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Model reduction for controller design for infinite-dimensional systems

Opmeer, Mark Robertus

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Chapter 4

Stabilizability

Stabilizability is an important concept in systems theory. In this chapter we consider several forms of stabilizability.

Definition 4.1. Let S be the system operator of a discrete-time system and $[F, G] \in \mathcal{L}(\mathscr{X} \times \mathscr{U}, \mathscr{U})$. Then $[F, G]$ is called an **admissible feedback pair** if $I - G$ is boundedly invertible. The corresponding **closed-loop system** is the discrete-time system with system operator

$$S_{[F,G]} := \left[\begin{array}{c|c} A + B(I-G)^{-1}F & B(I-G)^{-1} \\ \hline (I-G)^{-1}F & (I-G)^{-1} \\ C + D(I-G)^{-1}F & D(I-G)^{-1} \end{array}\right]. \tag{4.1}$$

Remark 4.2. Definition 4.1 is motivated by the following. We first add the equations $v_n = Fx_n + Gu_n$ to the equations $x_{n+1} = Ax_n + Bu_n$, $y_n = Cx_n + Du_n$ that describe the system. We then choose the input u to be $u_n = v_n + r_n$, i.e. we feed the additional output v back. We consider r as the input and $[u; y]$ as the output of a new system. This new system is described by the system operator given in Definition 4.1.

Remark 4.3. Let $[F, G]$ be an admissible feedback pair. Then $[(I - G)^{-1}F, 0]$ is also an admissible feedback pair and the state operator and output operator of the respective closed-loop systems are equal. This explains why it is often assumed in the literature that the second component of an admissible feedback pair equals zero. It turns out that for making the connection with continuous-time systems using the Cayley transform it is however useful to work with general admissible feedback pairs.

Definition 4.4. Let S be the system operator of a discrete-time system and let $[L; K] \in \mathcal{L}(\mathscr{Y}, \mathscr{X} \times \mathscr{Y})$. Then $[L; K]$ is called an **admissible injection pair** if $I - K$ is boundedly invertible. The corresponding **closed-loop**

system is the discrete-time system with system operator

$$S^{[L;K]} := \left[\begin{array}{c|cc} A + L(I-K)^{-1}C & L(I-K)^{-1} & B + L(I-K)^{-1}D \\ \hline (I-K)^{-1}C & (I-K)^{-1} & (I-K)^{-1}D \end{array}\right]. \quad (4.2)$$

The following result shows that the notions of admissible feedback pair and admissible injection pair are dual.

Lemma 4.5. *Let S be the system operator of a discrete-time system. The closed-loop system of S with the admissible feedback pair $[F, G]$ is the dual of the closed-loop system of S^* with the admissible injection pair $[L; K] := [F, G]^*$.*

Proof. This is immediate. □

Definition 4.6. A discrete-time system is called

- **exponentially stabilizable** if there exists an admissible feedback pair such that the closed-loop system is exponentially stable.
- **exponentially detectable** if there exists an admissible injection pair such that the closed-loop system is exponentially stable.
- **output stabilizable** if there exists an admissible feedback pair such that the closed-loop system is output stable.
- **input stabilizable** if there exists an admissible injection pair such that the closed-loop system is input stable.

Proposition 4.7. *A discrete-time system is exponentially stabilizable if and only if its dual system is exponentially detectable. It is output stabilizable if and only if its dual system is input stabilizable.*

Proof. This follows using Lemma 4.5. □

Proposition 4.8. *If a discrete-time system is exponentially stabilizable, then it is output stabilizable. If a discrete-time system is exponentially detectable, then it is input stabilizable.*

Proof. If the system is exponentially stabilizable, then there exists an admissible feedback pair such that the closed-loop system is exponentially stable. By Proposition 3.28 this closed-loop system is also output stable. The second statement follows by duality. □

Remark 4.9. Note that an exponentially stable system is exponentially stabilizable (take F and G equal to zero) and exponentially detectable (take K and L equal to zero). An output stable system is output stabilizable (take F and G equal to zero) and an input stable system is input stabilizable (take K and L equal to zero).

Proposition 4.10. *Let S be the system operator of a discrete-time system and $[F, G]$ an admissible feedback pair. Define $\mathfrak{F}(z) = F(I - zA)^{-1}$ and $\mathfrak{G}(z) = G + Fz(I - zA)^{-1}B$, then the generalized resolvents of the closed-loop system are*

$$\left[\begin{array}{c|c} \mathfrak{A}^{\mathrm{cl}} & \mathfrak{B}^{\mathrm{cl}} \\ \hline \mathfrak{C}^{\mathrm{cl}} & \mathfrak{D}^{\mathrm{cl}} \end{array}\right] = \left[\begin{array}{c|c} \mathfrak{A} + \mathfrak{B}(I - \mathfrak{G})^{-1}\mathfrak{F} & \mathfrak{B}(I - \mathfrak{G})^{-1} \\ \hline (I - \mathfrak{G})^{-1}\mathfrak{F} & (I - \mathfrak{G})^{-1} \\ \mathfrak{C} + \mathfrak{D}(I - \mathfrak{G})^{-1}\mathfrak{F} & \mathfrak{D}(I - \mathfrak{G})^{-1} \end{array}\right]. \tag{4.3}$$

Proof. This is easily computed. □

The following concerns a relationship between stabilizability and stability.

Proposition 4.11. *If a discrete-time system is input-output stable and*

- *output stabilizable, then it is output stable.*
- *input stabilizable, then it is input stable.*
- *exponentially stabilizable and detectable, then it is exponentially stable.*

Proof. It follows from (4.3) that $\mathfrak{C} = \mathfrak{C}_2^{\mathrm{cl}} - \mathfrak{D}\mathfrak{C}_1^{\mathrm{cl}}$. It follows that $\mathsf{C} = \mathsf{C}_2^{\mathrm{cl}} - \mathsf{D}\mathsf{C}_1^{\mathrm{cl}}$ in a neighbourhood of zero. Since the closed-loop system is output stable we have that for every $x \in \mathscr{X}$ the function $\mathsf{C}^{\mathrm{cl}}x$ restricts to a function in $H^2(\mathbb{D}; \mathscr{U} \times \mathscr{Y})$. By input-output stability D restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. It follows that for every $x \in \mathscr{X}$ the function $\mathsf{C}x$ restricts to a function in $H^2(\mathbb{D}; \mathscr{Y})$. Hence the system is output stable.

We note that in the case that the system is exponentially stabilizable we have that C^{cl} restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{X}, \mathscr{U} \times \mathscr{Y}))$ by Remark 3.29. The argumention above then leads to the stronger conclusion $\mathsf{C} \in H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{X}, \mathscr{Y}))$.

The second statement follows by duality.

From (4.3) we obtain that $\mathfrak{A} = \mathfrak{A}^{\mathrm{cl}} - \mathfrak{B}\mathfrak{C}_1^{\mathrm{cl}}$. It follows that $\mathsf{A} = \mathsf{A}^{\mathrm{cl}} - \mathsf{B}\mathsf{C}_1^{\mathrm{cl}}$ in a neighbourhood of zero. By exponential stabilizability A^{cl} restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{X}))$. That $\mathsf{C}_1^{\mathrm{cl}}$ restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{X}, \mathscr{U}))$ we already concluded in the second paragraph of this proof. Applying the second paragraph of this proof to the dual system, since the system is exponentially detectable this is justified, we see that B restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{U}, \mathscr{X}))$. It follows that A restricts to a function in $H^\infty(\mathbb{D}; \mathcal{L}(\mathscr{X}))$, which shows that the system is exponentially stable. □

Proposition 4.12. *Let Σ be a discrete-time system, $[F, G]$ an admissible feedback pair and $\Sigma_{[F,G]}$ the corresponding closed-loop system. If Σ is exponentially stabilizable, then $\Sigma_{[F,G]}$ is. If Σ is exponentially detectable, then $\Sigma_{[F,G]}$ is.*

Proof. Since Σ is exponentially stabilizable, there exists an admissible feedback pair $[\underline{F}, \underline{G}]$ such that $A + B(I - \underline{G})^{-1}\underline{F}$ is exponentially stable. Define the admissible feedback pair $[\tilde{G}, \tilde{F}] := [0, (I - G)(I - \underline{G})^{-1}\underline{F} - F]$. It is easily seen that the state operator of the closed-loop system of $\Sigma_{[F,G]}$ with this admissible feedback pair equals $A + B(I - \underline{G})^{-1}\underline{F}$. It follows that $\Sigma_{[F,G]}$ is exponentially stabilizable.

Since Σ is exponentially stabilizable, there exists an admissible injection pair $[\underline{L}; \underline{K}]$ such that $A + \underline{L}(I - \underline{K})^{-1}C$ is exponentially stable. Define the admissible injection pair $[\tilde{L}; \tilde{K}]$ by $\tilde{L} := [-(B + \underline{L}(I - \underline{K})^{-1}D), \underline{L}(I - \underline{K})^{-1}]$ and $\tilde{K} := 0$. It is easily computed that the state operator of the closed-loop system of $\Sigma_{[F,G]}$ with this admissible injection pair equals$A + \underline{L}(I - \underline{K})^{-1}C$. It follows that $\Sigma_{[F,G]}$ is exponentially detectable. □

Corollary 4.13. *Let Σ be a discrete-time system, $[F, G]$ an admissible feedback pair and $\Sigma_{[F,G]}$ the corresponding closed-loop system. If Σ is exponentially stabilizable and detectable and $\Sigma_{[F,G]}$ is input-output stable, then $\Sigma_{[F,G]}$ is exponentially stable.*

Proof. It follows from Proposition 4.12 that $\Sigma_{[F,G]}$ is exponentially stabilizable and detectable. Proposition 4.11 now shows that $\Sigma_{[F,G]}$ is exponentially stable. □

Proposition 4.14. *Let $\check{\Sigma}$ be a discrete-time system with input space $\mathscr{U}$ and output space $\mathscr{U} \times \mathscr{Y}$. Assume that $\check{D}_1$ is boundedly invertible. Define the system Σ as in Proposition 2.23. Then $[\check{C}_1; I - \check{D}_1]$ is an admissible feedback pair for Σ and the corresponding closed-loop system equals $\check{\Sigma}$.*

Proof. This is an easy computation. □

Corollary 4.15. *Use the notation and assumptions of Proposition 4.14. If $\check{\Sigma}$ is exponentially stable, then Σ is exponentially stabilizable. If $\check{\Sigma}$ is output stable, then Σ is output stabilizable.*

Proof. This follows immediately. □

Notes

The concept of stabilizability is classical. The notion of admissible feedback pair as given here is due to Staffans [89]; previously G was always taken equal to zero.