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On the symmetries of singular limits of spacetimes

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Abstract: We consider spacetime metrics with a given (but quite generic) dependence on a dimensionful parameter such that in the 0 and ∞ limits of that parameter the metric becomes singular. We study the isometry groups of the original spacetime metrics and of the singular metrics that arise in the limits and the corresponding symmetries of the motion of p-branes evolving in them, showing how the Killing vectors and their Lie algebras can be found in general. We illustrate our general results with several examples which include limits of anti-de Sitter spacetime in which the holographic screen is one of the singular metrics and of pp-waves.

Keywords: P-Branes, Space-Time Symmetries, Classical Theories of Gravity, Sigma Models

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Contents

1 Introduction

In the past few years, there has been a lot of activity in studying non-relativistic, or, more generally, non-Lorentzian gravity and string theories.[1] For some recent reviews, see [1–3]. The singular geometries underlying these gravity and string theories are quite different from the regular geometries underlying the relativistic parent theory in the sense that the regular geometry can be endowed with a single metric whereas the singular geometry is characterized by two separate metrics. A characteristic feature of taking the singular limit of a regular geometry leading to a degenerate geometry with two separate metrics is that divergences arise that need to be taken care of. These infinities are essential to make the transition between the two types of geometries possible.

A noteworthy feature of coupling extended objects, such as particles, strings, membranes etc., to a geometric background is that, whereas all extended objects couple to the same

[1] In this work we will generically call "non-relativistic" or "non-Lorentzian" theories which do not exhibit the full invariance corresponding to their space and time dimension. Some of these theories may be invariant under a lower-dimensional Lorentz group and, therefore, may still be "relativistic" or "Lorentzian" in a restricted sense.

Riemannian background in the relativistic case, in the non-Lorentzian case there are different non-relativistic backgrounds to which the extended objects of different worldvolume dimensions will naturally couple. Consider a p-dimensional extended object (a "p-brane") moving in a d-dimensional background, with $p+1$ ("longitudinal") spacetime directions parallel to the object's worldvolume and $d-p-1$ directions transverse to it. Then, a natural non-relativistic background to which such a p-brane may couple is a degenerate geometry foliated by spatial submanifolds of dimension $d-p-1$. The transverse and longitudinal directions are inequivalent in the sense that, under a boost transformation, a transverse direction transforms into a longitudinal direction but not the other way round. To obtain these different foliated geometries from the same Riemannian geometry one needs to define different so-called "p-brane limits" in which the longitudinal and transverse directions are treated in a different way. For a recent discussion of such p-brane limits, see [4].

In general, to define a limit, be it singular or not, we must start by redefining the fields (and possibly other constants) of the relativistic theory using a dimensionful "contraction parameter" ρ.[2] This contraction parameter can be anything: it could be the velocity of light c,[3] but it could also be some radial parameter R or the cosmological constant. The redefinition is done in such a way that no new fields are introduced and the redefinition is invertible. The redefined theory is still relativistic, but we will call the redefined fields the would-be non-Lorentzian fields in the sense that they will become the fields of the non-Lorentzian theory *after* taking the limit that the contraction parameter goes to infinity.

Several singular limits of geometries and/or solutions have been considered in the recent literature for different reasons. For a few examples, see [5–9]. It is the purpose of this work to develop a general framework that describes

1. What happens to the Lie algebra of the isometries of the original metrics after the singular limits are taken.

2. What happens to the Lie algebra of symmetries of the action of a p-brane moving in the original spacetime after the singular limit is taken.

(It is important to distinguish between these two symmetry algebras, because they only need to coincide in the relativistic case.)

Our setup will be quite general. In particular we will not impose any restrictions on the ranks of the singular metrics that arise in the limits. In most of the studies done in this field the focus is placed on the tangent space, which is assumed to split in a direct sum of two subspaces. The tangent space metric is also assumed to decompose into the sum of two singular metrics which are regular when restricted to one of the subspaces. The

[2]Strictly speaking the true contraction parameter λ should be dimensionless. Such a parameter can be obtained by redefining $\rho \to \lambda\rho$. With a slight misuse of language we will often call ρ the contraction parameter. Note that we are taking limits in this paper after which the parameter ρ disappears. Before taking this limit, we assume that all the redefined fields are independent of ρ so that all the ρ-dependence is explicit. The limit is now given by the leading term in ρ which, after a possible redefinition of a parameter, can always be made of the form ρ^0. The resulting limit is now independent of the contraction parameter ρ. The situation is different when one considers expansions. In that case the expansion is in terms of a dimensionless parameter σ/ρ where σ is another parameter of the same dimension as ρ that is particular to the process that one is considering.

[3]This is the conventional choice in the simplest non-relativistic limits.

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Vielbeins of these two subspaces are, then, used to construct the metrics of the two singular spacetime metrics in the standard fashion using the singular tangent space metrics. This construction is only possible when the ranks of the singular spacetime metrics add up to the total dimension of the spacetime, d and, therefore, it cannot be used in all the cases to which our setup applies and, in particular, it cannot be applied to the singular limit of 4-dimensional *pp*-wave metrics considered in section 5 that produces a metric of rank 3 and signature $+--$,[4] and a metric of rank 2 and signature $+-$.

On the other hand, in order to study the symmetries of the action of p-branes and its non-relativistic limits, we will make use of techniques that have been developed within the context of non-Lorentzian limits and apply them in a more general context. In particular, we will consider the limit of a sigma model action describing a p-brane in a generic curved background. We will control in three different ways the leading divergence that arises when performing in the action the redefinition that defines the limit as follows:

Option 1. One can neutralize the leading divergence by an appropriate rescaling of the string tension parameter and the worldvolume metric such that the leading term scales as ρ^0 leading to a finite answer. We will refer to this limit as the one 'with rescalings'.

Option 2. One can tame the leading divergence by performing a Hubbard-Stratonovich transformation which uses the fact that any term of the form $\rho^\alpha X^2$ for some X can be rewritten, by introducing an auxiliary field λ, in the equivalent form

$$\rho^\alpha \left[-\frac{1}{\rho^4}\lambda^2 - \frac{2}{\rho^2}\lambda X \right] . \tag{1.1}$$

Solving for λ and substituting this solution back one finds the original $\rho^\alpha X^2$ term. In this work, we will apply this rewriting for $\alpha = 2$.[5] In that case, after taking the limit $\rho \to \infty$, λ becomes a Lagrange multiplier imposing the constraint $X = 0$. The expression eq. (1.1) also applies if X and λ carry flat or curved spacetime indices. After rewriting the quadratic divergence according to eq. (1.1), one finds a finite answer at sub-leading order. We will refer to this limit as the one "with auxiliary fields".

Option 3. One can cancel the leading divergence by adding a Wess-Zumino (WZ) term to the sigma model action containing a new field and to redefine this new field introducing a new divergence in such a way that it neutralizes the leading divergence. A prime example of this occurs when taking the sigma model action describing a particle coupled to general relativity. The leading divergence can be cancelled by introducing a WZ term containing a vector field M_μ. After taking the limit, the non-Lorentzian version m_μ of this vector field, sometimes called the mass vector field, corresponds to a central extension of the Galilei algebra which is called the Bargmann algebra. It describes the property that, after taking the limit, energy and mass are two separately conserved quantities. We will refer to this limit as the one "with WZ term".

[4]Notice that in this paper we use mostly minus signature throughout.

[5]Note, however, that the same rewriting can also be applied for $\alpha = 0$ to eliminate a finite term. For an example, see [4].

In this work we will not only consider the $\rho \to \infty$ limit but also the $\rho \to 0$ limit. This option is a special case of the first one after replacing ρ by $\rho' = 1/\rho$. In terms of ρ' with $\rho' \to \infty$ there are again three ways to deal with the leading divergence. In option 1 one can take the limit straight-away without any rescalings since the leading "divergence" is already of the form ρ'^0. We will refer to this limit as one "without rescalings". In option 2 one first performs a rescaling such that one obtains a leading divergence of the form ρ'^2 and next tames this divergence by performing a Hubbard-Stratonovich transformation. We will refer to this limit as the one "with auxiliary fields".

One outcome of our general framework is that we will derive two dualities between the Lie algebras that arise after taking singular limits with $\rho \to \infty$ and $\rho \to 0$. One duality maps the Lie algebra that arises after taking a singular $\rho \to \infty$ "with auxiliary fields" to the Lie algebra that arises after taking a singular $\rho \to 0$ limit "with auxiliary fields". Having determined the Lie algebra that arises after taking one limit, one can apply this duality to determine the Lie algebra that arises after taking the other limit. A prime example of this duality is the map between the Galilei and Carroll algebras. For technical reasons, to be explained later, we will call this duality the "$1 \leftrightarrow -1$ duality". This is to be contrasted with a different, formal, "brane duality", which also follows from our general framework and was first noted in [10, 11], that relates the Lie algebra corresponding to one brane to the Lie algebra corresponding to a dual brane. A prime example of this duality, which also follows from our general framework, is the duality between a $p = 0$ Galilei algebra and a dual $p = d - 2$ Carroll algebra. This duality is formal in the sense that one should make a formal interchange between the single time direction of the Galilei particle and the single spatial transverse direction of the Carroll domain wall.

This work is organized as follows. First, in section 2, we will define the general sigma model action describing a p-brane in a special d-dimensional spacetime background that contains a parameter ρ whose limit we are going to take. We will focus on the symmetries and show how the Killing vectors describing the symmetries before taking the limit get transformed into the Killing vectors describing the symmetries of the foliated geometry after taking the limit and discuss the corresponding symmetry algebras. We will discuss both the $\rho \to \infty$ and $\rho \to 0$ limits for two of the three options discussed above, i.e. by neutralizing the leading divergence (option 1) and by performing a Hubbard-Stratonovich transformation introducing a Lagrange multiplier (option 2). The third option will only be applied when we discuss later in this paper the holographic limit of an Anti-de Sitter spacetime. Next, we will apply the general framework developed in section 2 to several examples. First, as a warming up exercise to elucidate our techniques, we will consider in section 3 the case that the background metric is that of a flat Minkowski spacetime. We will do this first for particles and next generalize this to p-branes. In section 4 we will consider the more involved case of an Anti-de Sitter (AdS) spacetime. We will consider both a so-called holographic limit as well as a p-brane limit. In the case of the holographic limit we will also consider option 3 mentioned above and add a Wess-Zumino term involving a new 1-form gauge field to the sigma model action. Finally, in section 5 we will consider a pp-wave background metric. We finish with a discussion of our results and have added a separate appendix collecting a few technical facts about AdS spacetimes that are used in the main text.

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2 General framework

In this section we construct the general framework that we will apply to specific examples in the next sections. Our starting point is the following $(p+1)$-dimensional σ-model (also known as the p-brane Polyakov-type action):

$$S\left[x^{\mu}(\zeta), \gamma_{ij}(\zeta)\right] = -\frac{T}{2}\int d^{p+1}\zeta\sqrt{|\gamma|}\left[\gamma^{ij}g_{ij} - (p-1)\right], \tag{2.1}$$

where T is the tension, with units of $[ML^{-p}]$, ζ^i, with $i = 0, 1, \cdots, p$, are the worldvolume coordinates, $\gamma_{ij}(\zeta)$ is the worldvolume metric, $|\gamma|$ is the absolute value of its determinant and γ^{ij} its inverse, $g_{\mu\nu}(x)$ is the d-dimensional spacetime metric and $g_{ij} = \partial_i x^{\mu}\partial_j x^{\nu} g_{\mu\nu}$, is its pullback to the worldvolume.

We are going to consider spacetime metrics that can be written in the form

$$g_{\mu\nu} = h_{\mu\nu} + \rho^2 k_{\mu\nu}, \tag{2.2}$$

where the ranks of the metrics $h_{\mu\nu}$ and $k_{\mu\nu}$ (assumed to be finite) is smaller than d and ρ is a parameter whose limit we will take to zero and infinity.[6] In these two limits the spacetime metric $g_{\mu\nu}$ becomes singular or ill-defined and must be replaced by the singular but well-defined, metrics $h_{\mu\nu}$ and $k_{\mu\nu}$ in terms of which the action eq. (2.1) takes the equivalent form

$$S\left[x^{\mu}(\zeta), \gamma_{ij}(\zeta)\right] = -\frac{T}{2}\int d^{p+1}\zeta\sqrt{|\gamma|}\left[\gamma^{ij}h_{ij} + \rho^2\gamma^{ij}k_{ij} - (p-1)\right]. \tag{2.3}$$

We want to study the symmetries of this action (specially the global ones) before and after taking the limits in which ρ goes to zero or infinity.

Before taking the limits, the global symmetries of the action (2.3) are generated by infinitesimal transformations of the embedding coordinates $x^{\mu}(\zeta)$ of the form

$$\delta x^{\mu} = \epsilon\xi^{\mu}(x), \tag{2.4}$$

where ϵ is an infinitesimal (constant) parameter and ξ^{μ} is a Killing vector of the metric $g_{\mu\nu}$, that is

$$\pounds_{\xi} g_{\mu\nu} = 0. \tag{2.5}$$

Thus, effectively, the study of the symmetries of the action is equivalent to the study of the isometries of the metric, at least before taking the limits. It is natural, then, to start our investigation by studying the Killing vectors of the original metric and the Lie algebra they generate.

[6]Although it seems natural to expect that the sums of the ranks of these metrics is d, as we have explained in the introduction, we are not going to make this assumption and, as a matter of fact, in section 5 we are going to study an example in which the sum is $d+1$. This is perfectly consistent in our setup, since the metric is a background field with no degrees of freedom attached but one has to bear in mind that, in a different context, this may turn out to be inconsistent.

2.1 The Lie algebra of isometries

We are going to assume that all the Killing vectors admit an expansion in the parameter ρ of the form

$$\xi = \frac{1}{\rho}\xi^{(-1)} + \xi^{(0)} + \rho\xi^{(1)}\,, \tag{2.6}$$

and, furthermore, that these Killing vectors generate a Lie algebra with ρ-independent structure constants. This assumption is backed by the fact that if we add terms of higher or lower orders in ρ to the expansion (2.6) they will correspond to Killing vectors which are already included in the above expansion, multiplied by powers of ρ. It is also backed by the examples that we are going to present in the following sections.

Observe that the components in the above expansion, $\xi^{(-1)}, \xi^{(0)}, \xi^{(1)}$, are (non-necessarily Killing) vector fields whose Lie brackets we can always compute. We can also compute the Lie derivatives of the singular metrics with respect to each of them.

Using the expansion eq. (2.6) and the main hypothesis about the spacetime metric eq. (2.2), we find at each order in ρ that the following ρ-independent equations must be satisfied:

$$\pounds_{\xi^{(1)}} k_{\mu\nu} = 0\,, \tag{2.7a}$$

$$\pounds_{\xi^{(0)}} k_{\mu\nu} = 0\,, \tag{2.7b}$$

$$\pounds_{\xi^{(-1)}} k_{\mu\nu} + \pounds_{\xi^{(1)}} h_{\mu\nu} = 0\,, \tag{2.7c}$$

$$\pounds_{\xi^{(0)}} h_{\mu\nu} = 0\,, \tag{2.7d}$$

$$\pounds_{\xi^{(-1)}} h_{\mu\nu} = 0\,. \tag{2.7e}$$

That is: the $\xi^{(0)}$ vector fields are Killing vectors of the two singular metrics and, therefore, they are Killing vectors of the original metric. The vector fields $\xi^{(1)}$ (resp. -1) are Killing vectors of the singular metric $k_{\mu\nu}$ (resp. $h_{\mu\nu}$) only. The isometry algebra of the singular metric $k_{\mu\nu}$ (resp. $h_{\mu\nu}$) is the Lie algebras generated by the vectors $\xi^{(0)}$ and $\xi^{(1)}$ (resp. $\xi^{(0)}$ and $\xi^{(-1)}$), which must be necessarily closed (see the discussion below.)

At first sight one may conclude that, if both singular metrics survive in the action after the limit, the algebra of symmetries of the action should be one generated by the $\xi^{(0)}$ vector fields, which should, therefore, generate a closed subalgebra of the original isometry algebra. If only $k_{\mu\nu}$ (resp. $h_{\mu\nu}$) survived in the action, the algebra of symmetries would be the one generated by the vectors $\xi^{(0)}$ and $\xi^{(1)}$ (resp. $\xi^{(0)}$ and $\xi^{(-1)}$). As we are going to see, though, this conclusion is wrong: the one-to-one relationship between Killing vectors of the metric that occurs in the action and the generators of symmetries of the action breaks down in the limits.[7] The missed ingredient in this analogy is the behaviour of the transformation parameters, which can be rescaled with powers of ρ so as to absorb divergences. Using these rescalings

[7]This is a well known fact that, perhaps, has not been formulated before in this way. The symmetry group of the action of the non-relativistic particle obtained in the $c \to \infty$ limit of the standard relativistic particle

$$S[t, x^i] \sim \int d\zeta \frac{\delta_{ij}\dot{x}^i\dot{x}^j}{\dot{t}}\,,$$

(the Galileo group) has the same dimension as the Poincaré group even though only the Euclidean metric δ_{ij}, and a trivial 1-dimensional metric in the time direction occur in it.

one can obtain a symmetry of the action for each of the original symmetries/isometries *after* taking the limits. We will see how this works out in the following sections, but we first need to study in more detail the brackets of all the vector fields involved.

Our assumptions for the expansion of the Killing vectors in powers of ρ and the ρ-independence of the structure constants have the following consequences for the Lie algebra:

1. The following Lie brackets vanish

$$[\xi_A^{(1)}, \xi_B^{(1)}] = [\xi_A^{(-1)}, \xi_B^{(-1)}] = 0\,, \tag{2.8}$$

   where the index A labels the different Killing vectors of which $\xi_A^{(1)}$ and $\xi_A^{(-1)}$ are components.

2. The Lie brackets of $\xi^{(0)}$ vectors close on $\xi^{(0)}$ vectors and, hence, they generate a subalgebra. Since they are Killing vectors of h and k simultaneously, one can always find a basis of the Lie algebra in which there are just two kinds of vectors: those which are independent of ρ, so $\xi = \xi^{(0)} \equiv \pi$ and those which do not contain any ρ-independent piece and are of the form $\xi = \rho^{-1}\xi^{(-1)} + \rho\xi^{(1)} \equiv \varpi$, where either $\xi^{(-1)}$ or $\xi^{(1)}$ may vanish. Henceforth we will always use this basis.

3. The Lie brackets of π vectors and ϖ vectors close on ϖ vectors only. Thus, the ϖ vectors span a representation of the subalgebra of the π vectors.

4. The Lie bracket of two ϖ vectors is always a π vector.

The last three statements can be summarized symbolically by the following Lie brackets:

$$[\pi, \pi] = \pi\,, \qquad [\pi, \varpi] = \varpi\,, \qquad [\varpi, \varpi] = \pi\,, \tag{2.9}$$

which describe a symmetric decomposition of the Lie algebra of the original isometry group of $g_{\mu\nu}$. Furthermore, all the ϖ vectors are of the form

$$\varpi = \frac{1}{\rho}\varpi^{(-1)} + \rho\varpi^{(1)}\,, \tag{2.10}$$

and the only non-trivial Lie brackets between the ϖ components are

$$[\varpi^{(-1)}, \varpi^{(1)}] = \pi\,, \tag{2.11}$$

and between the ϖ and π components are[8]

$$[\pi, \varpi^{(-1)}] = \varpi^{(-1)}\,, \quad \text{and} \quad [\pi, \varpi^{(1)}] = \varpi^{(1)}\,. \tag{2.12}$$

This means that we are free to rescale

$$\varpi^{(-1)} \to \rho^{-n}\varpi^{(-1)}\,, \qquad \varpi^{(1)} \to \rho^{n}\varpi^{(1)}\,, \tag{2.13}$$

[8]As we have remarked before, in general, these Lie brackets are not part of the Lie algebra because, in general, only the combination $\rho^{-1}\varpi^{(-1)} + \rho\varpi^{(1)}$ is actually a generator. Nevertheless, these Lie brackets are important because, depending on the limit taken, either $\varpi^{(-1)}$ or $\varpi^{(1)}$ will disappear and the surviving component will become a generator of the new (Wigner-Inönü-contracted) algebra of symmetries of our system even if they are not Killing vectors anymore.

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preserving the structure constants of the group. In particular, we could rescale with $n = -1$ so that

$$\varpi = \varpi^{(-1)} + \rho^2 \varpi^{(1)}\,, \tag{2.14}$$

which is a more standard expansion.

There are two special cases that need to be considered: when either $\varpi^{(1)} = 0$ or $\varpi^{(-1)} = 0$, the other non-zero component in the expansion ($\varpi^{(-1)}$ or $\varpi^{(1)}$, respectively) is a Killing vector of the whole metric and of each of the two singular metrics. These Killing vectors will always survive the limits because their dependence on ρ can always be removed and, therefore, they need to be taken into account.

We are now ready to study the limits of the sigma model action (2.3) and of its symmetries. We can take four different limits (there are two different ways of taking the $\rho \to 0, \infty$ limits) and we consider them separately in the following sections.

2.2 $\rho \to \infty$ with auxiliary fields

Performing a Hubbard-Stratonovich transformation eq. (1.1) for $\alpha = 2$, we rewrite the action eq. (2.3) in the following form

$$S\left[x^\mu(\zeta), \gamma_{ij}(\zeta)\right] = -\frac{T}{2}\int d^{p+1}\zeta\sqrt{|\gamma|}\left[\gamma^{ij}h_{ij} - \varepsilon\left(\frac{1}{4\rho^2}\lambda^2 + \lambda\sqrt{|\gamma^{ij}k_{ij}|}\right) - (p-1)\right]\,, \tag{2.15}$$

where we have introduced a parameter $\varepsilon = \pm 1$ with $\varepsilon = +1$ if the time coordinate is included in k and where $\varepsilon = -1$ otherwise. In this way we always have[9]

$$\varepsilon|\gamma^{ij}k_{ij}| = \gamma^{ij}k_{ij}\,. \tag{2.16}$$

The symmetries of the action eq. (2.15) are preserved provided λ transforms as the solution of its equation, namely $\lambda = -2\rho^2\sqrt{|\gamma^{ij}k_{ij}|}$. Thus,

$$\delta\lambda = \rho\frac{\varepsilon}{\sqrt{|\gamma^{ij}k_{ij}|}}\gamma^{ij}\partial_i x^\mu \partial_j x^\nu \pounds_{\xi^{(-1)}} k_{\mu\nu}\,. \tag{2.17}$$

Observe that, in the special case in which $\xi^{(1)} = 0$ so that $\xi^{(-1)}$ is a Killing vector of g, h and k, this variation vanishes automatically, λ is invariant and so is the full action.

It is important to realize that in the present case we cannot speak properly of isometries and Killing vectors because the variable λ is not a coordinate nor the derivative of a coordinate and has no known geometrical interpretation. We can still speak of the Lie algebra of symmetries of this action, though, and it is clear that all of them[10] are associated to the

[9] If the time direction is included in k (that is: if one of the signs of the signature of k is a $+$), then there should always be a coordinate system in which only the time coordinate is non-constant and, then $\gamma^{ij}k_{ij} > 0$ with our mostly minus signature convention for the metric. For point particles, this is equivalent to the assumption that the worldlines should always be timelike which is implicitly made in the Nambu-Goto- and Polyakov-type actions.

[10] Strictly speaking one can exclude additional emergent symmetries unrelated to the isometries of the original metric.

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Killing vectors of the original metric and their Lie brackets correspond to the commutators of the generators of the symmetries of the action.

Now we introduce infinitesimal transformation parameters a^m, b^x for every Killing vector π_m, ϖ_x, defining the following transformations of the variables of the action

$$\begin{aligned}\delta x^\mu &= a^m \pi_m^\mu + \frac{b^x}{\rho}\varpi_x^{(-1)\,\mu} + b^x \rho\, \varpi_x^{(1)\,\mu}\,,\\ \delta\lambda &= b^x \rho \frac{\varepsilon}{\sqrt{|\gamma^{ij}k_{ij}|}}\gamma^{ij}\partial_i x^\mu \partial_j x^\nu \pounds_{\varpi_x^{(1)}} h_{\mu\nu}\,.\end{aligned} \tag{2.18}$$

Assuming that $\varpi_x^{(1)} \neq 0$, in order to get a finite result in the $\rho \to \infty$ limit of these transformations, we must replace

$$b^x \to \frac{b^x}{\rho}\,. \tag{2.19}$$

After taking the $\rho \to \infty$ limit, we end up with the following action:

$$S\left[x^\mu(\zeta), \gamma_{ij}(\zeta)\right] = -\frac{T}{2}\int d^{p+1}\zeta \sqrt{|\gamma|}\left[\gamma^{ij}h_{ij} - \varepsilon\lambda\sqrt{|\gamma^{ij}k_{ij}|} - (p-1)\right]\,, \tag{2.20}$$

which is invariant under the transformation rules

$$\begin{aligned}\delta x^\mu &= a^m \pi_m^\mu + b^x \varpi_x^{(1)\,\mu}\,,\\ \delta\lambda &= b^x \frac{\varepsilon}{\sqrt{|\gamma^{ij}k_{ij}|}}\gamma^{ij}\partial_i x^\mu \partial_j x^\nu \pounds_{\varpi_x^{(1)}} h_{\mu\nu}\,.\end{aligned} \tag{2.21}$$

Notice that there is one symmetry generator for each of the isometry generators of the original metric and each of the symmetries of the original action.

This is clearly correct when the components $\varpi_x^{(1)} \neq 0$. When they vanish, we can make the opposite rescaling $b^x \to b^x\rho$ and the symmetry generated by $\varpi_x^{(-1)}$ will survive the limit.

It is convenient to split the index x as follows: x_{-1} which corresponds to ϖ generators with no $\varpi^{(1)}$ component, x_1 which corresponds to ϖ generators with no $\varpi^{(-1)}$ component, and x_2, which corresponds to ϖ generators with both components:

$$\{\varpi_x\} = \{\varpi_{x_{-1}}, \varpi_{x_1}, \varpi_{x_2}\}\,. \tag{2.22}$$

By definition, they are of the form

$$\varpi_{x_{-1}} = \frac{1}{\rho}\varpi_{x_{-1}}^{(-1)}\,, \qquad \varpi_{x_1} = \rho\varpi_{x_1}^{(1)}\,, \qquad \varpi_{x_2} = \frac{1}{\rho}\varpi_{x_2}^{(-1)} + \rho\varpi_{x_2}^{(1)}\,. \tag{2.23}$$

Using this notation, we only have to take into account the $\varpi_{x_{-1}}$ components at the very end, to compute the final symmetry algebra.

The algebra of symmetries is generated by

1. The vectors π_m, which were Killing vectors of the original metric g and remain Killing vectors of the two degenerate metrics h and k.

2. The vectors $\varpi_x^{(1)}$ coming from the $\varpi_{x_1}^{(1)}, \varpi_{x_2}^{(1)}$, which are Killing vectors of the degenerate metric k only. Their index x runs over all the values.

3. The vectors $\varpi_{x_{-1}}^{(-1)}$ which are Killing vectors of g, h and k.

The algebra of symmetries is determined by the non-vanishing brackets

$$[\pi_m, \pi_n] = f_{mn}{}^p \pi_p\,, \qquad [\pi_m, \varpi_x^{(1)}] = f_{mx}{}^y \varpi_y^{(1)}\,, \tag{2.24}$$

$$[\pi_m, \varpi_{x_{-1}}^{(-1)}] = f_{mx_{-1}}{}^{y_{-1}} \varpi_{y_{-1}}^{(-1)}\,, \qquad [\varpi_{x_{-1}}^{(-1)}, \varpi_y^{(1)}] = f_{x_{-1}y}{}^m \pi_m\,. \tag{2.25}$$

Although the specific details will differ, some patterns we have seen in this first limit (in particular, the rescaling of the transformation parameters and the survival after the limit of the same number of symmetries present in the original action, with some different commutators) will repeat themselves in the other three limits that we are going to study next.

2.3 $\rho \to 0$ with auxiliary fields

The $\rho \to 0$ limits with auxiliary fields are closely related to the $\rho \to \infty$ limits with auxiliary fields discussed above. Effectively, after replacing ρ by $1/\rho$ and a suitable rescaling, the two actions are related by an interchange of the degenerate metrics $h_{\mu\nu}$ and $k_{\mu\nu}$. It has, however, the effect, that one obtains a slightly different algebra of symmetries in this limit. One finds that the transformations of the coordinates after the limit is taken (with the necessary rescalings of the parameters) are given by

$$\delta x^\mu = a^m \pi_m^\mu + b^x \varpi_x^{(-1)\,\mu} + b^{x_1} \varpi_{x_1}^{(1)\,\mu}\,. \tag{2.26}$$

These symmetries are generated by

1. The vectors π_m, which were Killing vectors of the original metric g and remain Killing vectors of the two degenerate metrics h and k.

2. The vectors $\varpi_x^{(-1)}$ coming from the $\varpi_{x_{-1}}^{(-1)}, \varpi_{x_2}^{(-1)}$, which are Killing vectors of the degenerate metric h only. Their index x runs over all the values.

3. The vectors $\varpi_{x_1}^{(1)}$ which are Killing vectors of g, h and k.

The algebra of symmetries takes the form

$$[\pi_m, \pi_n] = f_{mn}{}^p \pi_p\,, \qquad [\pi_m, \varpi_x^{(-1)}] = f'_{mx}{}^y \varpi_y^{(-1)}\,, \tag{2.27}$$

$$[\pi_m, \varpi_{x_1}^{(1)}] = f'_{mn}{}^{x_1} \varpi_{x_1}^{(1)}\,, \qquad [\varpi_x^{(-1)}, \varpi_{x_1}^{(1)}] = f_{xx_1}{}^m \pi_m\,. \tag{2.28}$$

It is worth comparing this algebra with the algebra obtained in the $\rho \to \infty$ limit, given in eq. (2.24). The structure is very similar: one can be formally obtained from the other by exchanging all the 1 and -1 indices, except for those generators which have x_1, x_{-1} subindices, which survive both limits. It is worth stressing that the final structure constants $f'_{mx}{}^y$ are, in general, different from the final $f_{mx}{}^y$ structure constants obtained in the $\rho \to \infty$ limit and that neither of them has to coincide with the original structure constants in the $[\pi, \varpi]$ brackets. This can be seen explicitly in the examples (see, for instance, tables 1 and 3).

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2.4 $\rho \to \infty$ with rescalings

There is a second procedure for taking the $\rho \to \infty$ limit which does not require the use of auxiliary fields: removing the leading divergence by rescaling the worldvolume metric and the tension according to

$$\gamma_{ij} \to \rho^2 \gamma_{ij}\,, \qquad T \to \rho^{-(p+1)} T\,, \tag{2.29}$$

after which the action eq. (2.1) takes the form

$$S\left[x^\mu(\zeta), \gamma_{ij}(\zeta)\right] = -\frac{T}{2} \int d^{p+1}\zeta \sqrt{|\gamma|} \left[\frac{1}{\rho^2} \gamma^{ij} h_{ij} + \gamma^{ij} k_{ij} - (p-1)\right]. \tag{2.30}$$

We can now take the $\rho \to \infty$ limit directly without introducing an auxiliary field, obtaining

$$S\left[x^\mu(\zeta), \gamma_{ij}(\zeta)\right] = -\frac{T}{2} \int d^{p+1}\zeta \sqrt{|\gamma|} \left[\gamma^{ij} k_{ij} - (p-1)\right]. \tag{2.31}$$

Taking the limit in the transformation rules, we obtain the same rules as before:

$$\delta x^\mu = a^m \pi_m^\mu + b^{x-1} \varpi_{x-1}^{(-1)\,\mu} + b^x \varpi_x^{(1)\,\mu}\,. \tag{2.32}$$

These symmetries are generated by the vectors $\pi_m, \varpi_{x-1}^{(-1)}$, which are Killing vectors of g, h and k and by the vectors $\varpi_x^{(1)}$ which are Killing vectors of k only. The invariance of the action under these symmetries follows trivially. Furthermore, the algebra of symmetries is the same as the one we found before in eq. (2.24).

2.5 $\rho \to 0$ without rescalings

The $\rho \to 0$ limit the action eq. (2.3) can be taken without rescalings or auxiliary variables. We get directly eq. (2.31) with the degenerate metric $k_{\mu\nu}$ replaced by $h_{\mu\nu}$ and the same symmetry algebra we just found above.

It is worth mentioning that the action of some of the symmetry generators on the actions obtained by taking limits without the use of auxiliary fields can be trivial.

Having explained the general techniques, we are now going to apply them in the next sections to several examples.

3 Example 1: The standard limits of Minkowski spacetime

In this section we will recover some very well-known results for Minkowski spacetime using the machinery developed in the previous section, putting it to the test. We will first consider particle limits with $p = 0$ and then we will extend these limits to p-brane limits where p is general.

3.1 Particle limits

Our starting point is the Minkowski metric in Cartesian coordinates given by

$$ds^2 = c^2 dt^2 - dx^m dx^m\,, \qquad m = 1, \cdots, d-1\,. \tag{3.1}$$

$$h_{\mu\nu} dx^\mu dx^\nu = -dx^m dx^m\,, \qquad k_{\mu\nu} dx^\mu dx^\nu = dt^2\,, \qquad \rho = c\,. \tag{3.2}$$

The Killing vectors are given by

$$\xi_\mu = \partial_\mu \,, \qquad \xi_{\mu\nu} = 2\eta_{\mu\nu}{}^{\rho}{}_{\sigma} x^\sigma \partial_\rho \,, \tag{3.3}$$

and satisfy the Lie algebra

$$[\xi_{\mu\nu}, \xi_{\rho\sigma}] = +\eta_{\mu\rho}\xi_{\nu\sigma} + \eta_{\nu\sigma}\xi_{\mu\rho} - \eta_{\mu\sigma}\xi_{\nu\rho} - \eta_{\nu\rho}\xi_{\mu\sigma} \,, \tag{3.4a}$$

$$[\xi_\mu, \xi_{\rho\sigma}] = -2\eta_{\mu[\rho}\xi_{\sigma]} \,. \tag{3.4b}$$

They admit an expansion in the parameter c so that they can be identified with the vectors in the general framework in the following way

$$\xi_0 = \varpi_0 = \frac{1}{c}\partial_t \,, \tag{3.5a}$$

$$\xi_m = \pi_m = \partial_m \,, \tag{3.5b}$$

$$\xi_{m0} = \varpi_{m0} = ct\partial_m + \frac{1}{c}x^m \partial_t \,, \tag{3.5c}$$

$$\xi_{mn} = \pi_{mn} = x^m\partial_n - x^n\partial_m \,. \tag{3.5d}$$

The non-vanishing Lie brackets of these Killing vectors are[11]

$$[\pi_{mn}, \pi_{pq}] = -\delta_{mp}\pi_{nq} - \delta_{nq}\pi_{mp} + \delta_{mq}\pi_{np} + \delta_{np}\pi_{mq} \,, \tag{3.6a}$$

$$[\pi_m, \pi_{np}] = 2\delta_{m[n}\pi_{p]} \,, \tag{3.6b}$$

$$[\pi_{mn}, \varpi_{p0}] = -2\delta_{p[m}\varpi_{n]0} \,, \tag{3.6c}$$

$$[\pi_m, \varpi_{n0}] = \delta_{mn}\varpi_0 \,, \tag{3.6d}$$

$$[\varpi_0, \varpi_{m0}] = \pi_m \,, \tag{3.6e}$$

$$[\varpi_{(m0)}, \varpi_{(n0)}] = \pi_{mn} \,, \tag{3.6f}$$

which fits the general pattern of eq. (2.9). Furthermore, we see that the subalgebra generated by the πs (which is the algebra of the Euclidean group in $d-1$ dimensions) leaves invariant the two singular metrics

$$\pounds_\pi k_{\mu\nu} = \pounds_\pi h_{\mu\nu} = 0 \,. \tag{3.7}$$

Defining

$$\varpi_0^{(-1)} \equiv \partial_t \,, \tag{3.8a}$$

$$\varpi_{m0}^{(-1)} \equiv x^m\partial_t \,, \tag{3.8b}$$

$$\varpi_{m0}^{(1)} \equiv t\partial_m \,, \tag{3.8c}$$

we find

$$\pounds_{\varpi^{(-1)}} h_{\mu\nu} = \pounds_{\varpi^{(1)}} k_{\mu\nu} = 0 \,, \tag{3.9}$$

and

$$\pounds_{\varpi_{m0}^{(1)}} h_{\mu\nu} + \pounds_{\varpi_{m0}^{(-1)}} k_{\mu\nu} = 0 \,, \qquad \pounds_{\varpi_0^{(-1)}} k_{\mu\nu} = 0 \,. \tag{3.10}$$

The Killing vector $\varpi_0^{(-1)}$ belongs to the class of those which do not have a $\varpi^{(1)}$ component and generate symmetries of g, h and k. It will therefore survive all the limits.

[11] Remember that our Minkowski metric has mostly minus signature.

3.1.1 $c \to \infty$ with auxiliary fields

The $c \to \infty$ limit of the p-brane action in Minkowski spacetime takes the general form in eq. (2.20) with $\epsilon = 1$. For the particular case $p = 0$ in which the only component of the worldline metric γ is replaced by the Einbein e ($\gamma = e^2$) and the constant T takes the form $T = mc \to \tilde{m}$, we get

$$S\left[x^\mu(\zeta), e(\zeta)\right] = \frac{\tilde{m}}{2}\int d\zeta \left[e^{-1}\dot{x}^m\dot{x}^m + \lambda\dot{t} - e\right] . \tag{3.11}$$

This action is invariant under the global transformations

$$\delta x^\mu = a^m{\pi_m}^\mu + \tfrac{1}{2}\sigma^{mn}{\pi_{mn}}^\mu + b^m\varpi^{(1)}_{m0} + c\varpi^{(-1)}_0 . \tag{3.12}$$

The algebra of symmetries of this action is that generated by the $\{\pi_m, \pi_{mn}\}$ generators (the Euclidean group in $d-1$ dimensions) together with the $\varpi^{(-1)}_0$ and $\varpi^{(1)}_{m0}$ generators, which have the following non-trivial Lie brackets:

$$[\pi_{mn}, \varpi^{(1)}_{p0}] = -2\delta_{p[m}\varpi^{(1)}_{n]0} , \qquad [\varpi^{(-1)}_0, \varpi^{(1)}_{m0}] = \pi_m . \tag{3.13}$$

This is the algebra of the *Galilei group*.

The equation of motion of the Lagrange multiplier λ is $\dot{t} = 0$ and, therefore, we can eliminate the $\lambda\dot{t}$ term from the action. The solution to the equation of motion of the Einbein is

$$e = \sqrt{-\dot{x}^m\dot{x}^m} . \tag{3.14}$$

Following [12], the problematic negative sign of the square root argument problem can be fixed by starting with a tachyonic particle replacing $\tilde{m} \to i\tilde{m}$.[12] After that change we arrive at the action of a massless Galilean particle

$$S\left[x^m(\zeta)\right] = \tilde{m}\int d\zeta\sqrt{\dot{x}^m\dot{x}^m} . \tag{3.15}$$

3.1.2 $c \to 0$ with auxiliary fields

As indicated in section 2.3, the action for this case can be obtained by interchanging the degenerate metrics h and k in the original action, while at the same time replacing c by $1/c$. For $p = 0$, $Tc = mc^2 \to \tilde{m}$ and $\varepsilon = -1$, the resulting action is given by [14]

$$S\left[x^\mu(\zeta), e(\zeta)\right] = -\frac{\tilde{m}}{2}\int d\zeta \left[e^{-1}\dot{t}^2 + \lambda\sqrt{\dot{x}^m\dot{x}^m} + e\right] , \tag{3.16}$$

which is invariant under the global transformations

$$\delta x^\mu = a^m{\pi_m}^\mu + \tfrac{1}{2}\sigma^{mn}{\pi_{mn}}^\mu + b^m\varpi^{(-1)}_{m0} + c\varpi^{(-1)}_0 . \tag{3.17}$$

The algebra of symmetries is that generated by the $\{\pi_m, \pi_{mn}\}$ generators (the Euclidean group in $d-1$ dimensions) together with the $\{\varpi^{(-1)}_0, \varpi^{(-1)}_{m0}\}$ generators, which only have the following non-trivial Lie brackets with the πs:

$$[\pi_{mn}, \varpi^{(-1)}_{p0}] = -2\delta_{p[m}\varpi^{(-1)}_{n]0} , \qquad [\pi_m, \varpi^{(-1)}_{n0}] = \delta_{mn}\varpi^{(-1)}_0 . \tag{3.18}$$

This is the algebra of the *Carroll group*.

The equation of motion of λ eliminates any possible dynamics: $\dot{x}^m = 0$. The remaining terms in the action do not contain any degrees of freedom.

[12] For a tachyonic particle, $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu < 1$ in our conventions and $\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$ is imaginary. Therefore the constant in front of the action must also be imaginary in order for the action to be real.

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3.1.3 $c \to \infty$ with rescalings

The $c \to \infty$ limit can also be taken by rescaling the Einbein e and $\tilde{m}$ thus removing the leading divergence in the metric in the following way

$$\tilde{m} \to c^{-1}\tilde{m}\,, \qquad e \to c^2 e\,. \tag{3.19}$$

The particle action takes the form

$$S\left[x^\mu(\zeta), e(\zeta)\right] = -\frac{\tilde{m}}{2}\int d\zeta \left[\frac{1}{c^2}e^{-1}\dot{x}^m\dot{x}^m + e^{-1}\dot{t}^2 + e\right]. \tag{3.20}$$

We can take the $c \to \infty$ limit without introducing any auxiliary field, obtaining

$$S\left[x^\mu(\zeta), e(\zeta)\right] = -\frac{\tilde{m}}{2}\int d\zeta \left[e^{-1}\dot{t}^2 + e\right]. \tag{3.21}$$

The limit in the transformation laws leads to

$$\delta x^\mu = a^m {\pi_m}^\mu + \tfrac{1}{2}\sigma^{mn}{\pi_{mn}}^\mu + b^m \varpi^{(1)}_{m0} + c\varpi^{(-1)}_0\,, \tag{3.22}$$

and the algebra of the symmetries is the one of the Galilei group eq. (3.13).

3.1.4 $c \to 0$ without rescalings

The $c \to 0$ limit can be taken without any rescaling, so that the action is now given by [15]

$$S\left[x^\mu(\zeta), e(\zeta)\right] = -\frac{\tilde{m}}{2}\int d\zeta \left[e^{-1}\dot{x}^m\dot{x}^m + e\right]. \tag{3.23}$$

The action is invariant under the global transformations

$$\delta x^\mu = a^m {\pi_m}^\mu + \tfrac{1}{2}\sigma^{mn}{\pi_{mn}}^\mu + b^m \varpi^{(-1)}_{m0} + c\varpi^{(-1)}_0\,, \tag{3.24}$$

so that the algebra of the symmetries is the one of the Carroll group eq. (3.18).

3.2 The p-brane limits

Starting from the Minkowski metric we now define a p-brane limit by rescaling the first $p+1$ Cartesian coordinates x^A, $A = 0, \cdots, p$ with a parameter ρ as $x^A \equiv \rho y^A$ obtaining

$$ds^2 = \rho^2\eta_{AB}dy^A dy^B - dx^m dx^m\,, \qquad A = 0, \cdots p\,, m = 1, \cdots, d-(p+1)\,. \tag{3.25}$$

Here η_{AB} is the Minkowski metric in the $(p+1)$-dimensional subspace described by the coordinates y^A. The two singular metrics that occur in the $\rho \to 0$ and $\rho \to \infty$ limits are

$$h_{\mu\nu}dx^\mu dx^\nu = -dx^m dx^m\,, \qquad k_{\mu\nu}dx^\mu dx^\nu = \eta_{AB}dy^A dy^B\,. \tag{3.26}$$

The Killing vectors of Minkowski spacetime in Cartesian coordinates are given in eqs. (3.3) and their Lie algebra is given in eqs. (3.4). They admit an expansion in the parameter ρ, so

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that they can be identified with the Killing vectors in the general framework as

$$\varpi_A = \frac{1}{\rho}\partial_A\,, \tag{3.27a}$$

$$\pi_m = \partial_m\,, \tag{3.27b}$$

$$\pi_{AB} = 2\eta_{AB}{}^D{}_C y^C \partial_B = -\eta_{AC} y^C \partial_B + \eta_{BC} y^C \partial_A\,, \tag{3.27c}$$

$$\varpi_{mA} = \rho\eta_{AB} y^B \partial_m + \frac{1}{\rho} x^m \partial_A\,, \tag{3.27d}$$

$$\pi_{mn} = 2\eta_{mn}{}^p{}_q x^q \partial_p = x^m \partial_n - x^n \partial_m\,, \tag{3.27e}$$

The non-vanishing Lie brackets of these Killing vectors are

$$[\pi_{mn}, \pi_{pq}] = -\delta_{mp}\pi_{nq} - \delta_{nq}\pi_{mp} + \delta_{mq}\pi_{np} + \delta_{np}\pi_{mq}\,, \tag{3.28a}$$

$$[\pi_{AB}, \pi_{CD}] = +\eta_{AC}\pi_{BD} + \eta_{BD}\pi_{AC} - \eta_{AD}\pi_{BC} - \eta_{BC}\pi_{AD}\,, \tag{3.28b}$$

$$[\pi_m, \pi_{np}] = 2\delta_{m[n}\pi_{p]}\,, \tag{3.28c}$$

$$[\pi_{mn}, \varpi_{pA}] = -2\delta_{p[m}\varpi_{n]A}\,, \tag{3.28d}$$

$$[\pi_m, \varpi_{nA}] = \delta_{mn}\varpi_A\,, \tag{3.28e}$$

$$[\pi_{AB}, \varpi_C] = 2\eta_{C[A}\varpi_{B]}\,, \tag{3.28f}$$

$$[\pi_{AB}, \varpi_{mC}] = 2\eta_{C[A|}\varpi_{m|B]}\,, \tag{3.28g}$$

$$[\varpi_A, \varpi_{mB}] = \eta_{AB}\pi_m\,, \tag{3.28h}$$

$$[\varpi_{mA}, \varpi_{nB}] = \eta_{AB}\pi_{mn} - \delta_{mn}\pi_{AB}\,, \tag{3.28i}$$

which fits the general pattern given in eq. (2.9).

The subalgebra generated by the πs is the product of the Lorentz algebra in $(p+1)$ dimensions and the algebra of the Euclidean group in $d-(p+1)$ dimensions. It leaves invariant the two singular metrics

$$\pounds_\pi k_{\mu\nu} = \pounds_\pi h_{\mu\nu} = 0\,. \tag{3.29}$$

Defining

$$\varpi_A^{(-1)} \equiv \partial_A\,, \tag{3.30a}$$

$$\varpi_{mA}^{(-1)} = x^m \partial_A\,, \tag{3.30b}$$

$$\varpi_{mA}^{(1)} = \eta_{AB} y^B \partial_m\,, \tag{3.30c}$$

we find, in all cases,

$$\pounds_{\varpi^{(-1)}} h_{\mu\nu} = \pounds_{\varpi^{(1)}} k_{\mu\nu} = 0\,, \tag{3.31}$$

and

$$\pounds_{\varpi_{mA}^{(1)}} h_{\mu\nu} + \pounds_{\varpi_{mA}^{(-1)}} k_{\mu\nu} = 0\,, \qquad \pounds_{\varpi_A^{(-1)}} k_{\mu\nu} = 0\,. \tag{3.32}$$

All the Killing vectors $\varpi_A^{(-1)}$ belong to the class of those which do not have a $\varpi^{(1)}$ component and generate symmetries of g, h and k. They will therefore survive all the limits.

3.2.1 $\rho \to \infty$ with auxiliary fields

The $\rho \to \infty$ limit of the p-brane action in Minkowski spacetime takes the general form in eq. (2.20) with $h_{\mu\nu}$ given in eq. (3.26). This action is invariant under the global transformations

$$\delta x^\mu = a^m \pi_m{}^\mu + \tfrac{1}{2}\sigma^{mn}\pi_{mn}{}^\mu + \tfrac{1}{2}\sigma^{AB}\pi_{AB}{}^\mu + b^{mA}\varpi^{(1)}_{mA} + c^A \varpi^{(-1)}_A \,. \tag{3.33}$$

The algebra of symmetries of this action is that generated by the $\{\pi_m, \pi_{mn}, \pi_{AB}\}$ generators (those of the Lorentz group in $(p+1)$ dimensions and of the Euclidean group in $d-(p+1)$ dimensions) together with the $\varpi^{(-1)}_A$ and $\varpi^{(1)}_{mA}$ generators, which have the following non-trivial Lie brackets:

$$[\pi_{mn}, \varpi^{(1)}_{pA}] = -2\delta_{p[m}\varpi^{(1)}_{n]A}\,, \tag{3.34a}$$

$$[\pi_{AB}, \varpi^{(-1)}_{C}] = +2\eta_{C[A}\varpi^{(-1)}_{B]}\,, \tag{3.34b}$$

$$[\pi_{AB}, \varpi^{(1)}_{mC}] = +2\eta_{C[A|}\varpi^{(1)}_{m|B]}\,, \tag{3.34c}$$

$$[\varpi^{(-1)}_{A}, \varpi^{(1)}_{mB}] = \eta_{AB}\pi_m\,, \tag{3.34d}$$

This is the algebra of the *p-brane Galilei group.*

3.2.2 $\rho \to 0$ with auxiliary fields

As indicated in section 2.3, the action can be obtained by interchanging the degenerate metrics h and k, see eq. (3.26), in the action eq. (2.20), replacing simultaneously ρ by $1/\rho$. The resulting action is invariant under the global transformations

$$\delta x^\mu = a^m \pi_m{}^\mu + \tfrac{1}{2}\sigma^{mn}\pi_{mn}{}^\mu + \tfrac{1}{2}\sigma^{AB}\pi_{AB}{}^\mu + b^{mA}\varpi^{(-1)}_{mA} + c^A \varpi^{(-1)}_A \,. \tag{3.35}$$

The algebra of symmetries is that generated by the $\{\pi_m, \pi_{mn}, \pi_{AB}\}$ generators (those of the Lorentz group in $(p+1)$ dimensions and of the Euclidean group in $d-(p+1)$ dimensions) together with the $\{\varpi^{(-1)}_A, \varpi^{(-1)}_{mA}\}$ generators, which only have the following non-trivial Lie brackets with the πs:

$$[\pi_{mn}, \varpi^{(-1)}_{pA}] = -2\delta_{p[m}\varpi^{(-1)}_{n]A}\,, \tag{3.36a}$$

$$[\pi_{m}, \varpi^{(-1)}_{nA}] = \delta_{mn}\varpi^{(-1)}_{A}\,, \tag{3.36b}$$

$$[\pi_{AB}, \varpi^{(-1)}_{C}] = +2\eta_{C[A}\varpi^{(-1)}_{B]}\,, \tag{3.36c}$$

$$[\pi_{AB}, \varpi^{(-1)}_{mC}] = +2\eta_{C[A|}\varpi^{(-1)}_{m|B]}\,, \tag{3.36d}$$

This is the algebra of the *p-brane Carroll group*. It is worth comparing this algebra with the p-brane Galilei algebra in eq. (3.34). As mentioned at the end of section 2.3, the subalgebra generated by the πs is common to the two limiting algebras and the rest of the relevant commutators are related by the $1 \leftrightarrow -1$ duality (which acts on all indices except on those of $\varpi^{(-1)}_A$, which survive in both limits). In table 1 we compare the Lie brackets of the p-brane Galilei and Carroll algebras from the point of view of this $1 \leftrightarrow -1$ duality. For $p = 0$ a geometric interpretation of this duality has been discussed in [13].

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p-brane Galilei	p-brane Carroll
$[\pi_{mn}, \varpi^{(1)}_{pA}] = -2\delta_{p[m}\varpi^{(1)}_{n]A}\,,$	$[\pi_{mn}, \varpi^{(-1)}_{pA}] = -2\delta_{p[m}\varpi^{(-1)}_{n]A}\,,$
$[\pi_{AB}, \varpi^{(-1)}_{C}] = 2\eta_{C[A}\varpi^{(-1)}_{B]}\,,$	$[\pi_{AB}, \varpi^{(-1)}_{C}] = 2\eta_{C[A}\varpi^{(-1)}_{B]}\,,$
$[\pi_{AB}, \varpi^{(1)}_{mC}] = 2\eta_{C[A\|}\varpi^{(1)}_{m\|B]}\,,$	$[\pi_{AB}, \varpi^{(-1)}_{mC}] = 2\eta_{C[A\|}\varpi^{(-1)}_{m\|B]}\,,$
$[\varpi^{(-1)}_{A}, \varpi^{(1)}_{mB}] = \eta_{AB}\pi_m\,,$	$[\varpi^{(-1)}_{A}, \varpi^{(-1)}_{mB}] = 0\,,$
$[\pi_m, \varpi^{(1)}_{nA}] = 0\,,$	$[\pi_m, \varpi^{(-1)}_{nA}] = \delta_{mn}\varpi^{(-1)}_{A}\,.$

Table 1. This table compares the Lie brackets of the p-brane Galilei and p-brane Carroll algebras for the same p from the $1 \leftrightarrow -1$ duality point of view. The brackets in the first line are related by the $1 \leftrightarrow -1$ duality. The brackets in the second line are related by the same duality, which does not act on the $\varpi^{(-1)}_{A}$ generators, which are common to both algebras. Then, in the third line the right hand side of the Carroll bracket must vanish because it has two -1 indices. Something similar happens in the last line: if one starts with the Carroll bracket and applies the duality rule, then the dual Galilei bracket must vanish because it would have opposite $1, -1$ indices at both sides of the bracket relation.

p-brane Galilei	dual $(d-p-2)$-brane Carroll
$[\pi_{mn}, \pi_{pq}] = +\eta_{mp}\pi_{nq} + \eta_{nq}\pi_{mp} \dots$	$[\pi_{AB}, \pi_{CD}] = +\eta_{AC}\pi_{BD} + \eta_{BD}\pi_{AC} \dots\,,$
$[\pi_{AB}, \pi_{CD}] = +\eta_{AC}\pi_{BD} + \eta_{BD}\pi_{AC} \dots\,,$	$[\pi_{mn}, \pi_{pq}] = +\eta_{mp}\pi_{nq} + \eta_{nq}\pi_{mp} \dots$
$[\pi_{mn}, \varpi^{(1)}_{pA}] = 2\eta_{p[m}\varpi^{(1)}_{n]A}\,,$	$[\pi_{AB}, \varpi^{(-1)}_{mC}] = 2\eta_{C[A\|}\varpi^{(-1)}_{m\|B]}\,,$
$[\pi_{AB}, \varpi^{(1)}_{mC}] = 2\eta_{C[A\|}\varpi^{(1)}_{m\|B]}\,,$	$[\pi_{mn}, \varpi^{(-1)}_{pA}] = 2\eta_{p[m}\varpi^{(-1)}_{n]A}\,,$
$[\pi_{AB}, \varpi^{(-1)}_{C}] = 2\eta_{C[A}\varpi^{(-1)}_{B]}\,,$	$[\pi_{mn}, \pi_{p},] = 2\eta_{p[m}\pi_{n]}\,,$
$[\pi_{mn}, \pi_{p},] = 2\eta_{p[m}\pi_{n]}\,,$	$[\pi_{AB}, \varpi^{(-1)}_{C}] = 2\eta_{C[A}\varpi^{(-1)}_{B]}\,,$
$[\pi_m, \varpi^{(1)}_{nA}] = 0\,,$	$[\varpi^{(-1)}_{A}, \varpi^{(-1)}_{mB}] = 0\,,$
$[\varpi^{(-1)}_{A}, \varpi^{(1)}_{mB}] = \eta_{AB}\pi_m\,,$	$[\pi_m, \varpi^{(-1)}_{nA}] = -\eta_{mn}\varpi^{(-1)}_{A}\,.$

Table 2. This table compares the p-brane Galilei and Carroll algebras from a different so-called brane $\leftrightarrow$ dual brane point of view. This is a formal duality which relates the different generators by interchanging the $A, B, \dots$ and $m, n, \dots$ indices. This duality is one to one provided one compares a p-brane Galilei algebra with a dual $d-p-2$-brane Carroll algebra.

There exists a different formal duality between the p-brane Galilei algebra and the $d-p-2$-brane Carroll algebra [10, 11]. The whole set of non-vanishing Lie brackets of these two Lie algebras are re-arranged in table 2 so as to make manifest the formal duality, sometimes called brane $\leftrightarrow$ dual brane duality, that exists between these two algebras under the interchange of the $A, B, \dots$ and $m, n, \dots$ indices. In this case, in contrast to the $1 \leftrightarrow -1$ mapping, the correspondence between the generators of the algebra is not 1 to 1 unless we compare the p-brane Galilei and the $p' = d-(p+2)$-brane Carroll algebras in which case the range of the A, B indices is $p'+1 = d-p-1$ and the range of the m, n is $d-p'-1 = p+1$.

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3.2.3 $\rho \to \infty$ with rescalings

We can remove the divergence in the action by performing the rescaling

$$\gamma_{ij} \to c^2 \gamma_{ij}\,, \qquad T \to c^{-(p+1)} T\,, \tag{3.37}$$

and then taking the limit. The action involves the $(p+1)$-dimensional subspace with coordinates y^A and it turns to be invariant under

$$\delta x^\mu = a^m \pi_m{}^\mu + \tfrac{1}{2}\sigma^{mn}\pi_{mn}{}^\mu + \tfrac{1}{2}\sigma^{AB}\pi_{AB}{}^\mu + b^{mA}\varpi^{(1)}_{mA} + c^A \varpi^{(-1)}_A\,, \tag{3.38}$$

which corresponds to the p-brane Galilei group eq. (3.34).

3.2.4 $\rho \to 0$ without rescalings

The $\rho \to 0$ limit can be taken without any rescaling and the action involves the $d-(p+1)$-dimensional subspace described by the coordinates x^m. The action is now invariant under the p-brane Carroll symmetry eq. (3.36).

4 Example 2: Limits of Anti-de Sitter spacetime

In this section we are going to apply our general techniques to Anti-de Sitter (AdS) spacetime and explore several new and unconventional singular limits. We will first discuss the so-called holographic limits in which the Lorentz group of the holographic screen[13] together with dilatations survive as isometry algebra[14] and, after that, examples of p-brane limits.

4.1 Holographic limits

Our starting point is the metric of AdS_{p+2} in horospheric coordinates.[15] We use unhatted Greek indices $\mu, \nu, \ldots = 0, 1, \cdots, p$ for the Cartesian coordinates x^μ of the holographic screen and the holographic $(p+1)^{\text{th}}$ coordinate $x^{p+1} \equiv z$ normal to it. We use hatted Greek indices $\hat{\mu}, \hat{\nu}, \cdots = 0, \cdots, p+1$ to indicate the horospheric coordinates $x^{\hat{\mu}} = (x^\mu, z)$. Thus, the AdS_{p+2} spacetime is described by the metric

$$d\hat{s}^2_{p+2} = \left(\frac{R}{z}\right)^2 \left[\eta_{\mu\nu} dx^\mu dx^\nu - dz^2\right]\,, \tag{4.1}$$

which is invariant under the group $\mathrm{SO}(2, p+1)$, whose generators are labeled by an anti-symmetric pair of double-hatted Greek indices $\hat{\hat{\alpha}}, \hat{\hat{\beta}}, \ldots = +, -, \mu$. Hence, there are four kinds of Killing vectors $\hat{\hat{k}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}}$, which are

- The generators of the Lorentz group of the Minkowski metric associated to the holographic screen.
- $\hat{\hat{k}}_{+\alpha}$, which generate translations in the same metric.

[13] Since the holographic screen has codimension 1, the Lorentz group is not the one associated to the dimension of the original spacetime, but a smaller one.

[14] As usual, all the symmetries of the relativistic particle and p-brane actions "survive" in the sense that they are in one-to-one correspondence with a symmetry of the actions that arise after the limits.

[15] Some technical details about these coordinates can be found in appendix A, around eqs. (A.16).

- $\hat{\hat{k}}_{-\alpha}$, which generate conformal transformations of that metric compensated by rescalings of the holographic coordinate z.

- $\hat{\hat{k}}_{-+}$ which generate dilatations of all the coordinates.[16]

We are interested in taking limits in which the radius R, which here will play the role of the generic parameter ρ, goes to $(0, \infty)$. However, in order to apply our machinery, we first need to bring the metric in the desired form. This can be achieved by rescaling the holographic coordinate z defining a new rescaled coordinate w by

$$z = Rw\,. \tag{4.2}$$

After this rescaling, the metric takes form

$$d\hat{s}^2_{p+2} = \frac{1}{w^2}\eta_{\mu\nu}dx^\mu dx^\nu - R^2\frac{dw^2}{w^2}\,. \tag{4.3}$$

We can rewrite this metric in the form of eq. (2.2). In this case, the two singular metrics, $\hat{h}_{\hat{\mu}\hat{\nu}}$ and $\hat{k}_{\hat{\mu}\hat{\nu}}$ have ranks $p+1$ and 1 and signatures $(+, -, \cdots, -, 0)$ and $(0, \cdots, 0, -)$, respectively, with the AdS radius parameter R playing the role of the generic parameter ρ:

$$d\hat{s}^2_{p+2} = \hat{h}_{\hat{\mu}\hat{\nu}}d\hat{x}^{\hat{\mu}}d\hat{x}^{\hat{\nu}} + R^2\hat{k}_{\hat{\mu}\hat{\nu}}d\hat{x}^{\hat{\mu}}d\hat{x}^{\hat{\nu}}\,, \tag{4.4}$$

with

$$\hat{h}_{\hat{\mu}\hat{\nu}}d\hat{x}^{\hat{\mu}}d\hat{x}^{\hat{\nu}} = \frac{1}{w^2}\eta_{\mu\nu}dx^\mu dx^\nu\,, \tag{4.5a}$$

$$\hat{k}_{\hat{\mu}\hat{\nu}}d\hat{x}^{\hat{\mu}}d\hat{x}^{\hat{\nu}} = -\frac{dw^2}{w^2}\,. \tag{4.5b}$$

Notice that, since the metric $\hat{h}_{\mu\nu}$ is proportional to the $(p+1)$-dimensional Minkowski metric, it will be invariant under the $(p+1)$-dimensional Lorentz group $\mathrm{SO}(1,p)$ (actually, under the whole Poincaré group in $(p+1)$ dimensions). The metric is Lorentzian or relativistic in this restricted sense, but non-Lorentzian or non-relativistic when all the space and time dimensions are taken into account.

In this setup, in the $R \to 0$ for a constant value of the coordinate w, we will be getting close to the origin $(z \to 0)$ while the curvature of the AdS spacetime $\sim 1/R^2$ grows and the paraboloid degenerates into a sort of cone. In the $R \to \infty$ limit, for a constant value of the coordinate w, we will be getting close to the boundary $z \to \infty$ while the curvature decreases because the paraboloid grows in size.

In these new coordinates, the Killing vectors take the form

$$\hat{\hat{\xi}}_{\alpha\beta} = 2\eta_{\alpha\beta}{}^{\mu}{}_{\nu}x^\nu\partial_\mu\,, \tag{4.6a}$$

$$\hat{\hat{\xi}}_{+\alpha} = \frac{R}{2}\partial_\alpha\,, \tag{4.6b}$$

$$\hat{\hat{\xi}}_{-\alpha} = \frac{1}{2R}\left\{x\cdot x\partial_\alpha - 2\eta_{\alpha\mu}x^\mu(x^\nu\partial_\nu + w\partial_w)\right\} - \frac{R}{2}w^2\partial_\alpha\,, \tag{4.6c}$$

$$\hat{\hat{\xi}}_{-+} = \tfrac{1}{2}\left(x^\mu\partial_\mu + w\partial_w\right)\,. \tag{4.6d}$$

[16]See appendix A for the explicit form of the Killing vectors.

Following the general framework explained in section 2.1, we relabel these Killing vectors, according to their dependence on the parameter R, as follows:

$$\pi_{\alpha\beta} = 2\eta_{\alpha\beta}{}^{\mu}{}_{\nu}x^{\nu}\partial_{\mu}\,, \tag{4.7a}$$

$$\pi = \tfrac{1}{2}\left(x^{\mu}\partial_{\mu} + w\partial_{w}\right)\,, \tag{4.7b}$$

$$\varpi_{+\alpha} = \frac{R}{2}\partial_{\alpha}\,, \tag{4.7c}$$

$$\varpi_{-\alpha} = \frac{1}{2R}\left\{x\cdot x\partial_{\alpha} - 2\eta_{\alpha\mu}x^{\mu}(x^{\nu}\partial_{\nu} + w\partial_{w})\right\} - \frac{R}{2}w^{2}\partial_{\alpha}\,. \tag{4.7d}$$

The Lie brackets of these Killing vectors can be read from eqs. (A.23). They take the form

$$[\pi_{\mu\nu}, \pi_{\rho\sigma}] = \eta_{\mu\rho}\pi_{\nu\sigma} + \eta_{\nu\sigma}\pi_{\mu\rho} - \eta_{\mu\sigma}\pi_{\nu\rho} - \eta_{\nu\rho}\pi_{\mu\sigma}\,, \tag{4.8a}$$

$$[\pi_{\mu\nu}, \varpi_{\pm\alpha}] = 2\eta_{\alpha[\mu|}\varpi_{\pm|\nu]}\,, \tag{4.8b}$$

$$[\pi, \varpi_{\pm\alpha}] = \pm\tfrac{1}{2}\varpi_{\pm\alpha}\,, \tag{4.8c}$$

$$[\varpi_{+\alpha}, \varpi_{-\beta}] = \tfrac{1}{2}\pi_{\alpha\beta} - \eta_{\alpha\beta}\pi\,, \tag{4.8d}$$

which conforms to the general pattern given in eq. (2.9).

We can now consider several different limits. For simplicity, we restrict to the sigma model action of a massive point particle moving in the AdS_{p+2} spacetime. The transformations that leave invariant such an action are

$$\begin{aligned}
\delta x^{\mu} &= \tfrac{1}{2}\sigma^{\alpha\beta}\pi_{\alpha\beta}{}^{\mu} + 2a^{\alpha}\varpi_{+\alpha}{}^{\mu} + 2b^{\alpha}\varpi_{-\alpha}{}^{\mu} + 2c\pi^{\mu}\\
&= \sigma^{\mu}{}_{\nu}x^{\nu} + Ra^{\mu} + \frac{1}{R}\left(x\cdot xb^{\mu} - 2b\cdot xx^{\mu}\right) - Rw^{2}b^{\mu} + cx^{\mu}\,,\\
\delta w &= 2b^{\alpha}\varpi_{-\alpha}{}^{w} + c\pi^{w} = -\frac{2}{R}b\cdot xw + cw\,.
\end{aligned} \tag{4.9}$$

We need to rescale the parameters $\sigma^{\mu\nu}, a^{\mu}, b^{\mu}, c$ to keep these transformation rules finite in the $R \to (0, \infty)$ limits. We can always preserve the symmetries corresponding to the generators $\varpi_{+\alpha}$ by absorbing R into a^{μ} since these generators expand only in a term proportional to R:

$$\varpi_{+\alpha} = R\varpi^{(1)}_{+\alpha}\,, \qquad \varpi^{(1)}_{+\alpha} = \tfrac{1}{2}\partial_{\alpha}\,. \tag{4.10}$$

All the symmetries corresponding to the $\varpi_{-\alpha}$s, which can be written as

$$\begin{aligned}
\varpi_{-\alpha} &= \frac{1}{R}\varpi^{(-1)}_{-\alpha} + R\varpi^{(1)}_{-\alpha}\,,\\
\varpi^{(-1)}_{-\alpha} &= \tfrac{1}{2}\left\{x\cdot x\partial_{\alpha} - 2\eta_{\alpha\mu}x^{\mu}(x^{\nu}\partial_{\nu} + w\partial_{w})\right\}\,,\\
\varpi^{(1)}_{-\alpha} &= -\tfrac{1}{2}w^{2}\partial_{\alpha}\,,
\end{aligned} \tag{4.11}$$

will also be preserved but they will be generated by either $\varpi^{(1)}_{-\alpha}$ or $\varpi^{(-1)}_{-\alpha}$ which are Killing vectors of the metrics $k_{\mu\nu}$ and $h_{\mu\nu}$ respectively and not of the full AdS_{p+2} metric.

We will now discuss the different limits.

4.1.1 $R \to \infty$ with auxiliary fields

Following the general procedure, we obtain the following action including an auxiliary field λ:

$$S\left[x^\mu(\zeta), w(\zeta)\right] = -\frac{m}{2}\int d\zeta e\left\{e^{-2}\left[\frac{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{w^2} + \lambda\frac{\dot{w}}{w}\right] + 1\right\}. \tag{4.12}$$

This action is invariant under the following transformations:

$$\begin{aligned}\delta x^\mu &= \sigma^\mu{}_\nu x^\nu + a^\mu - w^2 b^\mu + c x^\mu\,,\\ \delta w &= c w\,,\\ \delta\lambda &= 4 b\cdot\dot{x}\,.\end{aligned} \tag{4.13}$$

The generators of these symmetries are given by $\{\pi_{\alpha\beta}, \pi, \omega^{(1)}_{+\alpha}, \omega^{(1)}_{-\alpha}\}$. The π generators (the only Killing vectors of the three metrics, the original and the two singular ones) correspond to the product of the $(p+1)$-dimensional Lorentz algebra with dilatations. The non-vanishing Lie brackets involving the ϖ generators are given by

$$[\pi_{\alpha\beta}, \omega^{(1)}_{+\gamma}] = 2\eta_{\gamma[\alpha|}\omega^{(1)}_{+|\beta]}\,, \qquad [\pi_{\alpha\beta}, \omega^{(1)}_{-\alpha}] = 2\eta_{\gamma[\alpha|}\omega^{(1)}_{-|\beta]}\,, \qquad [\pi, \omega^{(1)}_{\pm\alpha}] = \mp\tfrac{1}{2}\omega^{(1)}_{\pm\alpha}\,, \tag{4.14}$$

and, therefore, the $\omega^{(1)}_{+\alpha}, \omega^{(1)}_{-\alpha}$ generators can be interpreted as two momenta with opposite weights under dilatations which commute with each other. As usual, the number of generators of the symmetry algebra is the dimensional of the isometry group of the original metric AdS_{p+2}, $\mathrm{SO}(2, p+1)$, namely $(p+3)(p+2)/2$.

4.1.2 $R \to 0$ with auxiliary fields

If we use the rescaled action eq. (4.24), we have a divergent term in the $R \to 0$ limit. We can deal with this problem by performing a Hubbard-Stratonovich transformation introducing an auxiliary variable λ. We thus obtain

$$S\left[x^\mu(\zeta), w(\zeta), \lambda(\zeta)\right] = -\frac{m}{2}\int d\zeta e\left\{e^{-2}\left[-\frac{R^2}{4}\lambda^2 - \lambda\frac{\sqrt{\dot{x}\cdot\dot{x}}}{w} - \left(\frac{\dot{w}}{w}\right)^2\right] + 1\right\}. \tag{4.15}$$

The equation of the auxiliary field λ is solved by

$$\lambda = -\frac{2}{R^2}\frac{\sqrt{\dot{x}\cdot\dot{x}}}{w}\,. \tag{4.16}$$

The field λ must transform as the right-hand side of this equation if the action is going to enjoy the same symmetries as the original one. Thus,

$$\delta\lambda = -\frac{2}{R^2}\delta\frac{\sqrt{\dot{x}\cdot\dot{x}}}{w} = \frac{2}{R}\frac{b\cdot\dot{x}w}{\sqrt{\dot{x}\cdot\dot{x}}}\,. \tag{4.17}$$

If we now we take the $R \to 0$ limit we obtain the following action

$$S\left[x^\mu(\zeta), w(\zeta), \lambda(\zeta)\right] = \frac{m}{2}\int d\zeta e\left\{e^{-2}\left[\lambda\frac{\sqrt{\dot{x}\cdot\dot{x}}}{w} + \left(\frac{\dot{w}}{w}\right)^2\right] - 1\right\}, \tag{4.18}$$

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and transformation rules

$$\begin{aligned}\delta x^\mu &= \sigma^\mu{}_\nu x^\nu + a^\mu + (x\cdot x b^\mu - 2b\cdot x x^\mu) + \frac{c}{2}x^\mu\,,\\ \delta w &= -2b\cdot x w + \frac{c}{2}w\,,\\ \delta\lambda &= 4\frac{b\cdot\dot{x}\dot{w}}{\sqrt{\dot{x}\cdot\dot{x}}}\,.\end{aligned}\tag{4.19}$$

Thus, the action is invariant under so$(d-1)$ transformations, with generators $\{\pi_{\alpha\beta}, \pi, \omega^{(1)}_{+\alpha}, \omega^{(-1)}_{-\alpha}\}$ whose non-vanishing Lie brackets are given by

$$[\pi_{\alpha\beta}, \omega^{(1)}_{+\gamma}] = 2\eta_{\gamma[\alpha|}\omega^{(1)}_{+|\beta]}\,,\tag{4.20a}$$

$$[\pi_{\alpha\beta}, \omega^{(-1)}_{-\alpha}] = 2\eta_{\gamma[\alpha|}\omega^{(-1)}_{-|\beta]}\,,\tag{4.20b}$$

$$[\pi, \omega^{(\pm1)}_{\pm\alpha}] = \mp\tfrac{1}{2}\omega^{(\pm1)}_{\pm\alpha}\,,\tag{4.20c}$$

$$[\varpi^{(1)}_{+\alpha}, \varpi^{(-1)}_{-\beta}] = \tfrac{1}{2}\pi_{\alpha\beta} - \eta_{\alpha\beta}\pi\,.\tag{4.20d}$$

This is nothing but the AdS_{p+2} algebra so$(2,p+1)$ written as in eq. (4.8), which is fully preserved in this limit.

The invariance of the action eq. (4.26) under these transformations is manifest, except for the symmetries with parameters b^μ. Using the fact that under these symmetries we have

$$\delta\frac{\sqrt{\dot{x}\cdot\dot{x}}}{w} = 0\,,\tag{4.21}$$

one can easily show that the transformation of λ, which has become a Lagrange multiplier, is such that the new action obtained after taking the limit is invariant.

4.1.3 $R \to \infty$ with rescalings

In order to take the $R \to \infty$ limit of the action

$$S\left[x^\mu(\zeta), w(\zeta)\right] = -\frac{m}{2}\int d\zeta e\left\{e^{-2}\left[\frac{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{w^2} - R^2\left(\frac{\dot{w}}{w}\right)^2\right] + 1\right\}\,,\tag{4.22}$$

we make the following rescalings

$$e \to Re\,, \qquad m \to \frac{m}{R}\,,\tag{4.23}$$

such that the action takes the form

$$S\left[x^\mu(\zeta), w(\zeta)\right] = -\frac{m}{2}\int d\zeta e\left\{e^{-2}\left[\frac{1}{R^2}\frac{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{w^2} - \left(\frac{\dot{w}}{w}\right)^2\right] + 1\right\}\,.\tag{4.24}$$

Now taking the $R \to \infty$ limit leaves us with

$$S\left[w(\zeta)\right] = \frac{m}{2}\int d\zeta e\left\{e^{-2}\left(\frac{\dot{w}}{w}\right)^2 - 1\right\}\,,\tag{4.25}$$

which should be compared to eq. (4.12).

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$R \to \infty$	$R \to 0$
$[\pi_{\alpha\beta}, \omega^{(1)}_{+\gamma}] = 2\eta_{\gamma[\alpha\|}\omega^{(1)}_{+\|\beta]}\,,$	$[\pi_{\alpha\beta}, \omega^{(1)}_{+\gamma}] = 2\eta_{\gamma[\alpha\|}\omega^{(1)}_{+\|\beta]}\,,$
$[\pi_{\alpha\beta}, \omega^{(1)}_{-\alpha}] = 2\eta_{\gamma[\alpha\|}\omega^{(1)}_{-\|\beta]}\,,$	$[\pi_{\alpha\beta}, \omega^{(-1)}_{-\alpha}] = 2\eta_{\gamma[\alpha\|}\omega^{(-1)}_{-\|\beta]}\,,$
$[\pi, \omega^{(1)}_{+\alpha}] = -\frac{1}{2}\omega^{(1)}_{+\alpha}$	$[\pi, \omega^{(1)}_{+\alpha}] = -\frac{1}{2}\omega^{(1)}_{+\alpha}\,,$
$[\pi, \omega^{(1)}_{-\alpha}] = +\frac{1}{2}\omega^{(1)}_{-\alpha}\,,$	$[\pi, \omega^{(-1)}_{-\alpha}] = \frac{1}{2}\omega^{(-1)}_{-\alpha}\,,$
$[\omega^{(1)}_{+\alpha}, \omega^{(1)}_{-\beta}] = 0\,,$	$[\omega^{(1)}_{+\alpha}, \omega^{(-1)}_{-\beta}] = \frac{1}{2}\pi_{\alpha\beta} - \eta_{\alpha\beta}\pi\,,$

Table 3. In the left column we give the non-trivial Lie brackets of the ϖ generators obtained in the $R \to \infty$ limit, see eq. (4.14). They are related by the $1 \leftrightarrow -1$ duality to the Lie brackets obtained in the $R \to 0$ limit without rescalings. The duality does not apply to the $\omega^{(1)}_{+\alpha}$ generators, which are present in both algebras. The last bracket in the left column vanishes after the duality interchange because there would be two 1 indices in the left hand side. That is why there is no $[\varpi, \varpi]$ bracket in eq. (4.14).

The action eq. (4.25) is invariant under the transformations given in eqs. (4.13). The invariance under many of those transformations is due to the elimination of the $x^\mu(\zeta)$ coordinates which makes these symmetries trivial. In fact, the independence of the coordinate x^μ makes the action (4.25) invariant under any local shifts of the coordinate x^μ. Such additional emerging symmetries can happen when taking limits. They are hard to predict in the general procedure we are using here.

4.1.4 $R \to 0$ without rescalings

The $R \to 0$ limit can be taken directly in eq. (4.22) and gives

$$S\left[x^\mu(\zeta), w(\zeta)\right] = -\frac{m}{2}\int d\zeta e\left\{e^{-2}\frac{\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}{w^2} + 1\right\}. \tag{4.26}$$

This action is invariant under the transformations

$$\begin{aligned} \delta x^\mu &= \sigma^\mu{}_\nu x^\nu + a^\mu + (x\cdot x b^\mu - 2b\cdot x x^\mu) + c x^\mu\,,\\ \delta w &= -2b\cdot x w + c w\,. \end{aligned} \tag{4.27}$$

The generators of these symmetries are given by $\{\pi_{\alpha\beta}, \pi, \omega^{(1)}_{+\alpha}, \omega^{(-1)}_{-\alpha}\}$. The π generators correspond to the product of the $(p+1)$-dimensional Lorentz algebra with dilatations. To compare with the $R \to \infty$ limit result given in eq. (4.14), we give the non-vanishing Lie brackets involving the ϖ generators in the right column of table 3.

4.1.5 $R \to \infty$ plus WZ term

We finally discuss one more way of taking the $R \to \infty$ limit that corresponds to option 3 discussed in the introduction. We will cancel the divergence by coupling the particle moving in AdS_{p+1} to a 1-form field via a Wess-Zumino (WZ) term. First, we switch the sign of the "cosmological constant" term in the action so that it effectively describes a tachyon. Then we rescale $e \to Re$ and $m \to Rm$ and obtain

$$S\left[x^{\mu}(\zeta), w(\zeta)\right] = -\frac{m}{2}\int d\zeta e\left\{e^{-2}\left[\frac{\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{w^2} - R^2\left(\frac{\dot{w}}{w}\right)^2\right] - R^2\right\}. \tag{4.28}$$

Now we add the following pullback of 1-form $A_{\hat{\mu}}\dot{x}^{\hat{\mu}}$ over the worldline, i.e. a WZ term, with a judiciously chosen divergent R^2 term such that all R^2 terms in the action form a complete square (see below):

$$A_{\hat{\mu}}\dot{x}^{\hat{\mu}} = -\frac{m}{2}\left(\pm 2R^2\frac{\dot{w}}{w} + m_{\hat{\mu}}\dot{x}^{\hat{\mu}}\right). \tag{4.29}$$

The 1-form field $m_{\hat{\mu}}(\hat{x})$ must transform under the isometries of AdS_{p+1} according to

$$\begin{aligned} \delta m_{\mu} &= -m_{\nu}\sigma^{\nu}{}_{\mu} - \frac{c}{2}m_{\mu} + \frac{1}{R}\left(m_{\mu}b\cdot x + m_w b_{\mu} w - 2m_{\nu}b^{[\nu}x^{\rho]}\eta_{\rho\mu}\right) \pm 2Rb_{\mu}\,,\\ \delta m_w &= -\frac{c}{2}m_w + Rm_{\nu}b^{\nu}w + \frac{1}{R}m_w b\cdot x\,, \end{aligned} \tag{4.30}$$

making the WZ term invariant. In this way, the addition of the new WZ term preserves the symmetries of the original action.

The modified action whose limit we want to take can now be written as

$$S\left[x^{\mu}(\zeta), w(\zeta)\right] = -\frac{m}{2}\int d\zeta\left\{e^{-1}\left[\frac{\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{w^2} - R^2\left(\frac{\dot{w}}{w} \mp e\right)^2\right] + m_{\hat{\mu}}\dot{x}^{\hat{\mu}}\right\}. \tag{4.31}$$

Now we perform a Hubbard-Stratonovich transformation introducing an auxiliary variable λ thereby rewriting the action in the equivalent form

$$S\left[x^{\mu}(\zeta), w(\zeta)\right] = -\frac{m}{2}\int d\zeta\left\{e^{-1}\left[\frac{\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{w^2} + \frac{1}{4R^2}\lambda^2 + \lambda\left(\frac{\dot{w}}{w} \mp e\right)\right] + m_{\hat{\mu}}\dot{x}^{\hat{\mu}}\right\}. \tag{4.32}$$

The auxiliary field λ must transform as the solution of its equation in order to preserve the symmetries of the original action, that is

$$\delta\lambda = -2R^2\delta\frac{\dot{w}}{w} = 2Rb\cdot\dot{x}\,. \tag{4.33}$$

We next take the $R \to \infty$ in the action and in the transformations with the rescalings of the transformation parameters used before. This leads to the following action

$$S\left[x^{\mu}(\zeta), w(\zeta)\right] = -\frac{m}{2}\int d\zeta\left\{e^{-1}\left[\frac{\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{w^2} + \lambda\frac{\dot{w}}{w}\right] \mp \lambda + m_{\hat{\mu}}\dot{x}^{\hat{\mu}}\right\}, \tag{4.34}$$

and to the transformations eqs. (4.13) plus

$$\begin{aligned} \delta m_{\mu} &= -m_{\nu}\sigma^{\nu}{}_{\mu} - \frac{c}{2}m_{\mu} \pm 4b_{\mu}\,,\\ \delta m_w &= -\frac{c}{2}m_w + 2m_{\nu}b^{\nu}w\,. \end{aligned} \tag{4.35}$$

The action eq. (4.34) is the action eq. (4.12), which is invariant under all the transformations, plus a term involving the 1-form $\mp\lambda + m_{\hat{\mu}}\dot{x}^{\hat{\mu}}$ which also must be invariant under all the transformations. This implies that

$$\delta\lambda = \pm\delta(m_{\hat{\mu}}\dot{x}^{\hat{\mu}})\,. \tag{4.36}$$

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The equation for λ is solved by

$$e = \pm\frac{\dot{w}}{w}\,, \tag{4.37}$$

which we can substitute back in the action which now takes the form

$$S\left[x^{\mu}(\zeta), w(\zeta)\right] = \mp\frac{m}{2}\int d\zeta\left\{\frac{\eta_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu}}{\dot{w}w} \pm m_{\hat{\mu}}\dot{x}^{\hat{\mu}}\right\}. \tag{4.38}$$

This action is invariant under all the symmetries with the help of the 1-form $m_{\hat{\mu}}$.

4.2 p-brane limits

Horospheric coordinates make it easy to take in a very natural way p-brane-type limits of AdS using some of the Cartesian coordinates of the holographic screen. We will just give a very concise description of this example: we take the metric eq. (A.17) and rescale the first $(q+1)$ coordinates (we take $q < p$)

$$x^{A} \equiv \rho y^{A}\,, \qquad A = 0, \cdots, q\,, \tag{4.39}$$

obtaining

$$d\hat{s}^2_{p+2} = \hat{h}_{\hat{\mu}\hat{\nu}}d\hat{x}^{\hat{\mu}}d\hat{x}^{\hat{\nu}} + \rho^2\hat{k}_{\hat{\mu}\hat{\nu}}d\hat{x}^{\hat{\mu}}d\hat{x}^{\hat{\nu}}\,, \tag{4.40}$$

where the singular metrics are now

$$\hat{h}_{\hat{\mu}\hat{\nu}}d\hat{x}^{\hat{\mu}}d\hat{x}^{\hat{\nu}} = -\left(\frac{R}{z}\right)^2\left[\delta_{mn}dx^m dx^n + dz^2\right], \qquad m, n, \ldots = 1, \cdots, p-q\,, \tag{4.41a}$$

$$\hat{k}_{\hat{\mu}\hat{\nu}}d\hat{x}^{\hat{\mu}}d\hat{x}^{\hat{\nu}} = \left(\frac{R}{z}\right)^2\left[\eta_{AB}dx^A dy^B\right]. \tag{4.41b}$$

After rescaling and relabeling them, the Killing vectors eq. (A.22) become

$$\pi_{AB} = 2\eta_{AB}{}^{C}{}_{D}y^{D}\partial_C\,, \tag{4.42a}$$

$$\pi_{mn} = x^m\partial_n - x^n\partial_m\,, \tag{4.42b}$$

$$\pi_{+m} = \frac{R}{2}\partial_m\,, \tag{4.42c}$$

$$\pi_{-m} = \frac{1}{2R}\left\{\left(x\cdot x - z^2\right)\partial_m + 2x^m(y^C\partial_C + x^m\partial_m + z\partial_z)\right\}, \tag{4.42d}$$

$$\pi = \tfrac{1}{2}(y^C\partial_C + x^m\partial_m + z\partial_z)\,, \tag{4.42e}$$

$$\varpi_{mA} = \frac{1}{\rho}x^m\partial_A + \rho\eta_{AC}y^C\partial_m\,, \tag{4.42f}$$

$$\varpi_{+A} = \frac{1}{\rho}\frac{R}{2}\partial_A\,, \tag{4.42g}$$

$$\varpi_{-A} = \frac{1}{\rho}\frac{1}{2R}\left(x\cdot x - z^2\right)\partial_A - \rho\frac{1}{R}\eta_{AB}y^B(y^C\partial_C + x^m\partial_m + z\partial_z)\,. \tag{4.42h}$$

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The non-vanishing Lie brackets of these generators are given by

$$[\pi_{AB},\pi_{CD}] = \eta_{AC}\pi_{BD} + \eta_{BD}\pi_{AC} - \eta_{AD}\pi_{BC} - \eta_{BC}\pi_{AD}\,, \tag{4.43a}$$
$$[\pi_{mn},\pi_{pq}] = -\delta_{mp}\pi_{nq} - \delta_{nq}\pi_{mp} + \delta_{mq}\pi_{np} + \delta_{np}\pi_{mq}\,, \tag{4.43b}$$
$$[\pi_{mn},\pi_{\pm p}] = -2\eta_{p[m|}\pi_{\pm|n]}\,, \tag{4.43c}$$
$$[\pi_{+m},\pi_{-n}] = \tfrac{1}{2}\pi_{mn} + \delta_{mn}\pi\,, \tag{4.43d}$$
$$[\pi,\pi_{\pm m}] = \pm\tfrac{1}{2}\pi_{\pm m}\,, \tag{4.43e}$$
$$[\pi_{AB},\varpi_{mC}] = 2\eta_{C[A|}\varpi_{m|B]}\,, \tag{4.43f}$$
$$[\pi_{mn},\varpi_{pA}] = -2\delta_{p[m}\varpi_{n]A}\,, \tag{4.43g}$$
$$[\pi_{\pm m},\varpi_{nA}] = \delta_{mn}\varpi_{\pm A}\,, \tag{4.43h}$$
$$[\pi_{AB},\varpi_{\pm C}] = 2\eta_{C[A|}\varpi_{\pm|B]}\,, \tag{4.43i}$$
$$[\pi,\varpi_{\pm A}] = \pm\tfrac{1}{2}\varpi_{\pm A}\,, \tag{4.43j}$$
$$[\varpi_{mA},\varpi_{nB}] = -\delta_{mn}\pi_{AB} - \eta_{AB}\pi_{mn}\,, \tag{4.43k}$$
$$[\varpi_{mA},\varpi_{\pm B}] = -\eta_{AB}\pi_{\pm m}\,, \tag{4.43l}$$
$$[\varpi_{+A},\varpi_{-B}] = \tfrac{1}{2}\pi_{AB} - \eta_{AB}\pi\,. \tag{4.43m}$$

They are of the general form eq. (2.9).

The ϖ generators can be further decomposed as $\rho^{-1}\varpi^{(-1)} + \rho\varpi^{(1)}$ with

$$\varpi^{(-1)}_{mA} = x^m\partial_A\,, \tag{4.44a}$$
$$\varpi^{(1)}_{mA} = \eta_{AC}y^C\partial_m\,, \tag{4.44b}$$
$$\varpi^{(-1)}_{+A} = \frac{R}{2}\partial_A\,, \tag{4.44c}$$
$$\varpi^{(-1)}_{-A} = \frac{1}{2R}\left(x\cdot x - z^2\right)\partial_A\,, \tag{4.44d}$$
$$\varpi^{(1)}_{-A} = -\frac{1}{R}\eta_{AB}y^B(y^C\partial_C + x^m\partial_m + z\partial_z)\,. \tag{4.44e}$$

The generators $\varpi^{(-1)}_{+A}$ will survive both the $\rho\to\infty$ and $\rho\to 0$ limits. The Lie algebras obtained in these limits can be found in table 4, where it is explicitly shown how they are related by the general $1\leftrightarrow -1$ duality. This duality does not act on the $\varpi^{(-1)}_{+A}$ generators, which are common to both algebras. The Lie brackets involving the π generators are common to both algebras.

In this section we have seen how the general framework that we have developed allows us to explore systematically new, less conventional singular metrics of the AdS metric. In the next section we are going to study an even less conventional limit of a well-known family of solutions of the vacuum Einstein equations.

5 Example 3: *pp*-waves

In this last example we are going to consider a 4-dimensional *pp*-wave metric of the form

$$ds^2 = 2du(dv + \tfrac{1}{2}Hdu) - dx^m dx^m\,,\quad m=1,2,\qquad H = H_{mn}(u)x^m x^n\,. \tag{5.1}$$

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$\rho \to \infty$	$\rho \to 0$
$[\pi_{AB}, \varpi^{(1)}_{mC}] = 2\eta_{C[A|}\varpi^{(1)}_{m|B]}\,,$	$[\pi_{AB}, \varpi^{(-1)}_{mC}] = 2\eta_{C[A|}\varpi^{(-1)}_{m|B]}\,,$
$[\pi_{mn}, \varpi^{(1)}_{pA}] = 2\eta_{p[m}\varpi^{(1)}_{n]A}\,,$	$[\pi_{mn}, \varpi^{(-1)}_{pA}] = 2\eta_{p[m}\varpi^{(-1)}_{n]A}\,,$
$[\pi_{-m}, \varpi^{(1)}_{nA}] = -\eta_{mn}\varpi^{(1)}_{-A}\,,$	$[\pi_{-m}, \varpi^{(-1)}_{nA}] = -\eta_{mn}\varpi^{(-1)}_{-A}\,,$
$[\pi_{AB}, \varpi^{(-1)}_{+C}] = 2\eta_{C[A|}\varpi^{(-1)}_{+|B]}\,,$	$[\pi_{AB}, \varpi^{(-1)}_{+C}] = 2\eta_{C[A|}\varpi^{(-1)}_{+|B]}\,,$
$[\pi_{AB}, \varpi^{(1)}_{-C}] = 2\eta_{C[A|}\varpi^{(1)}_{-|B]}\,,$	$[\pi_{AB}, \varpi^{(-1)}_{-C}] = 2\eta_{C[A|}\varpi^{(-1)}_{-|B]}\,,$
$[\pi, \varpi^{(-1)}_{+A}] = \frac{1}{2}\varpi^{(-1)}_{+A}\,,$	$[\pi, \varpi^{(-1)}_{+A}] = \frac{1}{2}\varpi^{(-1)}_{+A}\,,$
$[\pi, \varpi^{(1)}_{-A}] = -\frac{1}{2}\varpi^{(1)}_{-A}\,,$	$[\pi, \varpi^{(-1)}_{-A}] = -\frac{1}{2}\varpi^{(-1)}_{-A}\,,$
$[\varpi^{(1)}_{mA}, \varpi^{(-1)}_{+B}] = -\eta_{AB}\pi_{+m}\,,$	$[\varpi^{(-1)}_{mA}, \varpi^{(-1)}_{+B}] = 0\,,$
$[\varpi^{(-1)}_{+A}, \varpi^{(1)}_{-B}] = \frac{1}{2}\pi_{AB} - \eta_{AB}\pi\,,$	$[\varpi^{(-1)}_{+A}, \varpi^{(-1)}_{-B}] = 0\,,$

Table 4. This table gives the non-vanishing Lie brackets that arise after taking the p-brane $\rho \to \infty$ and $\rho \to 0$ limits of the AdS algebra.

(We will also refer to the wavefront coordinates as x^1, x^2 as x and y respectively.) These metrics admit a null covariantly constant Killing vector ∂_v and solve the vacuum Einstein equations for any traceless u-dependent 2×2 matrix $H_{mn}(u)$. The most general matrix of this kind can be written in the form

$$H = a(u)\sigma^1 + b(u)\sigma^3 = \begin{pmatrix} b(u) & a(u) \\ a(u) & -b(u) \end{pmatrix}\,, \tag{5.2}$$

and, therefore, are determined by just two functions of u: $a(u)$ and $b(u)$.

Generically, all the metrics eq. (5.1) have the following 5 Killing vectors $\{X_A\}$ with $A = (a, 5)$ and $a = 1, \cdots, 4$:

$$X_a = (f'_a x + g'_a y)\partial_v + f_a\partial_x + g_a\partial_y\,, \tag{5.3a}$$

$$X_5 = \partial_v\,, \tag{5.3b}$$

where the prime denotes derivative with respect to u and the subindex $a = 1, \cdots, 4$ labels the 4 solutions of f and g of the following two differential equations

$$\begin{aligned} f'' &= -b(u)f - a(u)g\,, \\ g'' &= b(u)g - a(u)f\,. \end{aligned} \tag{5.4}$$

Furthermore, the commutators of these Killing vectors are given by

$$[X_5, X_a] = 0\,, \tag{5.5a}$$

$$[X_a, X_c] = (f'_c f_a - f'_a f_c + g'_c g_a - g'_a g_c)X_5\,. \tag{5.5b}$$

In the case that $a(u) = a$ and $b(u) = b$ with a, b constant, we can solve eqs. (5.4) obtaining

$$f_1 = a\frac{\cos\left[\left(a^2+b^2\right)^{1/4}u\right] - \cosh\left[\left(a^2+b^2\right)^{1/4}u\right]}{2\sqrt{a^2+b^2}}\,, \tag{5.6a}$$

$$g_1 = \frac{\left(-b+\sqrt{a^2+b^2}\right)\cos\left[\left(a^2+b^2\right)^{1/4}u\right]+\left(b+\sqrt{a^2+b^2}\right)\cosh\left[\left(a^2+b^2\right)^{1/4}u\right]}{2\sqrt{a^2+b^2}}\,, \tag{5.6b}$$

$$f_2 = a\frac{\sin\left[\left(a^2+b^2\right)^{1/4}u\right]-\sinh\left[\left(a^2+b^2\right)^{1/4}u\right]}{2\left(a^2+b^2\right)^{3/4}}\,, \tag{5.6c}$$

$$g_2 = \frac{\left(-b+\sqrt{a^2+b^2}\right)\sin\left[\left(a^2+b^2\right)^{1/4}u\right]+\left(b+\sqrt{a^2+b^2}\right)\sinh\left[\left(a^2+b^2\right)^{1/4}u\right]}{2\left(a^2+b^2\right)^{3/4}}\,, \tag{5.6d}$$

$$f_3 = \frac{\left(b+\sqrt{a^2+b^2}\right)\cos\left[\left(a^2+b^2\right)^{1/4}u\right]+\left(-b+\sqrt{a^2+b^2}\right)\cosh\left[\left(a^2+b^2\right)^{1/4}u\right]}{2\sqrt{a^2+b^2}}\,, \tag{5.6e}$$

$$g_3 = a\frac{\cos\left[\left(a^2+b^2\right)^{1/4}u\right]-\cosh\left[\left(a^2+b^2\right)^{1/4}u\right]}{2\sqrt{a^2+b^2}}\,, \tag{5.6f}$$

$$f_4 = \frac{\left(b+\sqrt{a^2+b^2}\right)\sin\left[\left(a^2+b^2\right)^{1/4}u\right]+\left(-b+\sqrt{a^2+b^2}\right)\sinh\left[\left(a^2+b^2\right)^{1/4}u\right]}{2\left(a^2+b^2\right)^{3/4}}\,, \tag{5.6g}$$

$$g_4 = a\frac{\sin\left[\left(a^2+b^2\right)^{1/4}u\right]-\sinh\left[\left(a^2+b^2\right)^{1/4}u\right]}{2\left(a^2+b^2\right)^{3/4}}\,, \tag{5.6h}$$

and there is an additional Killing vector

$$X_6 = \partial_u\,. \tag{5.7}$$

The non-vanishing commutators of the these 6 Killing vectors are given by

$$[X_1, X_2] = [X_3, X_4] = X_5\,, \tag{5.8a}$$
$$[X_6, X_1] = bX_2 - aX_4\,, \tag{5.8b}$$
$$[X_6, X_2] = X_1\,, \tag{5.8c}$$
$$[X_6, X_3] = -aX_2 - bX_4\,, \tag{5.8d}$$
$$[X_6, X_4] = X_3\,, \tag{5.8e}$$

and correspond to the so-called twisted Heisenberg algebra [19][17]

$$[X^{(k)}, X^{*(l)}] = \delta_{kl}Z\,, \tag{5.9a}$$
$$[X, X^{(k)}] = -H_{kl}X^{*(l)}\,, \tag{5.9b}$$
$$[X, X^{*(k)}] = X^{(k)}\,. \tag{5.9c}$$

If we rescale the coordinates of the wavefront, x^m as

$$x^m \to \rho x^m\,, \tag{5.10}$$

the metric and its Killing vectors take the form

$$ds^2 = 2dudv + \rho^2(Hdu^2 - dx^m dx^m)\,, \tag{5.11a}$$

[17]The correspondence between the generators X_A and the generators $X^{(k)}, X^{*(l)}$ in ref. [19] is $(X_3, X_1) \leftrightarrow (X^{(1)}, X^{(2)})$, $(X_4, X_2) \leftrightarrow (X^{*(1)}, X^{*(2)})$, $X_5 \leftrightarrow Z$ and $X_6 \leftrightarrow X$.

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$$X_a = \rho(f_a' x + g_a' y)\partial_v + \frac{1}{\rho}(f_a \partial_x + g_a \partial_y)\,, \tag{5.11b}$$

$$X_5 = \partial_v\,, \tag{5.11c}$$

$$X_6 = \partial_u\,, \tag{5.11d}$$

fitting in the general pattern studied in section 2 with the two singular metrics given by

$$h_{\mu\nu} dx^\mu dx^\nu = 2du dv\,, \tag{5.12a}$$

$$k_{\mu\nu} dx^\mu dx^\nu = H du^2 - dx^m dx^m\,. \tag{5.12b}$$

Observe that in this case, the ranks of these two metrics add up to 5, not just 4. While this is unusual, it is certainly not inconsistent in this framework, although it might be in a more general one. Another unusual feature of this particular split is that the two singular metrics are Lorentzian,[18] with signatures $(+-)$ and $+--$ respectively. The u coordinate is a lightcone coordinate of $h_{\mu\nu}$ but it is a time coordinate of $k_{\mu\nu}$. If the functions $a(u)$ and $b(u)$ are not constant, the latter is a dynamical, time-dependent metric.

Following the general procedure, we define

$$X_a \equiv \varpi_a = \rho^{-1}\varpi_a^{(-1)} + \rho\varpi_a^{(1)}\,, \tag{5.13a}$$

$$X_{5,6} \equiv \pi_{5,6}\,, \tag{5.13b}$$

with

$$\varpi_a^{(-1)} = f_a \partial_x + g_a \partial_y\,, \tag{5.14a}$$

$$\varpi_a^{(1)} = (f_a' x + g_a' y)\partial_v\,. \tag{5.14b}$$

We can apply our general results straightforwardly to this case finding that, in the $\rho \to 0, \infty$ limits, the non-vanishing commutators of the symmetry algebra of the point-particle action are given by

$$[\pi_6, \varpi_1^{(\pm1)}] = b\varpi_2^{(\pm1)} - a\varpi_4^{(\pm1)}\,, \tag{5.15a}$$

$$[\pi_6, \varpi_2^{(\pm1)}] = \varpi_1^{(\pm1)}\,, \tag{5.15b}$$

$$[\pi_6, \varpi_3^{(\pm1)}] = -a\varpi_2^{(\pm1)} - b\varpi_4^{(\pm1)}\,, \tag{5.15c}$$

$$[\pi_6, \varpi_4^{(\pm1)}] = \varpi_3^{(\pm1)}\,. \tag{5.15d}$$

Yet again, the dimension of the symmetry algebra remains unchanged in the two singular limits, even if in this particular case we have singular metrics whose ranks add up to 5. The physical interpretation of the actions obtained after taking the limits according to the different procedures we have described in section 2 is, generically, that of actions of particles moving in backgrounds described by one of the two singular metrics, although both may occur in the action. A detailed, case by case, study of the equations of motion, solving constraints and eliminating redundant variables may shed light and help us in getting a better understanding of the physics involved in these limits.

[18] Lorentzian in a restricted 2- and 3-dimensional sense, but not in the original 4-dimensional sense.

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6 Discussion

In this work we have constructed a general framework which allows one to study the fate of

1. the Killing vectors and the corresponding isometry algebra of metrics which depend on an arbitrary parameter ρ as in eq. (2.2)
2. the symmetries and the corresponding symmetry algebra of relativistic particles and branes moving in the background of metrics which depend on an arbitrary parameter ρ as in eq. (2.2)

in the $\rho \to 0, \infty$ limits in which the metric is singular or ill-defined.

In particular, this general framework allows us to determine the limiting symmetry algebras in a simple and systematic way, showing that the dimension of the symmetry algebra is not changed by taking the singular limits and leading to new relations or dualities between the algebras obtained after taking the $\rho \to 0$ limit and the algebras obtained after taking the $\rho \to \infty$ limit. This included two different dualities between p-brane Galilei and Carroll algebras.

At the level of the Lie algebra of symmetries our framework (excluding the option of adding a WZ term to the sigma model) uses techniques that resemble the ones that are used in a Lie algebra expansion providing a different perspective in which it is a background solution and not the structure group that plays a central role. This relation with Lie algebra expansions suggests several extensions of our framework including supersymmetric ones [16] and extending the limit to a general expansion order by order in the contraction parameter.[19]

An attractive feature of our general framework is that it can be applied to a wide range of situations. The parameter that we are using in taking the limit can be anything, a velocity, a time parameter, a radial parameter etc., as long as it is in agreement with the properties of the sigma model and solution we are taking the limit of. In particular, the "holographic" limits that we have been studying may be useful in the context of studying aspects of the AdS/CFT correspondence. Other limits, using the velocity of light, have been used to study holography in relation to non-relativistic theories of gravity in the bulk [5–9]. Furthermore, as we have seen in the 4-dimensional pp-wave example in which the ranks of the singular metrics add up to 5, this setup includes limits which cannot be handled by other current methods in which, however, the dimensoin of the symmetry algebra remains invariant in the limit. It would be interesting to see whether for those cases a Cartan-like formalism with "inverse" metrics, connection and curvature can be defined in this kind of cases.

A prominent question about our framework in need of an answer is the following: if we start with a metric of the form eq. (2.2) which is a classical solution of General Relativity, *are the singular metrics obtained in the limits solutions of some gravity theory that may also be obtained by taking related limits of the Lagrangian or equations of motion of General Relativity?*

Another important question concerning the use of the Hubbard-Stratonovich transformation when taking the limits of particle and p-brane actions is the following: *can we give a spacetime geometric meaning to the Lagrange multipliers that occur in the worldvolume actions after taking some of our limits?* One suggestion is that these extra fields naturally

[19] For a recent example of a Carroll expansion going beyond the limit, see [17].

arise by using a form of the sigma model where the equations of motion occur in a first-order or Hamiltonian formulation.

We hope to come back to these questions in a future publication.

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A Anti-De Sitter spacetime

In this appendix we are going to review the definition of the horospheric coordinates of AdS that we use in section 4 and we are also going to derive the explicit form of the Killing vectors in these coordinates using the isometric embedding of AdS space as a hyperboloid in a 1-dimension higher spacetime with signature $(++,-\cdots,-)$.

Following Gibbons [18], we are interested in Anti-De Sitter (AdS) spacetime in $p+2$ dimensions (AdS_{p+2}), which typically arises in the near-horizon limit of extremal p-brane solutions. This spacetime and its isometries can be better understood through its isometric embedding in a $(p+3)$-dimensional space with the metric

$$d\hat{\hat{s}}^2_{p+3} = (dX^{-1})^2 + (dX^0)^2 - d\vec{X}_p^2 - (dX^{p+1})^2\,, \qquad \vec{X}_p = (X^1,\cdots,X^p)\,, \tag{A.1}$$

of signature $(2,p+1)$ as the hyperboloid given by the equation

$$(X^{-1})^2 + (X^0)^2 - \vec{X}_p^2 - (X^{p+1})^2 = R^2\,, \tag{A.2}$$

where R if the AdS radius.

It is convenient to introduce the $(p+3)$-dimensional, constant, pseudo-Riemannian metric of signature $(2,p+1)$ $\hat{\hat{\eta}}$ =diag$(+,+,-,\cdots,-)$, and the $(p+3)$ coordinates $(\hat{\hat{X}}^{\hat{\hat{\mu}}}) = (X^{-1},X^0,\vec{X}_p,X^{p+1})$ in terms of which the above equations take the simple form

$$d\hat{\hat{s}}^2_{p+3} = \hat{\hat{\eta}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}} d\hat{\hat{X}}^{\hat{\hat{\mu}}} d\hat{\hat{X}}^{\hat{\hat{\nu}}}\,, \tag{A.3a}$$

$$R^2 = \hat{\hat{\eta}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}} \hat{\hat{X}}^{\hat{\hat{\mu}}} \hat{\hat{X}}^{\hat{\hat{\nu}}}\,, \tag{A.3b}$$

which make evident the $\mathrm{SO}(2,p+1)$ invariance of AdS_{p+2}. The finite $\mathrm{SO}(2,p+1)$ transformations act on the coordinates as

$$\hat{\hat{X}}'^{\hat{\hat{\mu}}} \hat{\hat{\Lambda}}^{\hat{\hat{\mu}}}{}_{\hat{\hat{\nu}}} \hat{\hat{X}}^{\hat{\hat{\nu}}}\,, \tag{A.4}$$

where

$$\hat{\hat{\eta}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}}\hat{\hat{\Lambda}}^{\hat{\hat{\alpha}}}{}_{\hat{\hat{\mu}}}\hat{\hat{\Lambda}}^{\hat{\hat{\beta}}}{}_{\hat{\hat{\nu}}} = \hat{\hat{\eta}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}} \,. \tag{A.5}$$

Infinitesimally,

$$\hat{\hat{X}}'^{\hat{\hat{\mu}}} \left(\delta^{\hat{\hat{\mu}}}{}_{\hat{\hat{\nu}}} + \hat{\hat{\sigma}}^{\hat{\hat{\mu}}}{}_{\hat{\hat{\nu}}}\right) \hat{\hat{X}}^{\hat{\hat{\nu}}} \,, \tag{A.6}$$

and the condition eq. (A.5) becomes

$$\hat{\hat{\sigma}}_{(\hat{\hat{\mu}}\hat{\hat{\nu}})} = 0 \,, \quad \text{with} \quad \hat{\hat{\sigma}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}} = \hat{\hat{\eta}}_{\hat{\hat{\mu}}\hat{\hat{\rho}}}\hat{\hat{\sigma}}^{\hat{\hat{\rho}}}{}_{\hat{\hat{\nu}}} \,. \tag{A.7}$$

It is natural to write the infinitesimal transformations as follows:

$$\delta_{\hat{\hat{\sigma}}}\hat{\hat{X}}^{\hat{\hat{\mu}}} = \tfrac{1}{2}\hat{\hat{\sigma}}^{\hat{\hat{\alpha}}\hat{\hat{\beta}}}\hat{\hat{k}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}}{}^{\hat{\hat{\mu}}} \,, \tag{A.8}$$

where

$$\hat{\hat{k}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}}{}^{\hat{\hat{\mu}}} = 2\hat{\hat{\eta}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}}{}^{\hat{\hat{\mu}}}{}_{\hat{\hat{\nu}}}\hat{\hat{X}}^{\hat{\hat{\nu}}} \,, \quad \text{with} \quad \hat{\hat{\eta}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}}{}^{\hat{\hat{\mu}}\hat{\hat{\nu}}} = \tfrac{1}{2}\left(\hat{\hat{\eta}}_{\hat{\hat{\alpha}}}{}^{\hat{\hat{\mu}}}\hat{\hat{\eta}}_{\hat{\hat{\beta}}}{}^{\hat{\hat{\nu}}} - \hat{\hat{\eta}}_{\hat{\hat{\alpha}}}{}^{\hat{\hat{\nu}}}\hat{\hat{\eta}}_{\hat{\hat{\beta}}}{}^{\hat{\hat{\mu}}}\right) \,, \tag{A.9}$$

are the infinitesimal generators of those transformations. $\hat{\hat{k}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}} = \hat{\hat{k}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}}{}^{\hat{\hat{\mu}}}\partial_{\hat{\hat{\mu}}}$ are Killing vectors of the $(p+3)$-dimensional metric eq. (A.3a).[20]

We can associate a generator $\hat{\hat{M}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}} = -\hat{\hat{M}}_{\hat{\hat{\nu}}\hat{\hat{\mu}}}$ of the $so(2, p+1)$ Lie algebra to (minus) each of the Killing vectors $\hat{\hat{k}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}}$.[21] The structure constants that occur in the commutators of the generators are minus the structure constants that occur in the Lie brackets of the vectors

$$[\hat{\hat{k}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}}, \hat{\hat{k}}_{\hat{\hat{\rho}}\hat{\hat{\sigma}}}] = -\hat{\hat{\eta}}_{\hat{\hat{\mu}}\hat{\hat{\rho}}}\hat{\hat{k}}_{\hat{\hat{\nu}}\hat{\hat{\sigma}}} - \hat{\hat{\eta}}_{\hat{\hat{\nu}}\hat{\hat{\sigma}}}\hat{\hat{k}}_{\hat{\hat{\mu}}\hat{\hat{\rho}}} + \hat{\hat{\eta}}_{\hat{\hat{\mu}}\hat{\hat{\sigma}}}\hat{\hat{k}}_{\hat{\hat{\nu}}\hat{\hat{\rho}}} + \hat{\hat{\eta}}_{\hat{\hat{\nu}}\hat{\hat{\rho}}}\hat{\hat{k}}_{\hat{\hat{\mu}}\hat{\hat{\sigma}}} \,, \tag{A.10}$$

and, in this case, one arrives at the following commutation relations between the generators:

$$[\hat{\hat{M}}_{\hat{\hat{\mu}}\hat{\hat{\nu}}}, \hat{\hat{M}}_{\hat{\hat{\rho}}\hat{\hat{\sigma}}}] = \hat{\hat{\eta}}_{\hat{\hat{\mu}}\hat{\hat{\rho}}}\hat{\hat{M}}_{\hat{\hat{\nu}}\hat{\hat{\sigma}}} + \hat{\hat{\eta}}_{\hat{\hat{\nu}}\hat{\hat{\sigma}}}\hat{\hat{M}}_{\hat{\hat{\mu}}\hat{\hat{\rho}}} - \hat{\hat{\eta}}_{\hat{\hat{\mu}}\hat{\hat{\sigma}}}\hat{\hat{M}}_{\hat{\hat{\nu}}\hat{\hat{\rho}}} - \hat{\hat{\eta}}_{\hat{\hat{\nu}}\hat{\hat{\rho}}}\hat{\hat{M}}_{\hat{\hat{\mu}}\hat{\hat{\sigma}}} \,. \tag{A.11}$$

A.1 Horospheric coordinates

First, we define the "lightcone" coordinates

$$\hat{\hat{X}}^{\pm} \equiv \hat{\hat{X}}^{-1} \pm \hat{\hat{X}}^{p+1} \,, \tag{A.12}$$

in terms of which the metric and the hyperboloid equation eqs. (A.3a) and (A.3b) take the form

$$d\hat{\hat{s}}^2_{p+3} = d\hat{\hat{X}}^{+}d\hat{\hat{X}}^{-} + \eta_{\mu\nu}dX^{\mu}dX^{\nu} \,, \tag{A.13a}$$

$$R^2 = \hat{\hat{X}}^{+}\hat{\hat{X}}^{-} + \eta_{\mu\nu}X^{\mu}X^{\nu} \,, \tag{A.13b}$$

[20]The $(p+3)$-dimensional metric is also invariant under constant translations of the coordinates, but these transformations do not leave invariant the equation of the hyperboloid eq. (A.3b).

[21]The minus sign is due to the relation between the transformations of the coordinates x^{μ} and the transformations of tensor fields T under the infinitesimal diffeomorphisms generated by some vector field ξ:

$$\delta_{\xi}x^{\mu} = \xi^{\mu} \,,$$

$$\delta_{\xi}T = -\pounds_{\xi}T = \pounds_{-\xi}T \,,$$

where $\pounds_{\xi}$ is the Lie derivative with respect to the vector fields ξ.

where we are using the $(p+1)$-dimensional indices $\mu, \nu = 0, \cdots, p$ and Lorentzian metric (η) =diag$(+,-,\cdots,-)$. We can use the hyperboloid equation to solve for the coordinate $\hat{\hat{X}}^+$

$$\hat{\hat{X}}^+ = R^2 - \eta_{\mu\nu} X^\mu X^\nu / \hat{\hat{X}}^- \,, \tag{A.14}$$

and eliminate it from the metric to obtain the $(p+2)$-dimensional metric induced on the hyperboloid

$$d\hat{s}^2_{p+2} = -\left(\frac{d\hat{\hat{X}}^-}{\hat{\hat{X}}^-}\right)^2 \left[R^2 - \eta_{\mu\nu} X^\mu X^\nu\right] - 2\frac{d\hat{\hat{X}}^-}{\hat{\hat{X}}^-} \eta_{\mu\nu} X^\mu dX^\nu + \eta_{\mu\nu} dX^\mu dX^\nu \,. \tag{A.15}$$

The transformation to $(p+2)$-dimensional *horospheric coordinates* $(\hat{x}^{\hat{\mu}}) = (x^\mu, z)$

$$\hat{\hat{X}}^- \equiv R^2/z \,, \qquad X^\mu \equiv R x^\mu / z \,, \tag{A.16}$$

diagonalizes the metric

$$d\hat{s}^2_{p+2} = \left(\frac{R}{z}\right)^2 \left[\eta_{\mu\nu} dx^\mu dx^\nu - dz^2\right] \,. \tag{A.17}$$

By construction, this metric has the same isometry group as the original $(p+3)$-dimensional one. The Killing vectors have a different form, though. The simplest way to find them is to take the pullback of the dual 1-forms over the hyperboloid and, then, change the coordinates. The dual 1-forms are given by

$$\tilde{\hat{\hat{k}}}_{\hat{\alpha}\hat{\hat{\beta}}} = 2\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},\hat{\mu}\hat{\nu}} \hat{\hat{X}}^{\hat{\nu}} d\hat{\hat{X}}^{\hat{\mu}} \,, \tag{A.18}$$

and their pullbacks over the hyperboloid are given by

$$\begin{aligned}
\tilde{\hat{\hat{k}}}_{\hat{\alpha}\hat{\hat{\beta}}} &= 2\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},\mu\nu} X^\nu dX^\mu + 2\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},\mu-} \left(X^\mu d\hat{\hat{X}}^- - \hat{\hat{X}}^- dX^\mu\right) \\
&\quad + 2\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},\mu+} \left(X^\mu d\hat{\hat{X}}^+ - \hat{\hat{X}}^+ dX^\mu\right) + 2\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},+-} \left(\hat{\hat{X}}^+ d\hat{\hat{X}}^- - \hat{\hat{X}}^- d\hat{\hat{X}}^+\right) \\
&= 2\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},\mu\nu} X^\nu dX^\mu + 2\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},\mu-} \left(X^\mu d\hat{\hat{X}}^- - \hat{\hat{X}}^- dX^\mu\right) \\
&\quad + 2\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},\mu+} \left[-\frac{X^\mu \left(R^2 - X\cdot X\right) d\hat{\hat{X}}^-}{(\hat{\hat{X}}^-)^2} - 2\frac{X^\mu X\cdot dX}{\hat{\hat{X}}^-} - \frac{\left(R^2 - X\cdot X\right) dX^\mu}{\hat{\hat{X}}^-}\right] \\
&\quad + 4\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},+-} \left[\frac{\left(R^2 - X\cdot X\right) d\hat{\hat{X}}^-}{\hat{\hat{X}}^-} + X\cdot dX\right] \,.
\end{aligned} \tag{A.19}$$

In horospheric coordinates

$$\begin{aligned}
\tilde{\hat{\hat{k}}}_{\hat{\alpha}\hat{\hat{\beta}}} &= 2\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},\mu\nu} \left(\frac{R}{z}\right)^2 x^\nu dx^\mu \\
&\quad - 2\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},\mu-} R \left(\frac{R}{z}\right)^2 dx^\mu \\
&\quad + \frac{2}{R}\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},\mu+} \left(\frac{R}{z}\right)^2 \left[\left(x\cdot x - z^2\right) dx^\mu - 2x^\mu x\cdot dx + 2z x^\mu dz\right] \\
&\quad + 4\hat{\hat{\eta}}_{\hat{\alpha}\hat{\hat{\beta}},+-} \left(\frac{R}{z}\right)^2 \left(x\cdot dx - z dz\right) \,.
\end{aligned} \tag{A.20}$$

Finally, we just have to raise the 1-form index using the inverse of the metric eq. (A.17) to find the vectors. Observe that this removes the overall $(R/z)^2$ factor. The result is

$$\begin{aligned}\hat{\hat{k}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}} &= 2\hat{\hat{\eta}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}}{}^{\mu}{}_{\nu}x^{\nu}\partial_{\mu} - 2R\hat{\hat{\eta}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}}{}^{\mu}{}_{-}\partial_{\mu} \\ &+ \frac{2}{R}\hat{\hat{\eta}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}}}{}^{\mu}{}_{+}\left(x\cdot x - z^2\right)\partial_{\mu} - \frac{4}{R}\hat{\hat{\eta}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}},\mu+}x^{\mu}x^{\nu}\partial_{\nu} - \frac{4}{R}\hat{\hat{\eta}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}},\mu+}zx^{\mu}\partial_z \\ &+ 4\hat{\hat{\eta}}_{\hat{\hat{\alpha}}\hat{\hat{\beta}},+-}\left(x^{\mu}\partial_{\mu} + z\partial_z\right) .\end{aligned} \tag{A.21}$$

Now, taking into account that the non-vanishing components of $\hat{\hat{\eta}}$ in the lightcone basis are $\hat{\hat{\eta}}_{+-} = 1/2$ and $\hat{\hat{\eta}}_{\mu\nu} = \eta_{\mu\nu}$, we find the following independent Killing vectors:

$$\hat{\hat{k}}_{\alpha\beta} = 2\eta_{\alpha\beta}{}^{\mu}{}_{\nu}x^{\nu}\partial_{\mu} , \tag{A.22a}$$

$$\hat{\hat{k}}_{+\alpha} = \frac{R}{2}\partial_{\alpha} , \tag{A.22b}$$

$$\hat{\hat{k}}_{-\alpha} = \frac{1}{2R}\left\{\left(x\cdot x - z^2\right)\partial_{\alpha} - 2\eta_{\alpha\mu}x^{\mu}(x^{\nu}\partial_{\nu} + z\partial_z)\right\} , \tag{A.22c}$$

$$\hat{\hat{k}}_{-+} = \tfrac{1}{2}\left(x^{\mu}\partial_{\mu} + z\partial_z\right) . \tag{A.22d}$$

The non-vanishing commutation relations of the corresponding generators are

$$[\hat{\hat{M}}_{\mu\nu}, \hat{\hat{M}}_{\rho\sigma}] = \eta_{\mu\rho}\hat{\hat{M}}_{\nu\sigma} + \eta_{\nu\sigma}\hat{\hat{M}}_{\mu\rho} - \eta_{\mu\sigma}\hat{\hat{M}}_{\nu\rho} - \eta_{\nu\rho}\hat{\hat{M}}_{\mu\sigma} , \tag{A.23a}$$

$$[\hat{\hat{M}}_{\pm\alpha}, \hat{\hat{M}}_{\mu\nu}] = -2\eta_{\alpha[\mu|}\hat{\hat{M}}_{\pm|\nu]} , \tag{A.23b}$$

$$[\hat{\hat{M}}_{+\alpha}, \hat{\hat{M}}_{-\beta}] = \tfrac{1}{2}\hat{\hat{M}}_{\alpha\beta} - \eta_{\alpha\beta}\hat{\hat{M}}_{-+} , \tag{A.23c}$$

$$[\hat{\hat{M}}_{\pm\alpha}, \hat{\hat{M}}_{-+}] = \mp\tfrac{1}{2}\hat{\hat{M}}_{\pm\alpha} . \tag{A.23d}$$

Defining[22]

$$\hat{\hat{M}}_{\mu\nu} \equiv M_{\mu\nu} , \qquad \hat{\hat{M}}_{+\alpha} \equiv \frac{R}{2}P_{\alpha} , \qquad \hat{\hat{M}}_{-\alpha} \equiv \frac{1}{2R}B_{\alpha} , \qquad \hat{\hat{M}}_{-+} \equiv \tfrac{1}{2}D , \tag{A.24}$$

the above algebra takes the form

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} , \tag{A.25a}$$

$$[P_{\alpha}, M_{\mu\nu}] = -2\eta_{\alpha[\mu|}P_{|\nu]} , \tag{A.25b}$$

$$[B_{\alpha}, M_{\mu\nu}] = -2\eta_{\alpha[\mu|}B_{|\nu]} , \tag{A.25c}$$

$$[P_{\alpha}, B_{\beta}] = 2M_{\alpha\beta} - 2\eta_{\alpha\beta}D , \tag{A.25d}$$

$$[P_{\alpha}, D] = -P_{\alpha} , \tag{A.25e}$$

$$[B_{\alpha}, D] = B_{\alpha} . \tag{A.25f}$$

[22] If P_{α} are rescaled with inverse factors the structure constants of the algebra remain invariant.

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