Global and local behavior of zeros of nonpositive type

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\textbf{Abstract}

A generalized Nevanlinna function $Q(z)$ with one negative square has precisely one generalized zero of nonpositive type in the closed extended upper half-plane. The fractional linear transformation defined by $Q_\tau(z) = (Q(z) - \tau)/(1 + \tau Q(z))$, $\tau \in \mathbb{R} \cup \{\infty\}$, is a generalized Nevanlinna function with one negative square. Its generalized zero of nonpositive type $\alpha(\tau)$ as a function of $\tau$ is being studied. In particular, it is shown that it is continuous and its behavior in the points where the function extends through the real line is investigated.

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\section{Introduction}

Let $M(z)$ be an ordinary Nevanlinna function, i.e., a function which is holomorphic in $\mathbb{C}^+$ and which maps the upper half-plane into itself. It is well known that $M(z)$ admits a representation

$$M(z) = a + bz + \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\sigma(t), \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{1}$$

with $a \in \mathbb{R}$, $b > 0$, and a measure $\sigma$ satisfying $\int_{\mathbb{R}} d\sigma(t)/(t^2 + 1) < \infty$; cf. \cite{7}. In the lower half-plane $\mathbb{C}^-$ the function $M(z)$ is defined by the symmetry principle $\overline{M(\bar{z})} = M(z)$. Then $M(z)$ is holomorphic on $\mathbb{C}^+ \cup \mathbb{C}^- \cup (\mathbb{R} \setminus \text{supp} \sigma)$. Note that if $\mathbb{R} \setminus \text{supp} \sigma$ contains some interval $I$, then the extension of $M(z)$ given on $\mathbb{C}^+$ to the set $\mathbb{C}^+ \cup \mathbb{C}^- \cup I$ is given by the Schwarz reflection principle.

However, the main interest in this paper is in the situation when $M(z)$, restricted to $\mathbb{C}^+$, has a holomorphic continuation $\bar{M}(z)$ across an interval $I \subset \mathbb{R}$ without $\bar{M}(z)$ being real for $z \in I$. In this case the symmetry

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property will be lost. For example the constant function

\[ M(z) = i = \frac{1}{\pi} \int_{\mathbb{R}} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) dt, \quad z \in \mathbb{C} \setminus \mathbb{R}, \]

as defined on \( \mathbb{C}^+ \), extends to a holomorphic function on all of \( \mathbb{C} \), although \( \text{supp} \sigma = \mathbb{R} \).

Let \( Q(z) \) be a generalized Nevanlinna function of class \( N_1 \), i.e., a meromorphic function in the upper half-plane such that the kernel

\[ N_Q(z, w) = \frac{Q(z) - \overline{Q(w)}}{z - \overline{w}}, \quad z, w \in \mathbb{C}^+, \]

has precisely one negative square. It has been shown that \( Q(z) \) has a unique factorization

\[ Q(z) = R(z)M(z), \tag{2} \]

with \( R(z) \) of one of the following three forms

\[ \frac{(z - \alpha)(z - \bar{\alpha})}{(z - \beta)(z - \bar{\beta})}, \quad (z - \alpha)(z - \bar{\alpha}), \quad \frac{1}{(z - \beta)(z - \bar{\beta})}, \tag{3} \]

with \( M(z) \) being a Nevanlinna function and \( \alpha, \beta \in \mathbb{C}^+ \cup \mathbb{R} \); cf. [4,6]. The point \( \alpha \) is called the \textit{generalized zero of nonpositive type (GZNT)} of \( Q(z) \) and the point \( \beta \) is called the \textit{generalized pole of nonpositive type (GPNT)} of \( Q(z) \); see e.g. [4,6,10] for a characterization of GZNT and GPNT in terms of nontangential limits. The extensions of \( N_1 \) functions that arise from not necessarily symmetric extensions of \( M(z) \) are the main objects of the paper.

A function \( Q(z) \) in \( N_1 \) generates of family of functions \( Q_\tau(z) \) via the linear fractional transformation

\[ Q_\tau(z) := \frac{Q(z) - \tau}{1 + \tau Q(z)}, \quad \tau \in \mathbb{R}, \]

and by

\[ Q_\infty(z) := -\frac{1}{Q(z)}, \quad \tau = \infty. \]

It is known that \( Q_\tau(z) \in N_1 \), which allows to define for \( \tau \in \mathbb{R} \cup \{ \infty \} \) the numbers \( \alpha(\tau) \) and \( \beta(\tau) \) as, respectively, GZNT and GPNT of the function \( Q_\tau(z) \). The local properties of \( \alpha(\tau) \) in the case when \( \alpha(\tau_0) \) lies in a spectral gap of \( M(z) \) were investigated in detail in [15]. In the present work these results are generalized to the case when \( Q(z) \) extends holomorphically to the lower half-plane around \( \alpha(\tau_0) \). The paper also contains some results of a global nature concerning the function \( \tau \to \alpha(\tau) \). In particular, in Theorem 3.2 it is shown that \( \alpha(\tau) \) forms a curve on the Riemann sphere which is homeomorphic to a circle. This problem was still open in [15] and is now solved by means of recent results concerning the convergence behavior of generalized Nevanlinna functions [11]. The problem is related to the convergence of poles in Padé approximation, see [3,14]. A related problem of tracking the eigenvalue of nonpositive type in the context of random matrices was considered in [13,16].

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2. Non-symmetric extensions of Nevanlinna functions

The following result, playing a crucial role in the paper, can be found in [8].

**Theorem 2.1.** Let \( M(z) \) be a Nevanlinna function of the form (1) and let \( \Omega \) be a simply connected domain, symmetric with respect to \( \mathbb{R} \). Then the following statements are equivalent:

(i) the restriction of \( M(z) \) to \( \mathbb{C}^+ \) extends to a holomorphic function in \( \Omega \cup \mathbb{C}^+ \);
(ii) the measure \( d\sigma \) in (1) satisfies
\[
d\sigma(t) = \phi(t) \, dt, \quad t \in \Omega \cap \mathbb{R},
\]
where \( \phi(z) \) is a real holomorphic function on \( \Omega \).

The above result will be now extended to \( N_1 \)-functions.

**Theorem 2.2.** Let \( Q(z) \) be in \( N_1 \) with the representation \( Q(z) = R(z)M(z) \) as in (2) and (3) with \( \alpha, \beta \in \mathbb{C}^+ \cup \mathbb{R} \). Let \( \Omega \) be a simply connected domain, symmetric with respect to \( \mathbb{R} \) and assume that \( \beta \notin \Omega \) and \( \alpha \in \Omega \cap \mathbb{R} \). Then the following statements are equivalent:

(i) the restriction of \( Q(z) \) to \( \mathbb{C}^+ \setminus \{\beta\} \) extends to a holomorphic function in \( \Omega \cup \mathbb{C}^+ \setminus \{\beta\} \);
(ii) the function \( M(z) \) is of the form
\[
M(z) = M_1(z) + \frac{m_0}{\alpha - z}, \quad t \in \Omega \cap \mathbb{R},
\]
where \( M_1(z) \) is a Nevanlinna function such that the restriction of \( M_1(z) \) to \( \mathbb{C}^+ \cup \Omega \) and \( m_0 \geq 0 \);
(iii) the measure \( d\sigma \) for \( M(z) \) in (1) satisfies
\[
d\sigma(t) = \phi(t) \, dt + m_0\delta_\alpha(t), \quad t \in \Omega \cap \mathbb{R},
\]
where \( \phi(z) \) is a real holomorphic function on \( \Omega \), \( m_0 \geq 0 \), and \( \delta_\alpha \) is the Dirac measure at \( \alpha \).

If instead of \( \alpha \in \Omega \cap \mathbb{R} \) one assumes \( \alpha \notin \Omega \), then the equivalences above hold with \( m_0 = 0 \) in statements (ii) and (iii).

**Proof.** The proof will be given for the case \( \alpha \in \Omega \cap \mathbb{R} \); the proof in the case \( \alpha \notin \Omega \) is left as an exercise.

(i) \(\Rightarrow\) (ii) Let \( \tilde{Q}(z) \) be a holomorphic extension of \( Q(z) \) to \( (\mathbb{C}^+ \cup \Omega) \setminus \{\beta\} \) and let
\[
\tilde{M}(z) = R_0^{-1}(z) \frac{1}{(z - \alpha)^2} \tilde{Q}(z),
\]
where \( R_0(z) = 1/(z - \beta)(z - \bar{\beta}) \) or \( R_0(z) \equiv 1 \), depending on the position of the GPNT \( \beta \). Note that \( \tilde{M}(z) \) is holomorphic in \( (\Omega \cup \mathbb{C}^+) \setminus \{\alpha\} \), \( \alpha \) is a pole of \( \tilde{M}(z) \) of order at most two of \( \tilde{M}(z) \) and \( \tilde{M}(z) = M(z) \) for \( z \in \mathbb{C}^+ \). Since \( M(z) \) is a Nevanlinna function the limit
\[
m_0 = \lim_{z \to \alpha} (\alpha - z)M(z)
\]
exists and is nonnegative. Consequently, \( \alpha \) is a pole of \( \tilde{M}(z) \) of order at most one, with the residuum \( m_0 \). Therefore, the function
is a Nevanlinna function, see [7, Chapter II.2]. Put

\[
\tilde{M}_1(z) = \tilde{M}(z) - \frac{m_0}{\alpha - z},
\]

and note that

\[
\lim_{z^\to\alpha} (\alpha - z) \tilde{M}_1(z) = 0.
\]

Thus, \( \tilde{M}_1(z) \) is holomorphic at \( \alpha \) and in consequence in \( \mathbb{C}^+ \cup \Omega \).

The implication (ii) \( \Rightarrow \) (i) is obvious and the equivalence (ii) \( \Leftrightarrow \) (iii) is a direct consequence of Theorem 2.1. \( \square \)

### 3. Global properties of the function \( \alpha(\tau) \)

Before continuing with extension properties across \( \mathbb{R} \) an open problem from [15] will be solved. For this aim consider the following definition. Let \( D \) be a nonempty open subset of the complex plane, and let \( (Q_n) \) be a sequence of functions which are meromorphic on \( D \). The sequence \( (Q_n) \) is said to converge locally uniformly on \( D \) to the function \( Q \), if for each nonempty open set \( D_0 \subseteq \mathbb{C} \) with compact closure \( D_0 \subseteq D \) there exists an index \( n_0(D_0) \) such that for \( n > n_0(D_0) \) the functions \( (Q_n) \) are holomorphic on \( D_0 \) and

\[
\lim_{n \to \infty} Q_n(z) = Q(z), \text{ uniformly on } D_0.
\]

(In other words, \( (Q_n) \) converges to \( Q \) in the compact-open topology on the space of holomorphic functions on \( D \) with values in the extended complex plane \( \mathbb{C} \), see [1,2].) The reader is referred to [11] for a treatment on locally uniform convergence of \( N_\kappa \)-functions. The proof of the following result can be derived from [1, Chapter 2.8]; an elementary argument is included anyway.

**Proposition 3.1.** Let \( Q(z) \) be an \( N_1 \) function and let \( \tau_n \in \mathbb{R} \) converge to \( \tau \in \mathbb{R} \). Then \( Q_{\tau_n}(z) \) converges locally uniformly to \( Q_{\tau}(z) \) on \( \mathbb{C}^+ \setminus \{\beta(\tau)\} \).

**Proof.** Let \( D \) be some open, bounded subset of \( \mathbb{C}^+ \) with \( \overline{D} \subset \mathbb{C}^+ \setminus \{\beta(0), \beta(\tau)\} \) and let \( \tau_n \to \tau \). Consider first the case \( \tau \in \mathbb{R} \). Since \( 1 + \tau Q(z) \) has no zero on \( \overline{D} \), it follows that

\[
\inf_{z \in \overline{D}} |1 + \tau Q(z)| = d > 0.
\]

The inverse triangle inequality

\[
|1 + \tau_n Q(z)| \geq |1 + \tau Q(z)| - |\tau - \tau_n||Q(z)|
\]

together with the fact that \( Q(z) \) is bounded on \( \overline{D} \) implies that for some \( n_0 \) the relation

\[
\inf_{z \in \overline{D}} |1 + \tau_n Q(z)| \geq d/2
\]

for all \( n > n_0 \) holds. Consequently

\[
|Q_{\tau_n}(z) - Q_{\tau}(z)| = \frac{|\tau - \tau_n|}{|1 + \tau Q(z)||1 + \tau_n Q(z)|} \leq 4d^{-2}|\tau - \tau_n|,
\]
which means that $Q_{\tau_n}(z)$ converges locally uniformly to $Q_\tau(z)$ on $D$. By [11, Theorem 1.4], $Q_{\tau_n}(z)$ converges locally uniformly to $Q_\tau(z)$ on $\mathbb{C}^+ \setminus \{\beta(\tau)\}$.

Now consider the case $\tau = \infty$. Since $\beta(\infty) = \alpha$, the function $Q(z)$ is bounded and bounded away from zero on $D$. Consequently, the locally uniform convergence $Q_{\tau_n}(z) \to Q_\infty(z)$ on $C^+ \{\beta(\tau)\}$ follows from

$$Q_{\tau_n}(z) - Q_\infty(z) = \frac{1}{1 + \tau_n Q(z)} \left( Q(z) + \frac{1}{Q(z)} \right) \to 0, \quad n \to \infty,$$

uniformly on $D$ and again [11, Theorem 1.4].

**Theorem 3.2.** The function $\tau \to \alpha(\tau)$ is continuous and the set

$$\{\alpha(\tau) : \tau \in \mathbb{R} \cup \{\infty\}\}

on the Riemann sphere is homeomorphic to a circle.

**Proof.** Let $(\tau_n)$ be some sequence which converges to $\tau$. By Proposition 3.1, the sequence $(Q_{\tau_n}(z))$ converges locally uniformly to $Q_\tau(z)$ on $\mathbb{C}^+ \setminus \{\beta(\tau)\}$. Now it follows from [11, Theorem 1.4] that $\alpha(\tau_n) \to \alpha(\tau)$ if $n \to \infty$. This shows that the function $\tau \to \alpha(\tau)$ is continuous. Since it is also injective [15, Corollary 3.5] and the extended real line is compact on the Riemann sphere, the inverse of $\tau \to \alpha(\tau)$ is continuous as well. □

Now the original topic about nonsymmetric extensions of Nevanlinna functions is taken up again.

**Proposition 3.3.** Let $Q(z)$ be an $N_1$ function. Assume that $\Omega$ is a simply connected domain with $\Omega \cap \mathbb{R} \neq \emptyset$ such that $\beta, \bar{\beta} \notin \Omega$, and assume that $Q(z)$ extends to a holomorphic function $\tilde{Q}(z)$ in $\Omega \cup \mathbb{C}^+$. If the set

$$A = \{\alpha(\tau) : \tau \in \mathbb{R} \cup \{\infty\}\} \cap \mathbb{R} \cap \Omega$$

has an accumulation point in $\Omega$, then $\Omega \cap \mathbb{R}$ is outside the support of $\sigma$.

**Proof.** Consider the function

$$W(z) = \frac{\tilde{Q}(z) + \bar{Q}(z)}{2},$$

which is holomorphic in $\Omega \cup \mathbb{C}^+$ and real on $\Omega \cap \mathbb{R}$. Furthermore,

$$\tilde{Q}(z) = W(z), \quad z \in A,$$

since $Q(z) \in \mathbb{R}$ for $z \in A$. Hence, $Q(z) = W(z)$ for $z \in \Omega \cap \mathbb{C}^+$. In particular,

$$\lim_{z \to x} \text{Im} Q(z) = 0, \quad x \in \mathbb{R} \cap \Omega$$

and therefore, $\Omega \cap \mathbb{R}$ is contained in the gap of $Q(z)$. □

**4. The behavior of $\alpha(\tau)$ meeting the real line**

Proposition 4.1 below is a generalization of [15, Proposition 2.4] for the case when $z_0 = \alpha \in \mathbb{R}$ and the function $Q(z)$ extends holomorphically to a holomorphic function $\tilde{Q}(z)$ in some simply connected neighborhood $\Omega$ of $\alpha$. Compared to [15], now it is not assumed that the extension satisfies the symmetry principle,
that is the $\alpha$ is not necessarily in the gap of the measure $\sigma$. The reasoning below is independent of [15], but in case $Q(z)$ satisfy the symmetry principle it reduces to the one in [15]. In what follows it will be frequently used that the $k$-th nontangential derivative of $Q(z)$ at $\alpha$ coincides with the derivative of the extension $\tilde{Q}(z)$ at $\alpha$.

**Proposition 4.1.** Let $Q(z) \in N_1$ with $\alpha \in \mathbb{R}$ being its GZNT. If $Q(z)$ extends to a holomorphic function $\tilde{Q}(z)$ in some neighborhood $\Omega$ of $\alpha$ then precisely one of the following cases occurs:

1. $\tilde{Q}'(\alpha) < 0$;
2. $\tilde{Q}'(\alpha) = 0$ and $\tilde{Q}''(\alpha) \neq 0$, in which case $\text{Im}\, \tilde{Q}''(\alpha) \geq 0$;
3. $\tilde{Q}'(\alpha) = 0$ and $\tilde{Q}''(\alpha) = 0$, in which case $\tilde{Q}'''(\alpha) > 0$.

**Proof.** According to Theorem 2.2(ii) the extension $\tilde{Q}(z)$ can be represented as follows:

$$\tilde{Q}(z) = (z - \alpha)^2 R_0(z) \left( \frac{m_0}{\alpha - z} + \tilde{M}_1(z) \right),$$

with $m_0 \geq 0$ and

$$R_0(z) = 1/(z - \beta)(z - \bar{\beta}) \quad \text{or} \quad R_0(z) \equiv 1,$$

depending on the position of the GPNT. The function $\tilde{M}_1(z)$ in (7) is a Nevanlinna function in $\mathbb{C}^+$ which is also holomorphic in a neighborhood $\Omega$ of $\alpha$ of the form

$$\Omega = [\alpha - \varepsilon, \alpha + \varepsilon] + i[-\varepsilon, \varepsilon],$$

where $\varepsilon > 0$ is sufficiently small.

In order to list the possible cases, observe that it follows from (7) that

$$\tilde{Q}'(\alpha) = -m_0 R_0(\alpha), \quad R_0(\alpha) > 0.$$ 

Hence, $Q'(\alpha) \leq 0$.

If $Q'(\alpha) < 0$ case (1) prevails. Assume now that $\tilde{Q}'(\alpha) = 0$, in which case it follows from (7) that

$$\tilde{Q}''(\alpha) = 2R_0(\alpha)\tilde{M}_1(\alpha).$$

Since $\tilde{M}_1(z)$ is a Nevanlinna function in $\mathbb{C}^+$ and it is continuous at $\alpha$ it follows that $\text{Im}\, \tilde{M}_1(\alpha) \geq 0$. Furthermore, note that $R_0(\alpha) > 0$. Therefore one has that

$$\tilde{Q}'(\alpha) = 0 \quad \Rightarrow \quad \text{Im}\, \tilde{Q}''(\alpha) \geq 0,$$

which takes care of (2).

Finally, consider the case $\tilde{Q}'(\alpha) = \tilde{Q}''(\alpha) = 0$. Then by (8) and the fact that $R_0(\alpha) > 0$ one has $\tilde{M}_1(\alpha) = 0$. Consequently,

$$\tilde{Q}'''(\alpha) = 2R_0(\alpha)\tilde{M}_1(\alpha).$$

Recall that by Theorem 2.1 the function $\tilde{M}_1(z)$ can be represented as

$$\tilde{M}_1(z) = \int_{\alpha - \varepsilon}^{\alpha + \varepsilon} \frac{\phi(t)}{t - z} \, dt + M_2(z),$$
where $\phi(z)$ is a function holomorphic in $\Omega$, and $M_2(z)$ is a Nevanlinna function with a gap $[\alpha - \varepsilon, \alpha + \varepsilon]$. By the general theory of Nevanlinna functions [7] one has

$$\phi(\alpha) = \frac{1}{\pi} \text{Im} \tilde{M}_1(\alpha) = 0.$$ 

Furthermore, $\phi'(\alpha) = 0$, since $\phi(t)$ is positive on $[\alpha - \varepsilon, \alpha + \varepsilon]$. Hence, the function $\phi(t)/(\alpha - t)^2$ is integrable on $[\alpha - \varepsilon, \alpha + \varepsilon]$. By the dominated convergence theorem it follows that

$$\tilde{M}_1'(\alpha) = \int_{\alpha - \varepsilon}^{\alpha + \varepsilon} \frac{\phi(t)}{(t - \alpha)^2} dt + M_2'(\alpha).$$  \hspace{1cm} (11)$$

It is clear that $M_2'(\alpha) \geq 0$, since $M_2(z)$ has a gap at $[\alpha - \varepsilon, \alpha + \varepsilon]$. Hence (11) shows that $\tilde{M}_1'(\alpha) \geq 0$. In fact, at least one of the terms in the right-hand side of (11) has to be positive, otherwise $Q(z) \equiv 0$, which is not an $\mathbb{N}_1$ function. Hence it follows that $M_2'(\alpha) > 0$ and, by (10), $\tilde{Q}''(\alpha) > 0$. This takes care of (3). $\Box$

In [15, Theorem 4.1] it is investigated what cases can occur if the curve $\{\alpha(\tau): \tau \in \mathbb{R} \cup \{\infty\}\}$ meets the real line in a spectral gap of the function $M(z)$. Either it approaches the spectral gap perpendicular and then continues through some subinterval of the spectral gap, or it approaches it with an angle of $\pi/3$, hits the spectral gap in a single point and leaves it with an angle of $2\pi/3$. However, if the curve meets the real line in a nonisolated point of $\text{supp}\sigma$ its behavior might be dramatically different, as Example 5.3 in [15] and Section 5 below show. The following theorem provides a characterization of the possible cases in the situation that $M(z)$ has a (not necessarily symmetric) holomorphic extension through the corresponding point of intersection as in Proposition 4.1. The proof is an extension of the proof of [15, Theorem 4.1].

Theorem 4.2. Let $Q(z) \in \mathbb{N}_1$ and assume that $Q(z)$ extends to a holomorphic function $\tilde{Q}(z)$ in some neighborhood $\Omega$ of $z_0 \in \mathbb{R}$. Furthermore assume that $\alpha(\tau_0) = z_0$ for $\tau_0 \in \mathbb{R}$. Then precisely one of the following cases occurs:

1. $Q'(z_0) < 0$. Then there exists $\varepsilon > 0$ such that the function $\alpha(\tau)$ is holomorphic on $(-\varepsilon, \varepsilon)$ and

$$\lim_{\tau \uparrow 0} \arg(\alpha(\tau) - z_0) = 0, \quad \lim_{\tau \downarrow 0} \arg(\alpha(\tau) - z_0) = \pi.$$ 

2. $Q'(z_0) = 0$ and $Q''(z_0) \neq 0$. Then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the function $\alpha(\tau)$ is holomorphic on each of the intervals $(-\varepsilon_1, 0)$ and $(0, \varepsilon_2)$. Moreover,

$$\lim_{\tau \uparrow 0} \arg(\alpha(\tau) - z_0) = \frac{2\pi - \theta_0}{2}, \quad \lim_{\tau \downarrow 0} \arg(\alpha(\tau) - z_0) = \frac{\pi - \theta_0}{2},$$  \hspace{1cm} (12)$$

where $\theta_0 = \arg Q''(z_0)$.

3. $Q'(z_0) = Q''(z_0) = 0$ and $Q'''(z_0) \neq 0$. Then there exist $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the function $\alpha(\tau)$ is holomorphic on each of the intervals $(-\varepsilon_1, 0)$ and $(0, \varepsilon_2)$. Moreover,

$$\lim_{\tau \uparrow 0} \arg(\alpha(\tau) - z_0) = \frac{\pi}{3}, \quad \lim_{\tau \downarrow 0} \arg(\alpha(\tau) - z_0) = \frac{2\pi}{3}.$$ 

Proof. Note that it is enough to consider the case $\tau = 0$. Indeed, if $Q(z)$ extends to some simply connected neighborhood $\Omega$ of $z_0$ then $Q_{\tau_0}(z)$ extends to $\Omega \setminus \{\beta(\tau_0), \beta(\tau_0)\}$. Since $z_0 = \alpha(\tau_0) \notin \{\beta(\tau_0), \beta(\tau_0)\}$ one can choose a sufficiently small $\Omega$ for the function $Q_{\tau_0}(z)$. In this situation the cases (1)–(3) correspond precisely to the classification in Proposition 4.1. Furthermore, without loosing generality, it is assumed that $z_0 = 0$. 

\hspace{1cm}
Case (1). According to the standard inverse function theorem, there exists a function $\phi(w)$ satisfying $Q(\phi(w)) = w$, cf. [15, Proof of Theorem 4.1]. Then define $\alpha(\tau) = \phi(\tau)$ for $\tau$ sufficiently small. The power series of $\hat{Q}(z)$ at zero does need to have all its coefficients real, as was the case in [15, Proof of Theorem 4.1].

Case (2). $\hat{Q}(0) = \tilde{Q}(0) = 0$, and $\text{Im} \tilde{Q}'(0) > 0$. According to the generalized inverse function theorem, see e.g. [9, Theorem 9.4.3], the equation

$$\tilde{Q}(\phi^\pm(w)) = w^2,$$

has in some neighborhood of zero exactly two holomorphic solutions $\phi^+(w)$ and $\phi^-(w)$. The corresponding expansions

$$\phi^\pm(w) = \phi_1^\pm w + \phi_2^\pm w^2 + \cdots,$$

satisfy $\phi_1^\pm = \pm(\hat{Q}''(0)/2)^{-1/2}$, where the square root is chosen in such way that it transforms $\mathbb{C}^+$ onto itself. Recall that $\text{Im} \tilde{Q}''(0) \geq 0$ and, hence, it follows that $\pm \text{Im} \phi_1^\pm \geq 0$.

In the case $\tau > 0$ one has the identity

$$\tilde{Q}(\phi^{-}(\tau^{1/2})) = \tau \quad (13)$$

and $\arg \phi^- = \pi - \theta_0/2$. Hence $\phi^{-}(\tau^{1/2})$ is in $\mathbb{C}^+ \cup \mathbb{R}$ for small $\tau > 0$. As a consequence one sees that

$$\alpha(\tau) = \phi^{-}(\tau^{1/2}), \quad 0 < \tau < +\infty.$$}

The expansion of $\phi^{-}(\tau^{1/2})$ implies the limit of $\arg(\alpha(\tau))$ as $\tau \downarrow 0$:

$$\lim_{\tau \downarrow 0} \tan(\arg(\alpha(\tau))) = \lim_{\tau \downarrow 0} \frac{\text{Im} \alpha(\tau)}{\text{Re} \alpha(\tau)} = \frac{\text{Im} \phi_1^-}{\text{Re} \phi_1^-} = \tan \arg \phi_1^- = \tan((2\pi - \theta_0)/2).$$

Since the tangent function is injective on the interval $[0, \pi]$, the first part of (12) follows.

Similarly, in the case $\tau < 0$ one has the identity

$$\tilde{Q}(\phi^+(i\tau^{1/2})) = -|\tau| = \tau, \quad (14)$$

and $\arg(\phi_1^+ i) = (\pi - \theta_0)/2$. Hence, $\phi^+(i\tau^{1/2})$ is in $\mathbb{C}^+$ for small $\tau < 0$. As a consequence one sees that

$$\alpha(\tau) = \phi^+(\tau^{1/2}), \quad -\infty < \tau < 0.$$}

The expansion of $\phi^+(i\tau^{1/2})$ implies the right limit of $\arg(\alpha(\tau))$ at zero:

$$\lim_{\tau \uparrow 0} \tan(\arg(\alpha(\tau))) = \lim_{\tau \uparrow 0} \frac{\text{Im} \alpha(\tau)}{\text{Re} \alpha(\tau)} = \tan(\arg(i\phi_1^+)) = \tan((\pi - \theta_0)/2).$$

Case (3) follows exactly along the same lines as in [15]. \(\Box\)

5. Classification of GZNT

Let $Q(z)$ belong to $N_1$ and assume for simplicity that its GPNT lies at infinity. Then the integral representation of $Q(z)$ has the following form:

$$Q(z) = (z - \alpha)(z - \bar{\alpha}) \left( a + bz + \int_\mathbb{R} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\sigma(t) \right), \quad (15)$$
with $\alpha \in \mathbb{C}$, $a \in \mathbb{R}$, $b \geq 0$, and a measure $\sigma$ satisfying $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t^2 + 1)} < \infty$. If the GZNT $\alpha$ belongs to $\mathbb{R}$, then there is the following classification of $\alpha$ in terms of the integral representation (15):

(A) $\delta_{\alpha} := \int_{\{\alpha\}} 1 \, d\sigma > 0$;
(B) $\delta_{\alpha} = 0$, $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-a)^2} = \infty$;
(C) $\delta_{\alpha} = 0$, $\gamma_{\alpha} := \lim_{z \to \alpha} \frac{Q(z)}{(z-\alpha)^2} \in \mathbb{R} \setminus \{0\}$, $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-a)^2} < \infty$;
(D) $\delta_{\alpha} = \gamma_{\alpha} = 0$, $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-a)^2} < \infty$, $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-a)^4} = \infty$;
(E) $\delta_{\alpha} = \gamma_{\alpha} = 0$, $\int_{\mathbb{R}} \frac{d\sigma(t)}{(t-a)^4} < \infty$.

cf. [5]. This classification has an interpretation in terms of a corresponding operator model, see [5, Theorem 5.1]. Furthermore, note that Proposition 4.1 provides an alternative classification of GZNT if $Q(z)$ has a (not necessarily symmetric) extension through $\alpha$. Clearly, that classification is coarser than (A)–(E). Below the possible pairings are listed and examples of functions with symmetric and nonsymmetric extensions are indicated:

(A) and (1): an example with a nonsymmetric extension is given by

$$Q(z) = z^2 \left( i - \frac{1}{z} \right),$$

see Example 5.1, while an example in the case of a spectral gap is given by $Q(z) = -z$.

(B) and (2): an example with a nonsymmetric extension is given by

$$Q(z) = z^2 e^{i\theta_0}, \quad \theta_0 \in [0, \pi],$$

see Example 5.2, while examples in the case of a spectral gap do not exist.

(C) and (2): an example with a nonsymmetric extension is given by

$$Q(z) = z^2 \left( 1 + \int_{-1}^{1} \frac{t^2 \, dt}{t - z} \right),$$

see Example 5.3, while an example in the case of a spectral gap is given by $Q(z) = z^2$.

(D) and (3): an example with a nonsymmetric extension is given by

$$Q(z) = z^2 \left( \int_{-1}^{1} \frac{t^2 \, dt}{t - z} \right),$$

see Example 5.4, while examples in the case of a spectral gap do not exist.

(E) and (3): an example with a nonsymmetric extension is given by

$$Q(z) = z^2 \left( \int_{-1}^{1} \frac{t^4 \, dt}{t - z} \right),$$

see Example 5.5, while an example in the case of a spectral gap is given by $Q(z) = z^3$.

The rest of this section is devoted to the treatment of these and other examples.
**Example 5.1.** To illustrate Case (A) consider the $N_1$ function

$$Q(z) = z^2 \left( i - \frac{1}{z} \right).$$

The plot in Fig. 1 (all plots obtained with Maple [12]) shows all the points $z$ from the upper half-plane where $\text{Im} Q(z) = 0$. From Theorem 4.2 case (1) one can determine, that $\alpha(\tau)$ moves along the plotted curve from the right to the left hand side, i.e $\text{Im Re}(\alpha(\tau))$ is decreasing in $\tau$. Note that although $\alpha(\tau)$ approaches the origin horizontally, $\alpha(\tau) \notin \mathbb{R}$ for $\tau \neq 0$, in contrast to the case of a spectral gap described in [15]. This behavior agrees with [15, Theorem 3.6]. An example, which is simpler to compute, however not in the form (15), is

$$Q(z) = \frac{z^2}{(z-i)(z+i)} \left( i - \frac{1}{z} \right) = \frac{iz}{z-i}.$$

Solving

$$\frac{iz}{z-i} = \tau$$

one gets

$$\alpha(\tau) = \frac{-\tau + \tau^2 i}{\tau^2 + 1},$$

and the same effect of approaching the origin tangentially from both sides is obtained.

**Example 5.2.** To illustrate Case (B) consider for $\theta_0 \in [0, \pi]$ the function

$$Q(z) = z^2 e^{i\theta_0}.$$

Solving $Q(z) = \tau$ with $z \in \mathbb{C}^+$ one obtains

$$\alpha(\tau) = \begin{cases} \sqrt{|\tau|} \cdot e^{i(\pi - \theta_0)/2}, & \tau \leq 0, \\ \sqrt{\tau} \cdot e^{i(2\pi - \theta_0)/2}, & \tau > 0. \end{cases}$$

As another example of Case (B) consider

$$Q(z) = z^2 \int_{-1}^{1} \frac{dt}{t-z}.$$

Fig. 2 contains the plot of points $z \in \mathbb{C}^+$ satisfying $\text{Im} Q(z) = 0$. Since $\alpha(\infty) = \infty$ one sees that for sufficiently small $\tau < 0$ the point $\alpha(\tau)$ moves with increasing $\tau$ along the real line to the left until it reaches...
Fig. 2. Case (B), $Q(z) = z^2 \int_{-1}^{1} \frac{dt}{t-z}$.

Fig. 3. Case (C), $Q(z) = z^2 (1 + \int_{-1}^{1} \frac{t^2 dt}{t-z})$.

Fig. 4. Case (D), $Q(z) = z^2 (\int_{-1}^{1} \frac{t^2 dt}{t-z})$.

the point near 1.7. There it leaves the real line to the upper half-plane and continues along the plotted path until it reaches the real line again at approximately −1.7. From that point it continues along the real line.

**Example 5.3.** To illustrate Case (C) consider the function

$$Q(z) = z^2 \left(1 + \int_{-1}^{1} \frac{t^2 dt}{t-z}\right).$$

Fig. 3 contains the plot of points $z \in \mathbb{C}^+$ satisfying $\text{Im} \ Q(z) = 0$. One may observe that with $\tau \downarrow 0$ the point $\alpha(\tau)$ approaches the real line approximately vertically and it leaves the origin approximately horizontally. It is known from Theorem 3.6 of [15] that the only point in a neighborhood of zero where $\alpha(\tau) \in \mathbb{R}$ is the origin itself.

**Example 5.4.** To illustrate Case (D) consider the function

$$Q(z) = z^2 \int_{-1}^{1} \frac{t^2 dt}{t-z}.$$ 

Fig. 4 contains the plot of points $z \in \mathbb{C}^+$ satisfying $\text{Im} \ Q(z) = 0$. Note the essential difference between Fig. 2 and Fig. 4, in Fig. 2 the angle between the left and right limit of $\alpha(\tau)$ at the origin is $\pi/2$, while in Fig. 4 it is $\pi/3$. The movement of $\alpha(\tau)$ along the plotted line and the real line is the same as in Example 5.2.

**Example 5.5.** Finally, to illustrate Case (E) consider the function

$$Q(z) = z^2 \left(\int_{-1}^{1} \frac{t^4 dt}{t-z}\right).$$
Fig. 5 contains the plot of points $z \in \mathbb{C}^+$ satisfying $\text{Im} Q(z) = 0$. The angle between the left and right limit of $\alpha(\tau)$ at the origin is $\pi/3$ and the movement of $\alpha(\tau)$ along the plotted line and the real line is the same as in Examples 5.2 and 5.3.

References