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Asymptotic Output-Feedback Stabilization of Linear Evolution Equations with Uncertain Inputs via Equivalent Control Method

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Abstract— **The paper studies the problem of minimax control design for linear evolution equations in Hilbert spaces with measurement noise and additive exogenous disturbances. The key result of the paper is an algorithm, generating a control in an output-feedback form, which steers the state of the system as close as possible to a given sliding hyperplane, asymptotically as time goes to infinity. The control is designed in the state space of the minimax filter, and guarantees that the state of the filter will be exactly on the sliding surface, and the state of the plant will belong to an ellipsoid centered at the filter's state vector for large enough time. The optimality of the designed feedback and estimation error is proven. The feedback is represented by means of the unique solution of an algebraic Riccati equation. The theory is then applied to design a minimax control for linear hereditary systems subject to noise and disturbances. This is achieved by projecting the hereditary system onto a finite dimensional subspace of the corresponding state space by means of a finite-volume approximation method, designing feedback in the state space of the resulting finite dimensional system. The solution of the operator Riccati equation is obtained using a (modified) Kleinman-Newton method. The efficacy of the proposed algorithm is illustrated by a numerical example for a time-delay linear systems with constant point delays.**

I. INTRODUCTION

The control engineering practice always looks for simple, precise and robust output-based feedback control algorithms. Being introduced in the 1960s, the sliding mode (SM) method is the first robust model-based control methodology allowing, at least theoretically, to reject a certain class of so-called matched uniformly bounded perturbations provided that the control magnitude is sufficiently high (see, e.g. [3]). The output-based SM control design methodology is well-developed for finite dimensional systems (see, for example, [1], [2], [3], [4] and references therein). Infinite dimensional (distributed parameter) systems are widely used in control engineering, e.g., to model flexible robots, controlled turbulent flows, combustion and other chemical processes. Since the 1980s it also is known that theoretically the SM methodology can also be utilized to design controllers for such complicated systems [5], [6]. We refer the reader to [7], [8], [9], [10], [11], [12] for an extensive overview of the achievements in this field. The recent practical application of SM principles in turbulent flow control problems shows its high efficiency and robustness (see, https://www.youtube.com/watch?v=b5NnAV2qeno) even in the case of large perturbations and measurement noise. The mathematical model utilized for control design in the latter case is infinite dimensional due to state delays [41]. Necessity of further developments of active flow control systems motivate the research activities in the field of output-based sliding mode control technique for infinite dimensional systems.

The sliding mode control methodology stands on the following two step design procedure. First, an appropriate (sliding) manifold should be selected in the state space of the system such that enforcing the system motion in the sliding manifold implies a required behaviour (e.g. set-point tracking). Next, a feedback control law steering all trajectories of the system to the sliding manifold must be synthesized using some well-developed procedures [2], [3], [4]. The conventional sliding mode algorithms assume exact measurement of the system state or at least the output. In practice, it is quite difficult to apply the state of the art SM methods in the case of noisy measurements (see, [13], [14]) and/or mismatched disturbances (see, [15], [16], [17]). The solution of the classical SM control problem does not exists in this case, i.e., it is impossible to ensure the ideal/exact sliding mode (even in the finite dimensional case) due to the noise in the measurements. This paper deals with a mathematically sound extension of the sliding mode control methodology allowing one to treat the aforementioned cases efficiently. Following [18], [19], [20] we propose to generalize the notion of the solution of the classical sliding model control problem for linear evolution equations, i.e., to construct a control law keeping the state's motion as close as possible (in the minimax sense) to the selected sliding surface. In [21] it has been proven that for ODE models the conventional (first order) sliding mode control methodology remains consistent provided that the feedback control law is designed in terms of the minimax observer variables.

In this paper, we consider sliding mode control principles and study the problem of observer-based sliding mode control design for a plant described by a linear evolution equation in a Hilbert space with additive exogenous disturbances and L^2-bounded deterministic measurement noises. To develop the SM controller, we first provide a dual description of the reachability set of a linear evolution equation, and then solve the following minimax control problem: find a feedback control steering the minimax center of the reachability set towards the sliding surface. The dual description of the reachability set relies upon the minimax framework [22], [23], [24], [25] and a duality argument [26],

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[27]. A solution to the mentioned minimax control problem on a finite horizon has been derived in [28] by means of the equivalent control method [13] applied to the observer equation. The practical implementation of this result is a non-trivial task since obtaining the feedback law requires solution of a differential operator Riccati equation. For the infinite dimensional case, online solution of such equations with high precision is infeasible, at least with the current state of the art. To overcome this difficulty, the present paper studies the problem on an infinite horizon, requiring only *algebraic* operator Riccati equations to be solved; the minimax-optimal reaching of the sliding surface is then only guaranteed asymptotically, as t goes to infinity. A solution to the algebraic operator equation within required precision can be found offline and used in the controller. Once additional assumptions (convergence, dual convergence, uniform stabilizability and uniform detectability) are imposed, we present how one can apply the output-stabilization method proposed in this paper to hereditary systems. The presented approach uses techniques similar to [37], [34]. The solution of the operator Riccati equation is obtained using a finite volume approximation method (known as "AVE") and the (modified) Kleinman-Newton method proposed in [32]. To shorten the exposition, we restrict to the particular case of delayed differential equations and illustrate the theory by working out a numerical example.

Notation. Given an abstract Hilbert space H we denote by $\langle\cdot,\cdot\rangle_H$ its inner product with values in $\mathbb{R}$, and set $\|x\|_H^2 := \langle x,x\rangle_H$ for any $x \in H$. The space of all linear continuous operators from Hilbert space H_1 to Hilbert space H_2 is denoted by $\mathscr{L}(H_1,H_2)$, and $[x,y]$ is an element of $X\times Y$, the Cartesian product of two Hilbert spaces X and Y. We let I denote the identity operator, and $\mathbf{1}_S(\zeta)$ the indicator function of the set S, which is 1 if $\zeta \in S$ and equal to 0 else. Throughout, we attempt to let calligraphic letters denote operators on infinite dimensional spaces, whereas regular letters denote operators on finite dimensional spaces and their matrix representations. For a point $p \in X$ and a set $S \subseteq X$, we let $p+S=\{x=p+s : s\in S\}$ denote the shift of S by p.

Outline. This paper is organized as follows: the formal problem statement is given in Section II; control design is described in Section III; approximation of the algebraic Riccati equation and the control design for hereditary systems are described in Sections IV and V, respectively. Finally, Section VI contains the numerical example.

II. Problem statement

Assume that $\mathcal{A}: H\to H$ generates a strongly continuous semigroup and $t\mapsto x(t)\in H$ is the mild solution (see [35, Def. 3.1.4]) of the following linear evolution equation:

$$\frac{dx}{dt} = \mathcal{A}x(t) + \mathcal{B}u(t) + \mathcal{D}d(t)\,, x(0)=x_0\,, \tag{1}$$

where $x_0 \in H$ is the initial condition, and $u \in L^2_{loc}(0,+\infty,H_u,)$ the control input, $d\in L^2(0,+\infty,H_d)$ an uncertain disturbance, and $\mathcal{B}\in\mathscr{L}(H_u,H)$, $\mathcal{D}\in\mathscr{L}(H_d,H)$ are given operators. The output of (1), $y(t)\in H_y$ is given by

$$y(t) = \mathcal{C}x(t) + w(t)\,, \tag{2}$$

where $\mathcal{C} \in \mathscr{L}(H,H_y)$ is the observation operator and $w\in L^2(0,+\infty,H_y)$ is unknown deterministic measurement noise. We further assume that $[x_0,d,w] \in \mathscr{E}_\infty$ are some elements of the set $\mathscr{E}_\infty := \bigcap_{T>0}\mathscr{E}(T)$, where

$$\mathscr{E}(T) := \{[x_0,d,w] : \rho_T(x_0,d,w;\mathcal{S},\mathcal{Q},\mathcal{R})\le 1\} \tag{3}$$

and

$$\rho_t(x,d,w;\mathcal{S},\mathcal{Q},\mathcal{R}) := \langle \mathcal{S}x,x\rangle_H + \int_0^t \langle \mathcal{Q}d(s),d(s)\rangle_{H_d}\mathrm{d}s + \int_0^t \langle \mathcal{R}w(s),w(s)\rangle_{H_y}\mathrm{d}s,$$

and $\mathcal{S},\mathcal{Q},\mathcal{R}$ are given self-adjoint positive definite bounded linear operators in H, H_d and H_y respectively with bounded inverse operators; ρ_T defines a new norm in the space $H\times L^2(0,T,H_d)\times L^2(0,T,H_y)$, and $\mathscr{E}(T)$ represents the unit ball of this space w.r.t. ρ. In what follows we suppose that H_u and H_y are abstract Hilbert spaces.

The aim of this paper is: given a finite-rank linear operator $\mathcal{F}: H\to H_u$ defining a sliding surface $\{x : \mathcal{F}x = 0\}\subset H$, to construct an *output* feedback control law $u\in L^2_{loc}(0,+\infty,H_u)$ minimizing the infinite-horizon worst-case deviation $\|\mathcal{F}x\|_{H_u}$ from the sliding surface. Formally, for

$$\tilde J(u) := \lim_{T\to+\infty}\sup_{[x_0,d,w]\in\mathscr{E}(T)} \|\mathcal{F}x(T)\|_{H_u} \quad \text{s.t. (1) and (2) hold.} \tag{4}$$

we find the minimax control $u\in L^2_{loc}(0,+\infty,H_u)$ such that for every $u'\in L^2_{loc}(0,+\infty,H_u)$:

$$\tilde J(u) \le \tilde J(u'), \tag{5}$$

provided $\mathcal{F}\mathcal{B}$ is in $\mathscr{L}(H_u,H_u)$ and invertible.

This problem statement extends the results of [18], [19], [20], [21], [28] on sliding mode control design for uncertain finite-dimensional LTI systems to the case of abstract evolution equations on an infinite horizon.

III. Main results

The results of this paper are based on the following simple geometric idea. The semigroup of (1) transforms the ellipsoid $\mathscr{E}(T)$ into another ellipsoid $\mathscr{R}(T)$ in H: for each $T>0$ the ellipsoid $\mathscr{R}(T)$ contains all the states $x(T)$ which are compatible with the measured output $y(t)$ on $[0,T]$, and that correspond to some $[x_0,d,w]\in\mathscr{E}(T)$. Note that the following representation holds: $\mathscr{R}(T) = \hat x(T) + \{z\in H : \langle \mathcal{P}(T)z,z\rangle \le 1\}$. Here $\hat x(T)$ represents the center of $\mathscr{R}(T)$, and a linear self-adjoint positive definite operator $\mathcal{P}(T)$ defines the axes of $\mathscr{R}(T)$. For a class of $\mathcal{A},\mathcal{D}$ and $\mathcal{C}$, the ellipsoid $\mathscr{R}(T)$ converges to some ellipsoid $\mathscr{R}^\infty = x^\infty(T) + \{z\in H : \langle \mathcal{P}^\infty z,z\rangle \le 1\}$ for $T\to\infty$ in that the center of $\mathscr{R}(T)$ approaches the center of $\mathscr{R}^\infty$, $\lim_{T\to+\infty}\|x^\infty(T)-\hat x(T)\| = 0$, and $\mathcal{P}(T)$ converges to

$\mathcal{P}^\infty$. The idea is to design the control u so that the center of $\mathscr{R}^\infty$ belongs to the null-space of $\mathcal{F}$: $\mathcal{F}x^\infty(T) = 0$. In this case, for large enough $T > 0$ the distance between every state of (1) and the null-space of $\mathcal{F}$ will be bounded by the norm of $\mathcal{P}^\infty$. In other words, for large enough $T > 0$ the states of (1) will belong to an ellipsoid with center within the null-space of $\mathcal{F}$ and axes defined by the eigenvectors of $\mathcal{P}^\infty$. This idea is formalised in the following proposition.

Definition 1: A pair $(\mathcal{A}, \mathcal{D})$ is said to be *$\mathcal{C}$-stabilizable* if for every $x_0 \in H$ there exists $v \in L^2(0, +\infty, H_d)$ such that $\mathcal{C}x(\cdot; x_0, v) \in L^2(0, +\infty, H_y)$ where $x(\cdot; x_0, v)$ denotes the unique mild solution of $\dot{x} = \mathcal{A}x + \mathcal{D}v$, $x(0) = x_0$. If $\mathcal{C}$ equals identity then $(\mathcal{A}, \mathcal{D})$ is said to be *stabilizable*.

Theorem 1: Assume that $(\mathcal{A}^\star, \mathcal{C}^\star)$ is $\mathcal{D}^\star$-stabilizable, $x(t)$ is the mild solution of (1), $y(t)$ is as in (2), and let $\mathcal{P}^\infty$ be a bounded non-negative self-adjoint solution of the algebraic Riccati equation

$$\mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^\star + \mathcal{D}\mathcal{Q}^{-1}\mathcal{D}^\star - \mathcal{P}\mathcal{C}^\star\mathcal{R}\mathcal{C}\mathcal{P} = 0. \tag{6}$$

Then let $x^\infty(t)$ be the mild solution of the observer equation

$$\frac{dx^\infty}{dt} = (\mathcal{A} - \mathcal{P}^\infty\mathcal{C}^\star\mathcal{R}\mathcal{C})x^\infty(t) + \mathcal{P}^\infty\mathcal{C}^\star\mathcal{R}y + \mathcal{B}u(t), \tag{7}$$

$x^\infty(0) = 0$. If $\mathcal{A} - \mathcal{P}^\infty\mathcal{C}^\star\mathcal{R}\mathcal{C}$ generates an exponentially stable semigroup, and $\mathcal{S}^{-1} \geq \mathcal{P}^\infty$ then:

A) $\forall l \in H$ it holds that

$$\lim_{t\to+\infty} \sup_{[x_0,d,w]\in\mathscr{E}(t)} |\langle l, x(t) - x^\infty(t)\rangle_H| = \langle l, \mathcal{P}^\infty l\rangle_H^{\frac{1}{2}}. \tag{8}$$

B) any control u such that

$$\hat{\sigma}(T) := \mathcal{F}x^\infty(T) = 0 \text{ for almost all } T \in (0, +\infty) \tag{9}$$

solves the minimax control problem (5).

Remark 1: The worst-case error $\langle l, \mathcal{P}^\infty l\rangle_H^{\frac{1}{2}}$ is also the best achievable by any filter/state estimate (in the form of a continuous functional of observations) on an infinite time horizon. This is shown in e.g. [21].

Proof sketch: From results in [29], we establish $\mathcal{P}^\infty$ as the limit as $t \to \infty$ of $\mathcal{P}(\cdot; \mathcal{P}_0)$ solving

$$\frac{d\mathcal{P}}{dt} = \mathcal{A}\mathcal{P} + \mathcal{P}\mathcal{A}^\star + \mathcal{D}\mathcal{Q}^{-1}\mathcal{D}^\star - \mathcal{P}\mathcal{C}^\star\mathcal{R}\mathcal{C}\mathcal{P}, \mathcal{P}(0) = \mathcal{P}_0.$$

Statement *A)* is then obtained from results for finite t in [28] by letting $t \to \infty$. For statement *B)*, we exhibit the worst-case $[x_0, d, w] \in \mathscr{E}(T)$ and show that (9) guarantees minimal $\tilde{J}$ for this worst case. The detailed proof is contained in a preprint posted on arXiv. ■

Corollary 1: If $(\mathcal{A}^\star, \mathcal{C}^\star)$ is $\mathcal{D}^\star$-stabilizable, and $(\mathcal{A}, \mathcal{D})$ is stabilizable then $\mathcal{A} - \mathcal{P}^\infty\mathcal{C}^\star\mathcal{R}\mathcal{C}$ generates an exponentially stable semigroup, and all the claims of Theorem 1 hold true if $\mathcal{S}^{-1} \geq \mathcal{P}^\infty$.

Corollary 2: The feedback control law

$$u(t) = u_{eq}(t) := - (\mathcal{F}\mathcal{B})^{-1}\mathcal{F}\left[\mathcal{A}x^\infty(t) + \mathcal{P}^\infty\mathcal{C}^\star\mathcal{R}(y(t) - \mathcal{C}x^\infty(t))\right] \tag{10}$$

is minimax optimal, i.e. it satisfies (5).

Proof: For any $v \in H_u$ we obtain

$$\langle \mathcal{F}x^\infty(t), v\rangle_{H_u} = \int_0^t \langle \mathcal{F}\frac{dx^\infty}{dt}(s), v\rangle_{H_u} \mathrm{d}s = 0,$$

where the last equality follows from inserting (10) into (7) and a bit of algebra. Hence, $\langle \mathcal{F}x^\infty(t), v\rangle_{H_u} = 0$ $\forall v \in H_u$ as required by condition (9) of Theorem 1. ■

Remark 2: The problem (1)-(5) assumes that the initial condition belongs to an ellipsoid centred at the origin. If the initial condition in (1) is given by $x(0) = x_0 + x^*$, where $x^* \in H$ is such that $\mathcal{F}x^* = 0$, Theorem 1 and Corollaries 1 and 2 remain true provided that the initial condition in (7) is $x(0) = x^*$.

IV. Approximation of the Solution of the Operator Riccati Equation

In order to be able to implement the output-feedback stabilization method proposed in this paper, one needs to approximate $\mathcal{P}^\infty$, the solution of the operator Riccati equation

$$\mathcal{P}\mathcal{A}^\star + \mathcal{A}\mathcal{P} + \mathcal{D}\mathcal{D}^\star - \mathcal{P}\mathcal{C}^\star\mathcal{C}\mathcal{P} = 0\,. \tag{11}$$

This equation can be obtained from (6) by setting $\mathcal{D} := \mathcal{D}\mathcal{Q}^{-\frac{1}{2}}$, $\mathcal{C} := \mathcal{R}^{\frac{1}{2}}\mathcal{C}$. To simplify the presentation we set $\mathcal{Q} = I$ and $\mathcal{R} = I$. The approach taken in this section to approximate $\mathcal{P}^\infty$ is based on the framework presented in [38].

Consider the system (1) and (2) with parameters $(H, \mathcal{A}, \mathcal{D}, \mathcal{C})$. Let $(H^N)_N$ be a sequence of subspaces of H with finite dimensions, $(\Pi^N)_N$ the corresponding sequence of orthogonal projections $\Pi^N : H \to H^N$ satisfying $\lim_{N\to\infty} \|\Pi^N x - x\| = 0$, $\forall x \in H$. Let also $(A^N)_N$, $(D^N)_N$ and $(C^N)_N$ be the sequences of approximating linear and bounded operators where $A^N : H^N \to H^N$, $D^N : H_d \to H^N$ and $C^N : H^N \to H_y$. Denote by $T^N(t)$ the semigroup generated by A^N. (H^N, A^N, D^N, C^N) is the N-th approximating system for $(H, \mathcal{A}, \mathcal{D}, \mathcal{C})$ with the corresponding Riccati equation

$$P^N(A^N)^\star + A^N P^N + D^N(D^N)^\star - P^N(C^N)^\star C^N P^N = 0\,. \tag{12}$$

For the existence of a sequence $(P^N)_N$ of unique and non-negative solutions of (12) such that $(P^N\Pi^N)_N$ converges strongly to $\mathcal{P}^\infty$, the unique non-negative solution of the Riccati equation (11), it is sufficient to have that the following convergence, dual convergence, uniform stabilizability and uniform detectability assumptions are satisfied.

Assumption 1 (convergence and dual convergence): For every $x \in H$, $y \in H_y$, and $d \in H_d$:

(i) $T^N(t)\Pi^N x \to T(t)x$ uniformly in t on bounded subintervals of $[0, \infty)$,
(ii) $(T^N(t))^\star\Pi^N x \to T^\star(t)x$ uniformly in t on bounded subintervals of $[0, \infty)$,
(iii) $C^N\Pi^N x \to \mathcal{C}x$,
(iv) $(C^N)^\star y \to \mathcal{C}^\star y$,
(v) $(D^N)d \to \mathcal{D}d$, and

(vi) $(D^N)^\star \Pi^N x \to \mathcal{D}^\star x$.

Assumption 2 (uniform stabilizability and detectability):

(vii) $(A^N, C^N)_N$ is uniformly detectable and

(viii) $(A^N, D^N)_N$ is uniformly stabilizable.

The following result follows from [38] by duality.

Theorem 2: Suppose that Assumptions 1 and 2 hold. If $(\mathcal{A}^\star, \mathcal{C}^\star)$ is stabilizable and $(\mathcal{A}^\star, \mathcal{D}^\star)$ detectable, then the sequence $(P^N)_N$ of unique and non-negative solutions of (12) converges strongly to $\mathcal{P}^\infty$, the unique non-negative solution of the Riccati equation (11). Moreover, $(T^N(t))_N$ converges strongly to $T(t)$.

Note that if the operators $\mathcal{D}$ and $\mathcal{C}$ are taken to be compact, then one can obtain convergence of the sequence $(P^N)_N$ to $\mathcal{P}^\infty$ in uniform operator topology (i.e. convergence in the operator norm) – see for example [34, Thm. 7.6] and the references therein. Conditions of Theorem 2 coincide with the conditions of Corollary 1 hence the approximation method described above is in full agreement with Theorem 1.

V. Output-Feedback Stabilization for Hereditary Systems

In this section we apply the proposed method to hereditary differential systems:

$$\begin{aligned} \dot{z}(t) &= A_0 z(t) + A_1 z(t-h) + \int_{-h}^{0} A(\theta) z(t+\theta) \mathrm{d}\theta \\ &\quad + B_0 u(t) + D_0 d(t), \quad t \geq 0, \\ z(0) &= r, \\ z(\theta) &= f(\theta), \quad -h \leq \theta < 0, \\ y(t) &= C_0 x(t) + w(t), \end{aligned} \tag{13}$$

where $h > 0$ represents the time delay, $A_1 \in \mathbb{R}^{n\times n}$, $B_0 \in \mathbb{R}^{n\times m}$, $D_0 \in \mathbb{R}^{n\times d}$, $C_0 \in \mathbb{R}^{m\times n}$, $r, z(t) \in \mathbb{R}^n$, $f \in L^2(-h, 0; \mathbb{R}^n)$ and $u \in L^2(0, h; \mathbb{R}^m)$. Hereditary systems can be represented in the form (1), (2) (see [37] for details). To shorten the presentation, we restrict ourselves to delayed differential equations with point-delays only, namely:

$$\begin{aligned} \dot{z}(t) &= A_0 z(t) + A_1 z(t-h) + B_0 u(t) + D_0 d(t), \quad t \geq 0, \\ z(0) &= r, \\ z(\theta) &= f(\theta), \quad -h \leq \theta < 0, \\ y(t) &= C_0 x(t) + w(t). \end{aligned} \tag{14}$$

On the space M_2 (see [35, Chapter 2]), the system (14) can be represented as the abstract evolution equation

$$\begin{aligned} \frac{dx}{dt}(t) &= \mathcal{A}x(t) + \mathcal{B}u(t) + \mathcal{D}d(t), \ t \geq 0, \quad x(0) = x_0, \\ y(t) &= \mathcal{C}x(t) + w(t), \end{aligned} \tag{15}$$

with the state vector $x(t) = \begin{bmatrix} z(t) \\ z(t+\cdot) \end{bmatrix}$. As $M_2([-h, 0]; \mathbb{R}^n)$ is isometrically isomorphic with $\mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n)$ (see also [36]), one may define $H = \mathbb{R}^n \times L^2(-h, 0; \mathbb{R}^n)$ and $\mathcal{A}$, the infinitesimal generator of the corresponding C_0-semigroup, as

$$\mathcal{A} \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} = \begin{bmatrix} A_0 r + A_1 f(-h) \\ \frac{df}{d\theta}(\cdot) \end{bmatrix},$$

with domain

$$\mathscr{D}(\mathcal{A}) = \left\{ \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} \in H \mid f \text{ abs. cont.}, \frac{df}{d\theta}(\cdot) \in L^2(-h, 0; \mathbb{R}^m) \text{ and } f(0) = r \right\}.$$

The disturbance operator $\mathcal{D} : \mathbb{R}^d \to H$ is defined by

$$\mathcal{D}d := \begin{bmatrix} D_0 d \\ 0 \end{bmatrix},$$

the measurement operator $\mathcal{C} : H \to \mathbb{R}^d$ is given by

$$\mathcal{C}x = \mathcal{C} \begin{bmatrix} r \\ f(\cdot) \end{bmatrix} = C_0 r,$$

and $\mathcal{B} : \mathbb{R}^u \to H$ is $\mathcal{B}u := \begin{bmatrix} B_0 u \\ 0 \end{bmatrix}$.

To approximate $\mathcal{P}^\infty$, the unique non-negative solution of the operator Riccati equation (11), consider the sequence of finite dimensional spaces $(H^N_{AVE})_N$, and $(A^N_{AVE})_N$, $(B^N_{AVE})_N$, $(D^N_{AVE})_N$ and $(C^N_{AVE})_N$, the sequences of approximating linear and bounded operators obtained using averaging approximations (AVE) as in [34]: Define $t^N_j := \frac{jh}{N}$, for $j = 0, ..., N$, χ^N_j the normalized characteristic functions on $[-t^N_j, -t^N_{j-1})$ such that $\|\chi^N_j\|_{L^2} = 1$, $H^N := \{[\xi, \phi^N] \in H; \phi^N(\tau) = \sum_{j=1}^N v^N_j \chi^N_j(\tau), v^N_j \in \mathbb{R}^n\}$, the projection $\Pi^N : H \to H^N$, $\Pi^N[\xi, \phi] := [\xi, \sum_{j=1}^N \phi^N_j \chi^N_j]$, with $\phi^N_j = \sqrt{\frac{N}{h}} \int_{-jhNn}^{-(j-1)h/N} f(\tau) \mathrm{d}\tau$,

$$A^N_{AVE}[\xi, \phi^N] := \Big[A_0 \xi + \sqrt{\frac{N}{h}} A_1 v^N_N, \frac{N}{h} \sum_{j=1}^{N} (v^N_{j-1} - v^N_j) \chi^N_j \Big],$$

where we take $v^N_0 = \sqrt{h/N}\xi$, $B^N_{AVE} u := \Pi^N \mathcal{B} u = \mathcal{B} u$, $D^N_{AVE} d := \Pi^N \mathcal{D} d = \mathcal{D} d$, $C^N_{AVE}[\xi, \phi]^T := \mathcal{C}\Pi^N [\xi, \phi]^T = \mathcal{C}[\xi, \phi]^T$. The operators $\mathcal{C}$ and $\mathcal{D}$ are compact. The semigroup $T(t)$, the operators $\mathcal{D}$ and $\mathcal{C}$ in combination with averaging approximations, satisfy Assumptions 1 and 2, even though these assumptions seem to be very restrictive (see [34]).

To compute the filter gain $L^N := -P^N (C^N_{AVE})^\star$, where P^N is the solution of the Riccati equation (12) corresponding to the AVE discretization, one can use the Newton-Kleinman method (see [39]). Assume that a L^N_0 is given such that $A^N_{AVE} - L^N_0 (C^N_{AVE})^\star$ is Hurwitz. Define $S^N_i := A^N_{AVE} - L^N_i (C^N_{AVE})^\star$, find a solution to the Lyapunov equation

$$S^N_i P^N_i + P^N_i (S^N_i)^\star = -D^N (D^N)^\star - L^N_i (L^N_i)^\star \tag{16}$$

and then update the filter operator $L^N_{i+1} := -P^N_i (C^N_{AVE})^\star$. Instead of the standard Kleinman-Newton method, one can also use a modified Kleinman-Newton (see [32]) by rewriting (16) as

$$S^N_i P^N_i + P^N_i (S^N_i)^\star = -E^N_i (E^N_i)^\star, \tag{17}$$

where $E^N_i := L^N_i - L^N_{i-1}$ and the update of the filter gain is $L^N_{i+1} := L^N_i - P^N_i (C^N_{AVE})^\star$. Note that in this case one needs

L_0^N and L_1^N given such that the corresponding filter matrices $A_{AVE}^N - L_0^N (C_{AVE}^N)^\star$ and $A_{AVE}^N - L_1^N (C_{AVE}^N)^\star$ are Hurwitz. However, this can reduce the complexity of the computations (see [40]).

Consider also $F_{AVE}^N : H^N \to H_u$ corresponding to $F : H \to H_u$. Then the sequence of approximating controls which solves the problem (5) is given by

$$u_{eq,AVE}^N(t) = -(F_{AVE}^N B_{AVE}^N)^{-1} F_{AVE}^N \cdot \left[A_{AVE}^N \hat{x}^N(t) - L^N(y^N(t) - C_{AVE}^N \hat{x}^N(t))\right]. \quad (18)$$

Note that $\mathcal{FB}$ is invertible, $B_{AVE}^N u = \Pi^N \mathcal{B} u = \mathcal{B} u$, so $\mathcal{B}$ is compact and $\mathcal{F}$ is of finite rank. Then one can show by using the results of Section IV that the sequence of finite dimensional controllers $u_{eq,AVE}^N$ constructed using the finite dimensional observations $\hat{x}^N \in H^N$ based on the AVE converges to a solution of problem (5) as $N \to \infty$.

VI. Numerical Example

We now demonstrate the presented theory by working out a numerical example. For this, we chose $A_0 = \left[\begin{smallmatrix} -1 & 1 \\ 1 & -3 \end{smallmatrix}\right]$ and $A_1 = -I_2/2$, $B_0 = \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, $C_0 = [1 \;\; 0]$, $D_0 = B_0$, where I_2 denotes the 2×2 identity matrix. Following the AVE procedure outlined in the previous section, the matrix representation of the operator A_{AVE}^N on H^N, if we're using the orthonormal basis of indicator functions $\chi_j^N(t) = \sqrt{N/h} \cdot \mathbf{1}_{[-\frac{jh}{N}, -\frac{(j-1)h}{N}]}(t)$, $j = 1, \ldots, N$ for the "L^2-part," is given by

$$A_{AVE}^N = \begin{bmatrix} A_0 & 0 & \cdots & \cdots & 0 & \sqrt{\frac{N}{h}} A_1 \\ \sqrt{\frac{N}{h}} I_2 & -\frac{N}{h} I_2 & 0 & \cdots & 0 & 0 \\ 0 & \frac{N}{h} I_2 & -\frac{N}{h} I_2 & \ddots & & \vdots \\ 0 & 0 & \ddots & \ddots & & \\ \vdots & & & \ddots & & 0 \\ 0 & \cdots & & & \frac{N}{h} I_2 & -\frac{N}{h} I_2 \end{bmatrix},$$

and $B_{AVE}^N = [B_0^\intercal \; 0 \cdots]^\intercal$, $C_{AVE}^N = [C_0 \; 0 \cdots]$. The initial state is given by

$$\xi^N = \begin{bmatrix} r \\ \sqrt{\frac{N}{h}} \int_{-h/N}^{0} f(\tau) \mathrm{d}\tau \\ \vdots \\ \sqrt{\frac{N}{h}} \int_{-h}^{-(N-1)h/N} f(\tau) \mathrm{d}\tau \end{bmatrix},$$

and we denote the vector representing the state $[x, \phi^N]$ by x^N. Finally, we select $h = 2$, $d(t) = \sin(2\pi t) \mathbf{1}_{[9,11]}(t)$, $w(t) = e^{-t/2} \sin(\pi t)/2$ and $f(t) = [1 \;\; -1]^\intercal (t/2)^2$, $\xi = f(0)$. This selection satisfies (3) with e.g. $\mathcal{S} \approx 0.42I$, $\mathcal{R} \approx 2.73$ and $\mathcal{Q} = 1/3$.

We then obtain the (discretization) of the filter (7) in the two ways described in the last section: First, by solving the Riccati equation (12) using the Newton-Kleinman (NK) procedure to obtain P^N, the matrix representation of the solution of the Riccati equation, then by using the modified NK procedure to compute $L^N = -P^N (C_{AVE}^N)^\intercal$ directly. We note here, that both converge to a relative error of 10^{-5} extremely fast: the NK procedure takes 7 iterations, whereas the modified NK procedure converges in 6 for all N that we have tested. The obtained values for L^N are, of course, virtually identical.

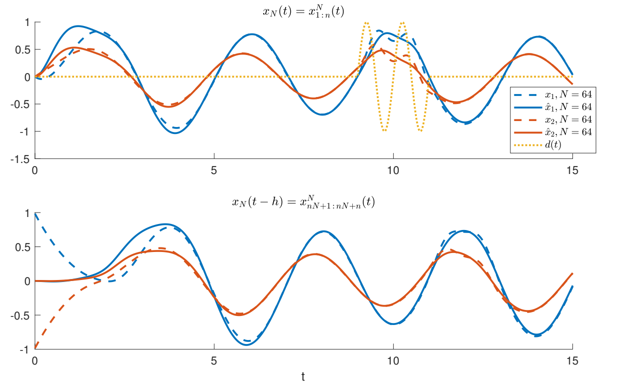

Fig. 1. The top panel shows the true states as dashed lines, and their estimates $\hat{x}_i$ as solid lines. The disturbance d is also shown: since d is unknown to the filter, the estimation error increases while the disturbance is active. The bottom filter shows the estimates of $x(t-2)$ to illustrate that a) even though the initial value $\hat{\xi}$ of the filter equals ξ, the full initial state deviates due to the initial profile $f(\cdot)$ being nonzero; and b) as is particularly obvious around $t = 10$, the estimates of $x(t-2)$ are not an exact shift of $x(t)$ but slightly smoothed due to numerical diffusion.

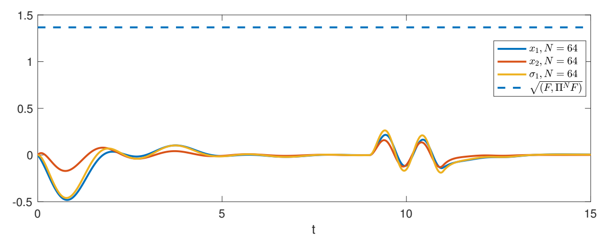

Fig. 2. With the feedback control law $u(t) = -(F_{AVE}^N B_{AVE}^N)^{-1} F_{AVE}^N \left[A_{AVE}^N x^N(t) - L^N(y^N(t) - C_{AVE}^N \hat{x}^N(t))\right]$, the states and sliding variable σ converge to zero and never even come close to the worst-case bound $\sqrt{F_{AVE}^N P^N (F_{AVE}^N)^\intercal}$.

In Figure 1, we demonstrate the validity of the filter with $\dot{\hat{x}} = (A_{AVE}^N - L^N R C_{AVE}^N)\hat{x} + B_{AVE}^N u + L^N R y^N$; for this experiment only, the system was driven by $u(t) = \sin(\pi t/2)$, since there is no feedback yet.

We then proceeded to implement the feedback $u_{eq,AVE}^N$ as in (18). We select $\sigma_1 = (x_1 + x_2)/\sqrt{2}$, i.e. $F_{AVE}^N = [\sqrt{1/2} \; \sqrt{1/2} \; 0 \cdots]$ and simulate the full system, consisting of the plant (13), the filter (7) and the control law (18). The results are shown in Figure 2 and show in particular, that the control is effective in steering σ close to zero, and that the bound in (8), shown as a dashed line, is very conservative – recall that it is a worst-case bound.

Finally, we check the convergence rate of the AVE approximation scheme. There are now 3 different trajectories: x^N of the system driven by $u(t)$, $\hat{x}^N$ of the filter driven by $y^N = C_{AVE}^N x^N$ and u, and x_{cl}^N, the state of the plant (13) controlled by $u_{eq,AVE}^N$. We computed each for several values

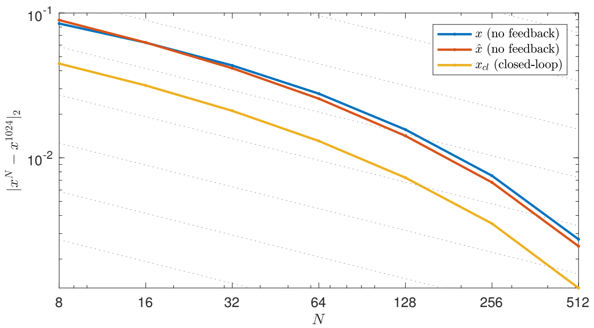

Fig. 3. Convergence of the approximation as $N \to \infty$. The grid lines in the log-log plot correspond to inverse square-root relationships. We used $N = 1024$ as a proxy for ∞, which explains the seemingly faster convergence as N approaches 1024.

of N and recorded the error norm $\|z^N(\cdot) - z^\infty(\cdot)\|_{L^2}$. Of course, trajectories for $N = \infty$ are not available, and we used the largest N we ran instead. Figure 3 illustrates the well-known $(1/\sqrt{N})$ convergence rate of the AVE scheme, and we remind the reader that, while there are schemes with faster convergence, those schemes, as pointed out in e.g. [31], do not generally provide for convergence and dual convergence as required in Assumption 1.

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