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Balanced Model Reduction for Linear Time-Varying Symmetric Systems

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***Abstract*—The goal of this paper is to develop balancing theory for the linear time-varying (LTV) *symmetric* systems. To this end, first, we extend the concept of symmetry in terms of the dual system. Then, we define the cross Gramian for the LTV systems. For LTV symmetric systems, we establish a connection among the controllability, observability, and cross Gramians. In particular, if one of these three Gramians is obtained, the other two Gramians can be constructed. Based on this fact, we show that the symmetry structure is preserved under balanced truncation if the Hankel singular values are pointwise distinct.**

***Index Terms*—Balanced truncation, model reduction, symmetric systems, time-varying systems.**

I. Introduction

In this paper, we develop the model reduction technique with balanced truncation of linear time-varying (LTV) symmetric systems. For LTV systems, the concept of balanced realizations [2] is extended into various problem settings depending on the considered time-intervals [3]–[5]. The typical one is the uniformly balanced realization [3] or sometimes called infinite-interval balanced realization [5] for exponentially stable and uniformly completely controllable and observable systems. For this realization, stability of the reduced order model is studied in [4], and error bounds [6] originally provided in the finite-interval case are applicable. Generally, the balancing procedure involves computing both the controllability and observability Gramians, i.e., solving a couple of differential Lyapunov equations [3], [5], which is not always easy, and developing efficient algorithms is another interest of research [7].

It is known that for specific linear time-invariant (LTI) systems, the controllability Gramian can be constructed from the observability Gramian, and vice versa [8]–[10]. Such systems are called symmetric [11], [12], since their transfer function matrices are symmetric. Symmetry structures naturally arise in a number of contexts such as RLC electrical circuits [13] and certain dissipative systems. For nonlinear (time-invariant) systems, several extensions of the symmetry notion are found in [14] and [15], and one is used for balanced truncation [15]. In contrast, until the preliminary version of this paper [1], the symmetry concept has never been extended to LTV systems.

In this paper, we study balancing of *LTV symmetric systems*. First, we extend the symmetry concept to LTV systems based on another interpretation of the LTI symmetric concept that the impulse responses of the system and its dual [16] are equivalent. As a byproduct, we also extend the cross Gramian [11], [12], [17] to the LTV symmetric systems. The cross Gramian is originally introduced for SISO LTI systems to study their minimality and is related to the controllability and observability Gramians and as such is related to balancing [17]. Then, it is extended to MIMO symmetric systems [11], [12] and useful for model reduction [8]–[10]. Similarly to the LTI symmetric case, minimality can be checked based on the cross Gramian. This fact follows from the main contribution of this paper that if one of the controllability, observability, or cross Gramians is obtained for the LTV symmetric systems, the other two Gramians can be constructed from it. That is, uniformly balanced truncation is performed only by computing one of these three Gramians. Based on this fact, we prove that the symmetry structure is preserved under balanced truncation if the Hankel singular values are pointwise distinct, where this requirement can be relaxed.

The preliminary results of this paper have been reported in [1], which presents first steps toward defining the symmetry concept and cross Gramian for the LTV system. However, in [1], no proof is provided, and deeper analysis of the relation with controllability and observability operators is lacking. In contrast, our current paper provides these items. All of the analysis in this paper is provided for the infinite interval case but can be extended to the fixed and sliding interval cases [3], [5] and also to a method for unstable systems. We briefly discuss these extensions and also study generalized balancing with Lyapunov inequalities [6].

The remainder of this paper is organized as follows. In Section II, we define LTV symmetric systems and provide their characterizations. In Section III, we extend the cross Gramian to LTV symmetric systems and investigate the relationship among the controllability, observability, and cross Gramians. In Section IV, we show that the symmetry structure is preserved under balanced truncation if the Hankel singular values are pointwise distinct.

Notation: When it is clear from the context, the argument $(\cdot)$ of time-varying operators is left out.

II. Linear Time-Varying Symmetric Systems

In this section, our goal is to extend the symmetry concept to LTV systems. As preliminaries, first, we state the class of LTV systems and basic concepts used in this paper such as controllability/observability operators and dual systems. After that, we extend the symmetry concept to LTV systems and study their properties.

A. Preliminaries

Consider a square LTV system

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t) \\ y(t) = C(t)x(t). \end{cases} \tag{1}$$

where $x : \mathbb{R} \to \mathbb{R}^n$, $u, y : \mathbb{R} \to \mathbb{R}^m$ are the state, input, and output, respectively, and $A : \mathbb{R} \to \mathbb{R}^{n \times n}$, $B : \mathbb{R} \to \mathbb{R}^{n \times m}$ and $C : \mathbb{R} \to \mathbb{R}^{m \times n}$. A system representation of this type is denoted by the triplet (A, B, C). A realization (A, B, C) is said to be bounded if there exists $K > 0$ such that $\|A(t)\| \leq K$, $\|B(t)\| \leq K$, and $\|C(t)\| \leq K$ for all $t \in \mathbb{R}$. If A is piecewise continuous, the system $\dot{x}(t) = A(t)x(t)$ always has a unique solution $x(t) = \Phi(t, \tau)x(\tau)$ for any $t \geq \tau$ and $x(\tau) \in \mathbb{R}^n$, where $\Phi(t, \tau)$ is called the (state) transition matrix [18].

For balancing theory, controllability and observability are essential concepts. A bounded realization (A, B, C) is said to be uniformly completely controllable and observable if there exists $\delta > 0$ such that for all $t \in \mathbb{R}$

$$W_c(t - \delta, t) = \int_{t-\delta}^t \Phi(t, \tau)B(\tau)B^\top(\tau)\Phi^\top(t, \tau)d\tau$$
$$> \alpha_c(\delta)I_n \tag{2}$$

$$W_o(t, t + \delta) = \int_t^{t+\delta} \Phi^\top(\tau, t)C^\top(\tau)C(\tau)\Phi(\tau, t)d\tau$$
$$> \alpha_o(\delta)I_n \tag{3}$$

for some positive constants $\alpha_c(\delta)$ and $\alpha_o(\delta)$, respectively [3]. These definitions of controllability and observability are equivalent to the more standard definitions by Kalman [19] as mentioned in [3]. The above-mentioned two matrices $W_c(t - \delta, t)$ and $W_o(t, t + \delta)$ are referred as the controllability (or called reachability) and observability Gramians, respectively. Throughout this paper, we assume the following.

Assumption 2.1: The triplet (A, B, C) is uniform [3], [4]. That is, A, B, and C are continuous and bounded, and the triplet is uniformly completely controllable and observable. ◁

Assumption 2.2: The system (1) is exponentially stable [18], i.e., there exist positive constants c and λ such that $\|\Phi(t, \tau)\| \leq ce^{-\lambda(t-\tau)}$ for any $t \geq \tau$. ◁

For LTV systems, there are several balanced realizations [3], [5], [6]. In this paper, we first develop theory for the infinite-interval case and briefly discuss extensions to the other cases. In the infinite-interval case, we use the following controllability and observability Gramians $W_c(t) := \lim_{\delta \to \infty} W_c(t - \delta, t)$ and $W_o(t) := \lim_{\delta \to \infty} W_o(t, t + \delta)$ for all $t \in \mathbb{R}$. Under Assumptions 2.1 and 2.2, these Gramians exist and are bounded and uniformly positive definite [3]. Moreover, W_c and W_o satisfy the following differential Lyapunov equations [3]:

$$-\dot{W}_c + W_c A^\top + A W_c = -BB^\top \tag{4}$$

$$\dot{W}_o + W_o A + A^\top W_o = -C^\top C \tag{5}$$

where $\lim_{t \to -\infty} W_c(t) = 0$ and $\lim_{t \to \infty} W_o(t) = 0$.

Next, we consider the impulse response of the triplet (A, B, C), which is given by

$$H(t, \tau) = C(t)\Phi(t, \tau)B(\tau), \ t \geq \tau \tag{6}$$

and exists under Assumptions 2.1 and 2.2. Thus, the state-space model (1) can be viewed as a bounded operator from $L_2^m[t_0, \infty) \to L_2^m[t_0, \infty)$ given by

$$y(t) = (\Sigma_{t_0} u)(t) := \int_{t_0}^t H(t, \tau)u(\tau)d\tau, \ t \geq t_0$$

which is the zero initial state response $x(t_0) = 0$ of the system (1). Also, the Hankel operator $\mathcal{H}_{t_0} : L_2^m(-\infty, t_0] \to L_2^m[t_0, \infty)$ can be written as

$$(\mathcal{H}_{t_0} u)(t) = \int_{-\infty}^{t_0} H(t, \tau)u(\tau)d\tau, \ t \geq t_0.$$

Under Assumptions 2.1 and 2.2, this is bounded.

B. Controllability and Observability Operators

In the LTI case, the symmetry concept and the cross Gramian [11], [12], [17] are provided in terms of the controllability and observability operators and their adjoint operators. Here, we summarize these operators for LTV systems.

The controllability operator $\mathcal{C}_{t_0} : L_2^m(-\infty, t_0] \to \mathbb{R}^n$ and the observability operator $\mathcal{O}_{t_0} : \mathbb{R}^n \to L_2^m[t_0, \infty)$ of the triplet (A, B, C) are, respectively, defined as follows:

$$\mathcal{C}_{t_0} u := \int_{-\infty}^{t_0} \Phi(t_0, \tau)B(\tau)u(\tau)d\tau$$

$$(\mathcal{O}_{t_0} x_0)(t) := C(t)\Phi(t, t_0)x_0, \ t \geq t_0$$

which are natural generalization of the LTI case [20]. Similarly to the LTI case, from the definition of the Hankel operator $\mathcal{H}_{t_0}$, it follows that $\mathcal{H}_{t_0} = \mathcal{O}_{t_0} \circ \mathcal{C}_{t_0}$.

Next, we compute the adjoint operators of controllability operator $\mathcal{C}_{t_0}$ and observability operator $\mathcal{O}_{t_0}$, which are denoted by $\mathcal{C}_{t_0}^*$ and $\mathcal{O}_{t_0}^*$, respectively. From

$$\langle \mathcal{C}_{t_0} u, x_{a0} \rangle_{\mathbb{R}^n} = \langle u, \mathcal{C}_{t_0}^* x_{a0} \rangle_{L_2^m(-\infty, t_0]}$$

$$\langle \mathcal{O}_{t_0} x_0, u_a \rangle_{L_2^m[t_0, \infty)} = \langle x_0, \mathcal{O}_{t_0}^* u_a \rangle_{\mathbb{R}^n}$$

the adjoint operators $\mathcal{C}_{t_0}^* : \mathbb{R}^n \to L_2^m(-\infty, t_0]$ and $\mathcal{O}_{t_0}^* : L_2^m[t_0, \infty) \to \mathbb{R}^n$ are obtained as

$$(\mathcal{C}_{t_0}^* x_{a0})(t) = B^\top(t)\Phi^\top(t_0, t)x_{a0}, \ t \leq t_0$$

$$\mathcal{O}_{t_0}^* u_a = \int_{t_0}^\infty \Phi^\top(\tau, t_0)C^\top(\tau)u_a(\tau)d\tau$$

respectively. Moreover, the adjoint of the Hankel operator $\mathcal{H}_{t_0}^* : L_2^m[t_0, \infty) \to L_2^m(-\infty, t_0]$ is

$$(\mathcal{H}_{t_0}^* u_a)(t) = \int_{t_0}^\infty B^\top(t)\Phi^\top(\tau, t)C^\top(\tau)u_a(\tau)d\tau$$

$$= (\mathcal{C}_{t_0}^* \mathcal{O}_{t_0}^* u_a)(t), \ t \leq t_0.$$

Similarly to the LTI case [20], the controllability/observability Gramians and operators have strong connections. It follows that $\mathcal{W}_c(t_0) = \mathcal{C}_{t_0} \circ \mathcal{C}_{t_0}^*$ and $\mathcal{W}_o(t_0) = \mathcal{O}_{t_0}^* \circ \mathcal{O}_{t_0}$ at each $t_0 \in \mathbb{R}$, where $\mathcal{W}_c(t_0), \mathcal{W}_o(t_0) : \mathbb{R}^n \to \mathbb{R}^n$ are linear operators defined by the controllability and observability Gramians

$$\mathcal{W}_c(t_0) : v \mapsto W_c(t_0)v, \ \mathcal{W}_o(t_0) : v \mapsto W_o(t_0)v.$$

Moreover, it can be shown that $\mathcal{H}_{t_0}^* \circ \mathcal{H}_{t_0}$ and $W_c(t_0)W_o(t_0)$ have the same nonzero eigenvalues at each $t_0 \in \mathbb{R}$. Therefore, we call the square roots of eigenvalues of $W_c W_o$ the Hankel singular values. Note that the Hankel singular values are invariant under the coordinate transformations.

C. Dual Systems

In this paper, we use the so called dual system [16] to extend the symmetry concept to the LTV system. In this subsection, we introduce

the dual system and explain the connection with adjoint operators shown in the previous subsection.

The dual system of (1) is given by

$$\begin{cases} \dot{\lambda}(t) = A^\top(-t)\lambda(t) + C^\top(-t)\nu(t) \\ \eta(t) = B^\top(-t)\lambda(t). \end{cases} \tag{7}$$

This system is similar to the modified adjoint system of (1) given in [5] and [21] but different because the sign in front of $C^\top(-t)$ is different. We can immediately see that the dual system of (7) is the original LTV system (1). Moreover, it is shown in [16] that the triplet $(A^\top(-\cdot), C^\top(-\cdot), B^\top(-\cdot))$ is uniform, and the dual system (7) is exponentially stable if and only if the triplet $(A(\cdot), B(\cdot), C(\cdot))$ is uniform, and the original system (1) is exponentially stable.

This dual system has a strong connection with the adjoint operators of the controllability, observability, and Hankel operators. By substituting $t := -\tilde{t}$ into (7), we have the time-reverse expression

$$\begin{cases} \frac{d\lambda(-\tilde{t})}{d\tilde{t}} = -A^\top(\tilde{t})\lambda(-\tilde{t}) - C^\top(\tilde{t})\nu(-\tilde{t}) \\ \eta(-\tilde{t}) = B^\top(\tilde{t})\lambda(-\tilde{t}) \end{cases} \tag{8}$$

which is called the adjoint system of (1), see, e.g., [5], [21]. For $\lambda(-\infty) = 0$ and $\nu(t) = 0$ for all $t \geq t_0$, the adjoint system (8) is a state-space representation of the adjoint of the Hankel operator $\mathcal{H}_{t_0}^*$ with the state $\tilde{\lambda}(\cdot) := \lambda(-\cdot)$, input $\tilde{\nu}(\cdot) := \nu(-\cdot)$, and output $\tilde{\eta}(\cdot) := \eta(-\cdot)$. This connection is the analogy of the fact that the original system (1) is a state-space representation of the Hankel operator $\mathcal{H}_{t_0}$ for $x(-\infty) = 0$ and $u(t) = 0$ for all $t \geq t_0$.

Remark 2.3: Under the exponential dichotomy assumption, which is weaker than the exponential stability Assumption 2.2, the dual system is introduced and studied in an operator theoretic framework [22]. This dual system is nothing but (7). The analysis in [22] is based on the fact that the adjoint of the L_2 input–output operator [22] associated with an exponential dichotomy system (1) is bounded, which corresponds to the adjoint operators of the controllability, observability, and Hankel operators being bounded under the exponential stability Assumption 2.2. ◁

D. Symmetric Systems

In this subsection, we generalize the symmetry concept to LTV systems via dual systems and study symmetry properties. In the LTI case, a system is said to be symmetric if its transfer function matrix is symmetric [9], [15], i.e., the transfer function matrices of the system (1) and its dual (7) are equal. In other words, their impulse responses are equal [11]. Since impulse responses are defined also for LTV systems, the impulse response is used in the LTV case. After the extension of the symmetry concept, we provide its characterizations in term of the state-space representation and operators.

The impulse response of system (1) is given by (6). Next, we compute the impulse response of its dual system (7). Since $d\Phi^\top(\tau, t)/dt = -A^\top(t)\Phi^\top(\tau, t)$, $t \geq \tau$ [18], the transition matrix of $\dot{\lambda}(t) = A^\top(-t)\lambda(t)$ satisfies

$$\frac{d\Phi^\top(-\tau, -t)}{dt} = A^\top(-t)\Phi^\top(-\tau, -t), \ t \geq \tau$$

i.e., $\Phi^\top(-\tau, -t)$ is the transition matrix of the dual system (7). Therefore, its impulse response is given by

$$H_d(t, \tau) = B^\top(-t)\Phi^\top(-\tau, -t)C^\top(-\tau), \ t \geq \tau. \tag{9}$$

Now, we are ready to define the LTV symmetric system.

Definition 2.4: The triplet (A, B, C) is said to be symmetric if

$$H(t, \tau) = H_d(t, \tau), \ t \geq \tau \tag{10}$$

holds. ◁

In the LTI case, (10) implies that the transfer function matrix of (A, B, C) is symmetric, and vice versa. Here, we aim for extending the results for LTI systems to LTV systems. For LTV systems, it is convenient to extend those results through the state-space interpretation of the symmetry notion, along with the use of the controllability and observability operators. For that, we need to introduce the so-called Lyapunov transformations [3] as follows.

Definition 2.5: A coordinate transformation $z(t) = T(t)x(t)$ $\forall t \in \mathbb{R}$ is said to be a Lyapunov transformation if T, T^{-1}, and $\dot{T}$ are continuous and bounded. ◁

Theorem 2.6: Under Assumptions 2.1 and 2.2, the triplet (A, B, C) is symmetric if and only if there exists a Lyapunov transformation $T(\cdot) = T^\top(-\cdot)$ such that

$$\dot{T}(\cdot) + T(\cdot)A(\cdot) = A^\top(-\cdot)T(\cdot) \tag{11}$$

$$T(\cdot)B(\cdot) = C^\top(-\cdot). \tag{12}$$

The triplet (A, B, C) is said to be symmetric with respect to T $(T(\cdot) = T^\top(-\cdot))$ if it satisfies (11) and (12).

Proof: (If part) After a change of coordinate $z = Tx$ for the system (1), we have

$$\begin{cases} \dot{z} = (\dot{T} + TA)T^{-1}z + TBu \\ y = CT^{-1}z. \end{cases}$$

From conditions (11) and (12), we have

$$\begin{cases} \dot{z}(\cdot) = A^\top(-\cdot)z(\cdot) + C^\top(-\cdot)u(\cdot) \\ y(\cdot) = B^\top(-\cdot)z(\cdot). \end{cases}$$

This is nothing but the dual system (7). Therefore, the impulse responses of the system (1) and its dual system are equivalent.

(Only if part) The system (1) and its dual (7) are uniform and exponentially stable. Thus, if these two systems have the same impulse response, there exists a Lyapunov transformation T such that (11) and (12) hold [3]. It thus suffices to show that T can be chosen as $T(\cdot) = T^\top(-\cdot)$. Hereafter, we focus on constructing a Lyapunov transformation $\hat{T}(\cdot) = \hat{T}^\top(-\cdot)$ satisfying (11) and (12). From (11), $T^\top(-\cdot)$ satisfies

$$-\frac{dT^\top(-\cdot)}{dt} + A^\top(-\cdot)T^\top(-\cdot) = T^\top(-\cdot)A(\cdot). \tag{13}$$

From (10), we also have

$$C(t)\Phi(t, \tau)B(\tau) = B^\top(-t)\Phi^\top(-\tau, -t)C^\top(-\tau), \ t \geq \tau. \tag{14}$$

Note that from (11), $T(t)\Phi(t, \tau)T^{-1}(\tau)$ is the transition matrix of $\dot{z}(\cdot) = A^\top(-\cdot)z(\cdot)$ [18]. On the other hand, $\Phi^\top(-\tau, -t)$ is also its transition matrix. Therefore, we have

$$T(t)\Phi(t, \tau)T^{-1}(\tau) = \Phi^\top(-\tau, -t), \ t \geq \tau. \tag{15}$$

From (12), (14), and (15), we have

$$C(t)\Phi(t, \tau)B(\tau) = B^\top(-t)T(t)\Phi(t, \tau)B(\tau), \ t \geq \tau.$$

That is, the impulse response has two representations $(A(\cdot), B(\cdot), C(\cdot))$ and $(A(\cdot), B(\cdot), B^\top(-\cdot)T(\cdot))$. Since the system is uniformly completely controllable and observable, for the same $A(\cdot)$ and $B(\cdot)$ matrices, the corresponding output matrix should be the same, i.e.,

$$T^\top(-\cdot)B(\cdot) = C^\top(-\cdot). \tag{16}$$

Define $\hat{T}(\cdot) := \frac{1}{2}(T^\top(-\cdot) + T(\cdot))$. From (11)–(13) and (16), we have

$$\dot{\hat{T}}(\cdot) + \hat{T}(\cdot)A(\cdot) = A^\top(-\cdot)\hat{T}(\cdot), \ \hat{T}(\cdot)B(\cdot) = C^\top(-\cdot).$$

Moreover, $\hat{T}^\top(-\cdot) = \frac{1}{2}(T(\cdot) + T^\top(-\cdot)) = \hat{T}(\cdot)$. ∎

Remark 2.7: Note that $T^\top(-\cdot)$ can be viewed as dual of a Lyapunov transformation $T(\cdot)$ because $T^\top(-\cdot)$ is the time-reverse of $T^\top(\cdot)$, i.e., the adjoint of $T(\cdot)$. Also, the dual system (7) can be interpreted as the time-reverse of the adjoint of the system (1). ◁

Theorem 2.6 extends the state-space interpretation of symmetry [9], [17] from LTI to LTV systems based on the impulse response. Originally, the symmetry concept is introduced via the transfer function matrices for LTI systems [11], [12]. However, for extending the concept to LTV (or even nonlinear) systems, it is more useful to use the corresponding property of the impulse responses.

Remark 2.8: It is not clear whether the differential equation (11) has a unique solution under the boundary conditions (12) and $T(\cdot) = T^\top(-\cdot)$ from its structure. We will develop this in the next section; see Remark 3.5. ◁

Next, we consider the connection between the coordinate transformation and symmetry notion. Similarly to the LTI case, we have the following invariance of the symmetry notion.

Proposition 2.9: The symmetry property is invariant under a Lyapunov coordinate transformation.

Proof: For a Lyapunov coordinate transformation $x = Sz$, denote the representation in the z-coordinates as $(\hat{A}, \hat{B}, \hat{C})$. Under the Lyapunov transformation, the impulse response is invariant [3], and thus $(\hat{A}, \hat{B}, \hat{C})$ and (A, B, C) have the same impulse response. Since the system is symmetric, $(A(\cdot), B(\cdot), C(\cdot))$ and $(A^\top(-\cdot), C^\top(-\cdot), B^\top(-\cdot))$ have the same impulse response. Therefore, it only suffices to show that there exists a Lyapunov coordinate transformation between $(A^\top(-\cdot), C^\top(-\cdot), B^\top(-\cdot))$ and $(\hat{A}^\top(-\cdot), \hat{C}^\top(-\cdot), \hat{B}^\top(-\cdot))$, where the state of the later representation is denoted by λ_z. From $A = (\dot{S} + S\hat{A})S^{-1}$ and $B = S\hat{B}, C = \hat{C}S^{-1}$, we have

$$A^\top(-\cdot) = S^{-\top}(-\cdot)(\dot{S}^\top(-\cdot) + \hat{A}^\top(-\cdot)S^\top(-\cdot))$$

$$= \left(\frac{dS^{-\top}(-\cdot)}{dt} + S^{-\top}(-\cdot)\hat{A}^\top(-\cdot)\right)S^\top(-\cdot)$$

and

$$C^\top(-\cdot) = S^{-\top}(-\cdot)\hat{C}^\top(-\cdot), \ B^\top(-\cdot) = \hat{B}^\top(-\cdot)S^\top(-\cdot).$$

Therefore, the Lyapunov coordinate transformation is $\lambda(\cdot) = S^\top(-\cdot)\lambda_z(\cdot)$. ∎

Remark 2.10: Based on Proposition 2.9, it is possible to show that (A, B, C) is symmetric with respect to T if and only if $(\hat{A}, \hat{B}, \hat{C})$ is symmetric with respect to $S^\top(-\cdot)T(\cdot)S(\cdot)$. ◁

In [15], the symmetry concept is generalized to nonlinear time-invariant systems via controllability/observability operators. For our notion of symmetry, we have similar properties.

Theorem 2.11: Under Assumptions 2.1 and 2.2, the system is symmetric with respect to T if and only if

$$\mathcal{O}_{t_0}x_0 = (\mathcal{F}_{+,t_0}\mathcal{C}^*_{-t_0})T(t_0)x_0, \quad \forall t_0 \in \mathbb{R},$$

$$(\mathcal{O}^*_{t_0}u_a)(t) = T^\top(t_0)(\mathcal{C}_{-t_0}\mathcal{F}_{-,t_0}u_a)(t), \quad \forall t_0 \in \mathbb{R}$$

where $\mathcal{F}_{-,t} : L_2^m[t, \infty) \to L_2^m(-\infty, -t]$ and $\mathcal{F}_{+,t} : L_2^m(-\infty, -t] \to L_2^m[t, \infty)$.

Proof: (Only if part) By using (12) and (15), for the symmetric system, we have

$$\mathcal{O}_{t_0}x_0 = B^\top(-t)\Phi^\top(-t_0, -t)T(t_0)x_0$$

$$= (\mathcal{F}_{+,t_0}\mathcal{C}^*_{-t_0})T(t_0)x_0.$$

$$(\mathcal{O}^*_{t_0}u_a)(t) = T^\top(t_0)\int_{t_0}^\infty \Phi(-t_0, -\tau)B(-\tau)u_a(\tau)d\tau$$

$$= T^\top(t_0)(\mathcal{C}_{-t_0}\mathcal{F}_{-,t_0}u_a)(t).$$

(If part) If $\mathcal{O}_{t_0}x_0 = (\mathcal{F}_{+,t_0}\mathcal{C}^*_{-t_0})T(t_0)x_0$, we have

$$C(t)\Phi(t, t_0) = B^\top(-t)\Phi^\top(-t_0, -t)T(t_0), \ t \geq t_0.$$

When $t = t_0$, we have $C(\cdot) = B^\top(-\cdot)T(\cdot)$, i.e., (12). From this equality and (15)

$$B^\top(-t)T(t)\Phi(t, t_0) = B^\top(-t)\Phi^\top(-t_0, -t)T(t_0), \ t \geq t_0.$$

Since the system is uniformly completely controllable and observable, we have (15), and consequently (11). Therefore, the system is symmetric. ∎

III. Cross Gramians for Linear Time-Varying Symmetric Systems

For LTI systems, the cross Gramian can be seen as a composition of the controllability and observability operators and helps in the analysis of symmetric systems. For instance, its nonsingularity is equivalent to minimality [11], which relates to the fact that if one of the controllability, observability, or cross Gramians is obtained, the other two Gramians can be constructed from it [9]. In this section, we extend the concept of the cross Gramian to LTV systems by using the LTV version of the controllability and observability operators and develop the connection among the three Gramians.

A. Cross Gramians

We define the cross Gramian as the composition of the controllability and observability operators as follows.

Definition 3.1: The cross Gramian W_x is defined as

$$W_x(t) = \int_t^\infty \Phi(-t, -\tau)B(-\tau)C(\tau)\Phi(\tau, t)d\tau, \quad \forall t \in \mathbb{R}$$

i.e., $\mathcal{W}_x(t) := \mathcal{C}_{-t} \circ \mathcal{F}_{-,t} \circ \mathcal{O}_t$ with $\mathcal{W}_x(t) : \mathbb{R}^n \to \mathbb{R}^n$, $v \mapsto W_x(t)v$. ◁

The cross Gramian exists under Assumption 2.2 and boundedness of triplet (A, B, C). In the LTI case, the eigenvalues of the cross Gramian of a symmetric system are invariant under the coordinate transformations. However, this is not always true in the LTV case.

Proposition 3.2: Consider a Lyapunov coordinate transformation $z = Sx$. Let W_x be the cross Gramian for (A, B, C). Then, $S(-\cdot)W_x(\cdot)S^{-1}(\cdot)$ is the cross Gramian of $((\dot{S} + SA)S^{-1}, SB, CS^{-1})$.

Proof: Since the transition matrix of $(\dot{S} + SA)S^{-1}$ is $S(t)\Phi(t, \tau)S^{-1}(\tau), \ t \geq \tau$ [18], the cross Gramian of realization

$((\dot{S} + SA)S^{-1}, SB, CS^{-1})$ is

$$\bar{W}_x(t) = \int_t^\infty (S(-t)\Phi(-t, -\tau)S^{-1}(-\tau))(S(-\tau)B(-\tau))$$

$$\times (C(\tau)S^{-1}(\tau))(S(\tau)\Phi(\tau, t)S^{-1}(t))d\tau$$

$$= \int_t^\infty S(-t)\Phi(-t, -\tau)B(-\tau)C(\tau)\Phi(\tau, t)S^{-1}(t)d\tau$$

$$= S(-t)W_x(t)S^{-1}(t), \qquad \forall t \in \mathbb{R}.$$

∎

From this proposition, it is clear that the cross Gramian is transformed into a new cross Gramian after a coordinate transformation. That is, in contrast to the LTI case, the eigenvalues of the transformed cross Gramian at each $t \in \mathbb{R}$ are different from the original cross Gramian unless $S(-\cdot) = S(\cdot)$. Therefore, the eigenvalues of the cross Gramian for LTV systems depend on the coordinates. Note that not all results for LTI systems can be extended to LTV systems. For instance, the eigenvalues of the A matrix depend on coordinates and are not always helpful for checking stability as demonstrated in [23]. The fact that eigenvalues of the cross Gramian depend on the coordinates is another example of a property that does not hold for LTV systems anymore.

B. Cross Gramians and Symmetric Systems

As shown in the following theorem, similar to the LTI case, if one of the controllability, observability, or cross Gramians of an LTV symmetric system is obtained, the other two Gramians can be constructed from it. Based on this theorem, it is possible to show that minimality of the LTV symmetric system can be studied based on the cross Gramian.

Theorem 3.3: Under Assumptions 2.1 and 2.2, for the symmetric system with respect to T, we have

$$W_x(\cdot) = W_c(-\cdot)T(\cdot) = T^{-\top}(\cdot)W_o(\cdot).$$

Proof: First, we show $W_x(\cdot) = W_c(-\cdot)T(\cdot)$. The symmetric system satisfies (15) and (16) for $T(\cdot) = T^\top(-\cdot)$. By using them, the cross Gramian can be rearranged as follows:

$$W_x(t) = \int_{-\infty}^{-t} \Phi(-t, \tilde{\tau})B(\tilde{\tau})C(-\tilde{\tau})\Phi(-\tilde{\tau}, t)d\tilde{\tau}$$

$$= \int_{-\infty}^{-t} \Phi(-t, \tilde{\tau})B(\tilde{\tau})B^\top(\tilde{\tau})T(-\tilde{\tau})\Phi(-\tilde{\tau}, t)d\tilde{\tau}$$

$$= W_c(-t)T(t), \qquad \forall t \in \mathbb{R}.$$

Next, we have

$$W_x(t) = \int_t^\infty \Phi(-t, -\tau)T^{-\top}(\tau)C^\top(\tau)C(\tau)\Phi(\tau, t)d\tau$$

$$= T^{-\top}(t)W_o(t), \qquad \forall t \in \mathbb{R}.$$

∎

In fact, for LTV systems, there are several Gramians defined, such as controllability, reachability (that we call controllability Gramian in this paper), observability, and constructivity Gramians [5]. By combination of a different pair of Gramians such as controllability and constructivity Gramians, different balanced realizations are defined [5]. Depending on the pairs, one can define different cross Gramians satisfying similar equalities in Theorem 3.3. In this paper, we study the most standard balanced realization.

Since T is a Lyapunov transformation, nonsingularity of the three Gramians are equivalent, which yields the following corollary. This is the LTV version of a well-known LTI result for minimality [11].

Corollary 3.4: Under Assumption 2.2, a symmetric system with continuous and bounded triplet (A, B, C) is uniformly completely controllable if and only if it is uniformly completely observable, or equivalently, if and only if all of the singular values of the cross Gramian W_x are bounded and uniformly positive.   ◁

Remark 3.5: In Remark 2.8, we mentioned uniqueness of a nonsingular solution T to the differential equation (11) under the boundary conditions (12) and $T(\cdot) = T^\top(-\cdot)$. Under Assumptions 2.1 and 2.2, any nonsingular solution satisfies $T(\cdot) = W_c^{-1}(-\cdot)W_x(\cdot)$. This implies uniqueness of the solution T to the differential equation, since both the controllability and cross Gramians uniquely exist and are nonsingular.   ◁

Next, we obtain a different property from the LTI case.

Corollary 3.6: For an LTV symmetric system, we have $W_x(-\cdot)W_x(\cdot) = W_c(\cdot)W_o(\cdot)$.   ◁

Remark 3.7: Differently from the LTI case, $W_x^2 = W_c W_o$ does not hold unless $W_x(\cdot) = W_x(-\cdot)$. That is, the set of singular values of W_x and the Hankel singular values are different in general.   ◁

IV. UNIFORMLY BALANCED TRUNCATION FOR SYMMETRIC SYSTEMS

In this section, first, we show the definition of a uniformly balanced realization studied in this paper. Then, we investigate its properties for symmetric systems to show that the symmetry structure is preserved under uniformly balanced truncation and under reduction via the singular perturbation method if the Hankel singular values are pointwise distinct. Next, our results are extended to a relaxation of uniform balancing that is based on solutions to Lyapunov inequalities.

A. Uniformly Balanced Realization

In this paper, we consider the following uniformly balanced realization.

Definition 4.1 ([3]): The triplet $(\hat{A}, \hat{B}, \hat{C})$ is said to be uniformly balanced if $(\hat{A}, \hat{B}, \hat{C})$ is uniform and if

$$W_c(t) = W_o(t) = \Sigma(t) := \text{diag}\{\sigma_1(t), \ldots, \sigma_n(t)\}$$

for all $t \in \mathbb{R}$.   ◁

Remark 4.2: Under Assumptions 2.1 and 2.2, the system (1) has a uniformly balanced realization with uniformly positive definite and bounded Σ, and its computational method is given in [3].   ◁

We analyze uniformly balanced realization for the symmetric system. Since by Proposition 2.9, the symmetry property is preserved under Lyapunov coordinate transformations, from Theorem 2.6, there exists a Lyapunov transformation $\hat{T}$ satisfying $\hat{T}(\cdot) = \hat{T}^\top(-\cdot)$ such that

$$\dot{\hat{T}}(\cdot) + \hat{T}(\cdot)\hat{A}(\cdot) = \hat{A}^\top(-\cdot)\hat{T}(\cdot) \tag{17}$$

$$\hat{T}(\cdot)\hat{B}(\cdot) = \hat{C}^\top(-\cdot). \tag{18}$$

Moreover, from Theorem 3.3, we have

$$\hat{T}^{-\top}(\cdot)\Sigma(\cdot)\hat{T}^{-1}(\cdot) = \Sigma(-\cdot) \tag{19}$$

in the uniformly balanced coordinates. This $\hat{T}$ has a specific structure.

Lemma 4.3: For a uniformly positive definite, bounded, and diagonal Σ, the matrix $\hat{T}$ in (19) satisfying $\hat{T}(\cdot) = \hat{T}^\top(-\cdot)$ is an orthogonal matrix.

Proof: By substituting $t = -\tilde{t}$ into (19), we obtain

$$\hat{T}^{-\top}(-\cdot)\Sigma(-\cdot)\hat{T}^{-1}(-\cdot) = \Sigma(\cdot).$$

Since $\hat{T}(\cdot) = \hat{T}^\top(-\cdot)$, we have

$$\hat{T}^{-1}(\cdot)\Sigma(-\cdot)\hat{T}^{-\top}(\cdot) = \Sigma(\cdot). \tag{20}$$

By substituting (19) into $\Sigma(-\cdot)$ of (20), we have

$$\hat{T}^{-1}\hat{T}^{-\top}\Sigma\hat{T}^{-1}\hat{T}^{-\top} = \Sigma.$$

On the other hand, for a Lyapunov transformation $\hat{T}$, $\hat{T}^\top\hat{T} = (\hat{T}^\top\hat{T})^\top$ is positive definite. So is its inverse $(\hat{T}^\top\hat{T})^{-1} = \hat{T}^{-1}\hat{T}^{-\top} > 0$. Therefore, it follows that

$$\hat{T}^{-1}\hat{T}^{-\top} = I_n. \tag{21}$$

∎

Theorem 4.4: Under Assumptions 2.1 and 2.2, consider a uniformly balanced realization. The set of eigenvalues of $\Sigma(\cdot)$ and $\Sigma(-\cdot)$ are equal. That is, for any σ_i, there exists σ_j such that $\sigma_i(\cdot) = \sigma_j(-\cdot)$. Moreover, if $\sigma_1 > \cdots > \sigma_n$, then $\Sigma(\cdot) = \Sigma(-\cdot)$, and $\hat{T}$ is a signature matrix.

Proof: From Remark 4.2, (19) and (21), we have

$$\hat{T}(\cdot)\Sigma(\cdot)\hat{T}^{-1}(\cdot) = \Sigma(-\cdot). \tag{22}$$

That is, the set of eigenvalues of $\Sigma(\cdot)$ and $\Sigma(-\cdot)$ are equal. If $\sigma_1 > \cdots > \sigma_n$, then $\sigma_1(-\cdot) > \cdots > \sigma_n(-\cdot)$. Therefore, $\Sigma(\cdot) = \Sigma(-\cdot)$, and (22) becomes $\hat{T}\Sigma\hat{T}^{-1} = \Sigma$, and consequently $\hat{T}\Sigma = \Sigma\hat{T}$. Therefore, $\hat{T}$ should be a diagonal matrix. Since $\hat{T}$ is an orthogonal matrix, each its diagonal element $\hat{t}_i$ $(i = 1, \ldots, n)$ satisfies $\hat{t}_i^2 = 1$. Therefore, $\hat{t}_i = \pm 1$, i.e., $\hat{T}$ is a signature matrix. ∎

Remark 4.5: As commented in Remark 3.7, the set of singular values of the cross Gramian W_x and the Hankel singular values are different in general. However, in the balanced coordinates, they are the same. According to Lemma 4.3 and Theorems 3.3 and 4.4, in the balanced coordinates

$$\hat{W}_x(\cdot) = \Sigma(-\cdot)\hat{T}(\cdot) = \hat{T}^{-\top}(\cdot)\Sigma(\cdot) = \hat{T}(\cdot)\Sigma(\cdot)$$

$$\hat{W}_x(-\cdot) = \Sigma(\cdot)\hat{T}(-\cdot) = \Sigma(\cdot)\hat{T}^\top(\cdot).$$

Thus, $\hat{W}_x^\top(\cdot) = \hat{W}_x(-\cdot)$. Corollary 3.6 implies $\hat{W}_x^\top\hat{W}_x = \Sigma^2$. Moreover, if $\sigma_1 > \cdots > \sigma_n$, then since $\hat{T}$ is a signature matrix, $\hat{W}_x$ is a diagonal matrix whose elements are $\pm\sigma_i$. ◁

B. Uniformly Balanced Truncation

In this subsection, based on analysis of the previous section, we prove that the symmetry structure is preserved under model reduction if the Hankel singular values are pointwise distinct. For model reduction, we partition the system as follows:

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \ \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}, \ \hat{C} = \begin{bmatrix} \hat{C}_1 & \hat{C}_2 \end{bmatrix} \tag{23}$$

where $\hat{A}_{11} : \mathbb{R} \to \mathbb{R}^{k \times k}$, $\hat{B}_1 : \mathbb{R} \to \mathbb{R}^{k \times m}$, and $\hat{C}_1 : \mathbb{R} \to \mathbb{R}^{m \times k}$. Then, the truncated system is given by

$$\begin{cases} \dot{x}_r = \hat{A}_{11}x_r + \hat{B}_1 u \\ y_r = \hat{C}_1 x_r. \end{cases} \tag{24}$$

We next prove that the symmetry property is preserved under balanced truncation if the Hankel singular values are distinct.

Theorem 4.6: Under Assumptions 2.1 and 2.2, a uniformly balanced truncated model (24) of an LTV symmetric system is again symmetric if $\sigma_1 > \cdots > \sigma_n$.

Proof: If $\sigma_1 > \cdots > \sigma_n$, then from Theorem 4.4, $\hat{T}$ is a signature matrix, i.e., a constant matrix. Therefore, (17) and (18), respectively, reduce to

$$\hat{T}\hat{A}(\cdot) = \hat{A}^\top(-\cdot)\hat{T}, \ \hat{T}\hat{B}(\cdot) = \hat{C}^\top(-\cdot).$$

We denote the diagonal matrix as $\hat{T} = \mathrm{diag}\{\hat{T}_1, \hat{T}_2\}$ with suitable diagonal matrices $\hat{T}_1$ and $\hat{T}_2$. From (23), we have

$$\begin{bmatrix} \hat{T}_1\hat{A}_{11}(\cdot) & \hat{T}_1\hat{A}_{12}(\cdot) \\ \hat{T}_2\hat{A}_{21}(\cdot) & \hat{T}_2\hat{A}_{22}(\cdot) \end{bmatrix} = \begin{bmatrix} \hat{A}_{11}^\top(-\cdot)\hat{T}_1 & \hat{A}_{21}^\top(-\cdot)\hat{T}_2 \\ \hat{A}_{12}^\top(-\cdot)\hat{T}_1 & \hat{A}_{22}^\top(-\cdot)\hat{T}_2 \end{bmatrix},$$

$$\begin{bmatrix} \hat{T}_1\hat{B}_1(\cdot) \\ \hat{T}_2\hat{B}_2(\cdot) \end{bmatrix} = \begin{bmatrix} \hat{C}_1^\top(-\cdot) \\ \hat{C}_2^\top(-\cdot) \end{bmatrix}$$

and consequently

$$\hat{T}_1\hat{A}_{11}(\cdot) = \hat{A}_{11}^\top(-\cdot)\hat{T}_1, \ \hat{T}_1\hat{B}_1(\cdot) = \hat{C}_1^\top(-\cdot).$$

Since $\hat{T}_1$ is a signature matrix, the reduced order model is also symmetric. ∎

Next to model reduction by truncation, also model reduction by singular perturbation can be considered, i.e., consider the reduced order system obtained by singular perturbation as follows:

$$\begin{cases} \dot{x}_s = (\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})x_s + (\hat{B}_1 - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{B}_2)u \\ y_s = (\hat{C}_1 - \hat{C}_2\hat{A}_{22}^{-1}\hat{A}_{21})x_s - \hat{C}_2\hat{A}_{22}^{-1}\hat{B}_2 u. \end{cases} \tag{25}$$

The reduced order model obtained by singular perturbation also preserves symmetry. The proof follows from direct computations and thus is omitted.

Theorem 4.7: Under Assumptions 2.1 and 2.2, a reduced order model (25) obtained by singular perturbation of an LTV symmetric system is again symmetric[1] if $\sigma_1 > \cdots > \sigma_n$.

Remark 4.8: The condition $\sigma_1 > \cdots > \sigma_n$ in Theorems 4.6 and 4.7 can be weakened as

$$\inf_{t \in \mathcal{T}}\{\sigma_1(t), \ldots, \sigma_k(t)\} > \sup_{t \in \mathcal{T}}\{\sigma_{k+1}(t), \ldots, \sigma_n(t)\} \tag{26}$$

for some interval $\mathcal{T} \subset \mathbb{R}$. In any interval $\mathcal{T}$ satisfying (26), a reduced order model (24) or (25) is symmetric. The computations are technically more involved but otherwise the proofs are similar. In this case, $\hat{T}$ is not necessarily a signature matrix; see [10] in the LTI case. ◁

C. Generalized Balanced Realization

In [6], instead of Lyapunov equations (4) and (5), Lyapunov inequalities are employed for balanced truncation of LTV systems. Model reduction based on these solutions is referred to as generalized balanced truncation. In this subsection, we show that generalized balanced truncation with a specific pair of solutions preserves the symmetry structure.

Consider bounded, symmetric, and uniformly positive definite solutions $\bar{W}_c$ and $\bar{W}_o$ to the following Lyapunov inequalities:

$$-\dot{\bar{W}}_c + \bar{W}_c A^\top + A\bar{W}_c \leq -BB^\top \tag{27}$$

$$\dot{\bar{W}}_o + \bar{W}_o A + A^\top\bar{W}_o \leq -C^\top C \tag{28}$$

where $\lim_{t \to -\infty}\bar{W}_c(t) = 0$ and $\lim_{t \to \infty}\bar{W}_o(t) = 0$. Each solution is called a generalized controllability/observability Gramian. Note that controllability and observability Gramians (2) and (3) are specific generalized Gramians. By using a pair of generalized Gramians, generalized balanced truncation can be achieved in a similar manner as the standard case.

We investigate properties of generalized Gramians and balanced truncation for LTV symmetric systems. First, we show that generalized controllability and observability Gramians can be constructed from one another.

---

[1]The definition of a symmetric system is easily extended to systems with a feedthrough term Du by additionally requiring that $D(\cdot) = D^\top(-\cdot)$.

Theorem 4.9: Consider a symmetric system with respect to T. Let $\bar{W}_c(\cdot)$ be a solution to (27). Then, $T^\top(\cdot)\bar{W}_c(-\cdot)T(\cdot)$ is a solution to (28). Conversely, if $\bar{W}_o(\cdot)$ is a solution to (28), then $T^{-1}(\cdot)\bar{W}_o(-\cdot)T^{-\top}(\cdot)$ is a solution to (27).

Proof: Since the proofs are similar, we only show the first part. By substituting $t = -\tilde{t}$ into (27), we have

$$\frac{d\bar{W}_c(-\cdot)}{d\tilde{t}} + \bar{W}_c(-\cdot)A^\top(-\cdot) + A(-\cdot)\bar{W}_c(-\cdot)$$
$$\leq -B(-\cdot)B^\top(-\cdot).$$

By premultiplying $T^\top$ and postmultiplying T by (27)

$$T^\top(\cdot)\frac{d\bar{W}_c(-\cdot)}{d\tilde{t}}T(\cdot) + T^\top(\cdot)\bar{W}_c(-\cdot)A^\top(-\cdot)T(\cdot)$$
$$+ T^\top(\cdot)A(-\cdot)\bar{W}_c(-\cdot)T(\cdot) \leq -T^\top(\cdot)B(-\cdot)B^\top(-\cdot)T(\cdot)$$

where we used the fact that positive semidefiniteness of W and $T^\top W T$ are equivalent for Lyapunov transformation T. Based on Theorem 2.6, the inequality can be rearranged as

$$\frac{d(T^\top(\cdot)\bar{W}_c(-\cdot)T(\cdot))}{d\tilde{t}} + T^\top(\cdot)\bar{W}_c(-\cdot)T(\cdot)A(\cdot)$$
$$+ A^\top(\cdot)T^\top(\cdot)\bar{W}_c(-\cdot)T(\cdot) \leq -C^\top(\cdot)C(\cdot).$$

This is nothing but (28) for $\bar{W}_o(\cdot) = T^\top(\cdot)\bar{W}_c(-\cdot)T(\cdot)$. ∎

This theorem is an extension of Theorem 3.3. Thanks to this generalization, the results for uniformly balanced truncation such as Theorems 4.6 and 4.7 can readily be extended. That is, the symmetry structure is preserved under generalized balanced truncation as follows. Since the proof is similar to Theorem 4.6, we omit it. Also, Remark 4.8 is applicable.

Theorem 4.10: Under Assumptions 2.1 and 2.2, consider a symmetric system with respect to T. Let $\bar{W}_c$ be a bounded, symmetric, and uniformly positive definite solution to (27). Suppose that the eigenvalues $\bar{\sigma}_i$, $i = 1, \ldots, n$ of $\bar{W}_c(\cdot)T^\top(\cdot)\bar{W}_c(-\cdot)T(\cdot)$ satisfy $\bar{\sigma}_1 > \cdots > \bar{\sigma}_n$. Then, a reduced order model of the symmetric system obtained by generalized balanced truncation with $\bar{W}_c(\cdot)$ and $\bar{W}_o(\cdot) = T^\top(\cdot)\bar{W}_c(-\cdot)T(\cdot)$ is again symmetric.

D. Academic Example

In this subsection, we give an academic example to demonstrate our results. Consider the following LTV system:

$$Aw(t) = \begin{bmatrix} a_{1,1}(t) & a_{1,2}(t) & -4/3 \\ 0 & -1+(4/3)\cos(t) & -4/3 \\ 0 & a_{3,2}(t) & -2-(4/3)\cos(t) \end{bmatrix}$$

$$a_{1,1}(t) = (-\cos^2(t)+5)/(-20+8\cos(t))$$

$$a_{1,2}(t) = (35\cos^2(t)-104\cos(t)+45)/(-60+24\cos(t))$$

$$a_{3,2}(t) = -(4/3)\sin^2(t)-\sin(t)+\cos(t)$$

$$B(t) = \begin{bmatrix} 2 & 4+\sin(t) \\ 2 & 0 \\ 2+\cos(t) & 0 \end{bmatrix}$$

$$C(t) = \begin{bmatrix} 1-(1/2)\sin(t) & -1+(1/2)\sin(t) & 0 \\ 0 & 2-2\cos(t) & 2 \end{bmatrix}.$$

This system is symmetric with respect to matrix T

$$T(t) = \begin{bmatrix} 1/4 & -1/4 & 0 \\ -1/4 & 5/4+\cos^2(t) & -\cos(t) \\ 0 & -\cos(t) & 1 \end{bmatrix}$$

i.e., (11) and (12) hold for $T(\cdot) = T^\top(-\cdot)$.

For this system, the cross Gramian W_x is obtained as

$$W_x(t) = \begin{bmatrix} 10-4\cos(t) & -8+4\cos(t) & 0 \\ 0 & 2 & 0 \\ 0 & \cos(t) & 1 \end{bmatrix}.$$

According to Theorem 3.3, by using the cross Gramian, we construct the controllability and observability Gramians

$$W_c(t) = W_x(-t)T^{-1}(-t)$$

$$= \begin{bmatrix} 42-16\cos(t) & 2 & 2\cos(t) \\ 2 & 2 & 2\cos(t) \\ 2\cos(t) & 2\cos(t) & 1+2\cos^2(t) \end{bmatrix}$$

$$W_o(t) = T^\top(t)W_x(t)$$

$$= \begin{bmatrix} 5/2-\cos(t) & -5/2+\cos(t) & 0 \\ -5/2+\cos(t) & 9/2-\cos(t)+\cos^2(t) & -\cos(t) \\ 0 & -\cos(t) & 1 \end{bmatrix}.$$

Based on these controllability and observability Gramians, we can compute the coordinate transformation $x = Sz$ for the uniformly balanced realization, where

$$S(t) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & \cos(t) & 1 \end{bmatrix}.$$

In the uniformly balanced coordinate, the triplet $(\hat{A}, \hat{B}, \hat{C})$ is

$$\hat{A}(t) = \begin{bmatrix} \frac{5-\cos^2(t)}{-20+8\cos(t)} & 0 & 0 \\ 0 & -1 & -4/3 \\ 0 & -4/3 & -2 \end{bmatrix}$$

$$\hat{B}^\top(t) = \hat{C}(t) = \begin{bmatrix} 0 & 2 & 2 \\ 2+\sin(t) & 0 & 0 \end{bmatrix}.$$

This system is symmetric with respect to $\hat{T} = I_3$. From Remark 4.5, the controllability, observability, and cross Gramians are equivalent in these coordinates. These Gramians are

$$\Sigma(t) = \text{diag}\{10-4\cos(t), 2, 1\}$$

where $6 \leq 10-4\cos(t) \leq 14$. According to Theorem 4.6, reduced order models are symmetric, which is clear from the structure of $(\hat{A}, \hat{B}, \hat{C})$.

Remark 4.11: In the above-mentioned example, we are able to provide Gramians. However, in general computing the Gramians analytically is hard. In contrast, the sliding-interval (SI) and finite-interval (FI) balanced realizations can be computed numerically. In the SI case, δ in (2) and (3) is taken as the same positive constant. Even though there exists an additional variable δ, the results of our paper can readily be extended to the SI case. Extensions to the FI case, i.e., when $\delta = t - t_-$, $t_- \geq 0$ in (2) and $\delta = t_+ > t_-$ in (3) is more technically involved but

doable by using the dual system with respect to $t_- + t_+$ [16]. For FI balancing of nonnecessarily symmetric systems, a numerical computation method is developed [7]. For symmetric systems, its numerical complexity may be reduced, since one of the solutions to the differential Lyapunov or Sylvester equation is sufficient for balanced truncation.◁

Remark 4.12: Dealing with model reduction of unstable LTI systems, linear quadratic gaussian (LQG) balancing is proposed, which is related to balancing with normalized coprime factorizations in [24]. Under Assumption 2.1, LQG balancing can be extended to LTV systems. Then, one can show that this extended balancing preserves the symmetry structure in a similar manner as for the uniformly balancing for stable systems. A similar statement holds for $\mathcal{H}_\infty$ balancing [25].◁

V. CONCLUSION

In this paper, we presented a symmetry concept and cross Gramian for LTV systems. For LTV symmetric systems, uniformly balanced realization is obtained by computing one of the controllability, observability, and cross Gramians. That is, we do not need to solve two differential Lyapunov equations but only one of the differential Lyapunov or Sylvester equation. By uniformly balanced truncation, the symmetry structure is preserved if the Hankel singular values are distinct.

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