Expansion and contraction of finite states

Tamminga, Allard

Allard Tamminga

Expansion and Contraction of Finite States[*]

Abstract. We present a theory that copes with the dynamics of inconsistent information. A method is set forth to represent possibly inconsistent information by a *finite state*. Next, finite operations for expansion and contraction of finite states are given. No extra-logical element – a choice function or an ordering over (sets of) sentences – is presupposed in the definition of contraction. Moreover, expansion and contraction are each other's duals. AGM-style characterizations of these operations follow.

Keywords: belief change, belief contraction, paraconsistent logic, first degree entailment.

1. Introduction

For the construction of philosophically satisfactory and computationally manageable belief change systems, three interrelated problems have to be addressed. Let us discuss them briefly. First, *infinite constructions* abound in the classical belief change literature: epistemic states are usually represented by deductively closed belief sets and, hence, operations of change are understood as transitions from one deductively closed set to another. Obviously, the appeal to infinite constructions is a serious hindrance to implementations of belief change systems.

Second, it is a widely held assumption that epistemic states are to be *consistent*,[1] since epistemic states are usually represented by deductively closed sets and since the underlying logic of almost all belief change systems is (supra)classical.[2] Thus, classical belief change systems unrealistically assume that our beliefs are inconsistent, only if we believe everything.

Third, *extra-logical elements*, such as choice functions or orderings over (sets of) sentences, were introduced to avoid triviality of belief change operations. Hence, in cases where there is no clue to the peculiarities of these extra-logical elements, classical systems are at a loss. Moreover, it is far from

clear how these extra-logical elements have to be adapted if an operation of change has been conducted. Therefore, iterated operations of belief change have as yet defied any straightforward solution.[3]

In the present paper, a system is developed that offers a solution to these three problems. Our belief change system uses finite representations of possibly inconsistent information states. Moreover, expansion and contraction are finite operations of change. Last, the system can do without, but does not rule out, extra-logical elements, so that iterated belief change loses its problematic character.

To set the stage, we distinguish *information* from *beliefs*. On the one hand, we shall set forth interrelated techniques for representing, expanding, and contracting *information*. Information may, of course, be inconsistent. Henceforth, the devices representing information can contain contradictory and even inconsistent sentences. On the other hand, operations must be defined to extract *beliefs* from the represented information.

Hansson makes a case for such a two-fold approach:

> [T]he dynamics of belief states involves two major types of operations. One is *operations of change*, transformations from one belief state to another. [...] The other major type can be called *operations of retrieval*. The task of such an operation is to find, for a given belief state, the set of sentences to which the agent has a certain epistemic attitude. (Hansson [11], p. 125)

The asymmetry between information and belief mirrors the fact that it is in our power to aspire to consistent beliefs (and perhaps even to attain them),[4] though the consistency of the information on which these beliefs are grounded is beyond our control: it is pointless to prescribe that information be consistent. Throughout the paper, we assume that all information incorporated in our present information state is on equal footing, that is, there is no a priori reason to assume that the sources from which the information stemmed were not equally reliable. In due course, it might turn out that (a part of) already accepted information was unreliable. Then, a contraction of this unreliable information from our information state is justified.

---

[3] All classical contraction functions, such as partial meet contraction, safe contraction, and contraction based on epistemic entrenchment, depend on an extra-logical element, such as a selection function or an ordering of the formulas in a belief set. The chief argument for adopting an extra-logical element is the fact that the only contraction functions defined by logical means alone, such as *full meet contraction* (Alchourrón, Gärdenfors and Makinson [1], p. 512) or *full meet Levi contraction* (Hansson and Olsson [12], p. 113), give rise to a trivial operation.

[4] Compare Rescher and Brandom [16], p. 25.

The present paper provides a method for finitely representing information states and two finite operations of change: expansion and contraction. As a starting point, we use propositional logic. Extensions to predicate logic have not been thoroughly studied yet. Operations of belief retrieval from possibly inconsistent information belong primarily to the field of reasoning from inconsistent information. Hence, we postpone a discussion of these operations to another paper, though the reader may find four proposals for retrieving a consistent set of beliefs from a finite state in Tamminga [19], p. 81–88.

2. Preliminaries

2.1. First degree entailment

Our basic logic will be first degree entailment (fde), which pertains to implications of the form $\phi \to \psi$, where ϕ and ψ are truth-functional, not containing any implications themselves. By Anderson and Belnap [2], this logic has been defined proof-theoretically. Later, formal semantics were provided by Routley and Routley [18], who propounded a two-valued semantics, and by Dunn [7], who propounded a four-valued semantics. Here, Dunn's four-valued semantics will be used. For a system of natural deduction for fde, the reader may have recourse to Tamminga and Tanaka [20].

Throughout the paper, we use a standard propositional language, denoted by $\mathfrak{F}$, built from an infinite set $\mathfrak{V} = \{p_1, p_2, p_3, \ldots\}$ of propositional variables using the connectives $\wedge$, $\vee$ and $\neg$. The usual formation rules apply. *Literals* are propositional variables and their negations.

Our basic semantics is just a rewriting of Dunn's four-valued semantics for fde [7]. In Dunn's sense, a valuation is a map $\nu : \mathfrak{V} \mapsto \wp(\{\text{TRUE}, \text{FALSE}\})$ from the set of propositional variables to subsets of the set of truth-values TRUE and FALSE. Hence, a propositional variable p can have both truth-values, only one truth-value, and no truth-value. In the present paper, we shall use an alternative, but equivalent definition.

DEFINITION 2.1. Let x be a set of literals. Then x is a *valuation*.

A valuation x defines a map $\nu : \mathfrak{V} \mapsto \wp(\{\text{TRUE}, \text{FALSE}\})$ as follows: $p \in x$ if and only if $\text{TRUE} \in \nu(p)$, and $\neg p \in x$ if and only if $\text{FALSE} \in \nu(p)$. As a consequence, given the valuation $\{p, \neg p, q\}$, it holds that $\nu(p) = \{\text{TRUE}, \text{FALSE}\}$, $\nu(q) = \{\text{TRUE}\}$, and $\nu(r) = \emptyset$ for all other propositional variables r.

The assigment of truth-values to literals is extended to all formulas of the language as follows:

DEFINITION 2.2 (Semantical Rules). Let x be a valuation. Then

(i)	$x \models \lambda$	iff	$\lambda \in x$,	if λ is a literal
(ii)	$x \models \phi \wedge \psi$	iff	$x \models \phi$ and $x \models \psi$	
(iii)	$x \models \phi \vee \psi$	iff	$x \models \phi$ or $x \models \psi$	
(iv)	$x \models \neg(\phi \wedge \psi)$	iff	$x \models \neg\phi$ or $x \models \neg\psi$	
(v)	$x \models \neg(\phi \vee \psi)$	iff	$x \models \neg\phi$ and $x \models \neg\psi$	
(vi)	$x \models \neg\neg\phi$	iff	$x \models \phi$.	

For instance, $\{p, \neg p, q\} \models p \wedge \neg p$, but $\{p, \neg p, q\} \not\models q \wedge \neg q$. Hence, fde keeps inconsistencies *local*. An inconsistency does not lead to triviality, in the sense that from an inconsistency everything follows.

DEFINITION 2.3 (Validity). Let $\phi, \psi \in \mathfrak{F}$. Then

$$\phi \models \psi \quad \text{iff} \quad \forall x(x \models \phi \to x \models \psi).$$

2.2. Minimal valuations

Every formula ϕ in $\mathfrak{F}$ defines a set of minimal valuations satisfying that formula: the set of ϕ-minimal valuations. A valuation x is ϕ-minimal, if $x \models \phi$ and for all valuations y such that $y \subset x$ it holds that $y \not\models \phi$. Hence, both $\{p, \neg q\}$ and $\{q\}$ are $(p \wedge \neg q) \vee q$-minimal valuations.

DEFINITION 2.4 (Min and Max). Let $\mathcal{X}$ be a set of valuations. Then

$$\begin{aligned} \min(\mathcal{X}) &= \{x \in \mathcal{X} : \forall y(y \in \mathcal{X} \to y \not\subset x)\} \\ \max(\mathcal{X}) &= \{x \in \mathcal{X} : \forall y(y \in \mathcal{X} \to y \not\supset x)\}. \end{aligned}$$

DEFINITION 2.5 (ϕ-Minimal Valuations). Let $\phi \in \mathfrak{F}$. Then *the set of ϕ-minimal valuations*, denoted by $[\![\phi]\!]$, is defined to be

$$[\![\phi]\!] \quad = \quad \min(\{x : x \models \phi\}).$$

FACT 2.6. *Let $\phi \in \mathfrak{F}$. Then*

$$x \models \phi \quad \text{iff} \quad \exists y(y \subseteq x \text{ and } y \in [\![\phi]\!]).$$

LEMMA 2.7 (Extensionality). *Let $\phi, \psi \in \mathfrak{F}$. Then*

$$[\![\phi]\!] = [\![\psi]\!] \quad \text{iff} \quad \phi \models \psi \text{ and } \psi \models \phi.$$

2.3. An algorithm for finding ϕ's minimal valuations

DEFINITION 2.8 (Sum and Product). Let $\phi, \psi \in \mathfrak{F}$. Then the *sum of* $[\![\phi]\!]$ *and* $[\![\psi]\!]$, denoted by $[\![\phi]\!] \oplus [\![\psi]\!]$, and the *product of* $[\![\phi]\!]$ *and* $[\![\psi]\!]$, denoted by $[\![\phi]\!] \otimes [\![\psi]\!]$, are defined to be

(i) $[\![\phi]\!] \oplus [\![\psi]\!]$ $=$ $\mathsf{min}([\![\phi]\!] \cup [\![\psi]\!])$
(ii) $[\![\phi]\!] \otimes [\![\psi]\!]$ $=$ $\mathsf{min}(\{x \cup y : x \in [\![\phi]\!] \text{ and } y \in [\![\psi]\!]\})$.

    The algorithm is based on the following Deconstruction Rules. We prove the correctness of the rules immediately. A direct definition of $[\![\neg\phi]\!]$ in terms of some set-theoretical operation on $[\![\phi]\!]$ is avoided by splitting cases according to the main connective of the negated formula.

LEMMA 2.9 (Deconstruction Rules). *Let* $\phi, \psi \in \mathfrak{F}$. *Then*

(i) $[\![\lambda]\!]$ $=$ $\{\{\lambda\}\}$,   *if* λ *is a literal*
(ii) $[\![\phi \wedge \psi]\!]$ $=$ $[\![\phi]\!] \otimes [\![\psi]\!]$
(iii) $[\![\phi \vee \psi]\!]$ $=$ $[\![\phi]\!] \oplus [\![\psi]\!]$
(iv) $[\![\neg(\phi \wedge \psi)]\!]$ $=$ $[\![\neg\phi]\!] \oplus [\![\neg\psi]\!]$
(v) $[\![\neg(\phi \vee \psi)]\!]$ $=$ $[\![\neg\phi]\!] \otimes [\![\neg\psi]\!]$
(vi) $[\![\neg\neg\phi]\!]$ $=$ $[\![\phi]\!]$.

PROOF. (i) is obvious.

    (ii) Suppose that $x \in [\![\phi \wedge \psi]\!]$. Then $x \models \phi$ and $x \models \psi$. Then, by Fact 2.6, there are $x_1 \in [\![\phi]\!]$ and $x_2 \in [\![\psi]\!]$ such that $x_1 \subseteq x$ and $x_2 \subseteq x$. Obviously, $x_1 \cup x_2 \subseteq x$. Now, suppose that $x_1 \cup x_2 \subset x$. Since $x_1 \cup x_2 \models \phi \wedge \psi$, we have that $x \notin [\![\phi \wedge \psi]\!]$. Contradiction. Therefore, there are $x_1 \in [\![\phi]\!]$ and $x_2 \in [\![\psi]\!]$ such that $x_1 \cup x_2 = x$. It remains to be shown that x is an element of the product of $[\![\phi]\!]$ and $[\![\psi]\!]$. Suppose it is not. Then there are $y_1 \in [\![\phi]\!]$ and $y_2 \in [\![\psi]\!]$, such that $y_1 \cup y_2 \subset x$. Since $y_1 \cup y_2 \models \phi \wedge \psi$, we have that $x \notin [\![\phi \wedge \psi]\!]$. Contradiction. Hence, $x \in [\![\phi]\!] \otimes [\![\psi]\!]$. Therefore, $[\![\phi \wedge \psi]\!] \subseteq [\![\phi]\!] \otimes [\![\psi]\!]$.

    Suppose that $x \in [\![\phi]\!] \otimes [\![\psi]\!]$. Then there are $x_1 \in [\![\phi]\!]$ and $x_2 \in [\![\psi]\!]$, such that $x = x_1 \cup x_2$. Obviously, $x \models \phi \wedge \psi$. Suppose that $x \notin [\![\phi \wedge \psi]\!]$. Then there is a y such that $y \models \phi \wedge \psi$ and $y \subset x$. Hence, by Fact 2.6, there are $y_1 \in [\![\phi]\!]$ and $y_2 \in [\![\psi]\!]$ such that $y_1 \subseteq y$ and $y_2 \subseteq y$. Obviously, $y_1 \cup y_2 \subseteq y \subset x$. Then $x \notin [\![\phi]\!] \otimes [\![\psi]\!]$. Contradiction. Hence, $x \in [\![\phi \wedge \psi]\!]$. Therefore, $[\![\phi]\!] \otimes [\![\psi]\!] \subseteq [\![\phi \wedge \psi]\!]$.

    (iii) Suppose that $x \in [\![\phi \vee \psi]\!]$. Then $x \models \phi \vee \psi$ and $\forall y(y \subset x \rightarrow y \not\models \phi \vee \psi)$, that is, $\forall y(y \subset x \rightarrow y \not\models \phi)$ and $\forall y(y \subset x \rightarrow y \not\models \psi)$. Then $x \in [\![\phi]\!] \cup [\![\psi]\!]$. It remains to be shown that x is in the sum of $[\![\phi]\!]$ and $[\![\psi]\!]$.

Suppose it is not. Then there is a $y \in [\![\phi]\!] \cup [\![\psi]\!]$, such that $y \subset x$. Since $y \models \phi \vee \psi$, we have that $x \notin [\![\phi \vee \psi]\!]$. Contradiction. Hence, $x \in [\![\phi]\!] \oplus [\![\psi]\!]$. Therefore, $[\![\phi \vee \psi]\!] \subseteq [\![\phi]\!] \oplus [\![\psi]\!]$.

Suppose that $x \in [\![\phi]\!] \oplus [\![\psi]\!]$. Then $x \in [\![\phi]\!]$ or $x \in [\![\psi]\!]$. Let us split cases.

Case 1: Suppose that $x \in [\![\phi]\!]$. Then $x \models \phi \vee \psi$. Suppose $x \notin [\![\phi \vee \psi]\!]$. Then there is a y, such that $y \models \phi \vee \psi$ and $y \subset x$. As x is ϕ-minimal, it must be that $y \not\models \phi$. Hence, $y \models \psi$. Then, by Fact 2.6, there is a $y_1 \in [\![\psi]\!]$, such that $y_1 \subseteq y$. Obviously, $y_1 \subseteq y \subset x$. Since $y_1 \in [\![\phi]\!] \cup [\![\psi]\!]$, we have that $x \notin [\![\phi]\!] \oplus [\![\psi]\!]$. Contradiction. Hence, $x \in [\![\phi \vee \psi]\!]$. Therefore, $[\![\phi]\!] \oplus [\![\psi]\!] \subseteq [\![\phi \vee \psi]\!]$.

Case 2: Analogous to the previous case.

(iv), (v) and (vi) follow from the fact that the De Morgan rules and the Law of Double Negation hold for fde, from Lemma 2.7, and from (ii) and (iii). ∎

DEFINITION 2.10 (Minimal Valuations Algorithm). Let $\phi \in \mathfrak{F}$. Then the *Minimal Valuations Algorithm applied to* ϕ is defined as follows:

1. Put ϕ between double brackets. Then apply Deconstruction Rules (ii), (iii), (iv), (v), and (vi), until no further application of one of these Deconstruction Rules is possible. Use brackets, in order to avoid confusion.

2. Apply Deconstruction Rule (i), and solve, bottom up, the operations $\otimes$ and $\oplus$ according to their definitions, until all occurrences of $\otimes$ and $\oplus$ have been treated.

THEOREM 2.11. *Let $\phi \in \mathfrak{F}$. Then the Minimal Valuations Algorithm applied to ϕ generates exactly all ϕ-minimal valuations.*

PROOF. By structural induction on ϕ. Use Lemma 2.9. ∎

3. Finite States

In standard epistemic logic, an agent's information is represented by a *state*: the set of all possible worlds that are consistent with the agent's information, where a possible world is a *consistent* and *total* valuation. For our present purposes, such an approach would be inappropriate.

First, differences between inconsistent sets of information, for example between $\{p, \neg p, q\}$ and $\{p, q, \neg q\}$, can not be accounted for in a standard possible worlds setting, as it can only represent inconsistent information by the empty state.

Second, in a standard approach using total valuations, partial information is usually represented by an infinite number of infinite valuations, if the underlying language has been built from an infinite set of propositional variables. Total valuations, however, not only validate the information to be represented, but are opinioned on every other formula of the language as well, regardless of its relevance to the represented information. Hence, a distinction that is crucial to our characterization of contraction can not be made in a standard setting: let w be a standard possible world in some information representing state. Suppose that neither of the state nor of the represented information further particularities are given. Then it is impossible to discriminate between (a) those truth assigments in w that are essential for validating the represented information and (b) those truth assignments in w that are *in*essential for validating the represented information.

Instead, we shall represent an agent's information by a *finite state*: a set of possibly inconsistent and partial valuations. As an agent's information will allways be finite, we shall only need finitely many finite valuations to represent it. Moreover, economy of representation is ensured by ruling out redundant valuations:[5]

DEFINITION 3.1 (Finite State). Let $\mathcal{K}$ be a set of valuations. Then $\mathcal{K}$ *is a finite state*, if

   (i) $\mathcal{K} \neq \emptyset$,
   (ii) $\mathcal{K}$ is finite,
  (iii)   every x in $\mathcal{K}$ is finite,
  (iv) $\forall x \forall y ((x \in \mathcal{K}$ *and* $y \in \mathcal{K}) \rightarrow x \not\subset y)$.

If $\mathcal{K} = \{\emptyset\}$, then $\mathcal{K}$ is *trivial*.

Note that a finite state can not be empty. In case we do not have any information at all, the finite state is *trivial* and should not impose any constraint on the choice of minimal valuations validating the incoming information. This situation is represented adequately by the trivial finite state $\{\emptyset\}$. Next to nothing is needed to represent nothing. In a standard approach, though, an infinite number of infinite valuations is needed to represent nothing, as the epistemic state of total ignorance is represented by the set of all possible worlds.

THEOREM 3.2. *Let* $\phi \in \mathfrak{F}$. *Then* $[\![\phi]\!]$ *is a finite state.*

PROOF. By structural induction on ϕ. Use Lemma 2.9. ■

---

[5] This definition is a refinement of the epistemic states in Anderson, Belnap and Dunn [4], p. 524–527. Anderson and Belnap [3], p. 169–170, discuss the differences between sets of consistent and total valuations and sets of possibly inconsistent and partial valuations.

3.1. A finite state's characteristic MDNF

In fde, every formula has a minimal disjunctive normal form (MDNF), denoted by $\bigvee_{i=1}^{n} \bigwedge x_i$, in which (a) every x_i is a finite set of literals and (b) $x_i \not\subseteq x_j$, if $i \neq j$.[6] Hence, $\{x_1, \ldots, x_n\}$ is a finite state. Conversely, every non-trivial finite state can be characterized by a MDNF:

DEFINITION 3.3. Let $\mathcal{K} = \{x_1, \ldots, x_n\}$ be a non-trivial finite state. Then the *characteristic MDNF of* $\mathcal{K}$, denoted by $\mathfrak{f}(\mathcal{K})$, is defined to be

$$\mathfrak{f}(\mathcal{K}) \quad = \quad \bigvee_{i=1}^{n} \bigwedge x_i.$$

FACT 3.4. *Let* $\mathcal{K}$ *be a non-trivial finite state. Then*

$$y \models \mathfrak{f}(\mathcal{K}) \quad \textit{iff} \quad \exists x (x \in \mathcal{K} \textit{ and } x \subseteq y).$$

THEOREM 3.5 (Characterization). *Let* $\mathcal{K}$ *be a non-trivial finite state. Then*

$$\llbracket \mathfrak{f}(\mathcal{K}) \rrbracket \quad = \quad \mathcal{K}.$$

PROOF. Suppose that $x \in \llbracket \mathfrak{f}(\mathcal{K}) \rrbracket$. Then $x \models \mathfrak{f}(\mathcal{K})$. Then, by Fact 3.4, there is a y in $\mathcal{K}$ such that $y \subseteq x$. Then $y \models \mathfrak{f}(\mathcal{K})$. Suppose that $y \subset x$. Then $\llbracket \mathfrak{f}(\mathcal{K}) \rrbracket$ is not a finite state. Contradiction. Hence, $y = x$. Therefore, $\llbracket \mathfrak{f}(\mathcal{K}) \rrbracket \subseteq \mathcal{K}$.

Suppose that $x \in \mathcal{K}$. Then $x \models \mathfrak{f}(\mathcal{K})$. Suppose there is a $y \in \llbracket \mathfrak{f}(\mathcal{K}) \rrbracket$ such that $y \subset x$. Then, by the reasoning in the first part of this proof, $y \in \mathcal{K}$. Then $\mathcal{K}$ is not a finite state. Contradiction. Hence, $x \in \llbracket \mathfrak{f}(\mathcal{K}) \rrbracket$. Therefore, $\mathcal{K} \subseteq \llbracket \mathfrak{f}(\mathcal{K}) \rrbracket$. ∎

3.2. A finite state's core

To probe the properties of expansion, the notion of a finite state's core will be instrumental. A formula ϕ is in the *core* of a finite state $\mathcal{K}$, if ϕ is validated by *all* valuations in $\mathcal{K}$.

DEFINITION 3.6 (Core). Let $\mathcal{K}$ be a finite state. Then the *core of* $\mathcal{K}$, denoted by $\mathsf{c}(\mathcal{K})$, is defined to be

$$\mathsf{c}(\mathcal{K}) \quad = \quad \{\phi : \forall x (x \in \mathcal{K} \rightarrow x \models \phi)\}.$$

Note that $\mathsf{c}(\mathcal{K}) = \emptyset$ iff $\mathcal{K}$ is trivial.

If two finite states have the same core, they are identical. We need the following lemma to prove this:

---

[6] For a definition of DNFs for fde, see Anderson and Belnap [3], p. 156. Moreover, as $\phi \vee \psi \vee (\psi \wedge \theta) \vee \chi$ is fde-equivalent to $\phi \vee \psi \vee \chi$, any DNF is fde-equivalent to a MDNF.

LEMMA 3.7. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be finite states. Suppose both*

(i) $\forall x(x \in \mathcal{K}_1 \rightarrow \exists y(y \in \mathcal{K}_2 \text{ and } y \subseteq x))$
(ii) $\forall y(y \in \mathcal{K}_2 \rightarrow \exists x(x \in \mathcal{K}_1 \text{ and } x \subseteq y)).$

Then $\mathcal{K}_1 = \mathcal{K}_2$.

PROOF. Suppose $x \in \mathcal{K}_1$. Then there is a y in $\mathcal{K}_2$ such that $y \subseteq x$. Then there is an x' in $\mathcal{K}_1$ such that $x' \subseteq y$. Suppose that $x \neq y$. Then $x' \subset x$. Then $\mathcal{K}_1$ is not a finite state. Contradiction. Hence, $x = y$. Hence, $x \in \mathcal{K}_2$. Therefore, $\mathcal{K}_1 \subseteq \mathcal{K}_2$. The other inclusion can be proved similarly. Therefore, $\mathcal{K}_1 = \mathcal{K}_2$. ∎

LEMMA 3.8. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be finite states. Then*

$$\mathsf{c}(\mathcal{K}_1) = \mathsf{c}(\mathcal{K}_2) \quad \textit{iff} \quad \mathcal{K}_1 = \mathcal{K}_2.$$

PROOF. If $\mathcal{K}_1$ or $\mathcal{K}_2$ is trivial, the proof is easy. Otherwise, suppose that $\mathsf{c}(\mathcal{K}_1) = \mathsf{c}(\mathcal{K}_2)$. It holds that $\mathsf{f}(\mathcal{K}_2) \in \mathsf{c}(\mathcal{K}_2)$. Hence, $\mathsf{f}(\mathcal{K}_2) \in \mathsf{c}(\mathcal{K}_1)$. Then $\forall x(x \in \mathcal{K}_1 \rightarrow x \models \mathsf{f}(\mathcal{K}_2))$. Hence, by Fact 3.4, $\forall x(x \in \mathcal{K}_1 \rightarrow \exists y(y \in \mathcal{K}_2 \text{ and } y \subseteq x))$. Conditon (ii) of Lemma 3.7 can be proved similarly. Therefore, by Lemma 3.7, $\mathcal{K}_1 = \mathcal{K}_2$.

The converse is obvious. ∎

FACT 3.9. *Let $\mathcal{K}$ be a non-trivial finite state and let $\phi \in \mathfrak{F}$. Then*

$$\phi \in \mathsf{c}(\mathcal{K}) \quad \textit{iff} \quad \mathsf{f}(\mathcal{K}) \models \phi.$$

3.3. A finite state's span

The notion of a finite state's span shall serve as a tool for analysing the properties of contraction. A formula ϕ is in the *span* of a finite state $\mathcal{K}$, if ϕ is validated by *at least one* valuation in $\mathcal{K}$. It should be noted that this concept does not function in a standard possible worlds setting: *every* formula that is consistent with an epistemic state σ would be, regardless of the state's content, in that state's span, for each world in σ is fully opinioned.

DEFINITION 3.10 (Span). Let $\mathcal{K}$ be a finite state. Then the *span of $\mathcal{K}$*, denoted by $\mathsf{s}(\mathcal{K})$, is defined to be

$$\mathsf{s}(\mathcal{K}) \quad = \quad \{\phi : \exists x(x \in \mathcal{K} \text{ and } x \models \phi)\}.$$

Note that $\mathsf{s}(\mathcal{K}) = \emptyset$ iff $\mathcal{K}$ is trivial.

From this definition it follows that if two finite states have the same span, they are identical. To prove this, we need the following lemma.

LEMMA 3.11. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be finite states. Suppose both*

(i) $\forall x(x \in \mathcal{K}_1 \rightarrow \exists y(y \in \mathcal{K}_2 \text{ and } x \subseteq y))$
(ii) $\forall y(y \in \mathcal{K}_2 \rightarrow \exists x(x \in \mathcal{K}_1 \text{ and } y \subseteq x)).$

Then $\mathcal{K}_1 = \mathcal{K}_2$.

PROOF. Analogous to the proof of Lemma 3.7.   ∎

LEMMA 3.12. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be finite states. Then*

$$\mathsf{s}(\mathcal{K}_1) = \mathsf{s}(\mathcal{K}_2) \quad \textit{iff} \quad \mathcal{K}_1 = \mathcal{K}_2.$$

PROOF. Suppose that $\mathsf{s}(\mathcal{K}_1) = \mathsf{s}(\mathcal{K}_2)$. If $\mathcal{K}_1$ or $\mathcal{K}_2$ is trivial, the proof is easy. Otherwise, suppose that $x \in \mathcal{K}_1$. Then $\bigwedge x \in \mathsf{s}(\mathcal{K}_1)$. Then $\bigwedge x \in \mathsf{s}(\mathcal{K}_2)$. Then there is a y in $\mathcal{K}_2$ such that $y \models \bigwedge x$. Then there is a y in $\mathcal{K}_2$ such that $x \subseteq y$. Then $\forall x(x \in \mathcal{K}_1 \rightarrow \exists y(y \in \mathcal{K}_2 \text{ and } x \subseteq y))$. Condition (ii) of Lemma 3.11 can be proved similarly. Therefore, by Lemma 3.11, $\mathcal{K}_1 = \mathcal{K}_2$.

The converse is obvious.   ∎

4. Operations of Change

In this section, expansions and contractions of a finite state $\mathcal{K}$ with a formula ϕ are defined. We first offer a direct definition of the operation to be explored. Then, a set of postulates for this operation is given. Subsequently, it is shown that this set of postulates characterizes the direct definition. Last, a theorem is proved on iterated operations.

4.1. Expansion

In the system to be propounded here, an expansion of a finite state $\mathcal{K}$ with a formula ϕ amounts to changing $\mathcal{K}$ to incorporate the information that ϕ. The result of this adaptation is a new finite state $\mathcal{K} + \phi$.

DEFINITION 4.1 (Expansion). Let $\mathcal{K}$ be a finite state and let $\phi \in \mathfrak{F}$. Then the *expansion of $\mathcal{K}$ with ϕ*, denoted by $\mathcal{K} + \phi$, is defined to be

$$\mathcal{K} + \phi \quad = \quad \mathsf{min}(\{y \supseteq x : x \in \mathcal{K} \text{ and } y \models \phi\}).$$

Note that $\mathcal{K} + \phi = [\![\phi]\!]$, if $\mathcal{K}$ is trivial. Otherwise,

LEMMA 4.2. *Let $\mathcal{K}$ be a non-trivial finite state and let $\phi \in \mathfrak{F}$. Then*

$$\mathcal{K} + \phi \quad = \quad [\![\mathsf{f}(\mathcal{K}) \wedge \phi]\!].$$

PROOF. By Fact 3.4, $\{y \supseteq x : x \in \mathcal{K} \text{ and } y \models \phi\} = \{z : z \models \mathfrak{f}(\mathcal{K}) \wedge \phi\}$. ∎

The following five postulates are similar to the first five (out of six) in Gärdenfors [9], p. 48–51. Compare Belnap's requirements for adding a formula to an epistemic state in Anderson, Belnap and Dunn [4], p. 529. The concept of a finite state's core is used in the formulation of the postulates for expansion.

DEFINITION 4.3 (Postulates for Expansion). Let $\mathcal{K}$ and $\mathcal{K}'$ be finite states and let $\phi \in \mathfrak{F}$. Then an *expansion operator* is any operator, $\dot{+}$, satisfying the following conditions:

[$\dot{+}$1] $\mathcal{K}\dot{+}\phi$ is a finite state
[$\dot{+}$2] $\mathsf{c}(\mathcal{K}) \subseteq \mathsf{c}(\mathcal{K}\dot{+}\phi)$
[$\dot{+}$3] $\phi \in \mathsf{c}(\mathcal{K}\dot{+}\phi)$
[$\dot{+}$4] If $\phi \in \mathsf{c}(\mathcal{K})$, then $\mathsf{c}(\mathcal{K}\dot{+}\phi) \subseteq \mathsf{c}(\mathcal{K})$
[$\dot{+}$5] If $\mathsf{c}(\mathcal{K}) \subseteq \mathsf{c}(\mathcal{K}')$, then $\mathsf{c}(\mathcal{K}\dot{+}\phi) \subseteq \mathsf{c}(\mathcal{K}'\dot{+}\phi)$.

The next lemma shows that, using finite states rather than belief sets, we do not need, unlike Gärdenfors, an additional postulate to secure uniqueness.

LEMMA 4.4 (Uniqueness of Expansion Operators). *Let $\mathcal{K}$ be a finite state and let $\phi \in \mathfrak{F}$. Suppose that $\dot{+}$ and $\ddot{+}$ are expansion operators, satisfying* [$\dot{+}$1] *through* [$\dot{+}$5]. *Then $\mathcal{K}\dot{+}\phi = \mathcal{K}\ddot{+}\phi$.*

PROOF. By [$\dot{+}$1], $\mathcal{K}\ddot{+}\phi$ is a finite state. By [$\dot{+}$3], $\phi \in \mathsf{c}(\mathcal{K}\ddot{+}\phi)$. Hence, by [$\dot{+}$4], $\mathsf{c}((\mathcal{K}\ddot{+}\phi)\dot{+}\phi) \subseteq \mathsf{c}(\mathcal{K}\ddot{+}\phi)$. Moreover, by [$\dot{+}$2], it holds that $\mathsf{c}(\mathcal{K}) \subseteq \mathsf{c}(\mathcal{K}\ddot{+}\phi)$. Hence, by [$\dot{+}$5], $\mathsf{c}(\mathcal{K}\dot{+}\phi) \subseteq \mathsf{c}((\mathcal{K}\ddot{+}\phi)\dot{+}\phi)$. Therefore, $\mathsf{c}(\mathcal{K}\dot{+}\phi) \subseteq \mathsf{c}(\mathcal{K}\ddot{+}\phi)$. The other inclusion can be proved similarly. By Lemma 3.8, $\mathcal{K}\dot{+}\phi = \mathcal{K}\ddot{+}\phi$. ∎

THEOREM 4.5 (Characterization of Expansion). *Let $\mathcal{K}$ be a finite state and let $\phi \in \mathfrak{F}$. Then $\mathcal{K}\dot{+}\phi$ satisfies* [$\dot{+}$1] *through* [$\dot{+}$5] *iff $\mathcal{K}\dot{+}\phi = \mathcal{K} + \phi$.*

PROOF. By Lemma 4.4, it is sufficient to show that $\mathcal{K} + \phi$ satisfies the Postulates for Expansion. If $\mathcal{K}$ is trivial, the proof is straighforward. Otherwise,
    [$\dot{+}$1] Directly from Lemma 4.2 and Theorem 3.2.
    [$\dot{+}$2] Suppose that $\psi \in \mathsf{c}(\mathcal{K})$. By Fact 3.9, $\mathfrak{f}(\mathcal{K}) \models \psi$. Suppose that $x \in \mathcal{K} + \phi$. By Lemma 4.2, $x \in [\![\mathfrak{f}(\mathcal{K}) \wedge \phi]\!]$. Then, $x \models \mathfrak{f}(\mathcal{K}) \wedge \phi$. Hence, $x \models \psi$. Therefore, $\psi \in \mathsf{c}(\mathcal{K} + \phi)$.
    [$\dot{+}$3] Directly from the definition of expansion.
    [$\dot{+}$4] Suppose that $\phi \in \mathsf{c}(\mathcal{K})$. By Fact 3.9, $\mathfrak{f}(\mathcal{K}) \models \phi$. Hence, $\mathfrak{f}(\mathcal{K}) \models \mathfrak{f}(\mathcal{K}) \wedge \phi$. Moreover, it holds that $\mathfrak{f}(\mathcal{K}) \wedge \phi \models \mathfrak{f}(\mathcal{K})$. By Lemma 2.7, $[\![\mathfrak{f}(\mathcal{K})]\!] =$

$[\![\mathfrak{f}(\mathcal{K}) \wedge \phi]\!]$. Hence, by Theorem 3.5 and Lemma 4.2, $\mathcal{K} = \mathcal{K} + \phi$. Lemma 3.8 does the job.

$[\dot{+}5]$ Suppose that $\mathsf{c}(\mathcal{K}) \subseteq \mathsf{c}(\mathcal{K}')$. As $\mathfrak{f}(\mathcal{K}) \in \mathsf{c}(\mathcal{K})$, it must be, by Fact 3.9, that $\mathfrak{f}(\mathcal{K}') \models \mathfrak{f}(\mathcal{K})$. By $[\dot{+}3]$ and Fact 3.9, $\mathfrak{f}(\mathcal{K}' + \phi) \models \phi$. By $[\dot{+}2]$ and Fact 3.9, $\mathfrak{f}(\mathcal{K}' + \phi) \models \mathfrak{f}(\mathcal{K}')$. Therefore, $\mathfrak{f}(\mathcal{K}' + \phi) \models \mathfrak{f}(\mathcal{K}) \wedge \phi$ ($\star$). Now, we prove that $\mathfrak{f}(\mathcal{K}) \wedge \phi \models \mathfrak{f}(\mathcal{K} + \phi)$. Suppose that $x \models \mathfrak{f}(\mathcal{K}) \wedge \phi$. By Fact 2.6, $\exists y (y \subseteq x \text{ and } y \in [\![\mathfrak{f}(\mathcal{K}) \wedge \phi]\!])$. By Lemma 4.2, $\exists y (y \subseteq x \text{ and } y \in \mathcal{K} + \phi)$. By Fact 3.4, $x \models \mathfrak{f}(\mathcal{K} + \phi)$. Hence, $\mathfrak{f}(\mathcal{K}) \wedge \phi \models \mathfrak{f}(\mathcal{K} + \phi)$. Therefore, by ($\star$), $\mathfrak{f}(\mathcal{K}' + \phi) \models \mathfrak{f}(\mathcal{K} + \phi)$. Fact 3.9 does the job. ∎

THEOREM 4.6 (Iterated Expansion). *Let $\mathcal{K}$ be a finite state and let $\phi, \psi \in \mathfrak{F}$. Then*

$$(\mathcal{K} + \phi) + \psi \quad = \quad \mathcal{K} + \phi \wedge \psi.$$

PROOF. By Lemma 3.8, it suffices to show that (*a*) $\mathsf{c}((\mathcal{K} + \phi) + \psi) \subseteq \mathsf{c}(\mathcal{K} + \phi \wedge \psi)$ and (*b*) $\mathsf{c}(\mathcal{K} + \phi \wedge \psi) \subseteq \mathsf{c}((\mathcal{K} + \phi) + \psi)$. We shall use our Postulates for Expansion. Obviously, by $[\dot{+}1]$, $\mathcal{K} + \phi$, $(\mathcal{K} + \phi) + \psi$, and $\mathcal{K} + \phi \wedge \psi$ are all finite states.

(*a*) By $[\dot{+}3]$, $\phi \wedge \psi \in \mathsf{c}(\mathcal{K} + \phi \wedge \psi)$. Hence, $\phi \in \mathsf{c}(\mathcal{K} + \phi \wedge \psi)$ and $\psi \in \mathsf{c}(\mathcal{K} + \phi \wedge \psi)$. By $[\dot{+}2]$, $[\dot{+}4]$, and Lemma 3.8, it holds that $(\mathcal{K} + \phi \wedge \psi) + \phi = (\mathcal{K} + \phi \wedge \psi) + \psi = \mathcal{K} + \phi \wedge \psi$. By $[\dot{+}2]$, it holds that $\mathsf{c}(\mathcal{K}) \subseteq \mathsf{c}(\mathcal{K} + \phi \wedge \psi)$. Our previously established equalities and a double application of $[\dot{+}5]$ do the job.

(*b*) By $[\dot{+}2]$ and $[\dot{+}3]$, $\phi \in \mathsf{c}((\mathcal{K} + \phi) + \psi)$ and $\psi \in \mathsf{c}((\mathcal{K} + \phi) + \psi)$. Hence, $\phi \wedge \psi \in \mathsf{c}((\mathcal{K} + \phi) + \psi)$. By $[\dot{+}2]$, $[\dot{+}4]$, and Lemma 3.8, it holds that $((\mathcal{K} + \phi) + \psi) + \phi \wedge \psi = (\mathcal{K} + \phi) + \psi$. By applying $[\dot{+}2]$ twice, we get $\mathsf{c}(\mathcal{K}) \subseteq \mathsf{c}((\mathcal{K} + \phi) + \psi)$. Our previously established equality and $[\dot{+}5]$ complete the proof. ∎

Commutativity of conjunction gives us the following:

COROLLARY 4.7. *Let $\mathcal{K}$ be a finite state and let $\phi, \psi \in \mathfrak{F}$. Then*

$$(\mathcal{K} + \phi) + \psi \quad = \quad (\mathcal{K} + \psi) + \phi.$$

4.2. Contraction

Intuitively, if we contract a finite state $\mathcal{K}$ with a formula ϕ, we skip all sufficient evidence for ϕ from the elements of our finite state, so that ϕ can not be among the formulas which are within the span of the resulting finite state, as there will be no residual evidence for ϕ left in it. Hence, ϕ is not within the span of $\mathcal{K} - \phi$. Moreover, all logical consequences that were in the span of $\mathcal{K}$ 'just because' ϕ was within the span of $\mathcal{K}$ will be removed as

well. Last, we need to retain as much evidence as is consistent with these goals:

DEFINITION 4.8 (Contraction). Let $\mathcal{K}$ be a finite state and let $\phi \in \mathfrak{F}$. Then the *contraction of $\mathcal{K}$ with ϕ*, denoted by $\mathcal{K} - \phi$, is defined to be

$$\mathcal{K} - \phi \quad = \quad \mathsf{max}(\{y \subseteq x : x \in \mathcal{K} \text{ and } y \not\models \phi\}).$$

In the present context, Gärdenfors's postulates for contraction cannot be straightforwardly applied, as the standard postulates have been devised for belief sets. Here, we translate them into the language of finite states. We only need four out of Gärdenfors's eight postulates plus an additional postulate for our characterization theorem.[7] The reader can easily check, however, that all translations of Gärdenfors's eight postulates hold, except for the hotly debated *Recovery* postulate.[8] Note that the concept of a finite state's span is used in the formulation of the postulates for contraction.

DEFINITION 4.9 (Postulates for Contraction). Let $\mathcal{K}$ and $\mathcal{K}'$ be finite states and let $\phi \in \mathfrak{F}$. Then a *contraction operator* is any operator, $\dot{-}$, satisfying the following conditions:

[$\dot{-}$1] $\mathcal{K} \dot{-} \phi$ is a finite state
[$\dot{-}$2] $\mathsf{s}(\mathcal{K} \dot{-} \phi) \subseteq \mathsf{s}(\mathcal{K})$
[$\dot{-}$3] $\phi \notin \mathsf{s}(\mathcal{K} \dot{-} \phi)$
[$\dot{-}$4]   If $\phi \notin \mathsf{s}(\mathcal{K})$, then $\mathsf{s}(\mathcal{K}) \subseteq \mathsf{s}(\mathcal{K} \dot{-} \phi)$
[$\dot{-}$5]   If $\mathsf{s}(\mathcal{K}) \subseteq \mathsf{s}(\mathcal{K}')$, then $\mathsf{s}(\mathcal{K} \dot{-} \phi) \subseteq \mathsf{s}(\mathcal{K}' \dot{-} \phi)$.

LEMMA 4.10 (Uniqueness of Contraction Operators). *Let $\mathcal{K}$ be a finite state and let $\phi \in \mathfrak{F}$. Suppose that $\dot{-}$ and $\ddot{-}$ are contraction operators, satisfying [$\dot{-}$1] through [$\dot{-}$5]. Then $\mathcal{K} \dot{-} \phi = \mathcal{K} \ddot{-} \phi$.*

PROOF. By [$\dot{-}$1], $\mathcal{K} \dot{-} \phi$ is a finite state. By [$\dot{-}$3], $\phi \notin \mathsf{s}(\mathcal{K} \dot{-} \phi)$. Hence, by [$\dot{-}$4], $\mathsf{s}(\mathcal{K} \dot{-} \phi) \subseteq \mathsf{s}((\mathcal{K} \dot{-} \phi) \ddot{-} \phi)$. Moreover, by [$\dot{-}$2], it holds that $\mathsf{s}(\mathcal{K} \dot{-} \phi) \subseteq$

---

[7] In Gärdenfors [9], p. 65, this additional postulate is discussed under the name (K$^-$M). It is dismissed, because it is, in the context of belief sets, equivalent to its counterpart (K*M), which was previously shown to be unsound (Gärdenfors [9], p. 59–60). In the present setting, monotonicity in contraction does not imply monotonicity in revision, if revision is defined via the Levi identity: $\mathcal{K} \times \phi = (\mathcal{K} - \neg\phi) + \phi$. Pais and Jackson [15] introduce a similar, but weaker postulate: *Partial Monotonicity*.

[8] In the present setting, *Recovery* would amount to $\mathsf{s}(\mathcal{K}) \subseteq \mathsf{s}((\mathcal{K} - \phi) + \phi)$. For a counterexample: let $\mathcal{K}$ be $[\![p \wedge q]\!]$, and let ϕ be $p \vee q$. As the other translations of Gärdenfors's postulates are satisfied by our contraction function, our contraction function is, following Makinson [13], a finite state based *withdrawal function*.

$\mathsf{s}(\mathcal{K})$. Hence, by [$\dot{-}$5], $\mathsf{s}((\mathcal{K}\dot{-}\phi)\ddot{-}\phi) \subseteq \mathsf{s}(\mathcal{K}\ddot{-}\phi)$. Therefore, $\mathsf{s}(\mathcal{K}\dot{-}\phi) \subseteq \mathsf{s}(\mathcal{K}\ddot{-}\phi)$. The other inclusion can be proved similarly. By Lemma 3.12, $\mathcal{K}\dot{-}\phi = \mathcal{K}\ddot{-}\phi$. ∎

THEOREM 4.11 (Characterization of Contraction). *Let $\mathcal{K}$ be a finite state and let $\phi \in \mathfrak{F}$. Then $\mathcal{K}\dot{-}\phi$ satisfies [$\dot{-}$1] through [$\dot{-}$5] iff $\mathcal{K}\dot{-}\phi = \mathcal{K} - \phi$.*

PROOF. By Lemma 4.10, it is sufficient to show that $\mathcal{K} - \phi$ satisfies the Postulates for Contraction. The first four postulates follow directly from the definition of contraction.

[$\dot{-}$5] Suppose that $\mathsf{s}(\mathcal{K}) \subseteq \mathsf{s}(\mathcal{K}')$. Suppose that $\psi \in \mathsf{s}(\mathcal{K} - \phi)$. Then there is a y in $\mathcal{K} - \phi$ such that $y \models \psi$. Hence, there is a x in $\mathcal{K}$ such that $y \subseteq x$ and $y \not\models \phi$ and $y \models \psi$. Hence, $\bigwedge x \in \mathsf{s}(\mathcal{K})$. Therefore, by assumption, $\bigwedge x \in \mathsf{s}(\mathcal{K}')$. Hence, there is a x' in $\mathcal{K}'$ such that $x \subseteq x'$. Summarizing, there is a x' in $\mathcal{K}'$ such that $y \subseteq x'$ and $y \not\models \phi$ and $y \models \psi$. Hence, $y \in \{y' \subseteq x' : x' \in \mathcal{K}' \text{ and } y' \not\models \phi\}$. As the maximalization of this set only skips y in favour of a superset of y, there will be an element in $\mathcal{K}' - \phi$ that validates ψ. Therefore, $\psi \in \mathsf{s}(\mathcal{K}' - \phi)$. ∎

THEOREM 4.12 (Iterated Contraction). *Let $\mathcal{K}$ be a finite state and let $\phi, \psi \in \mathfrak{F}$. Then*

$$(\mathcal{K} - \phi) - \psi \quad = \quad \mathcal{K} - \phi \vee \psi.$$

PROOF. By Lemma 3.12, it suffices to show that (*a*) $\mathsf{s}((\mathcal{K} - \phi) - \psi) \subseteq \mathsf{s}(\mathcal{K} - \phi \vee \psi)$ and (*b*) $\mathsf{s}(\mathcal{K} - \phi \vee \psi) \subseteq \mathsf{s}((\mathcal{K} - \phi) - \psi)$. We shall use our Postulates for Contraction. Obviously, by [$\dot{-}$1], $\mathcal{K} - \phi$, $(\mathcal{K} - \phi) - \psi$, and $\mathcal{K} - \phi \vee \psi$ are all finite states.

(*a*) By [$\dot{-}$2] and [$\dot{-}$3], $\phi \notin \mathsf{s}((\mathcal{K} - \phi) - \psi)$ and $\psi \notin \mathsf{s}((\mathcal{K} - \phi) - \psi)$. Hence, $\phi \vee \psi \notin \mathsf{s}((\mathcal{K} - \phi) - \psi)$. By [$\dot{-}$2], [$\dot{-}$4], and Lemma 3.12, it holds that $((\mathcal{K} - \phi) - \psi) - \phi \vee \psi = (\mathcal{K} - \phi) - \psi$. By applying [$\dot{-}$2] twice, we get $\mathsf{s}((\mathcal{K} - \phi) - \psi) \subseteq \mathsf{s}(\mathcal{K})$. Our previously established equality and [$\dot{-}$5] do the job.

(*b*) By [$\dot{-}$3], $\phi \vee \psi \notin \mathsf{s}(\mathcal{K} - \phi \vee \psi)$. Hence, $\phi \notin \mathsf{s}(\mathcal{K} - \phi \vee \psi)$ and $\psi \notin \mathsf{s}(\mathcal{K} - \phi \vee \psi)$. By [$\dot{-}$2], [$\dot{-}$4], and Lemma 3.12, it holds that $(\mathcal{K} - \phi \vee \psi) - \phi = (\mathcal{K} - \phi \vee \psi) - \psi = \mathcal{K} - \phi \vee \psi$. By [$\dot{-}$2], it holds that $\mathsf{s}(\mathcal{K} - \phi \vee \psi) \subseteq \mathsf{s}(\mathcal{K})$. Our previously established equalities and a double application of [$\dot{-}$5] complete the proof. ∎

Commutativity of disjunction gives us the following:

COROLLARY 4.13. *Let $\mathcal{K}$ be a finite state and let $\phi, \psi \in \mathfrak{F}$. Then*

$$(\mathcal{K} - \phi) - \psi \quad = \quad (\mathcal{K} - \psi) - \phi.$$

References

[1] ALCHOURRÓN, C. E., P. GÄRDENFORS, and D. MAKINSON, 'On the logic of theory change: partial meet contraction and revision functions', *Journal of Symbolic Logic* 50: 510–530, 1985.

[2] ANDERSON, A. R., and N. D. BELNAP, 'Tautological entailments', *Philosophical Studies* 13: 9–24, 1962.

[3] ANDERSON, A. R., and N. D. BELNAP, *Entailment: The Logic of Relevance and Necessity*, Volume 1, Princeton University Press, 1975.

[4] ANDERSON, A. R., N. D. BELNAP, and J. M. DUNN, *Entailment: The Logic of Relevance and Necessity*, Volume 2, Princeton University Press, 1992.

[5] CHOPRA, S., and R. PARIKH, 'Relevance sensitive belief structures', *Annals of Mathematics and Artificial Intelligence* 28: 259–285, 2000.

[6] DA COSTA, N. C. A., and O. BUENO, 'Belief change and inconsistency', *Logique & Analyse* 161-162-163: 31–56, 1998.

[7] DUNN, J. M., 'Intuitive semantics for first-degree entailments and "coupled trees"', *Philosophical Studies* 29: 149–168, 1976.

[8] FUHRMANN, A., 'Theory contraction through base contraction', *Journal of Philosophical Logic* 20: 175–203, 1991.

[9] GÄRDENFORS, P., *Knowledge in Flux*, The MIT Press, 1988.

[10] HANSSON, S. O., 'Reversing the Levi identity', *Journal of Philosophical Logic* 22: 637–669, 1993.

[11] HANSSON, S. O., 'Editorial: belief revision theory today', *Journal of Logic, Language, and Information* 7: 123–126, 1998.

[12] HANSSON, S. O., and E. J. OLSSON, 'Levi contractions and AGM contractions: a comparison', *Notre Dame Journal of Formal Logic* 36: 103–119, 1995.

[13] MAKINSON, D., 'On the status of the postulate of recovery in the logic of theory change', *Journal of Philosophical Logic* 16: 383–394, 1987.

[14] MARES, E. D., 'A paraconsistent theory of belief revision', *Erkenntnis* 56: 229–246, 2002.

[15] PAIS, J., and P. JACKSON, 'Partial monotonicity and a new version of the Ramsey test', *Studia Logica* 51: 21–47, 1992.

[16] RESCHER, N., and R. BRANDOM, *The Logic of Inconsistency*, Blackwell, 1980.

[17] RESTALL, G., and J. SLANEY, 'Realistic belief revision', Technical Report TR-ARP-2-95, Research School of Information Sciences and Engineering and Centre for Information Science Research, Australian National University, 1995.

[18] ROUTLEY, R., and V. ROUTLEY, 'The semantics of first degree entailment', *Noûs* 6: 335–359, 1972.

[19] Tamminga, A. M., *Belief Dynamics: (Epistemo)logical Investigations*, Ph.D. thesis, Universiteit van Amsterdam, 2001.

[20] Tamminga, A. M., and K. Tanaka, 'A natural deduction system for first degree entailment', *Notre Dame Journal of Formal Logic* 40: 258–272, 1999.

[21] Wassermann, R., *Resource-Bounded Belief Revision*, Ph.D. thesis, Universiteit van Amsterdam, 2000.

Allard Tamminga
Vakgroep Theoretische Filosofie
Faculteit der Wijsbegeerte
Rijksuniversiteit Groningen
A-weg 30
9718 CW Groningen, The Netherlands
a.m.tamminga@philos.rug.nl