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Capital accumulation and the sources of demographic change

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Abstract We develop a neoclassical growth model having a realistic demographic structure. We identify the critical channel of impact to be the intertemporal consumption allocation decision through the "generational turnover term". Expressing the aggregate dynamics as a generalization of the conventional neoclassical growth model provides important insights, enabling us to view in a unified way how alternative demographic structures impinge on the macrodynamic equilibrium. Using an approximation to the generational turnover term, we are able to characterize both the steady state and the local transitional dynamics. Through numerical simulations, we analyze the steady state as well as the transitional effects of structural and demographic changes.

Keywords Demography · Growth · Overlapping generations

JEL Classifications C62 · D91 · E13 · J10

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1 Introduction

In general, the theory of economic growth has downplayed the significance of an economy's demographic structure as a determinant of its growth potential. Standard benchmark growth models continue to assume infinitely lived representative agents and, therefore, lack the structure necessary to address demographic issues.[1] The two primary approaches incorporating demographic features into growth theory are (a) the original overlapping generations model, pioneered by Samuelson (1958) and Diamond (1965) [SD], and (b) the more recent "perpetual youth model". introduced by Blanchard (1985) and refined by Buiter (1988) and Weil (1989) [BBW].[2] Both approaches provide deep insights and have had profound impacts. But both are highly stylized, which limits their ability to examine demographic factors in a comprehensive way.

The canonical SD model usually adopts a two-period framework—a first period for working and a second period for retirement.[3] While the model can be used to analyze many intergenerational policy questions, the usual formulation is overly rigid with regard to its choice of time units for analyzing standard macroeconomic policy issues.[4] The BBW model is more flexible, but it assumes a mortality rate that is independent of the household's age. While this assumption has the advantage of analytical tractability, and captures the finite horizon aspect of life, it comes at the cost of being unable to incorporate changing behavior over the life cycle, a limitation that Blanchard himself originally acknowledged.

During the last decade, substantial progress has been made in extending the BBW model to incorporate more realistic demographic structures. Bommier and Lee (2003), d'Albis (2007), Lau (2009), and Gan and Lau (2010) employ very general mortality structures to study the existence and uniqueness of the steady-state equilibrium.[5] Complementing this approach, Boucekkine et al. (2002), Faruqee (2003), Heijdra and Romp (2008), Heijdra and Mierau (2012), and Bruce and Turnovsky (2012) adopt empirically plausible mortality functions as the basis for their

[1]This can be readily confirmed by consulting any one of the leading textbooks on modern growth theory (or macrodynamics) where some version of the Ramsey model or the Romer (1986) model—depending upon the underlying production structure—is the overwhelmingly dominant paradigm.

[2]A key component of the BBW model to deal with uncertainty of life-span, the existence of an actuarially fair insurance market, was originally introduced by Yaari (1965).

[3]Many variants of the model exist, including extensions to an initial third period, for education; see, e.g., Docquier and Michel (1999).

[4]For this reason, Auerbach and Kotlikoff (1987), in their comprehensive study of fiscal policy, introduced 55 periods in order to accommodate multiple generations, while employing a time unit of the order of one year. With the advent of cheap and powerful computing, the Auerbach–Kotlikoff approach has been extended to allow for further complexity; see, for instance, Krueger and Ludwig (2007). However, a caveat concerning these models is that, because of their complexity, the underlying mechanisms driving particular results become less transparent.

[5]Much of this growing literature analyzing continuous-time overlapping generations economies with finite horizons builds on the classic contributions of Cass and Yaari (1967) and Tobin (1967); see also Burke (1996), Demichelis and Polemarchakis (2007), and Edmond (2008) for related developments. This literature tends to focus on issues relating to existence and determinacy of equilibrium, as well as its characterization, in a more abstract form than is our intent here.

analysis of structural and demographic changes.[6] To a large degree, the issues addressed by these authors are motivated by the empirical findings of Modigliani and Brumberg (1954), who first argued that varying behavior over the life cycle has important consequences for the evolution of the aggregate economy. Similarly, recent empirical studies have acknowledged the existence of an intricate relationship between the demographic structure and the economic outcomes of developing and developed countries; see, e.g., Kelley (1988), Kelley and Schmidt (1995), Bloom et al. (2007), and Erlandsen and Nymoen (2008).

The objective of the current paper is to study the theoretical and quantitative aspects of a neoclassical growth model having a realistic demographic structure. The theoretical aspect focuses on two elements. The first is to highlight the mechanism whereby the demographic structure impinges on the macrodynamic equilibrium. This is through the "generational turnover term", which refers to the reduction in aggregate consumption arising from the addition of newborn agents having no accumulated assets, in conjunction with the departure of agents with accumulated lifetime assets. All demographic structures share the feature that they impact on the aggregate macrodynamic equilibrium through their effect on the aggregate consumption growth rate, so that differences between them reduce to differences in the specification of the generational turnover term.[7]

The second element is to analyze how the demographic structure impinges on the transitional dynamics of the aggregate economy. In general, the global dynamic analysis of an overlapping generations model having a realistic demographic structure is represented by a high-order transcendental system and is intractable with a neoclassical production structure; see, e.g., d'Albis and Augeraud-Véron (2009). In view of this, we linearize the macrodynamic equilibrium around its steady state. A crucial step in this process involves establishing that key measures of average mortality rates remain approximately constant over time. By proceeding in this way, we show how the equilibrium dynamics can be represented, at least locally, by an autonomous system expressed in terms of the evolution of aggregate physical capital, aggregate consumption, human wealth at birth, and the marginal propensity to consume at birth. We view the ability to provide even a local characterization of how a realistic demographic structure impinges on the dynamics of the aggregate economy to be the key contribution of our analysis.

Adopting the above procedure, we find that the local aggregate dynamics exhibit an essential feature of the conventional Ramsey model, namely that the internally generated stable transitional path is a one-dimensional manifold. However, the

[6]Boucekkine et al. (2002) adopt a generalization of the Blanchard mortality function, thereby embedding the latter as a special case. This formulation is also adopted by Heijdra and Mierau (2012). Heijdra and Romp (2008) use the Gompertz (1825) exponential mortality hazard function in a small open-economy overlapping generations model. Faruqee (2003) approximates the Gompertz function with an estimated hyperbolic function, which he introduces into the model of Blanchard (1985). Finally, Bruce and Turnovsky (2012) represent survival by a function of de Moivre (1725).

[7]The generational turnover term is a central element of overlapping generations models. Accordingly, it also features in the analyses of, amongst others, Heijdra and Ligthart (2006) and d'Albis and Augeraud-Véron (2009), who use the Blanchard and rectangular mortality functions, respectively. In contrast, in the current paper, we provide a general characterization of this term.

short-run transition to this path, which occurs through an initial jump in aggregate consumption, can contrast sharply with that of a representative agent economy in which the adjustment paths are generally monotonic. For example, in the case where it takes the form of a gradual decline in birth rate—characteristic of the experiences of many Western economies—it generates a nonmonotonic adjustment path for key aggregate quantities, in contrast to the corresponding dynamics in the conventional representative agent case, which tend to be monotonic.[8]

To further advance our understanding on the dynamics, we proceed numerically, and this requires the adoption of a parametric mortality function. For this purpose, we employ the Boucekkine et al. (2002) (BCL) mortality function, thereby generalizing their model to accommodate a constant elasticity of substitution utility function and a neoclassical production structure.[9] The BCL function has several attractive features for our purposes. First, it tracks the observed mortality data remarkably well, and in addition, we show that it validates the key requirement, noted above, regarding the approximate intertemporal constancy of the average mortality rates.[10] Second, it provides a very convenient vehicle for parameterizing changes in mortality. Finally, it also embeds the widely employed BBW model, the dynamics of which simplify dramatically due to the constant mortality assumption, which carries the implication that both human wealth and the marginal propensity to consume are independent of age.

Our numerical simulations focus on two aspects. First, we study the long-run equilibrium responses to an increase in productivity, illustrating its effects on aggregate quantities, as well as on the distributions of consumption and wealth across cohorts. Second, we study how the change in the demographic structure of the USA has affected its aggregate economic structure. Over the last half of the twentieth century, the USA experienced substantial declines in both birth rate and mortality. In this latter exercise, we decompose the total effect into the respective components attributable to these two demographic factors.

Our results suggest that the *decrease* in the birth rate, combined with the accompanying *decrease* in the mortality rate experienced in the USA between 1960 and 2006, has led to an increase in the aggregate per capita capital stock and aggregate per capita consumption of 7.8 and 4.2 %, respectively. These results are consistent with the empirical evidence provided by Blanchet (1988), Kelley and Schmidt (1995), and

[8]Elsewhere, we have introduced a more realistic demographic structure into a one-sector endogenous growth model. As in the benchmark model of Romer (1986), the economy is always on its balanced growth path, so that there are no transitional dynamics, either in aggregates or in the distributional measures; see Mierau and Turnovsky (2013). That paper has a very different focus in that it emphasizes the impact of the demographic structure on the endogenously determined equilibrium growth rate, as well as its impact on the natural rate of wealth inequality.

[9]Their analysis employs linear utility and a constant returns production that uses human capital as its only input.

[10]We employ US mortality data, but Heijdra and Mierau (2012) successfully apply the BCL function to Dutch data. However, like other functions, such as the de Moivre function, the BCL function fails in the extreme old age tail of the mortality distribution; see Fig. 2. Recently, in their study on human life-span, Strulik and Vollmer (2013) show how this convex extreme old age tail can be successfully tracked by the three-parameter Gompertz function.

Bloom et al. (2007), all of whom find that an increase in life expectancy leads to more individual and aggregate savings. When we decompose these aggregate increases into a birth rate- and a mortality rate-induced effect, we find that both these demographic changes have contributed to an increase in the aggregate per capita capital stock, with relatively more being due to the reduction in mortality. In contrast, for consumption, we find that the drop in the mortality rate has had a slightly negative effect, while the decline in the birth rate has had a strong, and more than compensating, positive effect.

Demographic changes occur slowly. Thus, in considering the transitional paths associated with the demographic changes, we focus particular attention on their gradual adjustments. In both cases, we find that a gradual adjustment leads to nonmonotonic transitional paths for the aggregate capital stock. In the case of the decline in the birth rate, the adjustment path is U-shaped, with the capital stock initially declining before eventually increasing. In contrast, for the decline in mortality, the adjustment path has an inverted U shape, so that during the transition the capital stock overshoots its long-run equilibrium increase. Taken together, the nonmonotonic components are approximately offsetting, and we find the overall adjustment paths for the aggregates to be essentially monotonic for the composite demographic change experienced in the USA. The qualitative characteristics of these paths for capital are reflected in those for consumption, although, due to "consumption smoothing", the nonmonotonicities are less pronounced.

The remainder of the paper is structured as follows. Section 2 sets out the underlying analytical framework, while Section 3 describes the corresponding macrodynamic equilibrium, local dynamics, and steady state. Section 4 performs the numerical simulations. The final section concludes and provides some suggestions for directions in which this research might be extended.

2 The analytical framework

2.1 Individual household behavior

Consider an individual born at time v. The probability that this agent lives to become $t-v$ years old is governed by the survival function $S(t-v)$, where $S'(s) \equiv \mathrm{d}S(s)/\mathrm{d}s < 0$, decreases with age. Given this function, the hazard rate, or instantaneous probability of death, is given by:

$$\mu(t-v) = -\frac{S'(t-v)}{S(t-v)} > 0. \tag{1}$$

The probability that an individual dies before reaching age $t-v$ is described by the cumulative mortality rate:

$$M(t-v) = \int_0^{t-v} \mu(\tau)\mathrm{d}\tau. \tag{2}$$

Combining (1) and (2), the survival function can be related to the mortality function by:

$$S(t-v)=e^{-M(t-v)}, \tag{3a}$$

where:

$$S(0)=e^{-M(0)}=1, \quad S(D)=e^{-M(D)}=0 \tag{3b}$$

so that D defines the maximum age that individuals can attain.[11]

Given this specification of the mortality function, the discounted expected lifetime utility of an individual newborn at time v is:

$$E\Lambda(v)=\int_{v}^{v+D}U(C(v,t))\cdot e^{-\rho(t-v)-M(t-v)}\mathrm{d}t, \tag{4a}$$

where $C(v,t)$ denotes the consumption at time t of an individual born at time v, and ρ is the pure rate of time preference of a newborn. Written in this way, the agent's discount rate, $\rho+\mu(t-v)$, varies with age.[12] The agent supplies a unit of labor inelastically and makes his consumption and asset accumulation decisions to maximize his expected utility, (4a), subject to his budget constraint:

$$A_t(v,t)\equiv\frac{\partial A(v,t)}{\partial t}=(r(t)+\mu(t-v))A(v,t)+w(t)-C(v,t), \tag{4b}$$

where $A(v,t)$ are assets held at time t of an individual born at time v, $w(t)$ is the wage rate, and $r(t)$ is the return to capital (see (25a) and (25b)).

Individuals are born without assets, have no bequest motive, and are not allowed to have debt upon reaching the maximum attainable age, D. Therefore, $A(v,v)=0$, and individuals fully annuitize all their assets. Annuities are life-insured financial assets that pay, conditional on the survival of the individual. Individuals receive a premium on these annuities equal to their instantaneous probability of death, $\mu(t-v)$, and in return, if an individual dies, his assets flow to the insurance company.[13] Thus, the overall rate of return received by an agent on his assets is $r(t)+\mu(t-v)$. Alternatively, an individual may engage in borrowing. In that case, he pays a premium of $\mu(t-v)$, and if he dies, his debts are canceled.

Optimizing (4a) subject to (4b) with respect to $C(v,t)$ and $A(v,t)$, yields:

$$U'(C(v,t))=\lambda(v,t), \tag{5a}$$

$$\rho-\frac{\lambda_t(v,t)}{\lambda(v,t)}-\frac{S'(t-v)}{S(t-v)}=r(t)+\mu(t-v). \tag{5b}$$

[11] Throughout this paper, we take D to be finite, though the extension to an infinite D is straightforward.

[12] From (1)–(3), we see that the discount increases with age if and only if $SS''<(S')^2$, which is certainly met if the mortality function is concave.

[13] This result follows from perfect competition between annuity firms. If competition between annuity firms is less than perfect, there is a load factor, $0\le\lambda\le1$, on the annuity premium, and individuals receive only $\lambda\mu(t-v)$ on their annuities. This is studied by Heijdra and Mierau (2012).

Equation 5a equates the marginal utility of consumption to the shadow value of financial wealth, while (5b) equates the rate of return on consumption, adjusted by the mortality hazard rate, to the rate of return on financial assets. In addition, the agent must satisfy the transversality condition:

$$A(v, v + D) = 0. \tag{5c}$$

In the absence of a bequest motive, individuals want to ensure that $A(v, v + D) \leq 0$, and annuity firms want to ensure that $A(v, v + D) \geq 0$. Thus, the only feasible solution is $A(v, v + D) = 0$.[14]

For analytical convenience, we follow the contemporary growth literature and assume an iso-elastic utility function:

$$U(C(v,t)) = \frac{C(v,t)^{1-1/\sigma} - 1}{1 - 1/\sigma},$$

where σ is the intertemporal elasticity of substitution. Combining (5a), (5b), and noting (1) enables us to write the Euler equation as follows:

$$\frac{\partial C(v,t)/\partial t}{C(v,t)} = \sigma(r(t) - \rho), \tag{6}$$

which expresses how the agent's consumption changes with age. Thus, as a consequence of assuming iso-elastic utility in combination with full annuitization of assets, Eq. 6 implies that consumption of all agents grows at a common rate, independent of their age or level of wealth.

Solving (6) forward from time t, the agent's consumption at an arbitrary time $\tau > t$ is:

$$C(v,\tau) = C(v,t)e^{\sigma(R(t,\tau)-\rho(\tau-t))}, \tag{7}$$

where $R(t,\tau) \equiv \int_t^\tau r(s)\mathrm{d}s$ is the cumulative interest rate over the period (t,τ). To express the agent's consumption in terms of his financial resources, we integrate the budget constraint (4b) forward from time t and impose the transversality condition, Eq. 5c, to yield the agent's intertemporal budget constraint operative from time t:

$$A(v,t) + e^{R(v,t)+M(t-v)} \int_t^{v+D} w(\tau)e^{-R(v,\tau)-M(\tau-v)}\mathrm{d}\tau = e^{R(v,t)+M(t-v)} \times \int_t^{v+D} C(v,\tau)e^{-R(v,\tau)-M(\tau-v)}\mathrm{d}\tau. \tag{8}$$

[14] Although annuity firms cancel debts of individuals, they will not take up debts of individuals who die indebted for sure. That is, at some time, D–ε, annuity firms will refuse to issue life insurance and recall all debts. Letting $\varepsilon \to 0$ gives the constraint implied by the annuity firms.

Substituting (7) into (8), we obtain the following expression for $C(v,t)$:

$$C(v,t) = \frac{A(v,t) + \int_t^{v+D} w(\tau) e^{-R(t,\tau)-(M(\tau-v)-M(t-v))} \mathrm{d}\tau}{\int_t^{v+D} e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-(M(\tau-v)-M(t-v))} \mathrm{d}\tau}$$

$$= \frac{A(v,t) + H(v,t)}{\Delta(v,t)}, \tag{9a}$$

where:

$$H(v,t) \equiv \int_t^{v+D} w(\tau) e^{-R(t,\tau)-(M(\tau-v)-M(t-v))} d\tau \tag{9b}$$

is the discounted future labor income (human wealth) at time t of an individual born at time v, and:

$$\Delta(v,t) \equiv \int_t^{v+D} e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-(M(\tau-v)-M(t-v))} \mathrm{d}\tau \tag{9c}$$

is the inverse of the marginal propensity to consume out of total wealth (i.e., financial wealth, $A(v,t)$, plus human wealth, $H(v,t)$) at age $t-v$. Expressions (9b) and (9c) show that an increase in mortality leads to a decline in human wealth and an increase in the marginal propensity to consume, as agents will have a shorter expected life-span over which to accumulate assets and to consume the income they derive. Setting $t = v$ yields the corresponding quantities at birth, $H(v,v)$ and $\Delta(v,v)$.

2.2 Aggregate household behavior

Let $P(t)$ denote the size of the total population at time t. The birth rate, β, is constant, so that at every instant v, a cohort of size $P(v,v) = \beta P(v)$ is born.[15] Given the mortality function, the number of individuals of cohort v still alive at time t is $P(v,t) = \beta P(v) e^{-M(t-v)}$. Similarly, at every instant v, a mass of $\overline{\mu} P(v)$ individuals dies, where $\overline{\mu}$ is the average mortality rate across cohorts:

$$\overline{\mu} \equiv \int_{t-D}^{t} \mu(t-v) \frac{P(v,t)}{P(t)} \mathrm{d}v, \tag{10}$$

which, assuming the demographic steady state, defined in Eq. 15 below, is constant. In the absence of migration, the growth rate of the population is equal to $n = \beta - \overline{\mu}$, which therefore is also constant. Hence, from the perspective of time v, the population at time t is equal to:

$$P(t) = P(v) e^{n(t-v)}. \tag{11}$$

[15] Note that a constant birth rate does not imply that all cohorts have the same fertility rate. It simply says that in the demographic steady state (see below), the birth rate of the entire population can be captured in a single quantity, just like we can express the average mortality in a single quantity (Lotka 1998).

The relative weight of a cohort is:

$$\frac{P(v,t)}{P(t)} = \beta e^{-n(t-v)-M(t-v)} \equiv p(t-v), \tag{12}$$

the dynamics of which are the following:

$$\frac{P_t(t-v)}{p(t-v)} \equiv \frac{\partial p(t-v)/\partial t}{p(t-v)} = -[n+\mu(t-v)]. \tag{13}$$

Thus, the decline in the relative size of each cohort reflects both its individual mortality rate and the overall population growth rate.

Aggregating over the surviving cohort members at each point of time, the total population at any time t is equal to:

$$P(t) = \beta \int_{t-D}^{t} P(v) e^{-M(t-v)} \mathrm{d}v. \tag{14}$$

Substituting (11) into (14) yields the relationship:

$$\beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} \mathrm{d}v = 1, \tag{15}$$

which defines the demographic steady state; see, Lotka (1998, p. 60). That is, Eq. 15 defines a constraint linking the birth rate, β, mortality structure, ($M(t-v)$ and D), and the overall population growth rate, n. For example, given the birth rate and mortality function, Eq. 15 yields the implied population growth rate. This relationship is an integral component of any consistently specified aggregate demographic structure.

To obtain aggregate per capita quantities, we sum across cohorts by employing the following generic aggregator function:

$$x(t) \equiv \int_{t-D}^{t} p(t-v) X(v,t) \mathrm{d}v = \beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)} X(v,t) \mathrm{d}v. \tag{16}$$

Taking the time derivative of (16), the evolution of $x(t)$ is given by:

$$\begin{aligned} \dot{x}(t) &= \beta X(t,t) + \int_{t-D}^{t} p(t-v) X_t(v,t) \mathrm{d}v - n x(t) \\ &\quad - \int_{t-D}^{t} \mu(t-v) p(t-v) X(v,t) \mathrm{d}v \end{aligned} \tag{17}$$

where we have used (13), and the fact that $p(0) = \beta$, $p(D) = 0$; see (3b) and (12).

Thus, aggregate per capita consumption is:

$$c(t) \equiv \int_{t-D}^{t} p(t-v) C(v,t) \mathrm{d}v. \tag{18}$$

Taking the time derivative of (18), and using (6) and (13), the dynamics of per capita consumption are described by:

$$\dot{c}(t) = (\sigma[r(t)-\rho]-n)c(t) + \beta C(t,t) - \int_{t-D}^{t} \mu(t-v) p(t-v) C(v,t) \mathrm{d}v. \tag{19}$$

Combining (19) with (6), we see that:

$$\frac{\dot{c}(t)}{c(t)} = \frac{\partial C(v,t)/\partial t}{C(v,t)} - \frac{\Phi(t)}{c(t)} \tag{20a}$$

where:

$$\Phi(t) \equiv \int_{t-D}^{t} \mu(t-v)p(t-v)C(v,t)\mathrm{d}v - \beta C(t,t) + nc(t) \tag{20b}$$

is the "generational turnover term". Thus, $\Phi(t)$ reflects the reduction in aggregate per capita consumption growth (below the common consumption growth rate of each cohort) due to the arrival of newborn agents with no accumulated assets and the departure of agents with assets. It depends upon the (a) total consumption given up by the dying relative to the average and (b) difference between the consumption of a newborn and the average per capita consumption due to growth.

The expression in Eq. 20b provides a general specification that encompasses all of the standard demographic models. With zero population growth, the textbook infinitely lived representative agent model is obtained by setting $\beta = \mu = 0$ (implying $D \to +\infty$). With a growing population, we need to take account of the fact that at each instant, each newborn is immediately endowed with the average capital stock, part of which he must immediately set aside for the individuals born at the next instant. Given the intertemporal elasticity of substitution σ, this reduces the per capita consumption growth rate by $\Phi(t)/c(t) = \sigma n$, so that (20a) reduces to the familiar Euler equation $\dot{c}(t) = \sigma(r(t) - \rho - n)c(t)$. If population growth is driven by the addition of newborns without assets, the structure of $\Phi(t)$ becomes more involved, as shown below.

Substituting for Eqs.1, 3, and 12 in Eq. 20b yields:

$$\Phi(t) = -\beta \int_{t-D}^{t} S'(t-v)e^{-n(t-v)}C(v,t)\mathrm{d}v - \beta C(t,t) + nc(t).$$

Integrating by parts and simplifying, yields:

$$\begin{aligned}\Phi(t) &= -\beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)}[nC(v,t) + C_v(v,t)]\mathrm{d}v + nc(t) \\ &= -\beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)}C_v(v,t)\mathrm{d}v,\end{aligned} \tag{20c}$$

where $C_v(v,t)$ represents the rate of change of consumption across cohorts at a given point in time. Hence, using (20c) in (19), the evolution of aggregate per capita consumption can be written as follows:

$$\dot{c}(t) = \sigma[r(t) - \rho]c(t) - \beta \int_{t-D}^{t} e^{-n(t-v)-M(t-v)}C_v(v,t)\mathrm{d}v. \tag{21}$$

To determine the sign of $\Phi(t)$, we use the fact that at any instant of time, the rate of change of consumption of agents of age $t-v$ is $\dot{C}(v,t) = C_v(v,t) + C_t(v,t)$.[16] Recalling (6), and letting $\gamma(v,t) \equiv \dot{C}(v,t)/C(v,t)$ denote the growth rate of consumption, this implies that:

$$C_v(v,t) = [\gamma(v,t) - \sigma[r(t) - \rho]]\, C(v,t).$$

Hence, a sufficient condition to ensure that $\Phi(t) > 0$ is that the growth rate of consumption with age exceeds the overall growth rate of consumption. In steady state, $\gamma(v,t) = 0$, implying that $C_v(v,t) = -C_t(v,t) = -\sigma\,(\tilde{r} - \rho)\,\tilde{C}$, and we immediately derive $\tilde{\Phi} = \sigma(\tilde{r} - \rho)\tilde{c} > 0$.[17]

Employing (16) again, the aggregate per capita assets are the following:

$$a(t) \equiv \int_{t-D}^{t} p(v,t)A(v,t)\mathrm{d}v. \tag{22}$$

Taking the time derivative of Eq. 22 and using (4b) and (13), the per capita asset accumulation is:

$$\begin{aligned}\dot{a}(t) &= \int_{t-D}^{t} p(t-v)\,[(r(t) + \mu(t-v))A(v,t) + w(t) - C(v,t)]\,\mathrm{d}v \\ &\quad - \int_{t-D}^{t} [n + \mu(t-v)] \cdot p(t-v)A(v,t)\mathrm{d}v\end{aligned}$$

so that:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t), \tag{23}$$

where we have used the fact that $A(t,t) = 0$. As in Blanchard (1985), the per capita rate of asset accumulation differs from the individual rate of asset accumulation, due to the fact that (a) an amount $\int_{t-D}^{t} p(t-v)\mu(t-v)A(v,t)\mathrm{d}v$ is transferred by annuity firms from those who die to those surviving and does not add to aggregate wealth and (b) an account must be taken of the growing population.

2.3 Firms

Output is produced by a representative firm in accordance with the neoclassical production function having constant returns to scale:

$$Y(t) = F(K(t), L(t)),\ \ F_K > 0, F_L > 0, F_{KK} < 0, F_{LL} < 0, F_{LK} > 0, \tag{24}$$

where $Y(t)$ is the output, $K(t)$ is the capital, and $L(t)$ is the aggregate labor supply. In per capita terms, this may be expressed as follows:

$$\frac{Y(t)}{L(t)} \equiv y(t) = F\left(\frac{K(t)}{L(t)}, 1\right) = f(k(t)). \tag{24a}$$

[16] Formally, the rate of change of consumption of t–v year old agents is $\lim_{h\to 0} \frac{C(v+h,t+h)-C(v,t)}{h} = C_v(v,t) + C_t(v,t)$.

[17] A tilde indicates a steady-state value.

Assuming that labor and capital are paid their marginal products, the equilibrium wage rate and return to capital are determined by:

$$w(t) = f(k(t)) - f'(k(t))k(t) \tag{25a}$$

$$r(t) = f'(k(t)) - \delta \tag{25b}$$

where δ denotes the depreciation rate of capital.

3 General equilibrium

In equilibrium, both the labor and the capital markets must clear. Labor market clearance is reflected by the fact that all agents are fully employed so that the total population equals the total labor force. Capital market equilibrium is imposed by setting aggregate assets equal to total capital $A(t) = K(t)$, so that in aggregate per capita terms, $a(t) = k(t)$, implying further that $\dot{a}(t) = \dot{k}(t)$.

Substituting the factor pricing relations (25) into (23) and (21) enables us to summarize the dynamics of the macroeconomic equilibrium in the following form:

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n)k(t) \tag{26a}$$

$$\dot{c}(t) = \sigma(f'(k(t)) - \delta - \rho)c(t) - \Phi(t) \tag{26b}$$

where:

$$\Phi(t) = \int_{t-D}^{t} \mu(t-v)p(t-v)C(v,t)\mathrm{d}v - \beta C(t,t) + nc(t). \tag{26c}$$

This pair of dynamic equations in $\dot{k}$ and $\dot{c}$ will be recognized as being a variant of the standard textbook neoclassical growth model. Equation 26a is the standard aggregate per capita accumulation of capital relationship, where the normalization of individual labor supply at unity implies that aggregate labor supply is equal to one, while (26b) is the aggregate Euler equation, determining the intertemporal allocation of consumption.

The key point to emphasize with regard to expressing the macroeconomic equilibrium in this way is that it highlights how the impact of the demographic structure on the economy occurs through the generational turnover term, $\Phi(t)$, and its impact on the aggregate Euler equation. It provides a general representation in which alternative demographic structures can be embedded.

We have already noted that, for the infinitely lived representative agent model, $\Phi(t) = \sigma nc(t)$, in which case (26a) and (26b) reduce to the standard equations relating capital and consumption. In the case of the BBW model, the constancy of the marginal propensity to consume across cohorts simplifies the evolution of $\Phi(t)$, and the full model can be described by a three-dimensional dynamic system; see Blanchard (1985, p. 234). However, the fact that for more general demographic structures, the marginal propensity to consume varies over the life cycle complicates the dynamics of $\Phi(t)$, leading to a higher-order differential-difference equation system.

In general, to analyze the global dynamics is intractable, making it necessary to impose constraints on the demography. In Section 3.1, we show how we are able to approximate the local dynamics in the neighborhood of the steady state.

While we take the demographic structure to be exogenous, there is an extensive and growing literature endogenizing this aspect, in the process appealing to different mechanisms. For example, building on the seminal contribution of Becker (1981), Becker and Barro (1988) develop a model in which parents choose fertility together with consumption by maximizing a dynastic utility function. Equilibrium is characterized by an arbitrage condition for consumption across generations and equality between the benefits of an extra child and child-raising costs. Cigno (1998) employs a similar framework to show how the co-movement between fertility and infant mortality depends upon the resources parents allocate to each child.

Using the Barro and Becker (1989) model, Manuelli and Seshadri (2009) show how fertility and mortality differences across countries can be accounted for by differences in productivity and in labor income tax rates. Because of the contrasting ways these two variables are related to the growth rate, this enables them to show how the relationship between demographic changes and growth depends upon the source of the demographic change. In contrast, Soares (2005) studies the relationship between the declining mortality rates at birth, leading to reductions in fertility and followed by increases in the rate of human capital accumulation. Another approach is taken by Doepke (2004), who emphasizes the importance of laws regulating child labor in accounting for the demographic transition from high to low fertility accompanying the process of industrialization.[18]

More recently, Chu et al. (2013) study how patent protection in Schumpeterian growth models affects the fertility rate. In a different growth context, Varvarigos and Zakaria (2013) highlight how adding endogenous fertility choice to a model of health expenditures can account for the fertility decline that has accompanied the decline in mortality in most developed countries. Although we treat demographic changes as exogenous, in Section 4, we analyze how their economic consequences depend on whether they are driven by mortality or fertility changes, which in turn may be the result of one of the mechanisms just described.

3.1 Aggregate equilibrium dynamics

To consider the dynamics of the aggregate economy summarized by (26), we must take account of $\Phi(t)$, which, as just noted, is generally intractable because it leads to a higher-order differential-difference equation system. We can, however, obtain a good local approximation to the dynamics by adopting the following procedure. We

[18] In a related study, Doepke (2005) shows that the original Barro–Becker model cannot account for the large decline in fertility rates observed in most developed countries over the past century. He then goes on to show that neither discrete nor sequential fertility choice can bring the model in line with the decline in fertility rates.

begin by applying the first mean value theorem for integration to the first term on the right-hand side of Eq. 26c, enabling us to express $\Phi(t)$ as follows:[19]

$$\Phi(t) = \mu(t-v_1)\int_{t-D}^{t} p(t-v)C(v,t)\mathrm{d}v - \beta C(t,t) + nc(t) \quad v_1 \in (t-D,t). \tag{27a}$$

Writing:

$$\mu_C(t-v_1) \equiv \mu(t-v_1) = \frac{\int_{t-D}^{t} \mu(t-v)p(t-v)C(v,t)\mathrm{d}v}{\int_{t-D}^{t} p(t-v)C(v,t)\mathrm{dv}}, \tag{27b}$$

we see that $\mu_C(t-v_1)$ is the ratio of the consumption given up by the dying to aggregate consumption and can be interpreted as the average mortality of consumers over the period $(t-D,t)$. We identify this measure as $\mu_C(t-v_1)$ so as to distinguish it from other measures of average μ computed from Eqs. 30a and 30b.[20] Using $\mu_C(t-v_1)$, we may write (27a) as follows:

$$\Phi(t) \equiv (\mu_C(t-\nu_1)+n)c(t) - \beta C(t,t). \tag{28}$$

In order to describe the dynamics of $C(t,t) = H(t,t)/\Delta(t,t)$, we take the time derivatives of $H(t) \equiv H(t,t)$ and $\Delta(t) \equiv \Delta(t,t)$ (see 9b and 9c) and apply the mean value theorem again to yield:

$$\dot{H}(t) = -w(t) + [r(t) + \mu_H(\tau_1 - t)]H(t) \quad \tau_1 \in (t, t+D), \tag{29a}$$

$$\dot{\Delta}(t) = -1 + [(1-\sigma)r(t) + \sigma\rho + \mu_\Delta(\tau_2 - t)]\,\Delta(t) \quad \tau_2 \in (t, t+D). \tag{29b}$$

μ_H and μ_Δ are defined analogously to μ_C, namely:

$$\mu_H(\tau_1 - t) = \frac{\int_{t}^{t+D} \mu\,(\tau - t)\,w(\tau)e^{-R(t,\tau)-M(\tau-t)}\mathrm{d}\tau}{\int_{t}^{t+D} w(\tau)e^{-R(t,\tau)-M(\tau-t)}\mathrm{d}\tau}, \tag{30a}$$

[19] Since there is some ambiguity with respect to the precise naming of the various mean value theorems, we are using the following specific result. For any real valued function $f(x)$ on the interval $[a,b]$ and function $g(x)$ that is integrable and does not change sign over the interval (a,b), there exists a value $c \in (a,b)$ such that $\int_a^b f(x)g(x)\mathrm{d}x = f(c)\int_a^b g(x)\mathrm{d}x$.

[20] We should note that the intermediate value $v_1 \in (t-D,t)$ will, in general, be a function of t, in which case $\mu(t-v_1)$ should be written as $\mu(t-v_1(t))$. We refrain from representing this explicitly, so as not to clutter notation.

$$\mu_{\Delta}(\tau_2 - t) = \frac{\displaystyle\int_t^{t+D} \mu(\tau - t)\, e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-M(\tau-t)} \mathrm{d}\tau}{\displaystyle\int_t^{t+D} e^{(\sigma-1)R(t,\tau)-\sigma\rho(\tau-t)-M(\tau-t)} \mathrm{d}\tau}, \tag{30b}$$

where τ_1 and τ_2 are the values of τ determined by the mean value theorem (see footnote (19)).

Using (28) and (29), together with the factor return expressions, (25), we can summarize the macrodynamic equilibrium, (26), by the autonomous fourth-order system:

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n)k(t) \tag{31a}$$

$$\dot{c}(t) = \left[\sigma\left(f'(k(t)) - \delta - \rho\right) - (\mu_C + n)\right] c(t) + \beta \frac{H(t)}{\Delta(t)} \tag{31b}$$

$$\dot{H}(t) = -\left[f(k(t)) - f'(k(t))k(t)\right] + \left[f'(k(t)) - \delta + \mu_H\right] H(t) \tag{31c}$$

$$\dot{\Delta}(t) = -1 + \left[(1-\sigma)(f'(k(t)) - \delta) + \sigma\rho + \mu_{\Delta}\right] \Delta(t) \tag{31d}$$

where μ_C, μ_H, and μ_{Δ} are generated by the integrals (27a), Eqs. 30a and 30b, respectively. Being functions of the wage rate and the return to capital, which vary over time, it is important to note that the μ_i terms are therefore also functions of time. However, as we show below, given the assumption of the demographic steady state, they in fact vary only slightly over time and, for practical purposes, can be treated as constants. Moreover, being estimates of mortality rates, they are uniformly small, and their equilibrium values can be computed as indicated in footnote 32. Accordingly, we approximate the dynamics (31), by assuming that μ_C, μ_H, and μ_{Δ} remain constant at these values. Thus, linearizing (31) around the steady state, $\left(\tilde{k}, \tilde{c}, \tilde{H}, \tilde{\Delta}\right)$, the local dynamics can be expressed as follows:

$$\begin{pmatrix} \dot{k} \\ \dot{c} \\ \dot{H} \\ \dot{\Delta} \end{pmatrix} = \begin{pmatrix} f'(\tilde{k}) - (\delta + n) & -1 & 0 & 0 \\ \sigma f''(\tilde{k})\tilde{c} & -\dfrac{\beta\tilde{H}}{\tilde{\Delta}\tilde{c}} & \dfrac{\beta}{\tilde{\Delta}} & \dfrac{-\beta\tilde{H}}{\tilde{\Delta}^2} \\ f''(\tilde{k})(\tilde{k} + \tilde{H}) & 0 & \dfrac{f\left(\tilde{k}\right) - \tilde{k} f'\left(\tilde{k}\right)}{\tilde{H}} & 0 \\ (1-\sigma) f''(\tilde{k})\tilde{\Delta} & 0 & 0 & \dfrac{1}{\tilde{\Delta}} \end{pmatrix} \begin{pmatrix} k - \tilde{k} \\ c - \tilde{c} \\ H - \tilde{H} \\ \Delta - \tilde{\Delta} \end{pmatrix} \tag{32}$$

where we have omitted the time indices to avoid cluttering the notation.

This is the system we employ to study the local dynamics. Since the per capita capital stock is constrained to evolve sluggishly, while per capita consumption, human wealth at birth, and the marginal propensity to consume at birth can respond instantaneously, this dynamic system will have a unique bounded path if and only if there are one negative and three positive eigenvalues, in which case the local stable manifold will be one-dimensional. In principle, one can establish the formal conditions that ensure this required configuration of eigenvalues, although in practice such conditions are uninformative to study. Instead, we proceed numerically using a specific mortality function, which, as we confirm in Section 4, generates the required stability characteristics.

Assuming that the system starts from an initial capital stock, k_0, the local stable transitional dynamics of the aggregate system, (32), can be described by:

$$k(t) = \tilde{k} + \left(k_0 - \tilde{k}\right) e^{\lambda_t t} \tag{33a}$$

$$c(t) = \tilde{c}\left(f'\left(\tilde{k}\right) - \delta - n - \lambda_1\right)\left(k(t) - \tilde{k}\right) \tag{33b}$$

$$H(t) = \tilde{H}\left[\lambda_1\left(\frac{f(\tilde{k}) - \tilde{k}f'(\tilde{k})}{\tilde{H}}\right)\right]^{-1} f''(\tilde{k})(\tilde{k} + \tilde{H})\left(k(t) - \tilde{k}\right) \tag{33c}$$

$$\Delta(t) = \tilde{\Delta} + \left(\lambda_1 - \frac{1}{\tilde{\Delta}}\right)^{-1} (1 - \sigma)\, f''(\tilde{k})\tilde{\Delta}\left(k(t) - \tilde{k}\right) \tag{33d}$$

where $\lambda_1 < 0$ is the unique negative eigenvalue. From these equations, we see that the demographic structure impinges on the transitional dynamics of the aggregate economy through two channels: (a) its impact on the steady state and (b) the speed of convergence of capital as measured by the size of the negative eigenvalue. In addition, (33b)–(33d) imply that consumption and human wealth at birth increase with capital, while the marginal propensity to consume, $\Delta(t)^{-1}$, varies inversely with capital.

To show that $\mu_i (i = C, H, \Delta)$ are approximately constant over time, we proceed as follows, focusing on μ_H, although analogous arguments apply to the other two quantities. First, recalling the definition of $R(t, \tau)$ and letting $\gamma_w(u)$ denote the time-varying growth rate of the wage rate, so that:

$$e^{-R(t,\tau)} = e^{-\int_t^\tau r(u)\mathrm{d}u}, \quad w(\tau) = w(t)e^{\int_t^\tau \gamma_w(u)\mathrm{d}u}$$

and letting $s = \tau - t$, we may write (30a) as follows:

$$\mu_H(\tau_1 - 1) = \frac{\int_0^D \mu(s)\, w(t)e^{-\int_t^{t+s}[r(u)-\gamma_w(u)]\mathrm{d}u} e^{-M(s)}\mathrm{d}s}{\int_0^D w(t)e^{-\int_t^{t+s}[r(u)-\gamma_w(u)]\mathrm{d}u} e^{-M(s)}\mathrm{d}s}$$

$$= \frac{\int_0^D \mu(s)e^{-\int_t^{t+s} r_w(u)\mathrm{d}u} e^{-M(s)}\mathrm{d}s}{\int_0^D e^{-\int_t^{t+s} r_w(u)\mathrm{d}u} e^{-M(s)}\mathrm{d}s} \tag{34a}$$

where for convenience $r_w(u) \equiv r(u) - \gamma_w(u)$ denotes the rate of return on capital net of the growth rate of wages. From (34a), we can immediately see that μ_H is independent of t (constant over time) if either (a) r_w is constant over time or (b) the mortality rate, $\mu(s)$, is constant over cohorts (as it is in the BBW model). Otherwise, it will be time-varying, although of the second order. To see this, we take the time derivative of (34a), which letting $F(s,t) \equiv e^{-M(s)}e^{-\int_t^{t+s} r_w(u)\mathrm{d}u} > 0$ yields:

$$\frac{d\mu_H(\tau_1-t)/\mathrm{d}t}{\mu_H(\tau_1-t)} = \frac{\int_0^D F(s,t)r_w(t+s)\,\mathrm{d}s}{\int_0^D F(s,t)\mathrm{d}s} - \frac{\int_0^D \mu(s)F(s,t)r_w(t+s)\,\mathrm{d}s}{\int_0^D \mu(s)F(s,t)\,\mathrm{d}s}. \tag{34b}$$

Written in this way, we see that the percentage rate of change of μ_H is the difference between two weighted averages of $r_w(t+s)$.[21] Now applying the mean value theorem to (34b), we obtain

$$\begin{aligned}\frac{\mathrm{d}\mu_H(\tau_1-t)/\mathrm{d}t}{\mu_H(\tau_1-t)} &= r_w(t+s_1(t))\frac{\int_0^D F(s,t)\mathrm{d}s}{\int_0^D F(s,t)\mathrm{d}s} - r_w(t+s_2(t))\frac{\int_0^D \mu(s)F(s,t)\mathrm{d}s}{\int_0^D \mu(s)F(s,t)\mathrm{d}s}\\ &= r_w(t+s_1(t)) - r_w(t+s_2(t)) \qquad 0<s_1(t)<D,\quad 0<s_2(t)<D\end{aligned}$$

which we can rewrite as follows:

$$d\mu_H(\tau_1-t) = \mu_H(\tau_1-t)\left[r_w(t+s_1) - r_w(t+s_2)\right]\mathrm{d}t \tag{35}$$

where (35) involves the difference in the average in the net return to capital at two points in time interacting with dt.

To consider (35), we adopt the following strategy. Assume initially that μ_H is constant at its steady-state value, computed from the numerical simulations reported in Section 5 to be $\tilde{\mu}_H = 0.0029$. With μ_H constant, the stable transitional dynamics is a one-dimensional path, along which all variables, including r_w, converge to their respective steady states, at the rate of the stable eigenvalue, $\lambda_1 < 0$. Thus, as a linear approximation, we obtain:

$$r_w(t+s_i) = \tilde{r}_w + (r_w(t) - \tilde{r}_w)\,e^{\lambda_1 s_i} \qquad i = 1, 2 \tag{36a}$$

where $\tilde{r}_w$ is the steady-state value, which in our simulations is 0.064, and the speed of convergence, $-\lambda_1$, is around 0.077. Using (36a), we obtain the following approximation:

$$r_w(t+s_2) - r_w(t+s_1) = (r_w(t) - \tilde{r}_w)\left[e^{\lambda_1 s_1} - e^{\lambda_1 s_2}\right] \approx \lambda_1\,(r_w\,(t) - \tilde{r}_w)\,(s_1 - s_2)$$

so that:

$$d\mu_H(\tau_1-t) \approx \tilde{\mu}_H\lambda_1(r_w\,(t) - \tilde{r}_w)\,(s_1 - s_2)\,\mathrm{d}t. \tag{36b}$$

This is clearly a *second-order* effect and is negligible relative to the linear approximations describing the local aggregate dynamics. To see this, take for example,

[21] Intuitively, since t enters the numerator and denominator of (34) in identical ways to the first order of approximation, their effects are offsetting.

$\tilde{\mu}_H = 0.0029$ and $\lambda_1 = -0.077$ (their equilibrium values) and an extreme starting point for $r_w(t) = 0.128$.[22] Noting that s_1, s_2 pertain to average time points over agents' life-spans and therefore satisfy $0 < s_1, s_2 < D = 95$, unless the weighted averages defined by (34b) reflect very different distributions, it would seem reasonable for $|s_1 - s_2| < 10$, say. In this case, (36b) implies that the coefficient of dt is of the order of 0.0001.[23] Therefore, μ_H can essentially be treated as constant and independent of t.

Analogous arguments apply to μ_C and μ_Δ, thereby enabling us to approximate them all as constants in addressing the dynamic adjustment of the aggregate economy. At the same time, we should stress that while the constant equilibrium values of μ_C, μ_H, and μ_Δ are reflected in the dynamics, their changes as described by Eq. 36b are assumed to have negligible effects on the linear approximation we are considering.[24] Thus, for example, the contribution of changing μ_H to Eq. 31c is described by the term $\tilde{H}(\mu_H - \tilde{\mu}_H)$ and is a second-order effect, which, for the adjustment approximated by Eq. 36b, is negligible.

Before proceeding, an additional point should be noted. For analytical convenience, we have adopted the standard assumption that the mortality function is independent of calendar time and, hence, remains fixed throughout the transition. But one of the more interesting issues—and one we address in our simulations in Section 4.2—is the impact of the demographic transition itself on the dynamics of the aggregate economy. To consider this, we need to allow the mortality function to vary with calendar time, as the transition progresses. In the Appendix, we show how the approximation to the dynamics we have been deriving can be modified to include the effect of the demographic transition. This is reflected in terms involving the rate of change of the mortality rate over time. There we show that for plausible values of the mortality rate and its rate of change, characteristic of those experienced by the USA over the period we study, the impact of the changing mortality on the transitional dynamics is also negligible. As further convincing evidence of this, we find that over the entire transitional period we consider, μ_H declines by just 0.001, from 0.0039 to 0.0029, the effects of such a small change on the aggregate macrodynamics surely being negligible.[25]

3.2 Steady state

In steady state, the distributions of consumption, asset accumulation, relative cohort size, survival, and mortality no longer depend on calendar time but only on age

[22] In fact, if we evaluate $r_w(t)$ at steady state, $\tilde{r}_w$, in which case $d\mu_H(\tau - t)$ is exactly zero.

[23] Since $0 < s_1, s_2 < 78$, the maximum $|s_1 - s_2| = 78$, in which case the coefficient is around 0.001 and is still negligible.

[24] We should also note that sgn $[r_w(t + s_1) = r_w(t + s_2)]$ may be either negative or positive. In the former case, $d\mu_H$ will dampen out over time. In the latter case, while it will eventually become nonnegligible, this will take thousands of periods to occur. For example, taking the coefficient of 0.0001, μ_H will increase by just 1 % over the first 100 periods.

[25] We should caution in situations where the demographic transition occurs at a rapid finite rate that this term might cease to be negligible. However, that is not so in the transition we consider.

($u \equiv t - v$). Hence, with no long-run per capita growth, per capita consumption, $\tilde{c}$, per capita capital stock, $\tilde{k}$, the wage rate, $\tilde{w}$, the return to capital, $\tilde{r}$, and the generational turnover term, $\tilde{\Phi}$, all remain constant over time.

3.2.1 Steady-state consumption–age and asset–age profiles

When the aggregate economy is in steady state, consumption grows at the constant rate $\sigma(\tilde{r} - \rho)$ with age, so that the consumption level of an individual of age u is equal to:

$$\tilde{C}(u) = \tilde{C}_0 e^{\sigma(\tilde{r}-\rho)u}. \tag{37}$$

Setting $t = v$ in Eq. 9a–9c, consumption at birth, $\tilde{C}_0$, can be expressed as follows:

$$\tilde{C}_0 = \frac{\tilde{H}}{\tilde{\Delta}} \tag{38a}$$

where

$$\tilde{H} = \tilde{w} \int_0^D e^{-\tilde{r}u - M(u)} \mathrm{d}u, \tag{38b}$$

$$\tilde{\Delta} = \int_0^D e^{-(\tilde{r}(1-\sigma)+\sigma\rho)u - M(u)} \mathrm{d}u. \tag{38c}$$

The steady-state consumption–age profile, described by (37), implies the following steady-state asset accumulation–age profile:

$$\dot{\tilde{A}}(u) = (\tilde{r} + \mu(u))\, \tilde{A}(u) + \tilde{w} - \tilde{C}(u) \tag{39a}$$

so that starting with a zero initial endowment, $\tilde{A}(0) = 0$, the agent's wealth at age u is:

$$\tilde{A}(u) = \int_0^u \left[\tilde{w} - \tilde{C}(u)\right] e^{-\tilde{r}u - M(u)} \mathrm{d}u \tag{39b}$$

with the transversality condition implying:

$$\int_0^D [\tilde{w} - \tilde{C}(u)] e^{-\tilde{r}u - M(u)} \mathrm{d}u = 0. \tag{39c}$$

Under weak conditions, d'Albis (2007) has shown that in steady state $\tilde{r} > p$, so that agents' consumption grows uniformly over their lifetimes. Using this fact, in conjunction with (37), (38), and (39a), one can show that, because $\tilde{A}(0) = \tilde{A}(D) = 0$, $\dot{\tilde{A}}(0) > 0$, and $\dot{\tilde{A}}(D) < 0$, the agent's assets reach a maximum at an age $\hat{u}$:

$$\tilde{A}(\hat{u}) = \frac{\tilde{C}\left(\hat{u}\right) - \tilde{w}}{\tilde{r} + \mu\left(\hat{u}\right)}.$$

Thus, the time profile of the agent's wealth over the life cycle is hump-shaped as illustrated in panel (iii) of Fig. 3a.

3.2.2 Steady-state aggregates

In steady state, $p(t-v) = p(u) = \beta e^{-nu-M(u)}$ implying that aggregate consumption per capita is:

$$\tilde{c} \equiv \int_0^D p(u)\tilde{C}(u)\mathrm{d}u = \beta\tilde{C}_0 \int_0^D e^{(\sigma(\tilde{r}-\rho)-n)u-M(u)}\mathrm{d}u. \tag{40}$$

Defining the function:[26]

$$\Xi(\lambda) \equiv \int_0^D e^{-\lambda s-M(s)}\mathrm{d}s,$$

we can use (38) to express the steady-state per capita consumption, Eq. 40, as follows:

$$\tilde{c} = \tilde{w}\frac{\Xi(\tilde{r})}{\Xi(\tilde{r}(1-\sigma)+\sigma\rho)}\frac{\Xi(n-\sigma(\tilde{r}-\rho))}{\Xi(n)}. \tag{41a}$$

Using the demographic steady-state condition,

$$\frac{1}{\beta} = \int_0^D e^{-nu-M(u)}\mathrm{d}u = \Xi(n),$$

we can write:

$$\tilde{c} = \beta\tilde{w}\frac{\Xi(\tilde{r})\cdot\Xi(n-\sigma(\tilde{r}-\rho))}{\Xi(\tilde{r}(1-\sigma)+\sigma\rho)}. \tag{41b}$$

Substituting for the steady-state factor prices, (25a), the steady-state equilibrium values of per capita consumption, $\tilde{c}$, and capital, $\tilde{k}$, are jointly determined by:

$$\tilde{c} = f\left(\tilde{k}\right) - (\delta+n)\tilde{k} \tag{42a}$$

$$\tilde{c} = \beta\left[f\left(\tilde{k}\right) - \tilde{k}f'\left(\tilde{k}\right)\right] \times\frac{\Xi\left(f'\left(\tilde{k}-\delta\right)\right)\cdot\Xi\left(n-\sigma\left(f'\left(\tilde{k}\right)-\delta-\rho\right)\right)}{\Xi\left(\left[f'\left(\tilde{k}\right)-\delta\right](1-\sigma)+\sigma\rho\right)} \tag{42b}$$

where the demographic characteristics are embedded in the function Ξ, and (42a) represents the steady-state analog of Eq. 26a. Letting $\tilde{s}\left(\tilde{k}\right) \equiv \tilde{k}f'\left(\tilde{k}\right)/\tilde{f}\left(\tilde{k}\right)$ denote the equilibrium share of capital, d'Albis (2007) shows that the pair of Eqs. 42a and 42b have a unique solution as long as $\lim_{\tilde{k}\to 0}\tilde{s}\left(\tilde{k}\right) = 1$, $\tilde{s} < \tilde{e}$, where $\tilde{e}$ is the elasticity of substitution in production, and $\sigma < 1$. Both conditions are mild and hold for the Cobb–Douglas production function, for example.[27] Figure 1 illustrates this

[26] The Ξ-function is very common in the overlapping generations literature and appears in some form in d'Albis (2007), Gan and Lau (2010), and Heijdra and Romp (2008).

[27] These conditions have been relaxed in a subsequent work by Gan and Lau (2010), who show further that uniqueness is still obtained if $\sigma = 1$.

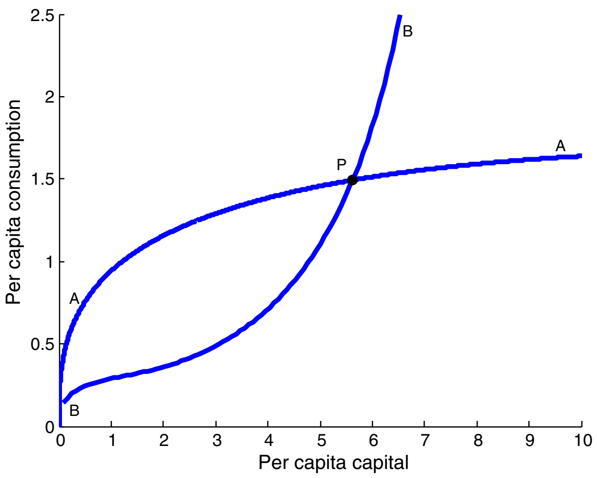

Fig. 1 Steady-state equilibrium

equilibrium for the calibrated model specified in Section 4, where AA represents (42a), BB depicts (42b), and the two intersect at the point P.

Finally, it is instructive to compare the equilibrium at P with the steady state obtained in the infinitely lived representative agent model. Denoting the corresponding steady-state per capita capital stocks by $\tilde{k}_P$ and $\tilde{k}_R$, these two quantities are determined, respectively, by:

$$\sigma\left(f'\left(\tilde{k}_P\right)-\delta-\rho\right)=\frac{\tilde{\Phi}}{\tilde{c}}, \tag{43a}$$

$$\sigma\left(f'\left(\tilde{k}_R\right)-\delta-\rho\right)=\sigma n. \tag{43b}$$

Recalling (20b), (43a), and (43b) imply that if (a) the total consumption given up by the dying exceeds the consumption of the newborn and if (b) the intertemporal elasticity of substitution, $\sigma<1$ that $\tilde{k}_P<\tilde{k}_R$, in which case the dynamic efficiency of the representative agent economy ensures that the demographic steady state is dynamically efficient as well.

3.3 Distributional dynamics

Having determined $\tilde{k}$ and $\tilde{c}$, and therefore the steady-state factor returns, $\tilde{r}$ and $\tilde{w}$, the steady-state values for $\tilde{H}$ and $\tilde{\Delta}$ immediately follow from Eqs. 38b and 38c, thereby enabling us to fully characterize the local aggregate dynamics as described by the system (32). Moreover, from Eq. 33a, we can express the local dynamics of factor returns, $r(t)$ and $w(t)$ in the form of $r(t)=\tilde{r}+f''(\tilde{k})(k(t)-\tilde{k})$, $w(t)=\tilde{w}-\tilde{k}f''(\tilde{k})(k(t)-\tilde{k})$. Substituting these expressions for $r(t)$ and $w(t)$ into (6)–(9)

permits us to spell out the local dynamics of consumption and asset accumulation across the distribution of agents of different age cohorts.

Thus, substituting for $r(t)$ into (6), in equilibrium, the agent's growth rate of consumption with age evolves over time in accordance with:

$$\frac{\partial C(v,t)/\partial t}{C(v,t)} = \sigma\left(f'\left(\tilde{k}\right) - \delta - \rho + f''\left(\tilde{k}\right)\left(k_0 - \tilde{k}\right)e^{\lambda_1 t}\right) \tag{44}$$

which, in principle, can be solved for $C(v,t)$ and will converge to Eq. 37 in steady state. From Eq. 44, we infer that a structural change that leads to an expansion in the overall capital stock will cause the rate at which consumption grows with age to decline over time. In addition, substituting for $r(t)$ and $w(t)$ into (9b) and for $r(t)$ into (9c), we can solve for the time path of $H(v,t)$ and $\Delta(v,t)$, and thence from Eq. 9a infer the time path of the distribution of individual asset holdings by age. Unfortunately, even for specific survival functions, such as the one we are about to introduce, these are extremely heavy computational calculations, necessitating further approximations, and instead, we address the distributional aspects only in the steady state; see Section 4.1.1 below.

4 Numerical simulations

To obtain further insights, we supplement the formal analysis with numerical simulations of the local dynamics and the steady-state demographic equilibrium, using the parameteric survival function suggested by Boucekkine et al. (2002):

$$S(t-v) \equiv e^{-M(t-v)} = \frac{\mu_0 - e^{\mu_1(t-v)}}{\mu_0 - 1}, \quad (\text{for } 0 \le t-v \le D), \quad \mu_0 > 1, \mu_1 > 0, \tag{45}$$

where μ_0 and μ_1 are parameters governing "youth" and "old age" mortality, respectively. The maximum attainable age D is determined by $S(t-v) = 0$ and equals ln μ_0/μ_1. We estimate the two parameters, μ_0 and μ_1, by nonlinear least squares, using US cohort data for 2006.[28] The estimates reported in Table 1 highlight how we obtain a remarkably tight fit (adjusted $R^2 = 0.996$), with highly significant parameter estimates. The resulting estimated survival function is illustrated in Fig. 2. Since we do not consider childhood and education, we normalize the function so that birth corresponds to age 18. As can be seen in the figure, it tracks the actual survival data for the USA closely from age 18 until around 90. Beyond that age, its concavity does not match the data particularly well. However, we do not view that as serious since only 0.7 % of the US population exceeds 90, and these individuals are generally

[28] Human Mortality Database. University of California, Berkeley (USA), and Max Planck Institute for Demographic Research, Rostock (Germany). Available at www.mortality.org or www.humanmortality.de.

Table 1 Estimated survival functions

$S(u) = I(u \leq D)\frac{\mu_0 - e^{\mu_1 u}}{\mu_0 - 1} + \varepsilon$ where $\varepsilon \sim$ i.i.d.$(0, \sigma^2)$[a]; $S(u) = e^{\mu u} + \varepsilon$ where $\varepsilon \sim$ i.i.d.$(0, \sigma^2)$[b]		
Demographic function	BCL[a]	BBW[b]
μ_0 (st. dev.)	78.3618 (6.0193)	
μ_1 (st. dev.)	0.0566 (0.0011)	0.0112 (0.0011)
Adj. R^2	0.9961	0.6157

[a]Boucekkine et al. (2002): $I(u \leq D)$ is an indicator function that is 1 for $u \leq D$ and 0 otherwise

[b]Blanchard (1985)–Buiter (1988)–Weil (1989)

retired and are relatively inactive in the economy.[29] For comparative purposes, we also estimate and illustrate the BBW survival function in Table 1 and Fig. 2. Being convex, rather than concave, it does not match the data well.

There is another possible approach that would avoid specifying a specific survival function, and that would be to use the observed mortality data. In order to adopt this approach, we would need to specify the framework in discrete time, replacing integrals such as $\tilde{H} = \tilde{w} \int_0^D e^{-\tilde{r}u - M(u)} du$, by the corresponding summations, $\tilde{w} \sum_{i=0}^{D} (1 + \tilde{r})^{-i} S_i$, which we could then compute by substituting directly for the survival rates, S_i, from the mortality tables. In this case, the entire macrodynamic equilibrium system would need to be specified in discrete time which, despite using actual mortality data, also has its problems. Some of these are generic, and the trade-offs involved in the choice between using discrete time versus continuous time in modeling macrodynamic systems have been extensively discussed and acknowledged long ago; see, e.g., May (1970), Turnovsky (1977, Section 3.6). In the present context, the resulting equilibrium system would be at least of fourth order (and likely higher), and it is well known that a fourth-order system of difference equations is far more cumbersome than the equivalent continuous-time system we are employing and, as a result, much less transparent. Moreover, its linearization will likely involve more approximations and, in the end, be no more accurate than the continuous-time system we are employing.

With the BCL function tracking the mortality data so closely, we can effectively view it as providing a smoothed version of the raw data, thereby enabling us to enjoy the advantages of continuous-time modeling. As such, the smoothed version also allows us to use a much finer grid when performing the simulations. That is, we now split the continuous period from 0 to D into 1,000 bits, while if we use the raw data,

[29]With this in mind, it might be more appropriate to refer to D as the maximum attainable economic age. We should also note that the ability of the BCL data to track mortality data closely is not restricted to the USA. It does just as well for The Netherlands; for example, see Heijdra and Mierau (2012). The reason why it fits so well is because, as Bruce and Turnovsky (2013, p.18) point out, it is a very good first-order approximation to the Gompertz function, which is widely accepted and adopted by demographers as a realistic representation of mortality. The Gompertz function itself, however, being doubly exponential, turns out to be computationally intractable in a model of this complexity.

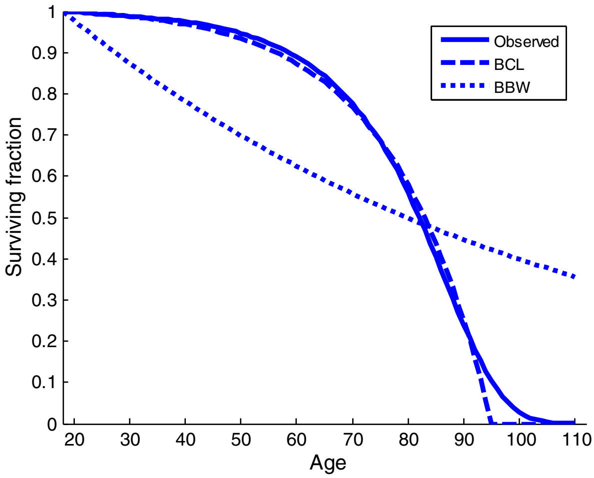

Fig. 2 Demography

we can use a maximum of 110 bits. If we would want to have a more precise simulation in that case, we would need to interpolate between different ages, which also involves approximations. One final, but not unimportant, advantage of employing the continuous-time formulation is that it facilitates comparison with the relevant literature, cited in the introduction and elsewhere, all of which uses continuous time with specific survival/mortality functions.[30]

Table 2 summarizes the key structural parameters for the baseline economy, all of which are quite standard. Output is produced by a Cobb–Douglas function, $y = Ak^{\alpha}\bar{l}^{1-\alpha}$, where $\bar{l}$ denotes inelastically supplied labor, with the elasticity of capital $\alpha = 0.35$ and depreciation rate $\delta = 0.05$. With respect to preferences, we set the intertemporal elasticity of substitution to 0.5, consistent with the consensus estimates reported by Guvenen (2006). As noted, the rate of time preference increases with age. Hence, we take $\rho = 0.035$ to be the rate of time preference at birth, implying a discount rate of 0.0388 for the individual of average age.

The baseline calibration adopts the demographic parameters of 2006. Thus, the estimates of the BCL function imply a maximum attainable age of 95.06 and life expectancy at age 18 of 78.38. These are a little low, reflecting the fact that, as Fig. 2 illustrates, the function fails to capture the outliers beyond age 90. We take the population growth rate to be 1.00 % which, given the survival function, implies a birth rate of 2.24 %.[31] This is a little high because the population growth rate also takes into account immigration. The implied equilibrium economic variables include an equilibrium capital–output ratio of 3.07 and a real net return on capital

[30] The most direct comparison is of course with Blanchard (1985); see footnote 6 for some of the other examples. Parameterizing mortality, as these studies do, also seems to us to be more convenient for investigating the effects of counterfactual changes in mortality.

[31] UN Population Predictions 2010. Available at www.un.org/esa/population

Table 2 Baseline parameters and benchmark equilibrium

Baseline model		BCL[a]	BBW[b]
Structural parameters			
Total factor productivity	A	1	1
Capital share of output	α	0.35	0.35
Depreciation rate	δ	5 %	5 %
Intertemporal substitution elasticity	σ	0.5	0.5
Time preference rate	ρ	3.5 %	3.5 %
Time preference of average individual	$\rho + \mu(\overline{u})$	3.88 %	4.61 %
Demographic parameters			
Youth mortality	μ_0	78.3618	N/A
Old age mortality	μ_1	0.0566	0.0112
Birth rate (implied)	β	2.24 %	2.12 %
Life expectation at 18 (age)	L_{18}	78.38	89.29
Maximum attainable age (implied)	D	95.06	∞
Population growth rate	n	1.00 %	1.00 %
Implied economic variables			
Per capita capital stock	$\tilde{k}$	5.6226	7.4044
Per capita output	$\tilde{y}$	1.8301	2.0152
Capital/output ratio	$\tilde{k}/\tilde{y}$	3.0722	3.6742
Real interest rate	$\tilde{r}$	6.39 %	4.52 %
Wage rate	$\tilde{w}$	1.1896	1.3099
Average per capita consumption	$\tilde{c}$	1.4928	1.5710
Marginal propensity to consume at birth	$[\Delta_B]^{-1}$	0.053	0.051

[a]Boucekkine et al. (2002)

[b]Blanchard (1985)–Buiter (1988)–Weil (1989)

of 6.39 %. The marginal propensity to consume at birth out of wealth is approximately 0.053 %, and each cohort's consumption grows at 1.45 % with age. We may also note the following dynamic characteristics. First, the linearized system does indeed have one stable eigenvalue $\lambda_1 = -0.077$ and, in addition, yields the values $\mu_C = 0.0189$, $\mu_H = 0.0029$, $\mu_\Delta = 0.0040$, confirming that $\mathrm{d}\mu_i(\tau_1 - t)/\mathrm{d}t$ will indeed be very small.[32]

The corresponding parameters and implied equilibrium values for the BBW model are also reported in Table 2. It yields a much higher life expectancy, due to the fact that the maximum attainable age in that model is infinite.

[32]The equilibrium values of μ_C, μ_H, and μ_Δ, approximated as constants during the transition, are obtained by substituting the steady-state aggregate quantities and consumption profile from Section 3.2, together with the hazard rate associated with the BCL mortality function, into (27b), (30a), and (30b).

From this initial baseline equilibrium, we analyze the transitional and steady-state effects of two types of structural changes: (a) an increase in productivity for a given demographic structure and (b) changes in the demographic structure.

4.1 Increase in productivity

We consider a neutral technological change, where A increases by 25 % from 1 to 1.25, focusing on the steady-state and transitional effects in turn.

4.1.1 Steady-state responses

As seen from row 2 in Table 3, this leads to a proportionate increase in capital and output, causing the capital–output ratio to remain unchanged.

Figure 3a illustrates the aggregate and the distributional effects for the BCL survival function. The locus BB in panel (i) depicts the pre-shock distribution of consumption over the different age cohorts (Eq. 10). The increase in productivity raises the wage rate, while the rate of return on capital remains unchanged. This causes the BB locus to shift up to B′B′, implying a uniformly higher consumption level for all ages, but growing at the unchanged rate. The AA locus presents average per capita consumption, which correspondingly jumps up to A′A′. Panel (ii) illustrates the long-run distributional changes across cohorts. Its mildly hump-shaped locus reflects the fact that the increase in consumption with age is offset by increasing mortality with age, leading to declining cohort-weighted consumption.

Panel (iii) illustrates the distribution of assets along the life cycle. Starting with zero assets at birth (18), agents accumulate wealth until around 65, after which they decumulate until assets run out at the maximum attainable age. This is reflected in the inverted-U locus EE which shifts out to E′E′ with the increase in productivity. The figures indicate that the greatest impact on wealth of the productivity increase accrues to individuals aged around 65. The upward shift in the distributional locus is also reflected in the horizontal line DD which illustrates the average per capita wealth, and which shifts up to D′D′ following the technological increase. Panel (iv) reflects assets weighted by the size of the cohorts. Due to the decline in survival with age, the greatest share of the benefits is enjoyed by the 55-year-old cohort.

Table 3 Structural changes

	Demography			Economic variables						
	L_{18}	$\beta(\%)$	$n(\%)$	$\tilde{k}$	$\tilde{y}$	$\tilde{k}/\tilde{y}$	$\tilde{r}(\%)$	$\tilde{w}$	$\tilde{c}$	$[\Delta_B]^{-1}$
Baseline model	78.38	2.24	1.00	5.623	1.830	3.072	6.39	1.190	1.493	0.053
Increase in productivity	78.38	2.24	1.00	7.926	2.580	3.072	6.39	1.677	2.104	0.053
Demographic analysis										
1960	71.68	2.92	1.71	5.217	1.783	2.926	6.96	1.159	1.433	0.058
1960 + birth rate 2006	71.68	2.24	0.66	5.365	1.800	2.980	6.74	1.170	1.497	0.057
1960 + mortality 2006	78.38	2.92	2.00	5.465	1.812	3.016	6.61	1.178	1.430	0.054

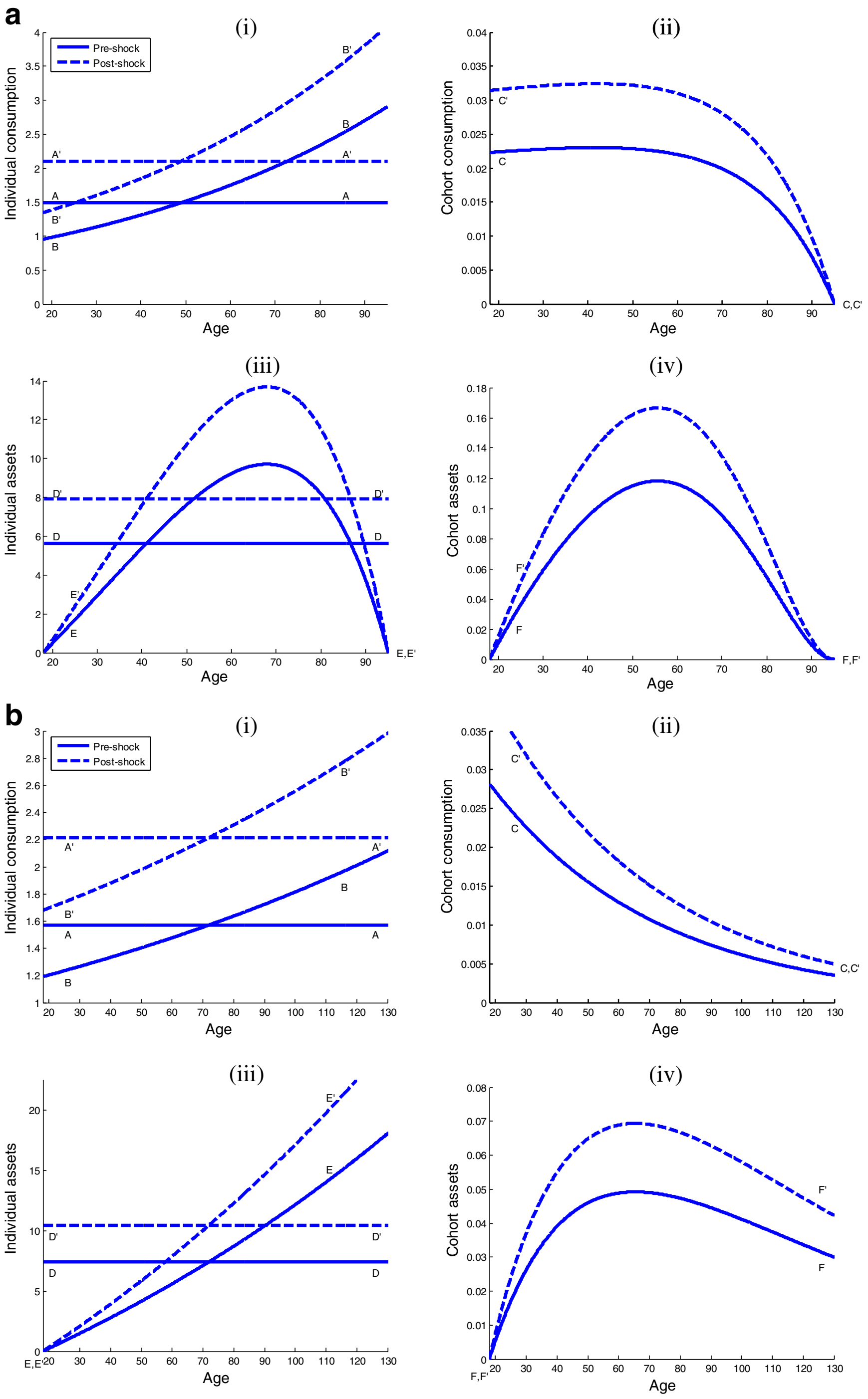

Fig. 3 **a** Increase in productivity: BCL. **b** Increase in productivity: BBW

Figure 3b illustrates the same exercise for the BBW demographic structure. It contrasts sharply, and is much less plausible, as a result of the convex survival function and the fact that agents may potentially live indefinitely (albeit with an arbitrarily low probability). For example, the perpetual upward slope of the assets accumulation locus EE in panel (iii) is unsatisfactory. However, with the dwindling cohort size, the implications for distributions across cohorts, as illustrated in panel (iv), is closer to the pattern implied by the more plausible BCL survival function.

4.1.2 Transitional responses

Figure 4 contrasts the dynamic time paths for the aggregate quantities, $k(t)$ and $c(t)$, for the BCL and BBW demographic structures, comparing both with that under the *representative agent* (RA) assumption. In all cases, with their respective stable adjustment paths being one-dimensional and with the (single) stable eigenvalues all being of comparable magnitudes, the aggregate quantities follow similar qualitative time paths.[33] In this respect, there are two comments worth making.

(a) The steady state and time paths under BBW track those of RA very closely. In fact, the steady-state capital stock and consumption under BBW may actually exceed that under RA. This is because with agents having potentially infinite lives under BBW, the generational turnover term can actually be less than that under RA, depending upon the relative sizes of the exogenously given birth and death rates.[34]

(b) The time path under the realistic BCL demographic structure lies substantially below that under RA. This is because with the growth rate of consumption over the life cycle exceeding the population growth rate ($\sigma(\tilde{r} - \rho) > n$), the generational turnover term exceeds that of the RA economy. Since the lost consumption it represents plays the role of a higher depreciation rate, the rate of convergence, as represented by the stable eigenvalue, is somewhat faster under BCL.

4.2 Changes in the demographic structure

In the course of the last half century, the birth rate, as well as the mortality rate, has declined substantially in the USA. In this section, we study how these changes have individually and jointly affected aggregate economic activity in the USA, particularly aggregate capital accumulation and consumption. While we take these changes as exogenously given, they presumably can be attributed to medical advances and

[33] The stable eigenvalues are $\lambda_1^{\mathrm{RA}} = -0.065$, $\lambda_1^{\mathrm{BBW}} = -0.066$, $\lambda_1^{\mathrm{BCL}} = -0.077$.

[34] Under BBW, we can show that the steady-state per capita capital stock satisfies $\sigma\left(f'(\tilde{k}_{\mathrm{BBW}}) - \delta - \rho\right) = \beta\tilde{k}_{\mathrm{BBW}}\tilde{\Delta}^{-1}\tilde{c}^{-1}$, where $\tilde{\Delta}$ is defined using the exponential mortality function, $e^{\mu u}$ Subtracting (43a), $f'(\tilde{k}_{\mathrm{BBW}}) - f'(\tilde{k}_{\mathrm{RA}}) = \left(\beta\tilde{k}_{\mathrm{BBW}}/\sigma\tilde{\Delta}\tilde{c}\right) - n$. If the population growth rate $n = 0$, as in Blanchard (1985), then $f'(\tilde{k}_{\mathrm{BBW}}) > f'(\tilde{k}_{\mathrm{RA}})$ and $\tilde{k}_{\mathrm{RA}} > \tilde{k}_{\mathrm{BBW}}$. Otherwise, the comparison depends upon the right-hand side of this relation.

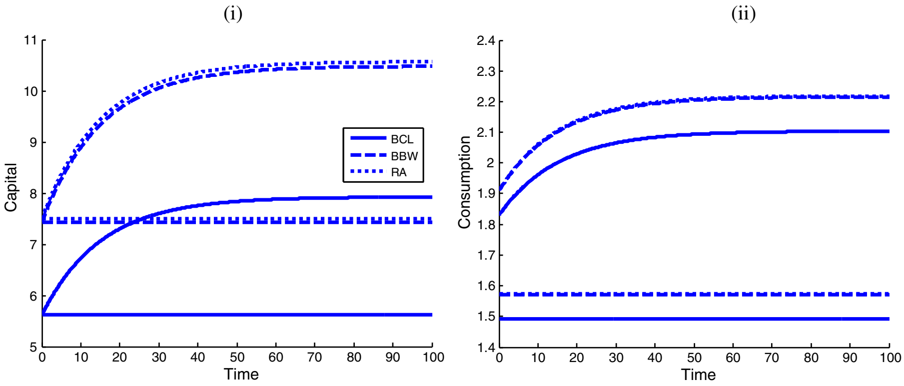

Fig. 4 Increase in A from 1 to 1.25: aggregate dynamics

enhanced health expenditures, as Varvarigos and Zakaria (2013) have suggested. To proceed, we first recalibrate the model by taking the demographic structure of 1960 as benchmark. From that starting point, we then consider how the decline in the birth rate alone, experienced between 1960 and 2006, has affected the economic structure. Next, we reset the birth rate at its 1960 value and consider instead the impact of the drop in mortality over that same period. Finally, we study the impact of both changes simultaneously.[35]

4.2.1 The 1960s benchmark

The 1960s benchmark equilibrium is set out in the third line in Table 3. Compared to 2006, the birth rate in 1960 was higher at 2.92 % and the life expectancy at 18 was substantially lower at 71.68.[36] Keeping the remaining parameters at their 2006 values, this implies an aggregate per capita capital stock of 5.217 and aggregate per capita consumption of 1.433. Comparing these with line 1, it indicates that between 1960 and 2006, capital and consumption increased by 7.8 and 4.2 %, respectively, purely from changes in the demographic structure alone. In the remainder of this section, we study how the two sources of demographic change have contributed to these economic shifts and what the transitional response was to these changes, both individually, and in concert.

[35] In an earlier version of this paper (available on request), we studied the impact of an increase in the population growth rate induced by either a change in the birth rate or a change in the mortality rate. There we showed that depending on the source of demographic change, an increase in the population growth rate can increase, decrease, or not affect the aggregate per capita capital stock.

[36] As before, our birth rate is a little high because it also includes immigration. Technically, the birth rate arises from the demographic steady state (15) in which we take the population growth rate and mortality structure as given and solve for a consistent birth rate. The BCL function used to estimate the mortality parameters fits equally well for 1960 as for 2006, yielding the estimates and standard errors: $\mu_0 = 43.9817(3.4183)$; $\mu_1 = 0.0535(0.0012)$; adjusted $R^2 = 0.996$. Since the BCL function fits the two endpoints of the transition so closely, we can be confident that it also approximates the average transition rate between 1960 and 2006 closely, as well.

4.2.2 Decrease in the birth rate

Comparing line 3 with line 4 in Table 3, we see that in the absence of any accompanying change in the mortality rate, the decline in the birth rate from 2.92 to 2.24 % would have reduced the population growth rate to 0.66 %. Moreover, this line reveals that of the 7.8 % increase in the aggregate per capita capital stock, 2.8 % (approximately 36 % of the increase) can be attributed to the reduction in the fertility rate. This response is consistent with the view emphasized by Kelley and Schmidt (1995) that a decline in the population growth rate resulting from a lower birth rate will have a positive effect on the level of economic activity. This is because it reduces the relative number of young who have not accumulated any capital stock to contribute to the productive capacity of the economy. The increase in the aggregate capital stock has several consequences. It leads to a 0.9 % increase in the wage rate (from 1.159 to 1.170) and a decrease in the rate of return on capital from 6.96 to 6.74 %. Regarding consumption, we see that the decline in the birth rate led to a 4.5 % increase in per capita consumption between 1960 and 2006, more than 100 % of the total 4.2 % increase that occurred over that period.[37] This is a reflection of the increase in the relative number of older people, and the fact that consumption increases with age.

4.2.3 Decrease in the mortality rate

From the fifth line in Table 3, we see that, taken in isolation, the decline in mortality experienced between 1960 and 2006 would have increased the population growth rate to 2.0 %. More importantly, the drop in the mortality rate accounted for a 4.8 % increase in the per capita stock of capital (from 5.217 to 5.465), representing around 62 % of the total increase. This response is also consistent with the evidence provided by Kelley and Schmidt (1995) and Bloom et al. (2007) that an increase in the population growth rate resulting from a reduction in mortality will have a positive effect on the level of economic activity. This is because it increases the relative number of old people who have accumulated capital stock to contribute to the productive capacity of the economy. Analogous to the lower birth rate, this increase in the aggregate capital stock resulting from the decline in mortality leads to a 1.6 % increase in the wage rate (from 1.159 to 1.178) together with a substantial decrease in the rate of return on capital from 6.96 to 5.61 %. However, it leads to a small (0.2 %) decline in per capital consumption. This is a consequence of its nonmonotonic distributional effects. On the one hand, the increase in the wage rate coupled with the anticipation of the future lower return to capital causes a slight increase in consumption at birth. However, the decrease in the rate of return on capital decreases the consumption growth rate over the life cycle. Hence, after a few years, agents experience a decrease in their consumption, and since this is the experience of most cohorts, average per capita consumption declines.

[37] In an earlier version of this paper we also studied the consequences of the demographic changes for the life cycle distribution of consumption and assets (as in Fig. 3). There we find that the life cycle patterns obtained by Fair and Dominguez (1991) and Attfield and Cannon (2003) are confirmed by our numerical simulations. These results are available on request.

4.2.4 Transitional dynamics

Figure 5 illustrates the transitional adjustment paths for aggregate capital, k, and consumption, c, following the declines in either the birth rate or the mortality rate outlined in Sections 4.2.2 and 4.2.3. Since demographic structures change slowly,

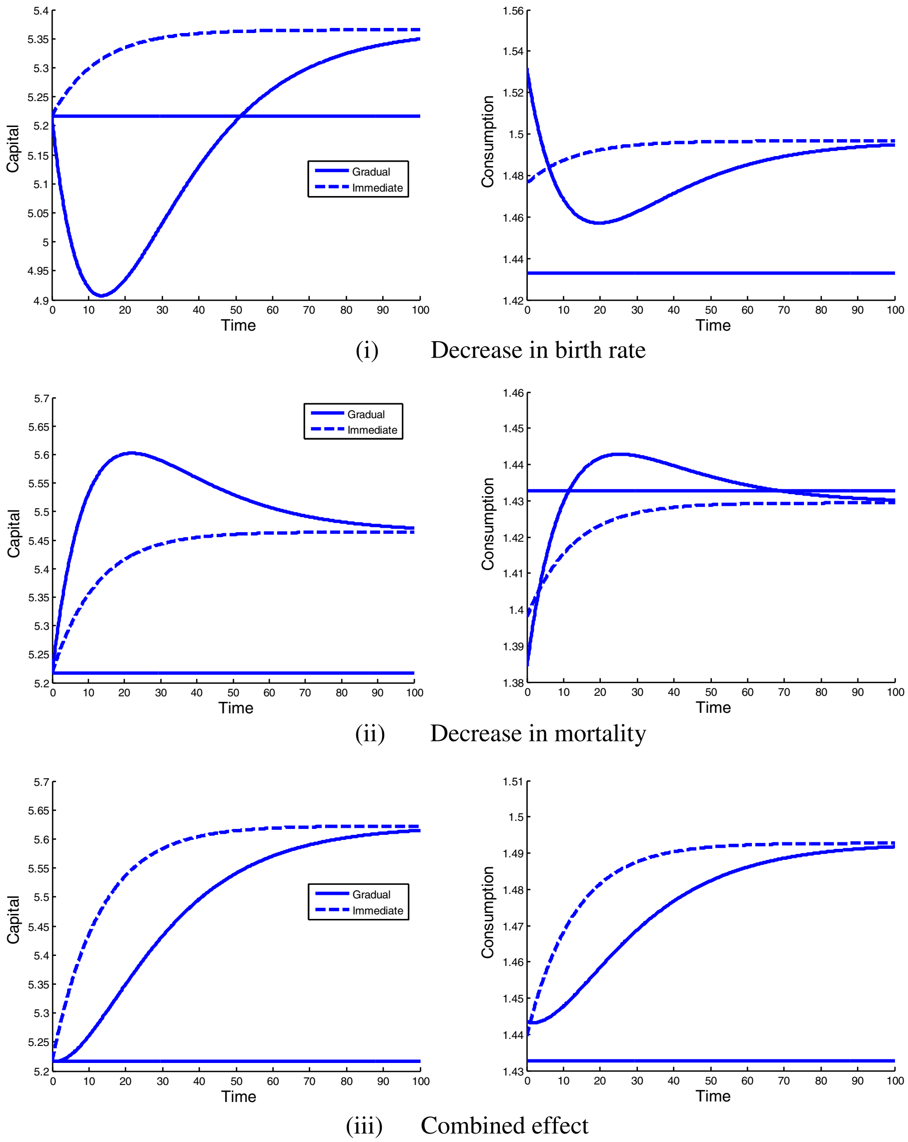

Fig. 5 Decomposition of demographic change in the USA, 1960–2006

it is important to introduce the induced change in the population growth rate gradually.[38] We do this by specifying the change from n_0 to $\tilde{n}$ by $n(t) = \tilde{n} + (n_0 - \tilde{n})e^{-\theta t}$, where θ denotes the rate of adjustment. Likewise, where the change in the population growth rate reflects the change in the birth rate, we specify the latter by $\beta(t) = \tilde{\beta} + \left(\beta_0 - \tilde{\beta}\right) e^{-\theta t}$.[39] In our simulations, we assume $\theta = 0.05$, implying that the population growth rate converges to its lower rate at the rate of 5 % per year. This is illustrated by the solid line in the figure. For comparative purposes, the dashed line illustrates the case where the full change in the growth rate occurs instantaneously ($\theta \to \infty$), and the horizontal solid line indicates the initial values of capital and consumption, respectively.

Figure 5(i) illustrates the dynamic adjustment for the decrease in the birth rate. In the case where the complete decrease in β occurs instantaneously, both capital and consumption increase monotonically, with consumption jumping up on impact and capital adjusting gradually. In the more plausible case where the decrease in the birth rate and accompanying population growth rate occur gradually, the adjustment paths are nonmonotonic. In that case, the adjustment paths for per capita stock of capital and the accompanying per capita consumption are U-shaped. The intuition behind this nonmonotonic behavior is seen from Eq. 31a. With the decrease in the population growth rate being gradual, on impact, the eventual lower population growth rate initially manifests itself as a decline in the rate of increase, $\dot{n}(0) < 0$, with $n(0) = n_0$ remaining unchanged. Knowing that the long-run effect of a decrease in n is to increase per capita consumption, in anticipation of this, agents immediately begin to increase consumption, so that $dc(0) > 0$. With the capital stock being predetermined, output is fixed, and for goods market equilibrium to prevail, the increase in aggregate consumption must be offset by a decrease in investment, i.e., $\dot{k}(0) < 0$. Over time, as the decrease in the population growth gradually takes effect, the need to devote resources to the growing population decreases, reducing aggregate demand and leaving more output for capital accumulation. Eventually, this expansionary effect dominates, leading to an increase in the per capita capital stock. The time path of capital and output is reflected in the time path of consumption.

Figure 5(ii) illustrates the dynamic adjustment path due to the decline in the mortality rate.[40] When the decrease in the mortality rate occurs instantaneously, with

[38] We may note that when studying the transitional dynamics of the demographic transition we do not consider the dynamics of the demographic system itself. In that sense, our results should be interpreted as though the economy where moving along a (continuous) sequence of demographic steady states in which the survival curve is gradually being shifted outward.

[39] In our simulations we treat 1960 and 2006 as steady-state equilibria, with all demographic parameters being set to satisfy the demographic steady state. Setting $\theta = 0.05$, implies that around 90 % of the transition from one steady state to the other is completed within the time horizon of 46 years. Since our approximation justifies treating the weighted average mortality rates μ_H, μ_C, μ_Δ as constant over the transition, it is compatible with the corresponding changes in the raw mortality rate μ implied by the parameterized mortality function. We also ignore any feedback from the economic structure to the demographic structure.

[40] In the case where the mortality change occurs instantaneously, it remains unchanged along the subsequent transitional path, and the approximation (36b) applies. When it occurs gradually, the approximation (50) becomes relevant.

more old wealthy agents, this generates a monotonic long-run increase in capital. As noted, the additional output this supports enables the long-run consumption demands of the growing population to be accommodated with only a negligible effect on long-run average consumption. Knowing that consumption will accumulate with capital during the transition, and that the long-run accumulation in capital occurs gradually, forces an initial decline in aggregate consumption of around 3.5 %, after which it increases steadily. In the case where the increase in the growth rate and accompanying decline in the death rate occur gradually, the time paths of capital and consumption are mildly hump-shaped, both over-shooting their respective long-run adjustments during the transition. Again, this is because of the sluggishness in the increase in the population growth rate and its resulting gradual impact on aggregate demand.

Finally, in Fig. 5(iii), we illustrate the joint effect of the demographic transition experienced in the USA over the last half century. This figure highlights that the nonmonotonicity induced by the decrease in the birth rate is effectively eliminated once the decrease in the mortality rate is also accounted for. Indeed, for the composite demographic change, both the gradual and immediate adjustment paths indicate a monotonic transition to the new steady-state values of capital and consumption, respectively. Unsurprisingly, the new steady state is approached more quickly in the immediate adjustment regime than when the adjustment is gradual.[41]

5 Conclusions

This paper has introduced an empirically realistic age-dependent demographic structure into a neoclassical growth model. In doing so, we have had two primary objectives. The first is to provide a general characterization of how the demographic structure impinges on the macrodynamic equilibrium. We show how this depends on the generational turnover term, which is an integral component of the intertemporal consumption allocation decision. Setting up the aggregate dynamics as a generalization of the conventional neoclassical growth model provides an important insight, enabling us to view alternative demographic specifications in a unified way. We are also able to approximate the local dynamics and see how the demographic structure impinges on the evolution of the aggregate economy.

The second objective has been to analyze the effect of structural changes—most importantly demographic structural changes—on both the aggregate macro-equilibrium, as well as the distributional life cycle implications. This is done by numerically analyzing the model using the very general mortality function proposed by Boucekkine et al. (2002). Here, we show how the shift in the demographic structure of the USA has affected its economy. When we decompose the economic effect

[41] We have also traced out the adjustment paths that would have prevailed in a representative agent (RA) model. These paths are very similar to the composite adjustment in Fig. 5iii.This indicates that while the RA model cannot account for the nonmonotonic adjustment induced by a birth rate driven change in the population growth alone, it does very well when composite changes in the population growth rate are considered.

into one component attributable to a reduction in the birth rate and a second component attributable to the decline in the mortality rate, we find that both have had positive impacts on capital accumulation. The transitional dynamics show that in isolation, a gradual drop in the birth rate may induce nonmonotonic dynamics. However, when taken in conjunction with the concurrent decline in the mortality rate, the net effect of the overall change in the demographic structure is to lead to a generally steady increase of both consumption and capital.

As a related point, we may note that our numerical simulations have been motivated by the US experience of simultaneous declines in both fertility and mortality, the long-run effects of which on the capital stock are reinforcing. Conversely, our results also show how the long-run effects of a given increase in the population growth rate differ sharply depending upon whether it occurs through an increase in the birth rate or a decrease in mortality. These differences (contraction in the former case, expansion in the latter) are consistent with the analyses of Sinha (1986) using a SD model, Heijdra and Ligthart (2006) using a BBW model, as well as empirical evidence provided by Blanchet (1988) and Kelley and Schmidt (1995). The latter summarize the difference in terms of children, having little accumulated wealth, being "resource users" and working adults with their accumulated capital being "resource creators".[42]

While we view our paper as being canonical, it clearly can be extended in various directions. First, the contrast between births and mortality in influencing the population growth rate and the resulting consequences for distribution across cohorts and for the aggregate economy raises interesting policy issues for a country seeking to influence its population growth rate. Second, a natural extension of the framework would be to endogenize labor supply and to allow for retirement, thereby addressing issues pertaining to social security and retirement benefits, issues that are of crucial importance for the USA and other countries with their aging populations.

Finally, we conclude with a caveat. While we find that the analysis of local dynamics we have analyzed provides important insights, we should not lose sight of its limitations, which are potentially severe in the OLG context. More specifically, by focusing on dynamics in the neighborhood of steady state, we are ignoring initial distributional allocations across the cohorts. Yet this may be particularly relevant in the case of a discrete change in the environment, such as the introduction of a social security system which affects different cohorts differently and will involve transitional dynamics as the change works through the existing demographic structure. To study the dynamics of these types of structural change will necessitate a more global approach.

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[42]It is also consistent with the related evidence from cross-country studies of fertility and growth. These have typically found the correlations between economic growth and population growth to be negative for less developed economies, having higher birth rates, and positive for developed economies, with their lower mortality rates (Kelley 1988).

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Appendix: Generalization of approximation to transitional dynamics to include changing mortality with calendar time

Throughout the text, we have assumed that the survival and mortality functions are independent of calendar time. In this Appendix, we show it is straightforward to generalize the approximation employed in the local dynamics, to allow for these functions to depend upon both age and calendar time. Specifically, the survival function, Eq. 3a, is modified to $S(s,t) = e^{-M(s,t)}$, where s denotes age, and t is calendar time. The corresponding mortality rate is thus $\mu(s,t) = -S_s(s,t)/S(s,t) = M_s(s,t)$, implying that $\mu_t(s,t) = M_{st}(s,t)$. With this notation, the average mortality rate defined by Eq. 34a may be written as follows:

$$\mu_H(\tau_1 - t) = \frac{\displaystyle\int_0^D \mu(s,t) e^{-\int_t^{t+s} r_w(u)\mathrm{d}u} e^{-M(s,t)} \mathrm{d}s}{\displaystyle\int_0^D e^{-\int_t^{t+s} r_w(u)\mathrm{d}u} e^{-M(s,t)} \mathrm{d}s}. \tag{46}$$

Differentiating (46) with respect to calendar time, t, yields:

$$\frac{d\mu_H(\tau_1 - t)/\mathrm{d}t}{\mu_H(\tau_1 - t)} = \frac{\displaystyle\int_0^D [r_w(t+s) + M_t(s,t)]\, F(s,t)\mathrm{d}s}{\displaystyle\int_0^D F(s,t)\mathrm{d}s} - \frac{\displaystyle\int_0^D \mu(s,t)\left\{[r_w(t+s) + M_t(s,t)] - \frac{\mu_t(s,t)}{\mu(s,t)}\right\} F(s,t)\mathrm{d}s}{\displaystyle\int_0^D \mu(s,t) F(s,t)\mathrm{d}s} \tag{47}$$

where:

$$M_t(s,t) = -\frac{S_t(s,t)}{S(s,t)} \tag{48}$$

specifies the proportionate rate of change in the survival rate over calendar time. Since our application focuses on an increase in the survival rate, $S_t(s,t) > 0$, and (48) implies $M_t(s,t) < 0$ Applying the mean value theorem to (47), precisely as we did to (34b), and regrouping terms, we can write (47) as follows:

$$\frac{\mathrm{d}\mu_H(\tau_1 - t)/\mathrm{d}t}{\mu_H(\tau_1 - t)} = [r_w(t+s_1) - r_w(t+s_2)] + [M_t(s_1,t) - M_t(s_2,t)] + \frac{\mu_t(s_2,t)}{\mu(S_2,t)} \tag{49}$$

where $0 < s_1(t) < D, 0 < s_2(t) < D$. The first term on the right-hand side of (49) remains as in the text and reflects the impact of the evolution of the economic variables on the weighted average mortality rate, μ_H. The remaining terms reflect the effects of the changing mortality over calendar time on that same quantity. Taking the first-order approximation yields:

$$M_t(s_1, t) - M_t(s_2, t) = M_{ts}(s_2, t)(s_1 - s_2)$$

which, using the relationship $\mu_t(s, t) = M_{st}(s, t)$, may be written as follows:

$$M_t(s_1, t) - M_t(s_2, t) = \mu_t(s_2, t)(s_1 - s_2).$$

Thus, analogous to (36b), we obtain:

$$\mathrm{d}\mu_H(\tau_1 - t) \approx \tilde{\mu}_H \left\{ \left[\lambda_1 (r_w(t) - \tilde{r}_w)(s_1 - s_2)\right] + \mu_t(s_2, t)(s_1 - s_2) + \frac{\mu_t(s_2, t)}{\mu(s_2, t)} \right\} \mathrm{d}t. \tag{50}$$

The first term, describing the impact of the changing economic structure on the weighted average, remains unchanged from (36b) and has been demonstrated to be small. The remaining terms describe the gradual impact of the change in mortality, and these too are small. For the USA, the average mortality rate during the second half of the twentieth century was about 0.01, declining at a rate of about 1 % per year, implying that $\mu = 0.01$, $\mu_t = -0.0001$. Taking $|s_1 - s_2| < 10$, the second and third expressions in the parenthesis total around -0.011, and for $\tilde{\mu}_H = 0.003$, they are of the order of 0.00003, even smaller than the first component.[43] Thus, the change in μ_H is negligible and can essentially be treated as constant and independent of t. Indeed, further support for treating μ_H as constant in our specific application to the change in the mortality structure in the USA is provided by the fact that we find the overall decline in μ_H over the period of 1960–2006 (treating both endpoints as steady states) to be from 0.0039 to 0.0029.

Analogous arguments apply to the other weighted mortality rates, μ_C, μ_Δ. Comparing their respective values at 1960 and 2006, we find that μ_C increases by 0.0001 from 0.0187 to 0.0188, while μ_Δ declines by 0.0013 from 0.0053 to 0.0040. With the three weighted mortality rates and their respective changes all being small, their accumulated effects on the dynamics of the aggregate variables are surely negligible.

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[43] Even if one takes $|s_1 - s_2| = 78$, the maximum possible value, these two expressions total -0.02 rather than -0.011, and their contribution remains negligible.

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