Model reduction for controller design for infinite-dimensional systems

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Chapter 3

Stability

The concept of stability plays a key role in systems theory. In this chapter we study different notions of stability for discrete-time systems.

Definition 3.1. A discrete-time system is called

- **exponentially stable** if for all sequences x with $[0; x; y] \in \mathbb{B}$ we have $x \in l^2(\mathbb{Z}^+, \mathscr{X})$.
- **strongly stable** if for all sequences x with $[0; x; y] \in \mathbb{B}$ we have $\|x_n\| \to 0$ as $n \to \infty$.
- **output stable** if for all sequences y with $[0; x; y] \in \mathbb{B}$ we have $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$.
- **input stable** if the dual system is output stable.
- **input-output stable** if $u \in l^2(\mathbb{Z}^+, \mathscr{U})$, $x_0 = 0$ and $[u; x; y] \in \mathbb{B}$ implies $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$.

We remark that exponential stability is often referred to as **power stability** in the literature.

We will first give alternative characterizations of the concepts just introduced. Then we will show that exponential stability implies all the other types of stability (Proposition 3.28).

The Hardy spaces H^2 and H^∞ play a role in this chapter. The reader is referred to Appendix A for the relevant background.

Proposition 3.2. *The following are equivalent.*

1. *The discrete-time system is output stable.*
2. *The output map $\mathcal{C}$ is an element of $\mathcal{L}(\mathscr{X}, l^2(\mathbb{Z}^+, \mathscr{Y}))$.*

3. *There exists a nonnegative self-adjoint operator* $L \in \mathcal{L}(\mathscr{X})$ *such that* $A^*LA - L + C^*C = 0$.

4. *We have* $r_{\mathsf{C}} \geq 1$ *and for all* $x \in \mathscr{X}$ *the restriction of* $\mathsf{C}(\cdot)x$ *to the open unit disc is in* $H^2(\mathbb{D}, \mathscr{Y})$.

Proof. We show that output stability is equivalent to 2, that 2 is equivalent to 3 and that 4 is equivalent to output stability.

(i) 2 implies output stability. The output for initial state x_0 and zero input is given by $\mathcal{C}x_0$ and since $\mathcal{C}$ maps into $l^2(\mathbb{Z}^+, \mathscr{Y})$, by assumption, we obtain the desired result.

(ii) output stability implies 2. Output stability shows that the range of the output map is contained in $l^2(\mathbb{Z}^+, \mathscr{Y})$. We show that $\mathcal{C}$ is closed. Assume that $x^n \to x$ in $\mathscr{X}$ and $\mathcal{C}x^n \to y$ in $l^2(\mathbb{Z}^+, \mathscr{Y})$ as $n \to \infty$. We have to show that $y = \mathcal{C}x$. Since $\mathcal{C}x^n \to y$ we have for all $k \in \mathbb{Z}^+$ that $(\mathcal{C}x^n)_k \to y_k$. Using the definition of output map we see that this is equivalent to $CA^k x^n \to y_k$. Since C and A are bounded operators and $x^n \to x$ we also have for all $k \in \mathbb{Z}^+$ that $CA^k x^n \to CA^k x$. This shows that for all $k \in \mathbb{Z}^+$ the following holds $(\mathcal{C}x)_k = y_k$, in other words $\mathcal{C}x = y$. This proves that $\mathcal{C}$ is closed. Since it is everywhere defined, it follows from the closed graph theorem that it is bounded.

(iii) 2 implies 3. Since $\mathcal{C}$ is bounded $L := \mathcal{C}^*\mathcal{C}$ is a nonnegative self-adjoint element of $\mathcal{L}(\mathscr{X})$. We show that it satisfies the given equation. We have for $x \in \mathscr{X}$

$$\langle LAx, Ax\rangle - \langle Lx, x\rangle + \langle Cx, Cx\rangle = \langle \mathcal{C}Ax, \mathcal{C}Ax\rangle - \langle \mathcal{C}x, \mathcal{C}x\rangle + \langle Cx, Cx\rangle =$$

$$\sum_{k=0}^{\infty} \|(\mathcal{C}Ax)_k\|^2 - \sum_{k=0}^{\infty} \|(\mathcal{C}x)_k\|^2 + \|Cx\|^2 = \sum_{k=0}^{\infty} \|(\mathcal{C}x)_{k+1}\|^2 - \|(\mathcal{C}x)_k\|^2 + \|Cx\|^2.$$

The reordering of terms is permitted, since the series involved converge absolutely. Noting that the last series above telescopes, we obtain that the above expression equals zero. Since this is true for all $x \in \mathscr{X}$, we see that L satisfies the above mentioned equation.

(iv) 3 implies 2. Multiply the given equation from the left with A^{*k} and from the right with A^k and sum from $k = 0$ to n to obtain

$$\begin{aligned} \sum_{k=0}^{n} A^{*k}C^*CA^k &= \sum_{k=0}^{n} A^{*k}LA^k - \sum_{k=0}^{n} A^{*k+1}LA^{k+1} \\ &= L - A^{*n+1}LA^{n+1} \leq L. \end{aligned}$$

From this we obtain for all $x \in \mathscr{X}$ and $n \in \mathbb{Z}^+$

$$\sum_{k=0}^{n} \|CA^k x\|^2 \leq \langle Lx, x\rangle. \tag{3.1}$$

Letting $n \to \infty$ shows that $\mathcal{C}$ is a bounded map.

(v) 4 implies output stability. This follows from the fact that the Z-transform maps $l^2(\mathbb{Z}^+, \mathscr{Y})$ one-to-one onto $H^2(\mathbb{D}, \mathscr{Y})$. Since by assumption, the Z-transform of the output with initial condition x_0 and zero input restricts to a function in $H^2(\mathbb{D}, \mathscr{Y})$ it follows that this output is in $l^2(\mathbb{Z}^+, \mathscr{Y})$.

(vi) Output stability implies 4. Using the fact that the Z-transform maps $l^2(\mathbb{Z}^+, \mathscr{Y})$ one-to-one onto $H^2(\mathbb{D}, \mathscr{Y})$, we obtain that the Z-transform of the output with initial condition x_0 and zero input restricts to a function in $H^2(\mathbb{D}, \mathscr{Y})$. This is equivalent to $\mathsf{C}x_0$ restricting to a function in $H^2(\mathbb{D}, \mathscr{Y})$. This shows that C is defined on the open unit disc and since an operator-valued function is holomorphic in the strong topology if and only if it is holomorphic in the uniform topology, it follows that C is holomorphic on the open unit disc, and this implies that we have $r_{\mathsf{C}} \geq 1$. □

Example 3.3. The backward shift realization and the restricted backward shift realization from Remark 2.13 are output stable. It is easily seen that the identity is a solution of the equation mentioned in part 3 of Proposition 3.2.

We formulate a corollary about input stability.

Corollary 3.4. *The following are equivalent.*

1. *The discrete-time system is input stable.*
2. *The input map $\mathcal{B}$ extends uniquely to an element of $\mathcal{L}(l^2(\mathbb{Z}^-, \mathscr{U}), \mathscr{X})$.*
3. *There exists a nonnegative self-adjoint operator $L \in \mathcal{L}(\mathscr{X})$ such that $ALA^* - L + BB^* = 0$.*
4. *We have $r_{\mathsf{B}} \geq 1$ and for all $x \in \mathscr{X}$ the restriction of $\mathsf{B}^\dagger(\cdot)x$ is in $H^2(\mathbb{D}, \mathscr{U})$.*

Proof. This follows by applying Proposition 3.2 to the dual system. □

From the above results on input and output stability we obtain the following result on boundedness of the Hankel map.

Proposition 3.5. *The Hankel map of an input and output stable discrete-time system has a unique extension to an element of $\mathcal{L}(l^2(\mathbb{Z}^-, \mathscr{U}), l^2(\mathbb{Z}^+, \mathscr{Y}))$.*

Proof. Proposition 3.2 shows that the output map is bounded and Corollary 3.4 that the input map is bounded. Lemma 2.4 shows that the Hankel map is the product of these two operators. Hence the Hankel map is bounded. □

The operator $\mathcal{C}^*\mathcal{C}$ that we encountered in the proof of Proposition 3.2 plays an important role.

Definition 3.6. The **observability gramian** L_C of an output stable system is defined as $L_C := \mathcal{C}^*\mathcal{C}$. Here $\mathcal{C}^*$ is the adjoint of $\mathcal{C}$ considered as an operator in $\mathcal{L}(\mathscr{X}, l^2(\mathbb{Z}^+, \mathscr{Y}))$.

The proof of proposition 3.2 shows that the observability gramian is a solution of the **observation Lyapunov equation**

$$A^*LA - L + C^*C = 0. \tag{3.2}$$

This equation may have several other bounded nonnegative self-adjoint solutions. The following result gives two additional properties that the observability gramian has, each of which identifies it uniquely in the set of bounded nonnegative self-adjoint solutions of the observation Lyapunov equation. This will be of use to us later to show that a certain bounded nonnegative self-adjoint operator is the observability gramian of the system.

Lemma 3.7. *The set of bounded nonnegative self-adjoint solutions of the observation Lyapunov equation of an output stable discrete-time system has a unique element $L_{\min}$ such that $L_{\min} \leq L$ for all other bounded nonnegative self-adjoint solutions L. This unique element $L_{\min}$ is the observability gramian.*

This set also has a unique element L such that $L^{1/2}A^n x \to 0$ for all $x \in \mathscr{X}$ as $n \to \infty$. This unique element is the observability gramian.

Proof. From (3.1) we obtain by letting $k \to \infty$ that $L_C \leq L$ for all bounded nonnegative self-adjoint solutions L of the observation Lyapunov equation. Obviously the smallest element is unique.

We have

$$\|L_C^{1/2}A^n x\|^2 = \langle L_C A^n x, A^n x\rangle = \|\mathcal{C}A^n x\|^2 = \sum_{k=0}^{\infty} \|CA^k A^n x\|^2 = \sum_{i=n}^{\infty} \|CA^i x\|^2.$$

For $n \to \infty$ this converges to zero, since $\mathcal{C}x \in l^2(\mathbb{Z}^+, \mathscr{Y})$. This shows that the observability gramian indeed satisfies the given convergence condition. Let L be a bounded nonnegative self-adjoint solution of the observation Lyapunov equation with the above mentioned convergence property. Multiply the Lyapunov equation from the left with A^{*k} and from the right with A^k and sum from $k = 0$ to n to obtain

$$\begin{aligned}\sum_{k=0}^{n} A^{*k}C^*CA^k &= \sum_{k=0}^{n} A^{*k}LA^k - \sum_{k=0}^{n} A^{*k+1}LA^{k+1} \\ &= L - A^{*n+1}LA^{n+1}\end{aligned}$$

From this we obtain for all $x \in \mathscr{X}$

$$\sum_{k=0}^{n} \|CA^k x\|^2 = \langle Lx, x\rangle - \|L^{1/2}A^{n+1}x\|^2.$$

Letting $n \to \infty$ the left-hand side converges to $\|\mathcal{C}x\|^2 = \langle L_C x, x\rangle$ while, since $\|L^{1/2}A^{n+1}x\|^2 \to 0$ by assumption, the right-hand side converges to $\langle Lx, x\rangle$. Hence we obtain $\langle L_C x, x\rangle = \langle Lx, x\rangle$ for all $x \in \mathscr{X}$, which implies $L = L_C$. □

Lemma 3.8. *Let Σ be output stable and strongly stable. Then the observability gramian is the unique nonnegative self-adjoint solution of the observation Lyapunov equation.*

Proof. According to Lemma 3.7 the observability gramian is a nonnegative self-adjoint solution of the observation Lyapunov equation, so we only have to show that it is the unique nonnegative self-adjoint solution. Let L be a nonnegative self-adjoint solution of the observation Lyapunov equation. Then, as in the proof of Lemma 3.7, we have for all $N \in \mathbb{N}$

$$\sum_{n=0}^{N} A^{*n}C^*CA^n = \sum_{n=0}^{N} A^{*n}LA^n - \sum_{n=0}^{N} A^{*n+1}LA^{n+1} = L - A^{*N+1}LA^{N+1}.$$

We then have for all $x, y \in X$

$$\left\langle \sum_{n=0}^{N} A^{*n}C^*CA^n x, y \right\rangle = \langle Lx, y\rangle - \langle LA^{N+1}x, A^{N+1}y\rangle.$$

Letting $N \to \infty$ and using the fact that A is strongly stable, we have for all $x, y \in X$

$$\langle L_C x, y\rangle = \langle Lx, y\rangle.$$

This implies that $L = L_C$. Since L was an arbitrary nonnegative self-adjoint solution, this implies that L_C is the unique nonnegative self-adjoint solution of the observation Lyapunov equation. □

Example 3.9. The backward shift realization and the restricted backward shift realization from Remark 2.13 have the identity as observability gramian. From Example 3.9 we obtain that the identity is a solution of the observation Lyapunov equation. Since the systems are strongly stable it follows from Proposition 3.8 that the identity is the observability gramian of both systems.

Proposition 3.10. *Consider an output stable discrete-time system with observability Gramian* L_C. *If* $[u; x; y] \in \mathbb{B}$ *with* u *finitely nonzero, then* $L_C^{1/2} x_n \to 0$ *as* $n \to \infty$.

Proof. Let N be such that $u_n = 0$ for $n \geq N$. We have $L_C^{1/2} A^n x_N \to 0$ by Lemma 3.7. Since for $k \geq N$ we have $x_{k+N} = A^k x_N$ this implies that $L_C^{1/2} x_n \to 0$. □

Proposition 3.11. *An output stable discrete-time system is approximately observable if and only if* $L_C > 0$.

Proof. This follows since $\langle L_C x, x \rangle = \|\mathcal{C} x\|^2$ and a discrete-time system is approximately observable if and only if $\mathcal{C}$ is one-to-one by Proposition 2.3. □

The following Lyapunov equation that we already encountered in Corollary 3.4 is called the **control Lyapunov equation**

$$ALA^* - L + BB^* = 0. \tag{3.3}$$

Definition 3.12. The **controllability gramian** L_B of an input stable system is defined as $L_B := \mathcal{B}\mathcal{B}^*$. Here $\mathcal{B}^*$ is the adjoint of $\mathcal{B}$ considered as an operator in $\mathcal{L}(l^2(\mathbb{Z}^-, \mathscr{U}), \mathscr{X})$.

By duality we obtain similar results for the control Lyapunov equation as we obtained for the observability Lyapunov equation.

Lemma 3.13. *The set of bounded nonnegative self-adjoint solutions of the control Lyapunov equation of an input stable discrete-time system has a unique element* $L_{\min}$ *such that* $L_{\min} \leq L$ *for all other bounded nonnegative self-adjoint solutions* L. *This unique element* $L_{\min}$ *is the controllability gramian.*

This set also has a unique element L *such that* $L^{1/2} A^{*n} x \to 0$ *for all* $x \in \mathscr{X}$ *as* $n \to \infty$. *This unique element is the controllability gramian.*

Proof. This follows from applying Lemma 3.7 to the dual system. □

Lemma 3.14. *Let* Σ *be input stable with a strongly stable dual system. Then the controllability gramian is the unique nonnegative self-adjoint solution of the control Lyapunov equation.*

Proof. This follows from applying Lemma 3.8 to the dual system. □

Proposition 3.15. *An input stable discrete-time system is approximately controllable if and only if* $L_B > 0$*.*

Proof. Assume the system is approximately controllable. Then the input map has dense range. It follows that its adjoint is injective. From this we obtain that $L_B = \mathcal{B}\mathcal{B}^*$ is a positive operator.

Assume $L_B > 0$. Then $\mathcal{B}^*$ is injective, from which it follows that $\mathcal{B}$ has dense range. This implies that the system is approximately controllable. □

The following lemma and its corollary will be used throughout the thesis.

Lemma 3.16. *Let* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *be Hilbert spaces,* $Z \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$*,* $T \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$ *and* $\lambda \neq 0$*. Then* $\lambda \in \sigma(ZT)$ *if and only if* $\lambda \in \sigma(TZ)$*.*

Proof. Suppose that $\lambda \in \rho(ZT)$. Then we have

$$\frac{1}{\lambda}(I + T(\lambda I - ZT)^{-1}Z)(\lambda I - TZ) = I$$

and

$$(\lambda I - TZ)\frac{1}{\lambda}(I + T(\lambda I - ZT)^{-1}Z) = I.$$

This implies $\lambda \in \rho(TZ)$. The converse follows from interchanging the role of Z and T. Since the spectrum is the complement of the resolvent set the result follows. □

Note that $\lambda \neq 0$ is essential for Lemma 3.16 to hold: the left and right shift on $l^2(\mathbb{Z}^+)$ offers a counterexample for the case $\lambda = 0$. Lemma 3.16 has the following obvious corollary on the spectral radius of a product.

Corollary 3.17. *Let* $\mathscr{H}_1$ *and* $\mathscr{H}_2$ *be Hilbert spaces,* $Z \in \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$*,* $T \in \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$*. Then* $r(ZT) = r(TZ)$*.*

Proof. This follows from Lemma 3.16. □

We use Corollary 3.17 to prove the following.

Lemma 3.18. *Let* Σ *be an input and output stable discrete-time system. Let* L_C *and* L_B *be its observability and controllabilty gramian, respectively, and* $\mathcal{H}$ *its Hankel map. Then* $\|\mathcal{H}\| = \sqrt{r(L_C L_B)}$*.*

Proof. We use Corollary 3.17 and Lemma 2.4 to obtain the following.

$$r(L_C L_B) = r(\mathcal{C}^*\mathcal{C}\mathcal{B}\mathcal{B}^*) = r(\mathcal{B}^*\mathcal{C}^*\mathcal{C}\mathcal{B}) = r(\mathcal{H}^*\mathcal{H}).$$

Since $\mathcal{H}^*\mathcal{H}$ is a self-adjoint operator its spectral radius equals its norm. Since the norm of a self-adjoint operator T can be computed as

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

we obtain

$$\|\mathcal{H}^*\mathcal{H}\| = \sup_{\|x\|=1} |\langle \mathcal{H}^*\mathcal{H}x, x\rangle| = \sup_{\|x\|=1} |\langle \mathcal{H}x, \mathcal{H}x\rangle| = \|\mathcal{H}\|^2.$$

Combing the above we obtain

$$\|\mathcal{H}\|^2 = \|\mathcal{H}^*\mathcal{H}\| = r(\mathcal{H}^*\mathcal{H}) = r(L_C L_B),$$

as desired. □

Lemma 3.19. *Let Σ be an input and output stable discrete-time system. Let L_C and L_B be its observability and controllabilty gramian, respectively, and let L_c and L_b be arbitrary nonnegative self-adjoint solutions of its observation and control Lyapunov equations, respectively. Then $r(L_C L_B) \le r(L_c L_b)$.*

Proof. Lemma 3.17 implies that

$$r(L_C L_B) = r(L_C^{1/2} L_B L_C^{1/2}).$$

By Lemma 3.7 we have $L_C \le L_c$ and by Lemma 3.13 we have $L_B \le L_b$. From $L_B \le L_b$ we conclude that $L_C^{1/2} L_B L_C^{1/2} \le L_C^{1/2} L_b L_C^{1/2}$. This implies that

$$r(L_C^{1/2} L_B L_C^{1/2}) \le r(L_C^{1/2} L_b L_C^{1/2}).$$

Using Lemma 3.17 we obtain

$$r(L_C^{1/2} L_b L_C^{1/2}) = r(L_b^{1/2} L_C L_b^{1/2}).$$

Since $L_C \le L_c$ we obtain $L_b^{1/2} L_C L_b^{1/2} \le L_b^{1/2} L_c L_b^{1/2}$, which implies that

$$r(L_b^{1/2} L_C L_b^{1/2}) \le r(L_b^{1/2} L_c L_b^{1/2}).$$

Using Lemma 3.17 again we obtain

$$r(L_b^{1/2} L_c L_b^{1/2}) = r(L_c L_b).$$

Combing the above obtained inequalities we arrive at $r(L_C L_B) \le r(L_c L_b)$. □

Output stability tells us the following about the transfer function.

Proposition 3.20. *For an output stable discrete time system we have the following:*

$$\begin{array}{rcll} \mathsf{D}(z) & = & D + \mathsf{C}(z)zB & \forall z \in \mathbb{D}, \\ \mathsf{D}(z) & = & \mathfrak{D}(z) & \forall z \in \rho(A) \cap \mathbb{D}, \end{array}$$

and for all $u \in \mathscr{U}$ we have that $\mathsf{D}(\cdot)u$ restricts to a function in $H^2(\mathbb{D}, \mathscr{Y})$.

Proof. Proposition 3.2 part 4 shows that $r_{\mathsf{C}} \geq 1$. Remark 2.5 and Proposition 2.9 now give the indicated equalities. The first of these equalities together with Proposition 3.2 part 4 (with $x = Bu$) shows the H^2 property. □

The dual result reads as follows.

Proposition 3.21. *For an input stable discrete time system we have the following:*

$$\begin{array}{rcll} \mathsf{D}(z) & = & D + C\mathsf{B}(z) & \forall z \in \mathbb{D}, \\ \mathsf{D}(z) & = & \mathfrak{D}(z) & \forall z \in \rho(A) \cap \mathbb{D}, \end{array}$$

and for all $y \in \mathscr{Y}$ we have that $\mathsf{D}^\dagger(\cdot)y$ restricts to a function in $H^2(\mathbb{D}, \mathscr{U})$.

Proof. This follows along similar lines as Proposition 3.20. □

We now give a necessary and sufficient condition for input-output stability.

Proposition 3.22. *A system is input-output stable if and only if $r_{\mathsf{D}} \geq 1$ and D restricts to a function in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$.*

Proof. We first prove the if part. Let $u \in l^2(\mathbb{Z}^+, \mathscr{U})$ and denote the output corresponding to this input and initial condition zero by y. By Lemma 2.6 the Z-transform of the output is given by $\mathsf{D}(z)\hat{u}(z)$. From Lemmas A.2 and A.4 we obtain that $y \in l^2(\mathbb{Z}^+, \mathscr{Y})$. Hence the system is input-output stable.

We now prove the only if part. We first show that the map from the input to the output (with initial condition zero) is closed from $l^2(\mathbb{Z}^+, \mathscr{U})$ to $l^2(\mathbb{Z}^+, \mathscr{Y})$. So assume that $u^n \to u$ in $l^2(\mathbb{Z}^+, \mathscr{U})$ and the corresponding outputs $y^n \to y$ in $l^2(\mathbb{Z}^+, \mathscr{Y})$. We have to show that y is the output for input u. For y^n we have

$$y_k^n = \sum_{i=0}^{k-1} CA^i B u_{k-i-1}^n + D u_k^n.$$

Since A, B, C and D are continuous and u_j^n converges to u_j, we have

$$y_k^n \to \sum_{i=0}^{k-1} CA^iBu_{k-i-1} + Du_k.$$

On the other hand, since $y^n \to y$, we have $y_k^n \to y_k$. This shows that

$$y_k = \sum_{i=0}^{k-1} CA^iBu_{k-i-1} + Du_k.$$

So y is indeed the output for input u. By the closed graph theorem the map that sends an input to the corresponding output is in $\mathcal{L}(l^2(\mathbb{Z}^+, \mathscr{U}), l^2(\mathbb{Z}^+, \mathscr{Y}))$. This map obviously commutes with right-translations: if y is the output for the input u then $[0; y]$ is the output for the input $[0; u]$. Since the Z-transform is an isometric isomorphism between $l^2(\mathbb{Z}^+, \mathscr{H})$ and $H^2(\mathbb{D}, \mathscr{H})$ the map that sends $\hat{u}$ to $\hat{y}$ is bounded from $H^2(\mathbb{D}, \mathscr{U})$ to $H^2(\mathbb{D}, \mathscr{Y})$. The shift-invariance in time-domain translates to commutation with multiplication by z in the frequency domain. Hence $\hat{u} \mapsto \hat{y}$ is a bounded linear map from $H^2(\mathbb{D}, \mathscr{U})$ to $H^2(\mathbb{D}, \mathscr{Y})$ that commutes with multiplication by z. By Lemma A.4 it is given by multiplication by an $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function. This function coincides with the input-output function restricted to the unit disc. □

Corollary 3.23. *The dual system of an input-output stable system is input-output stable.*

Proof. This follows from Proposition 3.22 since $\mathsf{D} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ if and only if $\mathsf{D}^\dagger \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{Y}, \mathscr{U}))$. □

Proposition 3.24. *The Hankel map of an input-output stable discrete-time system has a unique extension to an element of* $\mathcal{L}(l^2(\mathbb{Z}^-, \mathscr{U}), l^2(\mathbb{Z}^+, \mathscr{Y}))$.

Proof. By Proposition 3.22 the transfer function of the system is an element of $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. It follows from Definition A.24 that it has a bounded Hankel operator. By Lemma A.26 the Hankel operator and the Hankel map are similar with as similarity operator the Z-transform. It follows that the Hankel map extends to a bounded operator from $l^2(\mathbb{Z}^-, \mathscr{U})$ to $l^2(\mathbb{Z}^+, \mathscr{Y})$ as desired. □

Example 3.25. The backward shift realization and the restricted backward shift realization of a $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ function from Remark 2.13 are input stable. From Lemma 2.4 we obtain $\mathcal{H}^*\mathcal{H} = \mathcal{B}^*\mathcal{C}^*\mathcal{C}\mathcal{B} = \mathcal{B}^*L_C\mathcal{B}$. From Example 3.9 we obtain $L_C = I$. From Proposition 3.24 we obtain that the Hankel map is bounded. It follows that $\mathcal{B}$ is bounded. Hence the system is input stable by Proposition 3.4.

The following result gives necessary and sufficient conditions for exponential stability.

Proposition 3.26. *The following are equivalent.*

1. *The discrete-time system is exponentially stable.*
2. *There exists a nonnegative self-adjoint operator $L \in \mathcal{L}(\mathscr{X})$ such that $A^*LA - L + I = 0$.*
3. *The spectral radius of the state operator is strictly smaller than one.*
4. *There exist $M \geq 0$ and $r \in [0,1)$ such that for all sequences x with $[0; x; y] \in \mathbb{B}$ we have $\|x_n\| \leq Mr^n\|x_0\|$ for all $n \geq 0$.*
5. *We have $r_{\mathsf{A}} \geq 1$ and the restriction of the state function to the open unit disc is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{X}))$.*

Proof. We will show that exponential stability implies 2 implies 3 implies 4 implies 5 implies exponential stability.

(i) exponential stability implies 2: this follows from Proposition 3.2 with $C = I$.

(ii) 2 implies 3. We will show that it follows from the Lyapunov equation that the approximate eigenvalues of A must lie in the open unit disc. Since the boundary of the spectrum of an operator consists of approximate eigenvalues (see Taylor and Lay [91, Theorem V.4.1 page 282]), this shows that the spectrum of A is contained in the open unit disc which is equivalent with the spectral radius being strictly smaller than one. Suppose λ is an approximate eigenvalue and x_n is a sequence of approximate eigenvectors; that is, $\|x_n\| = 1$ and $\|(\lambda I - A)x_n\| \to 0$. Using the Lyapunov equation we obtain

$$(\lambda I - A)^*L(\lambda I - A) - \lambda(\lambda I - A)^*L - \bar{\lambda}L(\lambda I - A) = (1 - |\lambda|^2)L - I.$$

By applying this to x_n and taking the inner product with x_n we obtain $(1-|\lambda|^2)\langle Lx_n, x_n\rangle \to 1$. Since L is nonnegative this implies that $1-|\lambda|^2 > 0$.

(ii) 3 implies 4. From the Gelfand formula $r(A) = \lim_{n\to\infty} \sqrt[n]{\|A^n\|}$ it follows that for $r := (1 + r(A))/2 \in (0,1)$ there exists a $N \in \mathbb{Z}^+$ such that for all $n \geq N$ we have $\|A^n\| \leq r^n$. Define $\tilde{M} := \max_{i=0,\dots,N-1} \|A^i\|/r^i$ and $M = \max\{\tilde{M}, 1\}$. Then $\|A^n\| \leq Mr^n$ for all $n \geq 1$. Since $x_n = A^n x_0$ the assertion follows.

(iii) 4 implies 5. From the given inequality we conclude that Z-transform of x is holomorphic on the open disc with radius $1/r$. In particular, it follows that for all $x_0 \in \mathscr{X}$ the function $\mathsf{A}(\cdot)x_0$ is holomorphic in a neighborhood of the unit disc. It follows that for all $x_0 \in \mathscr{X}$ the function $z \mapsto \|\mathsf{A}(z)x_0\|^2$ is continuous on the closed unit disc. Since the closed unit disc is compact, this function is bounded. We conclude that for all $x_0 \in \mathscr{X}$ the function $\mathsf{A}(\cdot)x_0$ restricts to a function in $H^\infty(\mathbb{D}, \mathscr{X})$. It follows that $r_\mathsf{A} \geq 1$ and A restricted to the unit disc is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{X}))$.

(iv) 5 implies exponential stability. Let x be the state corresponding to initial state x_0 and zero input. Since the state function restricts to a function in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{X}))$ we have that the Z-transform of the state, $\hat{x}(z) = \mathsf{A}(z)x_0$, restricted to the unit disc is in $H^\infty(\mathbb{D}, \mathscr{X})$. Since $H^\infty(\mathbb{D}, \mathscr{X})$ is contained in $H^2(\mathbb{D}, \mathscr{X})$ and the Z-transform is isometric from $l^2(\mathbb{Z}^+, \mathscr{X})$ onto $H^2(\mathbb{D}, \mathscr{X})$ we obtain that the state is in $l^2(\mathbb{Z}^+, \mathscr{X})$ and so the system is exponentially stable. □

Corollary 3.27. *The dual system of an exponentially stable system is exponentially stable.*

Proof. This follows from Proposition 3.26 since the spectral radius of an operator and its dual are equal. □

After having established equivalent conditions for the types of stability we have introduced, we are now ready to study their relationships to each other.

The following proposition shows that exponential stability implies all the other types of stability.

Proposition 3.28. *If a discrete-time system is exponentially stable, then it is strongly stable, output stable, input stable and input-output stable.*

Proof. (i) Exponential stability implies strong stability: any square summable sequence tend to zero.

(ii) Exponential stability implies output stability: since the input is assumed to be zero we have $y_n = Cx_n$ and so $\|y_n\| \leq \|C\|\ \|x_n\|$. Since x is square summable it follows that y is.

(iii) Exponential stability implies input stability: by Corollary 3.27 the dual system is exponentially stable so it follows by (ii) that the dual system is output stable, which shows that the original system is input stable.

(iv) Exponential stability implies input-output stability: by Proposition 3.26 part 5 we have $r_\mathsf{A} \geq 1$ which by Remark 2.5 implies $r_\mathsf{D} \geq 1$. From the same proposition we obtain that A restricted to the open unit disc is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{X}))$, using Remark 2.5 again we obtain that the restriction of D to the unit disc is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. Proposition 3.22 now shows that the system is input-output stable. □

Remark 3.29. It follows similarly as in part (iv) of the proof of Proposition 3.28 that B and C restrict to functions in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{X}))$ and $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{X}, \mathscr{Y}))$, respectively, when the system is exponentially stable.

As the following example shows Proposition 3.28 is the only possible positive result on the connection between the different stability concepts.

Example 3.30.

1. An example of a system that is strongly stable, input stable, output stable and input-output stable, but not exponentially stable. Take $\mathscr{X} = l^2(\mathbb{Z}^+, \mathbb{C})$, $\mathscr{U} = \mathbb{C}$, $\mathscr{Y} = \mathbb{C}$, $B = 0$, $C = 0$, $D = 0$. It trivially follows that this system for any A is input stable, output stable and input-output stable. Define A as follows: $(Ax)_n := x_{n+1}$. Then A is strongly stable: we have

$$\|A^n x\|^2 = \sum_{k=0}^{\infty} \|(A^n x)_k\|^2 = \sum_{k=0}^{\infty} \|x_{k+n}\|^2 = \sum_{i=n}^{\infty} \|x_i\|^2,$$

and since $x \in l^2(\mathbb{Z}^+, \mathbb{C})$ this expression tends to zero as $n \to \infty$. Let $\{e_n\}$ be the standard basis of $l^2(\mathbb{Z}^+, \mathbb{C})$. The system is not exponentially stable: take e_0 as initial state, then the state at time n equals e_n. Since the state at any time instance has norm one it is not square summable over time.

2. An example of a system that is input stable, output stable and input-output stable, but not strongly stable. Take $\mathscr{X} = \mathscr{U} = \mathscr{Y} = \mathbb{C}$, $A = 1$, $B = 0$, $C = 0$, $D = 0$. It trivially follows that this system is input stable, output stable and input-output stable. Since $A^n x = x$ for all $n \in \mathbb{Z}^+$ the system is not strongly stable.

3. An example of a system that is strongly stable, input stable and input-output stable, but not output stable. Take $\mathscr{X} = l^2(\mathbb{Z}^+, \mathbb{C})$, $\mathscr{U} = \mathbb{C}$, $\mathscr{Y} = l^2(\mathbb{Z}^+, \mathbb{C})$, $B = 0$, $D = 0$. Define A as follows: $(Ax)_n := x_{n+1}$. Then, as in part 1, A is strongly stable. Since $B = 0$ the system is obviously input and input-output stable for any choice of C. Choose $C = I$. Then the state and the output coincide and it follows that the system is output stable if and only if it is exponentially stable. We saw in part 1 that A is not exponentially stable. It follows that the system is not output stable.

4. An example of a system that is strongly stable, output stable and input-output stable, but not input stable. The dual system of the system from part 3 provides such an example.

5. An example of a system that is strongly stable, input stable, output stable, but not input-output stable. The function $\mathsf{G} : \mathbb{D} \to \mathbb{C}$ defined by

$$\mathsf{G}(z) := \sum_{n=1}^{\infty} \frac{1}{n} z^n$$

is in $H^2(\mathbb{D}, \mathbb{C})$ since $\sum_{n=1}^{\infty} 1/n^2 < \infty$, but is not in $H^\infty(\mathbb{D}, \mathbb{C})$ since for $z \to 1$ we have that $|\mathsf{G}(z)|$ becomes arbitrarily large since $\sum_{n=1}^{\infty} 1/n$ diverges. It follows using Proposition 3.22 that any system with G as transfer function is not input-output stable. Define $\mathscr{X} := l^2(\mathbb{Z}^+, \mathbb{C})$, the operator $A \in \mathcal{L}(\mathscr{X})$ by $(Ax)_n = x_{n+1}$ and the operator $C \in \mathcal{L}(\mathscr{X}, \mathbb{C})$ by $Cx = x_0$. Define $B \in \mathcal{L}(\mathbb{C}, \mathscr{X})$ by $(Bu)_n = u/(n+1)$. This operator is bounded since

$$\|Bu\|_{\mathscr{X}}^2 = \left\| \frac{u}{n+1} \right\|_{l^2(\mathbb{Z}^+, \mathbb{C})}^2 = |u|^2 \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

It is easily seen that $CA^nB = 1/(n+1)$ for all $n \in \mathbb{Z}^+$, from which it follows that G is the transfer function of the discrete-time system Σ with system operator $[A, B; C, 0]$. As in part 1 A is strongly stable. It is easily seen that the identity is a solution of the observation Lyapunov equation, from which it follows using Proposition 3.2 that Σ is output stable. It is easily computed that the output map of the dual system has, with respect to the standard basis of $l^2(\mathbb{Z}^+, \mathbb{C})$, the following matrix represention.

$$\begin{bmatrix} 1 & 1/2 & 1/3 & \dots \\ 1/2 & 1/3 & \dots & \dots \\ 1/3 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

This matrix is called the infinite Hilbert matrix and is known to define a bounded operator on $l^2(\mathbb{Z}^+, \mathbb{C})$ with norm π (see Peller [75, page 6]). It follows that Σ is input stable.

Notes

Exponential stability and input-output stability have been the main stability concepts in systems and control theory in the last decades. Proposition 3.26 can be considered as the discrete-time version of a now classical continuous-time result of Datko [20]. Connections between Lyapunov equations and strong stability were investigated by Przyłuski [77]. Proposition 3.22 is classical, see for example Weiss [96] for more information on the continuous-time version.