On the role of dislocations in fatigue crack initiation

Brinckmann, Steffen

Appendix: Image force of a dislocation in half-space

In chapter 2 the analytical expression of Freund [30] for a dislocation in half-space was divided into the infinite space solution and the expression representing the effect of the free surface, i.e. image stress. This derivation is explained here. The definitions are given in figure 1.

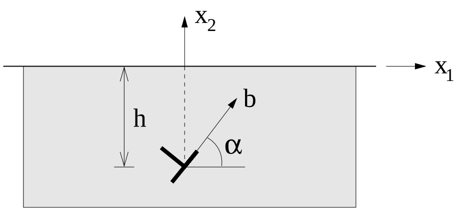

Figure 1: Coordinate system used for the elastic fields of a dislocation in half-space close to a free surface by Freund [30].

The complex coordinate ζ is defined as $x_1 + ix_2$. The complex notation for the Burgers vector is given by $b = b_{x_1} + ib_{x_2}$. In this appendix we will denote the half-space solution with σ, infinite space solution by Σ and the free-surface image stress by $\breve{\sigma}$:

$$\sigma = \Sigma + \breve{\sigma}$$

Freund [30] gave the following equations for the stress field of a dislocation in infinite space:

$$\frac{1}{2}(\Sigma_{11} + \Sigma_{22}) = 2\Re(\Phi'(\zeta)) \tag{1}$$

$$\frac{1}{2}(\Sigma_{11} + \Sigma_{22}) + i\Sigma_{xy} = \bar{\zeta}\Phi''(\zeta) + \Psi'(\zeta) \tag{2}$$

where

$$\Phi'(\zeta) = \frac{\mu}{4\pi(1-\nu)}\frac{ib}{\zeta+ih}$$

$$\Psi'(\zeta) = -\frac{\mu}{4\pi(1-\nu)}\left[\frac{i\bar{b}}{\zeta+ih}+\frac{bh}{(\zeta+ih)^2}\right]$$

For the half-space,

$$\sigma_{22}-i\sigma_{12} = \varphi'(\zeta)-\varphi'(\bar{\zeta})+(\zeta-\bar{\zeta})\overline{\varphi''(\zeta)} \quad (3)$$

$$\sigma_{11}+i\sigma_{12} = \varphi'(\zeta)+\varphi'(\bar{\zeta})-(\zeta-\bar{\zeta})\overline{\varphi''(\zeta)}+2\overline{\varphi'(\zeta)} \quad (4)$$

where, as mentioned in chapter 2,

$$\varphi'(\zeta) = \frac{\mu}{4\pi(1-\nu)}\left(\frac{ib}{\zeta+ih}-\frac{i\bar{b}(\zeta+ih)}{(\zeta-ih)^2}-\frac{i(b-\bar{b})}{\zeta-ih}\right)$$

$$\varphi''(\zeta) = \frac{\mu}{4\pi(1-\nu)}\left(-\frac{ib}{(\zeta+ih)^2}+2\frac{i\bar{b}(\zeta+ih)}{(\zeta-ih)^3}+\frac{i(b-2\bar{b})}{(\zeta-ih)^2}\right)$$

From (3) and (4) one gets

$$\frac{1}{2}(\sigma_{11}+\sigma_{22}) = 2\Re(\varphi'(\zeta)) \quad (5)$$

which is similar to equation (1) of the infinite space solution. Also

$$\frac{1}{2}(\sigma_{22}-\sigma_{11})+i\sigma_{xy} = \bar{\zeta}\varphi''(\zeta)-\left(\overline{\varphi'(\bar{\zeta})}+\varphi'(\zeta)+\zeta\varphi''(\zeta)\right) \quad (6)$$

Based on comparing this expression to the infinite space solution (2), the last part in the parenthesis will be called $-\psi'(\zeta)$.

We continue to identify the free surface image fields. These can be calculated form the difference of the infinite space and the half-space solution. Therefore, we have to identify $\psi'(\zeta)$ as

$$\begin{aligned}\psi'(\zeta) &= \zeta\varphi''(\zeta)+\varphi'(\zeta)+\overline{\varphi'(\bar{\zeta})}\\ &= \frac{\mu}{4\pi(1-\nu)}\left[\frac{i\bar{b}}{\zeta+ih}+\frac{bh}{(\zeta+ih)^2}-\frac{ib}{\zeta-ih}+\frac{i(b\zeta-3\bar{b}\zeta-i\bar{b})}{(\zeta-ih)^2}+\frac{2i\bar{b}\zeta(\zeta+ih)}{(\zeta-ih)^3}\right]\\ &= -\Psi'+\frac{\mu}{4\pi(1-\nu)}\left[-\frac{ib}{\zeta-ih}+\frac{i(b\zeta-3\bar{b}\zeta-i\bar{b})}{(\zeta-ih)^2}+\frac{2i\bar{b}\zeta(\zeta+ih)}{(\zeta-ih)^3}\right] \quad (7)\end{aligned}$$

If we use the separation of the function φ of the half-space into an infinite space function Φ and a free-surface solution $\check{\varphi}$,

$$\varphi'(\zeta) = \Phi'(\zeta)+\check{\varphi}'(\zeta). \quad (8)$$

Equation (7) leads to

$$\zeta\varphi''(\zeta) + \varphi'(\zeta) + \overline{\varphi'(\bar{\zeta})} = -\Psi(\zeta)' + \overline{\Phi'(\bar{\zeta})} + \zeta\breve{\varphi}''(\zeta) + \breve{\varphi}'(\zeta) \tag{9}$$

Finally we get with equations (2), (6), (8) and (9):

$$\begin{aligned} \frac{1}{2}(\sigma_{11} + \sigma_{22}) + i\sigma_{12} &= \frac{1}{2}(\Sigma_{11} + \Sigma_{22}) + i\Sigma_{12} + \frac{1}{2}(\breve{\sigma}_{11} + \breve{\sigma}_{22}) + i\breve{\sigma}_{12} \\ &= \bar{\zeta}\Phi''(\zeta) + \bar{\zeta}\breve{\varphi}''(\zeta) + \Psi' - \left[\overline{\Phi'(\bar{\zeta})} + \zeta\breve{\varphi}''(\zeta) + \breve{\varphi}'(\zeta)\right] \end{aligned} \tag{10}$$

Therefore, with equation (2) we get for the free surface fields

$$\begin{aligned} \frac{1}{2}(\breve{\sigma}_{11} + \breve{\sigma}_{22}) + i\breve{\sigma}_{12} &= \bar{\zeta}\breve{\varphi}''(\zeta) - \left[\overline{\Phi'(\bar{\zeta})} + \zeta\breve{\varphi}''(\zeta) + \breve{\varphi}'(\zeta)\right] \\ &= (\bar{\zeta} - \zeta)\breve{\varphi}''(\zeta) - \overline{\Phi'(\bar{\zeta})} - \breve{\varphi}'(\zeta) \end{aligned} \tag{11}$$

Rewriting this equation in the style of the half-space solution, one ends up with

$$\begin{aligned} \breve{\sigma}_{22} - i\breve{\sigma}_{12} &= \breve{\varphi}'(\zeta) - \Phi'(\bar{\zeta}) + (\zeta - \bar{\zeta})\overline{\breve{\varphi}''(\zeta)} \\ \breve{\sigma}_{11} + i\breve{\sigma}_{12} &= \breve{\varphi}'(\zeta) + \Phi'(\bar{\zeta}) + 2\overline{\breve{\varphi}'(\zeta)} - (\zeta - \bar{\zeta})\overline{\breve{\varphi}''(\zeta)} \end{aligned} \tag{12}$$

However, it is more convenient to separate the function φ':

$$\begin{aligned} \varphi'(\zeta) &= \frac{\mu}{4\pi(1-\nu)}\left(-\frac{i\bar{b}(\zeta + ih)}{(\zeta - ih)^2} - \frac{i(b - \bar{b})}{\zeta - ih}\right) \\ \varphi''(\zeta) &= \frac{\mu}{4\pi(1-\nu)}\left(2\frac{i\bar{b}(\zeta + ih)}{(\zeta - ih)^3} + \frac{i(b - 2\bar{b})}{(\zeta - ih)^2}\right) \end{aligned}$$

This analytic expression determines the elastic fields of the free surface on the dislocation.

Now we calculate the stress on the dislocation itself. We evaluate equation (12) at the dislocation $\zeta = -ih$:

$$\begin{aligned} \breve{\sigma}_{22} - i\breve{\sigma}_{12} &= -\frac{\mu}{4\pi(1-\nu)}\frac{b}{h} \\ \breve{\sigma}_{11} + i\breve{\sigma}_{12} &= \frac{\mu}{4\pi(1-\nu)}\frac{b}{h} \end{aligned}$$

and calculate with the definition of the Burgers vector, as shown in figure 1 the shear component:

$$\breve{\sigma}_{12} = \frac{\mu}{4\pi(1-\nu)}\frac{|b|}{h}\sin\alpha \tag{13}$$

where α is the angle of the dislocation with respect to the surface. When we rewrite equation (13) in terms of the distance $s = h/\sin\alpha$ from the dislocation to the surface along the slip plane, we recover the image stress, as it was given by Hirth and Lothe [31] for a dislocation perpendicular to the free surface:

$$\check{\sigma}_{12} = \frac{\mu}{4\pi(1-\nu)}\frac{|b|}{s}$$