In this chapter we consider energy preserving systems. We will not go very deeply into the theory. Only those results that will be used in later chapters are discussed. For more on energy preserving systems we refer to Staffans [89, Chapter 11] and the references therein. We start with the definition of an energy preserving system.

Definition 5.1. A discrete-time system is called **energy preserving** if there exists an $L = L^* \in \mathcal{L}(\mathscr{X})$ such that for any trajectory $[u; x; y] \in \mathbb{B}$ and any $n \in \mathbb{Z}^+$ we have

$$\langle Lx_n, x_n \rangle + \sum_{k=0}^{n-1} \|y_k\|^2 = \langle Lx_0, x_0 \rangle + \sum_{k=0}^{n-1} \|u_k\|^2. \tag{5.1}$$

The operator L is called the **storage operator**.

The idea of the definition is that the norms in the input and output spaces represent energy so that $\sum_{k=0}^{n-1} \|u_k\|^2$ is the amount of energy supplied to the system up to time n and $\sum_{k=0}^{n-1} \|y_k\|^2$ is the amount of energy extracted from the system up to time n; the quadratic form $\langle Lx, x \rangle$ is supposed to represent the energy stored in the system if it is in the state x; (5.1) then says that the amount of energy supplied to the system up to time n plus the energy stored inside the system initially is equal to the amount of energy extracted from the system up to time n plus the energy stored inside the system at time n. This physical interpretation will not be important in the sequel. Note that we allowed the storage operator L to be indefinite which means that the energy stored in the system can be negative.

Our first result concerns stability.

Proposition 5.2. *An energy preserving discrete-time system with nonnegative storage operator is both output stable and input-output stable.*

Proof. Let $x_0 \in \mathscr{X}$ and $u \in l^2(\mathbb{Z}^+, \mathscr{U})$. Adding the finite nonnegative quantity $\sum_{k=n}^{\infty} \|u_k\|^2$ to the right-hand side of (5.1) we obtain for all $n \in \mathbb{Z}^+$

$$\langle Lx_n, x_n \rangle + \sum_{k=0}^{n-1} \|y_k\|^2 \leq \langle Lx_0, x_0 \rangle + \|u\|^2.$$

Since L is nonnegative we obtain from this the inequality

$$\sum_{k=0}^{n-1} \|y_k\|^2 \leq \langle Lx_0, x_0 \rangle + \|u\|^2.$$

This inequality shows that for any initial condition and zero input the output is in $l^2(\mathbb{Z}^+, \mathscr{Y})$, i.e the system is output stable; it also shows that for initial condition zero and input in $l^2(\mathbb{Z}^+, \mathscr{U})$ the output is in $l^2(\mathbb{Z}^+, \mathscr{Y})$, i.e. the system is input-output stable. □

The next result gives algebraic conditions for a system to be energy preserving.

Proposition 5.3. *A discrete-time system is energy preserving if and only if there exists a* $L = L^* \in \mathcal{L}(\mathscr{X})$ *such that*

$$A^*LA - L + C^*C = 0, \quad B^*LB + D^*D = I, \quad B^*LA + D^*C = 0. \quad (5.2)$$

This L *is then a storage operator.*

Proof. We first note that a system is energy preserving with storage operator L if and only if for all $w \in \mathscr{X}$ and $v \in \mathscr{U}$ we have

$$\langle L(Aw + Bv), Aw + Bv \rangle + \langle Cw + Dv, Cw + Dv \rangle = \langle Lw, w \rangle + \langle v, v \rangle. \quad (5.3)$$

Indeed, this equation is (5.1) for $n = 1$ with $x_0 = w$ and $u_0 = v$ and so it is clearly implied by (5.1). It follows using induction that (5.1) is implied by (5.3). Equation (5.3) can be written in the following form

$$\left\langle \begin{bmatrix} A^*LA - L + C^*C & A^*LB + C^*D \\ B^*LA + D^*C & B^*LB + D^*D - I \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix}, \begin{bmatrix} w \\ v \end{bmatrix} \right\rangle = 0,$$

which is equivalent to (5.2). □

Note that we already met the first equation of (5.2) in Chapter 3 and called it the observation Lyapunov equation (Definition 3.6).

Proposition 5.4. *Consider a discrete-time system for which the observability gramian is a storage operator. Let* $[u; x; y] \in \mathbb{B}$ *with* $x_0 = 0$ *and* $u \in l^2(\mathbb{Z}^+, \mathscr{U})$. *Then* $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$.

Proof. First assume that $u : \mathbb{Z}^+ \to \mathscr{U}$ is finitely nonzero. We have

$$\langle L_C x_n, x_n\rangle + \sum_{k=0}^{n-1} \|y_k\|^2 = \sum_{k=0}^{n-1} \|u_k\|^2.$$

By Proposition 3.10 we have $\langle L_C x_n, x_n\rangle \to 0$ as $n \to \infty$. So we obtain $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$ in case u is finitely nonzero. Using that the finitely nonzero sequences are dense in $l^2(\mathbb{Z}^+, \mathscr{U})$ and that the system is input-output stable by Proposition 5.2 we obtain the general case. □

Lemma 5.5. *Consider an output stable discrete-time system with the property that if* $[u; x; y] \in \mathbb{B}$ *with* $x_0 = 0$ *and* $u \in l^2(\mathbb{Z}^+, \mathscr{U})$, *then* $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$. *Then* $B^*L_C B + D^*D = I$.

Proof. Define the sequence u by $u_0 = v$ and $u_i = 0$ if $i > 0$. Let y denote the corresponding output for initial condition zero. Then, since $(y_n)_{n\geq 1}$ is the output for initial condition Bv and zero input and $y_0 = Dv$, we have (here $\mathcal{C}$ is the output map of the system)

$$\|y\|^2_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|\mathcal{C}Bv\|^2 + \|Dv\|^2 = \langle B^*L_C Bv, v\rangle + \langle D^*Dv, v\rangle.$$

Since $\|u\|^2_{l^2(\mathbb{Z}^+,\mathscr{U})} = \|v\|^2$ we obtain the desired equality. □

Lemma 5.6. *Consider an approximately controllable output stable discrete-time system with the property that if* $[u; x; y] \in \mathbb{B}$ *with* $x_0 = 0$ *and* $u \in l^2(\mathbb{Z}^+, \mathscr{U})$, *then* $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$. *Then* $B^*L_C A + D^*C = 0$.

Proof. We first note that in a real Hilbert space we have

$$\langle h_1, h_2\rangle = \left(\|h_1 + h_2\|^2 - \|h_1\|^2 - \|h_2\|^2\right)/2,$$

and that in a complex Hilbert space a similar equation expressing the inner product in terms of the norm exists. This shows that from $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$ it follows that $\langle y^1, y^2\rangle_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \langle u^1, u^2\rangle_{l^2(\mathbb{Z}^+,\mathscr{U})}$, where y^i is the output for input u^i and initial condition zero. By shift-invariance we obtain

$$\langle \mathcal{D}u^1, \mathcal{D}u^2\rangle_{l^2(\mathbb{Z},\mathscr{Y})} = \langle u^1, u^2\rangle_{l^2(\mathbb{Z},\mathscr{U})} \tag{5.4}$$

when both u^1 and u^2 have support bounded to the left. Here $\mathcal{D}$ is the input-output map of the system.

Let $u^1 : \mathbb{Z} \to \mathscr{U}$ have support bounded to the left and be zero on $\mathbb{Z}^+$. Let $v \in \mathscr{U}$ and define $u^2 : \mathbb{Z} \to \mathscr{U}$ by $u^2_0 = v$, $u^2_i = 0$ for $i \neq 0$. Since obviously $\langle u^1, u^2 \rangle_{l^2(\mathbb{Z},\mathscr{U})} = 0$ we obtain using (5.4) that $\langle \mathcal{D}u^1, \mathcal{D}u^2 \rangle_{l^2(\mathbb{Z},\mathscr{Y})} = 0$. Since u^2 equals zero on $\mathbb{Z}^-$ we have that $\mathcal{D}u^2$ equals zero on $\mathbb{Z}^-$ and using this we see that

$$\langle \mathcal{D}u^1, \mathcal{D}u^2 \rangle_{l^2(\mathbb{Z}^+,\mathscr{Y})} = 0. \tag{5.5}$$

Since u^1 is zero on $\mathbb{Z}^+$, we have that $\mathcal{D}u^1$ restricted to $\mathbb{Z}^+$ equals $\mathcal{H}u^1$, where $\mathcal{H}$ is the Hankel map of the system. It follows that

$$\langle \mathcal{H}u^1, \mathcal{D}u^2 \rangle_{l^2(\mathbb{Z}^+,\mathscr{Y})} = 0. \tag{5.6}$$

Define $w := \mathcal{B}u^1$, where $\mathcal{B}$ is the input map of the system. Then we see that $\mathcal{H}u^1 = \mathcal{C}\mathcal{B}u^1$, where $\mathcal{C}$ is the output map of the system, using Lemma 2.4. Since u^2_n equals zero for $n \geq 1$ we have $(\mathcal{D}u^2)_n = (\mathcal{C}Bv)_n$ for $n \geq 1$. Separating the first term in (5.6) we obtain

$$\langle Cw, Dv \rangle_{\mathscr{Y}} + \langle \mathcal{C}Aw, \mathcal{C}Bv \rangle_{l^2(\mathbb{Z}^+,\mathscr{Y})} = 0.$$

Since this holds for all $v \in \mathscr{U}$ this implies $(B^*L_C A + D^*C)w = 0$. Since by approximate controllability $\mathcal{B}$ has dense range we obtain $B^*L_C A + D^*C = 0$ on a dense set. Hence $B^*L_C A + D^*C = 0$ by continuity. □

Combining the last two lemmas we obtain the following result.

Proposition 5.7. *An approximately controllable output stable discrete-time system with the property that if $[u; x; y] \in \mathbb{B}$ with $x_0 = 0$ and $u \in l^2(\mathbb{Z}^+, \mathscr{U})$, then $\|y\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}$ is energy preserving with the observability gramian as storage operator.*

Proof. This follows immediately from Lemmas 5.5 and 5.6 combined with the algebraic conditions for energy preservation from Proposition 5.3. □

Proposition 5.8. *A discrete-time system that is energy preserving and input stable is input-output stable.*

Proof. Let $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ and denote the output for this input and initial condition zero by y. Since the system is input stable the input map is bounded from $l^2(\mathbb{Z}^-, \mathscr{U})$ to $\mathscr{X}$ and we have

$$\|x_n\| \leq \|\mathcal{B}\| \; \|u\|_{l^2(\mathbb{Z}^+,\mathscr{U})}.$$

Using that the system is energy preserving we obtain

$$\sum_{k=0}^{n-1} \|y_k\|^2 = \sum_{k=0}^{n-1} \|u_k\|^2 - \langle Lx_n, x_n \rangle \leq (1 + \|L\| \; \|\mathcal{B}\|^2) \; \|u\|^2_{l^2(\mathbb{Z}^+,\mathscr{U})},$$

and so $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$. □

Notes

The concept of energy-preserving, or more generally passive or dissipative, systems is well-established within systems and control theory. We refer to Staffans [89, Chapter 11] for more information on energy-preserving infinite-dimensional systems. Propositions 5.2 and 5.3 are rather obvious and well-known. We took most of the other results in this chapter from Curtain and Opmeer [16] and Opmeer and Curtain [71], but we make no priority claim, these results may have appeared elsewhere earlier.