Model reduction for controller design for infinite-dimensional systems

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Part I

Discrete-time systems

Chapter 2

Basic objects

In this chapter we introduce the main concept of part I of this thesis, that of a discrete-time system. We also introduce several objects associated with a discrete-time system that will be used throughout this thesis. Finally, we study several ways in which we can obtain a new discrete-time system from one or two known ones.

We first introduce the concept of a dynamical system. Note that $\mathbb{W}^{\mathbb{T}}$ denotes the set of functions from $\mathbb{T}$ to $\mathbb{W}$.

Definition 2.1. A **dynamical system** Σ is a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{B})$ with $\mathbb{T}$ a set, called the time axis; $\mathbb{W}$ a set, called the signal space, and $\mathbb{B} \subset \mathbb{W}^{\mathbb{T}}$, the behavior of the system.

In Part I of this thesis we will be concerned with the following dynamical systems.

Let $\mathbb{T} = \mathbb{Z}^+$, the nonnegative integers, and $\mathbb{W} = \mathscr{U} \times \mathscr{X} \times \mathscr{Y}$, where $\mathscr{U}$, $\mathscr{X}$, $\mathscr{Y}$ are separable Hilbert spaces. Let $A \in \mathcal{L}(\mathscr{X})$, $B \in \mathcal{L}(\mathscr{U}, \mathscr{X})$, $C \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ and $D \in \mathcal{L}(\mathscr{U}, \mathscr{Y})$ be bounded operators between the given spaces. Define the behavior by

$$\mathbb{B} := \left\{ \begin{bmatrix} u \\ x \\ y \end{bmatrix} \in \mathbb{W}^{\mathbb{T}} : \begin{bmatrix} x_{n+1} \\ y_n \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x_n \\ u_n \end{bmatrix} \text{ for all } n \in \mathbb{Z}^+ \right\} \tag{2.1}$$

This type of dynamical system will be called a **discrete-time system** . The elements of the behavior are called **trajectories** of the system. It follows from (2.1) that for arbitrary $u : \mathbb{Z}^+ \to \mathscr{U}$ and $x_0 \in \mathscr{X}$ there exists a unique trajectory $[u; x; y] \in \mathbb{B}$. The sequence u is called the **input**, x_0 the **initial state**, x the **state** and y the **output**. The space $\mathscr{U}$ is called the **input space**, $\mathscr{X}$ is called the **state space** and $\mathscr{Y}$ the **output space**. Usually in control theory the goal is to choose for a given initial state an input such that the trajectory has some specified property.

Note that the operator

$$S := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{2.2}$$

is completely determined by the behavior in the following sense: if the behaviors corresponding to S_1 and S_2 are equal, then S_1 and S_2 are equal. This can be proven as follows. Let $x_0 \in \mathscr{X}$ and $u : \mathbb{Z}^+ \to \mathscr{U}$ be arbitrary. Since the behaviors corresponding to S_1 and S_2 are equal the trajectories corresponding to this initial state and input are equal. In particular the state at time one and the output at time zero are equal. Hence

$$S_1 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_0 \end{bmatrix} = S_2 \begin{bmatrix} x_0 \\ u_0 \end{bmatrix},$$

and since u_0 and x_0 where arbitrary this shows that $S_1 = S_2$.

The above shows that the following definitions are unambiguous. The operators appearing in (2.1) have the following names: A is called the **state operator**, B the **input operator**, C the **output operator**, and D the **feedthrough operator** of the discrete-time system. The operator S is called the **system operator**.

We say that a sequence $h : \mathbb{Z} \to \mathscr{H}$ is **finitely nonzero** if only a finite number of elements in the sequence is nonzero.

A discrete-time system is called **approximately observable** if $[0; x; y] \in \mathbb{B}$ and $[0; w; y] \in \mathbb{B}$ implies $x = w$, i.e. if the output with zero input uniquely determines the state. A discrete-time system is called **approximately controllable** if the set

$$\{w \in \mathscr{X} : \text{ there exist } [u; x; y] \in \mathbb{B} \text{ with } u \text{ finitely nonzero,} \\ N \in \mathbb{Z}^+ \text{ such that } x_0 = 0, x_N = w\}$$

is dense in $\mathscr{X}$. A discrete-time system is called **minimal** of it is both approximately controllable and approximately observable.

We define three maps on sequence spaces that will play an important role in this thesis. The **input map** of a discrete-time system is defined for finitely nonzero $u : \mathbb{Z}^- \to \mathscr{U}$ by (here $\mathbb{Z}^-$ is the set of negative integers)

$$\mathcal{B}u := \sum_{i=0}^{\infty} A^i B u_{-i-1},$$

the **output map** is defined for $x \in \mathscr{X}$ by

$$(\mathcal{C}x)_k := CA^k x \quad k \in \mathbb{Z}^+,$$

the **input-output map** is defined for finitely nonzero $u : \mathbb{Z} \to \mathscr{U}$ by

$$(\mathcal{D}u)_k := \sum_{i=0}^{\infty} CA^i Bu_{k-i-1} + Du_k, \quad k \in \mathbb{Z}.$$

Remark 2.2. Let $J \subset \mathbb{Z}$. Denote by $l_c(J, \mathscr{H})$ the set of sequences $J \to \mathscr{H}$ with compact support and by $l(J, \mathscr{H})$ the set of all sequences $J \to \mathscr{H}$. As indicated above we consider $\mathcal{B} : l_c(\mathbb{Z}^-, \mathscr{U}) \to \mathscr{X}$, $\mathcal{C} : \mathscr{X} \to l(\mathbb{Z}^+, \mathscr{Y})$, $\mathcal{D} : l_c(\mathbb{Z}, \mathscr{U}) \to l(\mathbb{Z}, \mathscr{Y})$. In this thesis we will not need to consider topologies on $l_c(J, \mathscr{H})$ and $l(J, \mathscr{H})$. In connection with stability of the system in Chapter 3 and the subsequent chapters we sometimes consider the extension of $\mathcal{B}$ to a bounded operator on $l^2(\mathbb{Z}^-, \mathscr{U})$ (which when it exists is unique), $\mathcal{C}$ as a bounded operator into $l^2(\mathbb{Z}^+, \mathscr{Y})$ (which in some cases it may not be) and the extension of $\mathcal{D}$ as a bounded operator from $l^2(\mathbb{Z}, \mathscr{U})$ to $l^2(\mathbb{Z}, \mathscr{Y})$ (which when it exists is unique). It should be clear from the context on which spaces we consider the input, output and input-output map.

To further study the above maps we introduce the maps τ, π_- and π_+ on the space of sequences $\mathbb{Z} \to \mathscr{H}$ where $\mathscr{H}$ is a separable Hilbert space

$$(\tau h)_k := h_{k+1}, \quad (\pi_- h)_k := \begin{cases} h_k & k \in \mathbb{Z}^- \\ 0 & k \in \mathbb{Z}^+ \end{cases}, (\pi_+ h)_k := \begin{cases} 0 & k \in \mathbb{Z}^- \\ h_k & k \in \mathbb{Z}^+ \end{cases}.$$

The significance of the above maps is apparent from the following result. If $[u; x; y]$ is a trajectory with u finitely nonzero, then

$$\begin{aligned} x_n &= A^n x_0 + \mathcal{B}\pi_- \tau^n u \\ y &= \mathcal{C}x_0 + \mathcal{D}u. \end{aligned}$$

The above follows from an easy computation.

Proposition 2.3. *A discrete-time system is approximately observable if and only if its output map $\mathcal{C}$ is one-to-one and approximately controllable if and only if its input map $\mathcal{B}$ has dense range.*

Proof. This follows easily from the above characterization of a trajectory in terms of the main operator, the input map, the output map and the input-output map. □

The fourth map on sequence spaces that will play an important role in this thesis is the following map. The **Hankel map** is defined for finitely nonzero $u : \mathbb{Z}^- \to \mathscr{U}$ by

$$(\mathcal{H}u)_k := \sum_{i=0}^{\infty} CA^i Bu_{k-i-1} \quad k \in \mathbb{Z}^+.$$

We state the following lemma on the Hankel map.

Lemma 2.4. *For the Hankel map of a discrete-time system we have*

$$\mathcal{C}\mathcal{B} = \mathcal{H} = \pi_+\mathcal{D}\pi_-,$$

where $\mathcal{B}$ is the input map, $\mathcal{C}$ the output map and $\mathcal{D}$ the input-output map of the discrete-time system.

Proof. The equality $\mathcal{H} = \pi_+\mathcal{D}\pi_-$ is immediate from the definitions. The equality $\mathcal{C}\mathcal{B} = \mathcal{H}$ is proven as follows. Let $u : \mathbb{Z}^- \to \mathscr{U}$ be finitely nonzero and $k \in \mathbb{Z}^+$, we then have

$$\begin{aligned}(\mathcal{C}\mathcal{B}u)_k &= CA^k \textstyle\sum_{i=0}^{\infty} A^i B u_{-i-1} \\ &= \textstyle\sum_{i=0}^{\infty} CA^{k+i} B u_{-i-1} && \text{using that } u_{-i-1} = 0 \text{ for } i < 0 \\ &= \textstyle\sum_{i=-k}^{\infty} CA^{k+i} B u_{-i-1} && \text{substituting } j := k + i \\ &= \textstyle\sum_{j=0}^{\infty} CA^j B u_{k-j-1} \\ &= (\mathcal{H}u)_k.\end{aligned}$$

This shows that $\mathcal{C}\mathcal{B} = \mathcal{H}$. □

We define four operator-valued holomorphic functions associated with a discrete-time system that will play an important role in this thesis. We define them through power series expansions. Note that, as in the scalar case, operator-valued power series have a radius of convergence and we consider the four operator-valued holomorphic functions as functions on an open disc with the radius of convergence as radius. The **state function** of a discrete-time system is defined by

$$\mathsf{A}(z) = \sum_{i=0}^{\infty} A^i z^i.$$

Note that this series converges for $|z| < 1/\|A\|$, since it forms a geometric series with common ratio Az. The radius of convergence of this series equals $1/r(A)$, where $r(A)$ denotes the spectral radius of A, and so $\mathsf{A} : \mathbb{D}_{1/r(A)} \to \mathcal{L}(\mathscr{X})$. We denote $1/r(A)$ by r_{A}. The **input function** of a discrete-time system is defined by

$$\mathsf{B}(z) = \sum_{i=0}^{\infty} A^i B z^{i+1}.$$

We have $\mathsf{B} : \mathbb{D}_{r_{\mathsf{B}}} \to \mathcal{L}(\mathscr{U}, \mathscr{X})$, where r_{B} is the radius of convergence of the power series. The **output function** $\mathsf{C} : \mathbb{D}_{r_{\mathsf{C}}} \to \mathcal{L}(\mathscr{X}, \mathscr{Y})$ of a discrete-time system is defined by

$$\mathsf{C}(z) = \sum_{i=0}^{\infty} CA^i z^i$$

and the **transfer function** $\mathsf{D} : \mathbb{D}_{r_\mathsf{D}} \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ by

$$\mathsf{D}(z) = D + \sum_{i=0}^{\infty} CA^i B z^{i+1}.$$

Note that

$$A = \mathsf{A}'(0), \quad B = \mathsf{B}'(0), \quad C = \mathsf{C}(0), \quad D = \mathsf{D}(0). \tag{2.3}$$

Remark 2.5. We remark that we have the following inequalities

$$r_\mathsf{A} \le r_\mathsf{B}, r_\mathsf{C} \le r_\mathsf{D}$$

and the equalities

$$\begin{aligned} \mathsf{A}(z)Bz &= \mathsf{B}(z) \quad \forall z \in \mathbb{D}_{r_\mathsf{A}}, \\ C\mathsf{A}(z) &= \mathsf{C}(z) \quad \forall z \in \mathbb{D}_{r_\mathsf{A}}, \\ D + C\mathsf{B}(z) &= \mathsf{D}(z) \quad \forall z \in \mathbb{D}_{r_\mathsf{B}}, \\ D + \mathsf{C}(z)Bz &= \mathsf{D}(z) \quad \forall z \in \mathbb{D}_{r_\mathsf{C}}. \end{aligned}$$

A sequence $h : \mathbb{Z}^+ \to \mathscr{H}$ is called Z**-transformable** if the power series

$$\sum_{i=0}^{\infty} h_i z^i$$

has a positive radius of convergence. The Z-transform of a Z-transformable sequence h is denoted by $\hat{h}$.

Lemma 2.6. *If the input of a discrete-time system is Z-transformable, then the state and output are Z-transformable and they satisfy*

$$\begin{aligned} \hat{x}(z) &= \mathsf{A}(z)x_0 + \mathsf{B}(z)\hat{u}(z), \\ \hat{y}(z) &= \mathsf{C}(z)x_0 + \mathsf{D}(z)\hat{u}(z), \end{aligned}$$

for z such that $|z| < r_\mathsf{A}$ and $|z|$ smaller than the radius of convergence of the power series corresponding to the sequence u.

Proof. Due to the linearity of the system we can prove this in two steps: in the first we can take $u = 0$ and in the second $x_0 = 0$.

First step ($u = 0$). Since $x_{n+1} = Ax_n$, we obtain $x_i = A^i x_0$ and so the Z-transform of the state is

$$\sum_{i=0}^{\infty} x_i z^i = \sum_{i=0}^{\infty} A^i x_0 z^i = \mathsf{A}(z)x_0$$

and this power series converges for $|z| < r_{\mathsf{A}}$. Since $y_n = Cx_n$ we obtain that y is Z-transformable and

$$\hat{y}(z) = \mathsf{C}(z)x_0.$$

Second step ($x_0 = 0$). The state is now given by

$$x_n = \sum_{i=0}^{n-1} A^i B u_{n-i-1},$$

and so its Z-transform is

$$\sum_{n=0}^{\infty} x_n z^n = \sum_{n=0}^{\infty} \sum_{i=0}^{n-1} A^i B u_{n-i-1} z^n.$$

On the other hand, we have

$$\mathsf{B}(z)\hat{u}(z) = \sum_{j=0}^{\infty} A^j B z^{j+1} \sum_{k=0}^{\infty} u_k z^k = \sum_{n=0}^{\infty} \sum_{i=0}^{n} A^i B u_{n-i-1} z^n,$$

where the rearranging of terms is justified, since the series converge absolutely. Hence x is Z-transformable and satisfies $\hat{x}(z) = \mathsf{B}(z)\hat{u}(z)$. The proof that $\hat{y}(z) = \mathsf{D}(z)\hat{u}(z)$ follows along the same lines.

Combining steps 1 and 2 and using linearity proves the lemma. □

We now define four other operator-valued holomorphic functions that will also play a role in this thesis. They are defined on a set that we denote by $1/\rho(A)$ and that is defined as follows:

$$1/\rho(A) := \{z \in \mathbb{C} : 1/z \in \rho(A)\} \cup \{0\}.$$

Here $\rho(A)$ denotes the resolvent set of the operator A. The **resolvent** $\mathfrak{A} : 1/\rho(A) \to \mathcal{L}(\mathscr{X})$ of a discrete-time system is defined by

$$\mathfrak{A}(z) := (I - zA)^{-1},$$

the **incoming wave function** $\mathfrak{B} : 1/\rho(A) \to \mathcal{L}(\mathscr{U}, \mathscr{X})$ of a discrete-time system is defined by

$$\mathfrak{B}(z) := z(I - zA)^{-1}B,$$

the **outgoing wave function** $\mathfrak{C} : 1/\rho(A) \to \mathcal{L}(\mathscr{X}, \mathscr{Y})$ of a discrete-time system is defined by

$$\mathfrak{C}(z) := C(I - zA)^{-1},$$

and the **characteristic function** $\mathfrak{D} : 1/\rho(A) \to \mathcal{L}(\mathscr{U}, \mathscr{Y})$ of a discrete-time system is defined by

$$\mathfrak{D}(z) := D + Cz(I - zA)^{-1}B.$$

It is easily seen that

$$\mathfrak{A}(z) = \mathsf{A}(z),\ \mathfrak{B}(z) = \mathsf{B}(z),\ \mathfrak{C}(z) = \mathsf{C}(z),\ \mathfrak{D}(z) = \mathsf{D}(z),\ \text{for } |z| < r_{\mathsf{A}}, \quad (2.4)$$

but the following examples show that these functions are not identical.

Example 2.7. Let $\mathscr{U} = \mathscr{X} = \mathscr{Y} = \mathbb{C}$ and $A = -1$, $B = C = D = 0$. Then both the transfer function and the characteristic function are zero, but the transfer function has domain $\mathbb{C}$ while the characteristic function has domain $\mathbb{C}\backslash\{-1\}$. This shows that the transfer function and the characteristic function are not identical. Similar arguments apply to the other functions.

The above example is somewhat pathological, since on the intersection of their domains the functions are equal. This example identifies the only possible difference when the state space $\mathscr{X}$ is finite-dimensional. In the case that $\mathscr{X}$ is infinite-dimensional, the transfer function and the characteristic function need not even be equal on the intersection of their domains, as the following example shows.

Example 2.8. Let $\mathscr{U} = \mathscr{Y} = \mathbb{C}$ and $\mathscr{X} = l^2(\mathbb{Z})$. Define the operators A, B, C by

$$(Ax)_k = x_{k-1}, \quad (Bu)_k = \begin{cases} u & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}, \quad Cx = x_{-1},$$

and $D = 0$. Then $CA^iB = 0$ for all $i \geq 0$ and so the transfer function is defined on the whole complex plane and equals zero. We calculate $\mathfrak{D}(2)$. We first note that the solution v of $B1 = (I - 2A)v$ has to satisfy

$$v_k - 2v_{k-1} = \begin{cases} 0 & \text{for } k \neq 0 \\ 1 & \text{for } k = 0. \end{cases}$$

The unique solution in $l^2(\mathbb{Z})$ is given by

$$v_k = \begin{cases} -2^k & \text{for } k < 0 \\ 0 & \text{for } k \geq 0. \end{cases}$$

So $v = (I - 2A)^{-1}B1$. It follows that $\mathfrak{D}(2)1 = C2(I - 2A)^{-1}B1 = 2v_{-1} = -1$. Hence $\mathfrak{D}(2) = -1$. We conclude that the transfer function and the characteristic function are both defined in 2, but that their values in this point are different. So the transfer function and the characteristic function are not equal on the intersection of their domains.

The following result shows on which domain we do have equality of the transfer function and the characteristic function.

Proposition 2.9. *For a discrete-time system we have the following equalities.*

$$\begin{aligned}
\mathsf{A}(z) &= \mathfrak{A}(z) \quad \forall z : |z| < r_{\mathsf{A}}, \\
\mathsf{B}(z) &= \mathfrak{B}(z) \quad \forall z : |z| < r_{\mathsf{A}}, \\
\mathsf{C}(z) &= \mathfrak{C}(z) \quad \forall z : |z| < r_{\mathsf{A}}, \\
\mathsf{D}(z) &= \mathfrak{D}(z) \quad \forall z : |z| < \max\{r_{\mathsf{B}}, r_{\mathsf{C}}\}, z \in 1/\rho(A).
\end{aligned}$$

Proof. The first three equalities were already mentioned in (2.4). We prove the fourth equality. From (2.4) we conclude that $\mathsf{C}(z)(I - zA) = C$ for all z with $|z| < r_{\mathsf{A}}$. Since both sides are holomorphic this equality extends to all z with $|z| < r_{\mathsf{C}}$. We now multiply both sides by $(I - zA)^{-1}zB$ which is well-defined on $1/\rho(A)$ and obtain $\mathsf{C}(z)zB = C(I - zA)^{-1}zB$ for all $z \in 1/\rho(A)$ with $|z| < r_{\mathsf{C}}$. This shows that $\mathsf{D}(z) = \mathfrak{D}(z)$ for these z and proves the fourth equality in the case that $r_{\mathsf{C}} \geq r_{\mathsf{B}}$. If $r_{\mathsf{B}} < r_{\mathsf{C}}$, then a similar argument with B instead of C proves the assertion. □

Example 2.10. We apply Proposition 2.9 to Example 2.8. It is easily seen that the spectral radius of A equals one. So we have $\mathsf{A} = \mathfrak{A}$, $\mathsf{B} = \mathfrak{B}$, $\mathsf{C} = \mathfrak{C}$, $\mathsf{D} = \mathfrak{D}$ on the open unit disc. It is not very difficult to show that $r_{\mathsf{B}} = r_{\mathsf{C}} = 1$, so that the in principle more precise condition for equality of the transfer function and the characteristic function from Proposition 2.9 in this case gives the same as the condition based on the spectral radius of A.

As we saw the transfer function of a discrete-time system is always holomorphic at zero. The following result shows that any function that is holomorphic in zero is the transfer function of some discrete-time system. We first give the relevant definition.

Definition 2.11. Let G be a $\mathcal{L}(\mathscr{U}, \mathscr{Y})$-valued function defined in a neigbourhood of zero. A discrete-time system Σ is called a **realization** of G if the transfer function of Σ coincides with G in a neighbourhood of zero.

Proposition 2.12. *Any $\mathcal{L}(\mathscr{U}, \mathscr{Y})$-valued function which is holomorphic at zero has a realization.*

Proof. In this proof we will use the Hardy spaces H^2 and H^∞ (see Appendix A). First assume that the given function G satisfies $\mathsf{G}(0) = 0$ and $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. Define $\mathscr{X} := l^2(\mathbb{Z}^+, \mathscr{Y})$ and for $x \in \mathscr{X}$ the operator $A \in \mathcal{L}(\mathscr{X})$ by $(Ax)_n = x_{n+1}$ and the operator $C \in \mathcal{L}(\mathscr{X}, \mathscr{Y})$ by $Cx = x_0$.

For $u \in U$ define $\mathsf{F}(z) := \mathsf{G}(z)u$. Since $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ we have $\mathsf{F} \in H^2(\mathbb{D}, \mathscr{Y})$. It follows from Lemma A.2 that $\mathsf{F}(z) = \sum_{n=0}^\infty F_n z^n$, with the sequence $F_n \in l^2(\mathbb{Z}^+, \mathscr{Y})$. Define $B \in \mathcal{L}(\mathscr{U}, \mathscr{X})$ by $(Bu)_n = F_n$. This operator is bounded since

$$\|Bu\|_{\mathscr{X}} = \|(F_n)_{n\geq 0}\|_{l^2(\mathbb{Z}^+,\mathscr{Y})} = \|\mathsf{F}\|_{H^2(\mathbb{D},\mathscr{Y})}\| \leq \|\mathsf{G}\|_{H^\infty(\mathbb{D},\mathcal{L}(\mathscr{U},\mathscr{Y}))} \|u\|_{\mathscr{U}},$$

where we have used Lemmas A.2 and A.4. It is easily seen that $CA^nB = F_n$ for all $n \in \mathbb{Z}^+$, from which it follows that G is the transfer function of the discrete-time system with system operator $[A, B; C, 0]$.

If $\mathsf{G} \in H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$, but $\mathsf{G}(0) \neq 0$, then by applying the above to $\mathsf{G}(z) - \mathsf{G}(0)$ we see that the discrete-time system with system operator $[A, B; C, \mathsf{G}(0)]$ has the transfer function G.

If G is holomorphic at zero, then there exists a $r > 0$ such that $\mathsf{G}_r(z) := \mathsf{G}(rz)$ is in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$. Applying the above to G_r we obtain a realization $[A_r, B_r; C_r, D_r]$ of G_r. Define $[A, B; C, D] := [A_r/r, B_r; C_r/r, D_r]$, then it is easily seen that this is a realization of G. □

Remark 2.13. The realization constructed in the proof of Proposition 2.12 for a function in $H^\infty(\mathbb{D}, \mathcal{L}(\mathscr{U}, \mathscr{Y}))$ is called the **backward shift realization**. If we define

$$\mathscr{X}_{\min} := \overline{\operatorname{span}}\{A^n Bu : u \in \mathscr{U}, n \geq 0\},$$

then obviously the system operator (see (2.2)) restricts to a bounded operator from $[\mathscr{X}_{\min}; \mathscr{U}]$ to $[\mathscr{X}_{\min}; \mathscr{Y}]$ and the resulting discrete-time system is approximately controllable and approximately observable and has the same transfer function as the backward shift realization. This realization is called the **restricted backward shift realization**. We will denote its system operator by S^{rs} and we will denote its components similarly.

Remark 2.14. In general, a function has infinitely many realizations. If $[A, B; C, D]$ is a realization of G and $S \in \mathcal{L}(\mathscr{X})$ has a bounded inverse, then $[SAS^{-1}, SB; CS^{-1}, D]$ is also a realization of G. In fact there are many more realizations. For a reasonably complete discussion of realization theory we refer to Staffans [89, Chapter 9].

Since in operator theory the adjoint (also known as conjugate or dual) of an operator plays an essential role, it might not come as a surprise that the following dual system plays an important role in systems theory.

Definition 2.15. The **dual system** of a discrete-time system with system operator S is the discrete-time system with system operator S^*.

Note that the state space of the dual system equals the state space of the system itself, but that the input and output spaces are interchanged.

The following results show how the holomorphic functions of a system are related to those associated with its dual system. To formulate this we need the following notation: let $f : \Lambda \subset \mathbb{C} \to \mathcal{L}(\mathscr{H}_1, \mathscr{H}_2)$ where $\mathscr{H}_1$ and $\mathscr{H}_2$ are separable Hilbert spaces, then $f^\dagger : \overline{\Lambda} \to \mathcal{L}(\mathscr{H}_2, \mathscr{H}_1)$ is defined by $f^\dagger(s) := f(\bar{s})^*$.

Proposition 2.16. *The resolvent, the wave functions and the characteristic function of the dual system satisfy*

$$\left[\begin{array}{c|c} \mathfrak{A}_{\mathrm{dual}} & \mathfrak{B}_{\mathrm{dual}} \\ \hline \mathfrak{C}_{\mathrm{dual}} & \mathfrak{D}_{\mathrm{dual}} \end{array}\right] = \left[\begin{array}{c|c} \mathfrak{A}^\dagger & \mathfrak{C}^\dagger \\ \hline \mathfrak{B}^\dagger & \mathfrak{D}^\dagger \end{array}\right].$$

Proof. This follows easily from the definitions. □

Proposition 2.17. *The state function, input function, output function and transfer function of the dual system are given by*

$$\left[\begin{array}{c|c} \mathsf{A}_{\mathrm{dual}} & \mathsf{B}_{\mathrm{dual}} \\ \hline \mathsf{C}_{\mathrm{dual}} & \mathsf{D}_{\mathrm{dual}} \end{array}\right] = \left[\begin{array}{c|c} \mathsf{A}^\dagger & \mathsf{C}^\dagger \\ \hline \mathsf{B}^\dagger & \mathsf{D}^\dagger \end{array}\right].$$

Proof. This follows easily from the definitions. □

Definition 2.18. The **series interconnection** of the system Σ_1 and the system Σ_2 is defined when $\mathscr{Y}_1 = \mathscr{U}_2$ by its system operator

$$S_{\mathrm{series}} = \left[\begin{array}{c|c} A_{\mathrm{series}} & B_{\mathrm{series}} \\ \hline C_{\mathrm{series}} & D_{\mathrm{series}} \end{array}\right] := \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ B_2C_1 & A_2 & B_2D_1 \\ \hline D_2C_1 & C_2 & D_2D_1 \end{array}\right].$$

Proposition 2.19. *The resolvent, the wave functions and the characteristic function of the series interconnection satisfy*

$$\left[\begin{array}{c|c} \mathfrak{A}_{\mathrm{series}} & \mathfrak{B}_{\mathrm{series}} \\ \hline \mathfrak{C}_{\mathrm{series}} & \mathfrak{D}_{\mathrm{series}} \end{array}\right] = \left[\begin{array}{cc|c} \mathfrak{A}_1 & 0 & \mathfrak{B}_1 \\ \mathfrak{B}_2\mathfrak{C}_1 & \mathfrak{A}_2 & \mathfrak{B}_2\mathfrak{D}_1 \\ \hline \mathfrak{D}_2\mathfrak{C}_1 & \mathfrak{C}_2 & \mathfrak{D}_2\mathfrak{D}_1 \end{array}\right].$$

Proof. This is an easy calculation. □

Proposition 2.20. *The state function, input function, output function and transfer function of the series interconnection are given by*

$$\left[\begin{array}{c|c} \mathsf{A}_{\mathrm{series}} & \mathsf{B}_{\mathrm{series}} \\ \hline \mathsf{C}_{\mathrm{series}} & \mathsf{D}_{\mathrm{series}} \end{array}\right] = \left[\begin{array}{cc|c} \mathsf{A}_1 & 0 & \mathsf{B}_1 \\ \mathsf{B}_2\mathsf{C}_1 & \mathsf{A}_2 & \mathsf{B}_2\mathsf{D}_1 \\ \hline \mathsf{D}_2\mathsf{C}_1 & \mathsf{C}_2 & \mathsf{D}_2\mathsf{D}_1 \end{array}\right].$$

Proof. This is an easy calculation. □

Lemma 2.21. *A realization of the transfer function of the series interconnection of Σ_1 and Σ_2 is given by the system operator*

$$\left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ A_2 + B_2C_1 - A_1 & A_2 & B_2D_1 - B_1 \\ \hline D_2C_1 + C_2 & C_2 & D_2D_1 \end{array}\right].$$

Proof. This follows from applying the state space transformation

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$$

to the system operator of the series interconnection given in Definition 2.18. □

Proposition 2.22. *Let $\mathsf{G} : D(\mathsf{G}) \subset \mathbb{C} \to \mathcal{L}(\mathscr{H})$ be holomorphic with $0 \in D(\mathsf{G})$. Let $[A, B; C, D]$ be a realization of G and assume that D is boundedly invertible. Then $\mathsf{G}(z)$ is invertible in a neighbourhood of zero and the inverse of G has a realization $[A - BD^{-1}C, BD^{-1}; -D^{-1}C, D^{-1}]$.*

Proof. This follows from writing down realizations of the transfer functions of the series interconnection of $[A, B; C, D]$ and $[A - BD^{-1}C, BD^{-1}; -D^{-1}C, D^{-1}]$ in both orders using Lemma 2.21. Since these are both equal to the identity the result follows. □

Proposition 2.23. *Let $\check{\Sigma}$ be a discrete-time system with input space $\mathscr{U}$ and output space $\mathscr{U} \times \mathscr{Y}$. Denote its transfer function by $[\check{\mathsf{D}}_1; \check{\mathsf{D}}_2]$. Assume that $\check{D}_1$ has a bounded inverse. Define the discrete-time system Σ by its system operator:*

$$S = \left[\begin{array}{c|c} \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1 & \check{B}\check{D}_1^{-1} \\ \hline \check{C}_2 - \check{D}_2\check{D}_1^{-1}\check{C}_1 & \check{D}_2\check{D}_1^{-1} \end{array}\right]. \tag{2.5}$$

Then the transfer function D of Σ satisfies $\mathsf{D}(z) = \check{\mathsf{D}}_2(z)\check{\mathsf{D}}_1(z)^{-1}$.

Proof. The operator $[\check{A}, \check{B}; \check{C}_2, \check{D}_2])$ is a realization of $\check{\mathsf{D}}_2$. A realization of $\check{\mathsf{D}}_1^{-1}$ can be obtained from Proposition 2.22. A realization of the transfer function of their series interconnection is provided by Lemma 2.21 as

$$\left[\begin{array}{cc|c} \check{A} - \check{B}\check{D}_1^{-1}\check{C}_1 & 0 & \check{B}\check{D}_1^{-1} \\ 0 & \check{A} & 0 \\ \hline \check{C}_2 - \check{D}_2\check{D}_1^{-1}\check{C}_1 & \check{C}_2 & \check{D}_2\check{D}_1^{-1} \end{array}\right].$$

It follows from Proposition 2.20 that this system is a realization of $\check{\mathsf{D}}_2\check{\mathsf{D}}_1^{-1}$. It is easily seen that the transfer function of this system equals that of Σ. □

Notes

The concept of dynamical system as defined in Definition 2.1 is taken from Polderman and Willems [76]. Discrete-time systems have been studied for quite some time. The first account in book form of the infinite-dimensional case seems to be Fuhrmann [31]. Chapter 12 of Staffans [89] contains some more recent developments. Example 2.8 is adapted from Curtain and Zwart [18, Example 4.3.8]. The backward shift realization mentioned in Remark 2.13 is due to Fuhrmann [30] and Helton [37].