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Linear and Quadratic Chabauty for Affine Hyperbolic Curves

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We give sufficient conditions for finiteness of linear and quadratic refined Chabauty–Kim loci of affine hyperbolic curves. We achieve this by constructing depth ≤ 2 quotients of the fundamental group, following a construction of Balakrishnan–Dogra in the projective case. We also apply Betts' machinery of weight filtrations to give unconditional explicit upper bounds on the number of S-integral points when our conditions are satisfied.

1 Introduction

Let $Y/\mathbb{Q}$ be a smooth affine hyperbolic curve and let $\mathcal{Y}/\mathbb{Z}_S$ be a regular model of Y over the ring of S-integers for some finite set of primes S. By the theorems of Siegel and Faltings, the set of S-integral points $\mathcal{Y}(\mathbb{Z}_S)$ is finite. However, this result is in general not effective. One approach towards effectivity is the method of Chabauty–Coleman [17, 18] and its nonabelian generalisation due to Minhyong Kim [21, 22], by which $\mathcal{Y}(\mathbb{Z}_S)$ is regarded as a subset of the p-adic integral points $\mathcal{Y}(\mathbb{Z}_p)$ for some prime $p \notin S$ of good reduction, and p-adic analytic functions on $\mathcal{Y}(\mathbb{Z}_p)$ are constructed that vanish on $\mathcal{Y}(\mathbb{Z}_S)$. More precisely, we are interested in the refined Chabauty–Kim method, as developed by Betts–Dogra [14], which produces a descending sequence of subsets

$$\mathcal{Y}(\mathbb{Z}_p) \supseteq \mathcal{Y}(\mathbb{Z}_p)_{S,1} \supseteq \mathcal{Y}(\mathbb{Z}_p)_{S,2} \supseteq \dots,$$

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all containing $\mathcal{Y}(\mathbb{Z}_S)$. We call these the *refined Chabauty–Kim loci*. (The refined Chabauty–Kim loci $\mathcal{Y}(\mathbb{Z}_p)_{S,n}$ are denoted $\mathcal{Y}(\mathbb{Z}_p)^{\min}_{S,n}$ in [14] to distinguish them from the non-refined, possibly larger Chabauty–Kim loci. In this paper, we only consider the refined variant, therefore we omit the superscript $(-)^{\min}$ from the notation.) It is conjectured that $\mathcal{Y}(\mathbb{Z}_p)_{S,n}$ is finite for sufficiently large $n \gg 0$, in which case it is given as the vanishing set of nontrivial Coleman functions. (In fact, we expect $\mathcal{Y}(\mathbb{Z}_p)_{S,n}$ to be equal to $\mathcal{Y}(\mathbb{Z}_S)$ for sufficiently large n; this is the refined version of *Kim's conjecture* [2, Conjecture 3.1].) Thus, computing the set $\mathcal{Y}(\mathbb{Z}_p)_{S,n}$, whenever it is finite, can serve as an approximation to computing the set of S-integral points.

In general, the refined Chabauty–Kim loci $\mathcal{Y}(\mathbb{Z}_p)_{S,n}$ are difficult to compute. Significant progress has only been made in the cases $n = 1$ and $n = 2$, which correspond to *linear* and *quadratic Chabauty*, respectively. In this paper, we give sufficient criteria on (Y, S, p) for finiteness of $\mathcal{Y}(\mathbb{Z}_p)_{S,1}$ (Theorem A(1)) and $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ (Theorems A(2) and C). In addition, we obtain bounds on the size of the quadratic Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$, which also bound $\#\mathcal{Y}(\mathbb{Z}_S)$. Our results (Theorems B and D) in this direction have the form that whenever a certain inequality holds, then $\#\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ is bounded in terms of invariants associated to $(\mathcal{Y}, S, p)$ that can often be computed explicitly.

Our theorems are affine analogues of the following results for the set of rational points on *projective* hyperbolic curves: The classical theorem of Chabauty [17] proves finiteness whenever the rank-genus inequality $g - r > 0$ is satisfied. More recently, Balakrishnan and Dogra showed finiteness whenever the inequality $g - r + \rho - 1 > 0$ involving the Picard number ρ of the Jacobian is satisfied by developing and applying quadratic Chabauty (see [4, Lemma 3.2], strengthened in [6, Proposition 2.2]). They also proved an effective version [5, Theorem 1.1] giving a bound on the number of rational points. (For different approaches to quadratic Chabauty, see [20] and [10].)

Previous finiteness results for Chabauty–Kim in depth ≤ 2 for affine hyperbolic curves of genus > 0 were restricted to $S = \varnothing$ (see [23] and [4, Remark 3.3]), and bounds for such curves are only known for $S = \varnothing$ and Y hyperelliptic (see [5, Theorem 1.3]).

We illustrate our results with several special cases and examples in Section 2:

- the rank equals genus case (Example 2.1);
- totally ramified superelliptic curves (Example 2.2);
- even degree hyperelliptic curves (Example 2.3);
- the thrice-punctured line (Example 2.4).

In order to state our main results precisely, we introduce some notation. Suppose that the smooth affine hyperbolic curve $Y/\mathbb{Q}$ is given as $X \setminus D$, where $X/\mathbb{Q}$ is a smooth projective curve and $D \neq \varnothing$ is the reduced boundary divisor. We call the points in D *cusps* or *points at infinity*. Let $\mathcal{X}$ be a regular model of X [24, Definition 10.1.1] over the ring $\mathbb{Z}_S$ of S-integers. Let $\mathcal{D}$ be the closure of D in $\mathcal{X}$ and set $\mathcal{Y} := \mathcal{X} \setminus \mathcal{D}$. Assume that $\mathcal{Y}$ admits an S-integral point. Fix a prime $p \notin S$ such that $\mathcal{X}_{\mathbb{F}_p}$ is smooth and $\mathcal{D}_{\mathbb{F}_p}$ is étale. We use the following notation, which will be kept throughout this paper:

- $r := \operatorname{rk} \operatorname{Jac}_X(\mathbb{Q})$ the Mordell–Weil rank of the Jacobian of X;
- $r_p := \operatorname{rk}_{\mathbb{Z}_p} \operatorname{Sel}_{p^\infty}(\operatorname{Jac}_X)$ the p^∞-Selmer rank of the Jacobian of X;
- g the genus of X;
- $\rho := \operatorname{rk} \operatorname{NS}(\operatorname{Jac}_X)$ the Picard number of the Jacobian of X; note that $\rho \geq 1$ if $g \geq 1$;
- $\rho_f := \rho + \operatorname{rk} \operatorname{NS}(\operatorname{Jac}_{X_{\overline{\mathbb{Q}}}})^{\sigma=-1}$, where σ denotes complex conjugation and $A^{\sigma=\pm 1}$ denotes the ± 1-eigenspace of a $\langle \sigma \rangle$-module A;
- $\#|D| > 0$ the number of closed points at infinity;
- $n := \#D(\overline{\mathbb{Q}}) > 0$ the number of geometric points at infinity;
- write $n = n_1 + 2n_2$ with $n_1 := \#D(\mathbb{R})$ the number of real points and n_2 the number of conjugate pairs of complex points at infinity;
- $b := \#|D| + n_2 - 1$;
- $s := \#S$.

The condition that Y is hyperbolic is equivalent to $2 - 2g - n < 0$. Note also that $D \neq \varnothing$ and thus $b \geq 0$ since we are assuming Y to be affine. Our first finiteness theorem for the linear and quadratic Chabauty–Kim loci now reads as follows.

Theorem A.

(1) If

$$\alpha_1(Y, s, p) := g - r_p + b - s > 0,$$

then $\mathcal{Y}(\mathbb{Z}_p)_{S,1}$ is finite.

(2) If

$$\alpha_2(Y, s, p) := \alpha_1(Y, s, p) + \rho_f > 0,$$

then $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ is finite.

If one assumes the Tate–Shafarevich conjecture, the p^∞-Selmer rank r_p appearing in $\alpha_1(Y, s, p)$ and $\alpha_2(Y, s, p)$ can be replaced with the Mordell–Weil rank r. There is also an unconditional variant of Theorem A using r instead of r_p; see Remark 1.4 below.

Once we have finiteness of the Chabauty–Kim loci, we know that they are defined by one or more Coleman functions. Under suitable assumptions, it is possible to get some control over these Coleman functions. This allows us to bound their number of zeros and hence the size of $\mathcal{Y}(\mathbb{Z}_S)$. The kind of control we are looking for is a bound on the *weight* of the Coleman functions, a notion introduced by Betts [12].

To state our main results in this direction, we consider the decomposition of the S-integral points

$$\mathcal{Y}(\mathbb{Z}_S) = \coprod_{\Sigma} \mathcal{Y}(\mathbb{Z}_S)_\Sigma$$

into points of given *reduction types* Σ (see [12, §6.2]), which control the mod-ℓ reduction for all primes ℓ. The refined Chabauty–Kim loci $\mathcal{Y}(\mathbb{Z}_p)_{S,n}$ are similarly a union over reduction types

$$\mathcal{Y}(\mathbb{Z}_p)_{S,n} = \bigcup_{\Sigma} \mathcal{Y}(\mathbb{Z}_p)_{S,n,\Sigma},$$

with $\mathcal{Y}(\mathbb{Z}_p)_{S,n,\Sigma}$ containing $\mathcal{Y}(\mathbb{Z}_S)_\Sigma$. For a prime ℓ, we denote by n_ℓ the number of irreducible components of the mod-ℓ special fibre of $\mathcal{X}$ (if $\ell \notin S$), respectively of the minimal regular normal crossings model of (X, D) over $\mathbb{Z}_\ell$ (if $\ell \in S$; see [15, Appendix B]). (The symbol n_2 has two different meanings: the number of conjugate pairs of complex points at infinity and the number of components of the mod-2 special fibre. This should not cause any confusion as the correct meaning will always be clear from the context.) Set $\kappa_p := 1 + \frac{p-1}{(p-2)\log(p)}$ if p is odd and $\kappa_2 := 2 + \frac{2}{\log(2)}$.

Theorem B. If

$$\beta(Y, s, p) := \frac{1}{2}g(g+3) - \frac{1}{2}r_p(r_p+3) + \rho_f + b - s > 0,$$

then for each reduction type Σ there exists a nonzero Coleman algebraic function of weight at most 2 vanishing on $\mathcal{Y}(\mathbb{Z}_p)_{S,2,\Sigma}$. Moreover, the size of the refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ (and thus the number of S-integral points of $\mathcal{Y}$) is bounded by

$$\#\mathcal{Y}(\mathbb{Z}_p)_{S,2} \le \kappa_p \cdot \prod_{\ell \in S}(n_\ell + n) \cdot \prod_{\ell \notin S} n_\ell \cdot \#\mathcal{Y}(\mathbb{F}_p) \cdot (4g + 2n - 2)^2(g+1)\,.$$

Remark 1.1 (Weight 2 Coleman algebraic functions). Coleman algebraic functions of weight at most 2 are the kind of functions showing up in quadratic Chabauty as in [1, 4–6, 10]. They are linear combinations of double Coleman integrals, single Coleman integrals, and rational functions. The precise form is given in [12, Lemma 4.1.13]. Coleman algebraic functions of general weight are constructed in [12, §4.1]; they form a subring of the algebra of all Coleman (analytic) functions defined by Besser [9]. The theory of Betts yields bounds for the number of zeros of Coleman algebraic functions of bounded weight. In this way, the bound on $\#\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ in Theorem B follows from the existence of the weight 2 Coleman algebraic functions. Namely, each $\#\mathcal{Y}(\mathbb{Z}_p)_{S,2,\Sigma}$ is bounded by $\kappa_p \cdot \#\mathcal{Y}(\mathbb{F}_p) \cdot (4g+2n-2)^2(g+1)$, and multiplying this by $\prod_{\ell \in S}(n_\ell + n) \cdot \prod_{\ell \notin S} n_\ell$, the number of reduction types Σ, yields the bound for $\#\mathcal{Y}(\mathbb{Z}_p)_{S,2}$.

Remark 1.2. Using essentially the same argument as in our proof of Theorem B, we can show an analogous statement for the *linear* Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,1}$. Namely, if the stronger condition $\beta(Y,s,p) - \rho_f > 0$ holds, then there are nonzero Coleman algebraic functions of weight at most 2 vanishing already on the $\mathcal{Y}(\mathbb{Z}_p)_{S,1,\Sigma}$, and the upper bound on $\#\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ from Theorem B already holds for $\#\mathcal{Y}(\mathbb{Z}_p)_{S,1}$.

Our main tool is Betts' theory of weight filtrations on (refined) Selmer schemes introduced in [12]. This machinery reduces statements about finiteness of and bounds for refined Chabauty–Kim loci to calculations of local and global Bloch–Kato Selmer groups. The general strategy is reviewed in Section 3 below. At this point, suffice it to say that the theory takes as input a $G_{\mathbb{Q}}$-equivariant quotient $U_Y \twoheadrightarrow U$ of the $\mathbb{Q}_p$-pro-unipotent étale fundamental group U_Y of Y at an S-integral base point, where we write $G_K = \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group of a field K. If one is able to calculate or at least bound the dimensions of the Bloch–Kato Selmer groups of the weight-graded pieces of U, one gets finiteness of and bounds on the size of the associated refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$. The sets $\mathcal{Y}(\mathbb{Z}_p)_{S,n}$ above correspond to the choice $U_Y \twoheadrightarrow U_{Y,n}$, the n-th quotient of U_Y along the lower central series. If one is willing to assume the Bloch–Kato conjecture, one can choose for U the full fundamental group $U_{Y,\infty} = U_Y$. The conditional estimates on the dimensions of the relevant Bloch–Kato Selmer groups can then be used to obtain conditional bounds on the size of $\mathcal{Y}(\mathbb{Z}_p)_{S,\infty}$ and hence on the number of S-integral points of $\mathcal{Y}$. This is one of the main results of Betts–Corwin–Leonhardt [13, Theorem 1.4]. In contrast with this, all our results are unconditional.

In order to achieve this, we work with rather small quotients of the fundamental group whose Bloch–Kato Selmer groups we are able to compute. For Theorem A(1), we choose the abelianisation $U_{Y,1} = U_Y^{\mathrm{ab}}$ of U_Y; the relevant calculations are done in Section 4. For Theorems A(2) and B, rather than working with all of $U_{Y,2}$, we construct a certain intermediate quotient

$$U_{Y,2} \twoheadrightarrow U \twoheadrightarrow U_{Y,1}.$$

The construction of this intermediate quotient, which is carried out in Section 5 below, is motivated by the analogous construction in the projective case given by Balakrishnan–Dogra in [6, Proposition 2.2] and generalises [4, Remark 3.3].

Finally, in Section 6, we also investigate the finiteness statements and bounds that we get from the weight ≥ -2 quotient of the fundamental group. This is another intermediate quotient between $U_{Y,2}$ and $U_{Y,1}$, which is in general larger than the one used for Theorems A(2) and B, so we expect to get finiteness and bounds under weaker conditions. The price to pay for this is that the conditions involve a term

$$h_{\mathrm{BK}} := \dim_{\mathbb{Q}_p} \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{Hom}(\bigwedge^2 V_p \operatorname{Jac}_X, \mathbb{Q}_p(1))), \tag{1.1}$$

which we do not understand well but which is conjectured to vanish as a consequence of the Bloch–Kato conjectures [13, Conjecture 2.8]:

Conjecture 1.3. $h_{\mathrm{BK}} = 0$.

We are not assuming Conjecture 1.3 for our results but rather make the dependence on the conjecture explicit by including the term h_{BK} in the statements. Our results obtained by working with the full weight ≥ -2 quotient of the fundamental group read as follows.

Theorem C. If

$$\gamma(Y,s,p) := g^2 - r_p + \rho + b - s - h_{\mathrm{BK}} > 0,$$

then $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ is finite.

Theorem D. If

$$\delta(Y,s,p) := \frac{1}{2}g(3g+1) - \frac{1}{2}r_p(r_p+3) + \rho + b - s - h_{\mathrm{BK}} > 0,$$

then for every reduction type Σ there exists a nonzero Coleman algebraic function of weight at most 2 that vanishes on $\mathcal{Y}(\mathbb{Z}_p)_{S,2,\Sigma}$. Moreover, the size of the refined Chabauty–Kim locus $\#\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ (and thus the number of S-integral points of $\mathcal{Y}$) is then bounded by the same bound as in Theorem B.

Having stated our main results, let us conclude with a few remarks.

Remark 1.4 (r versus r_p). The p^∞-Selmer rank r_p and the Mordell–Weil rank r of the Jacobian of X satisfy $r_p \geq r$, with equality if and only if the p-divisible part of the Tate–Shafarevich group of Jac_X is trivial, as predicted by the Tate–Shafarevich conjecture. One way to replace r_p with r in Theorems A and C without assuming the conjecture is to modify the definition of the Selmer scheme using the "Balakrishnan–Dogra trick" (see [4, Definition 2.2] for the case of a projective curve, and [12, §6.3] for the affine variant). The modified refined Chabauty–Kim loci $\mathcal{Y}(\mathbb{Z}_p)_{S,n}^{\mathrm{BD}}$ are potentially smaller than $\mathcal{Y}(\mathbb{Z}_p)_{S,n}$ but still contain $\mathcal{Y}(\mathbb{Z}_S)$. The analogous finiteness statements are as follows: let

$$\alpha'_1(Y,s) := g - r + b - s;$$

$$\alpha'_2(Y,s) := \alpha'_1(Y,s) + \rho_f;$$

$$\gamma'(Y,s,p) := g^2 - r + \rho + b - s - h_{\mathrm{BK}}.$$

If $\alpha'_1(Y,s) > 0$ (resp. $\alpha'_2(Y,s) > 0$ or $\gamma'(Y,s,p) > 0$), then the Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,1}^{\mathrm{BD}}$ (resp. $\mathcal{Y}(\mathbb{Z}_p)_{S,2}^{\mathrm{BD}}$) is finite.

The modified loci $\mathcal{Y}(\mathbb{Z}_p)_{S,n}^{\mathrm{BD}}$ can also be written as a union of $\mathcal{Y}(\mathbb{Z}_p)_{S,n,\Sigma}^{\mathrm{BD}}$ over all reduction types, and the analogues of Theorems B and D are: let

$$\beta'(Y,s) := \frac{1}{2}g(g+3) - \frac{1}{2}r(r+3) + \rho_f + b - s;$$

$$\delta'(Y,s,p) := \frac{1}{2}g(3g+1) - \frac{1}{2}r(r+3) + \rho + b - s - h_{\mathrm{BK}}.$$

If $\beta'(Y,s) > 0$ or $\delta'(Y,s,p) > 0$, then for every reduction type Σ, there exists a nonzero Coleman algebraic function of weight at most 2 that vanishes on $\mathcal{Y}(\mathbb{Z}_p)_{S,2,\Sigma}^{\mathrm{BD}}$, and the size of the locus $\mathcal{Y}(\mathbb{Z}_p)_{S,2}^{\mathrm{BD}}$ is bounded by the same bound as in Theorem B. Note that this is also a bound for the number of S-integral points thanks to the inclusion $\mathcal{Y}(\mathbb{Z}_S) \subseteq \mathcal{Y}(\mathbb{Z}_p)_{S,2}^{\mathrm{BD}}$. As in Remark 1.2, we also get the same bound on $\#\mathcal{Y}(\mathbb{Z}_p)_{S,1}^{\mathrm{BD}}$ when $\beta'(Y,s) - \rho_f > 0$.

Remark 1.5 (Dependence on p). The conditions in Theorems A–D depend on p only through r_p and h_{BK}. Therefore, if the Tate–Shafarevich conjecture and the Bloch–Kato conjecture Conjecture 1.3 are known to hold, then we have $r_p = r$ and $h_{\mathrm{BK}} = 0$, and the conditions are in fact independent of p. As explained in Remark 1.4, the dependence on r_p can be avoided by using the Balakrishnan–Dogra trick, which is why $\alpha'_1(Y,s)$, $\alpha'_2(Y,s)$, and $\beta'(Y,s)$ are independent of p.

Remark 1.6 (Dependence on $\mathcal{Y}$ and S). Note that the conditions in Theorems A–D do not depend on the choice of S-integral model $\mathcal{Y}/\mathbb{Z}_S$, only on the generic fibre $Y/\mathbb{Q}$. They also do not depend on the set S but only on its cardinality $s = \#S$. The bounds on $\#\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ in Theorems B and D do depend on $\mathcal{Y}/\mathbb{Z}_S$ through the invariants n_ℓ, that is, the number of irreducible components of the special fibres.

Remark 1.7. It would be interesting to make the results of this paper explicit. For a projective curve $X/\mathbb{Q}$ satisfying $r = g$ and $\rho > 1$, explicit methods for the computation of $X(\mathbb{Q}_p)_U^{\mathrm{BD}}$ based on p-adic heights have been developed (and applied) in [3, 4, 6, 7], where U is the fundamental group quotient constructed in [4, §3]. We expect that one could use similar methods to compute $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$ (or at least a finite superset thereof) for the quotient U constructed in Lemma 5.1, at least in some special cases.

2 Examples

In this section, we give some sample applications of our theorems. We keep the notation of the previous section.

Example 2.1 (Rank equals genus case). Assume that $r_p = r = g$ and assume for simplicity that all points at infinity are rational. Then Theorems A and B simplify as follows:

1. If $n - 1 - s > 0$, then $\mathcal{Y}(\mathbb{Z}_p)_{S,1}$ is finite.
2. If $n - 1 - s + \rho_f > 0$, then $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ is finite, for every reduction type Σ there exists a nonzero Coleman algebraic function of weight at most 2 vanishing on $\mathcal{Y}(\mathbb{Z}_p)_{S,2,\Sigma}$, and $\#\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ is bounded as in Theorem B.

Example 2.2 (Totally ramified superelliptic curves). Let $Y/\mathbb{Q}$ be an affine superelliptic curve given by an equation $y^m = f(x)$, where $m > 1$, $f \in \mathbb{Z}[x]$ is squarefree of degree $d > 2$ and $\gcd(d, m) = 1$. Then we have $n = n_1 = \#|D| = 1$, so that $b = 0$. Suppose that $r = g$. Then the Balakrishnan–Dogra variant (see Remark 1.4) of Theorem A(2) shows that $\mathcal{Y}(\mathbb{Z}_p)_{\varnothing,2}^{\mathrm{BD}}$ is finite. The variant of Theorem B shows that for every reduction type Σ, there exists a nonzero Coleman algebraic function of weight at most 2 vanishing on $\mathcal{Y}(\mathbb{Z}_p)_{\varnothing,2,\Sigma}$ and that we have the simple bound

$$\#\mathcal{Y}(\mathbb{Z}) \leq \#\mathcal{Y}(\mathbb{Z}_p)_{\varnothing,2}^{\mathrm{BD}} \leq \kappa_p \cdot \prod_\ell n_\ell \cdot \#\mathcal{Y}(\mathbb{F}_p) \cdot 16g^2(g+1)\,. \tag{2.1}$$

In particular, when $m = 2$ and $d = 2g+1 \geq 3$, then Y is an affine hyperelliptic curve of genus g and odd degree. In this case, the finiteness of $\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{BD}}_{\varnothing,2}$ was previously shown in [1, Theorem 1.1], and then again in [4, Theorem 1.1]. An upper bound for $\#\mathcal{Y}(\mathbb{Z})$ of order g^3 was given by Balakrishnan–Dogra in [5, Theorem 1.3], whereas the bound we obtain from specialising (2.1) is of order g^4 due to Hasse–Weil. Their bound is better because their construction is based on differential operators that are special to hyperelliptic curves, whereas we use the general approach of Betts. (Compare with the upper bound for rational points in [5, Theorem 1.1]; it is of order g^3 for hyperelliptic curves, but of order g^4 for non-hyperelliptic curves.)

Example 2.3 (Even degree hyperelliptic curves). Now let $Y/\mathbb{Q}$ be an affine hyperelliptic curve given by an equation

$$y^2 = a_{2g+2}x^{2g+2} + \ldots + a_0, \quad a_i \in \mathbb{Z}\,,\ a_{2g+2} \neq 0\,,$$

where $a_{2g+2}x^{2g+2} + \ldots + a_0$ is squarefree. Then we have $n = 2$. Suppose that a_{2g+2} is either a square of an integer or is negative. Then $b = 1$, and hence the Balakrishnan–Dogra variant of Theorem A(2) gives an unconditional proof that $\mathcal{Y}(\mathbb{Z})$ is finite using non-abelian Chabauty when $r \leq g+1$. Suppose in addition that $r = g$. Then $\alpha_2'(Y, s) = \beta'(Y, s) = \rho_f + 1 - s$, so whenever $s < \rho_f + 1$, we obtain the upper bound

$$\#\mathcal{Y}(\mathbb{Z}_S) \leq \#\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{BD}}_{S,2} \leq \kappa_p \cdot \prod_{\ell \in S}(n_\ell + 2) \cdot \prod_{\ell \notin S} n_\ell \cdot \#\mathcal{Y}(\mathbb{F}_p) \cdot (4g+2)^2(g+1)\,.$$

In fact, when $r = g$, then $\alpha_1'(Y, 0) > 0$, and hence Theorem A(1) and Remark 1.4 imply that the depth 1 locus $\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{BD}}_{\varnothing,1}$ is already finite. In this case, the conclusions of Theorem B also hold for $\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{BD}}_{\varnothing,1}$ by Remark 1.2. In particular, we get the bound

$$\#\mathcal{Y}(\mathbb{Z}) \leq \#\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{BD}}_{\varnothing,1} \leq \kappa_p \cdot \prod_{\ell} n_\ell \cdot \#\mathcal{Y}(\mathbb{F}_p) \cdot (4g+2)^2(g+1)\,.$$

Example 2.4 (S-integral points on the thrice-punctured line). Let $\mathcal{Y} := \mathbb{P}^1_{\mathbb{Z}_S} \smallsetminus \{0, 1, \infty\}$ be the thrice-punctured line. Assume that $2 \in S$; otherwise, all refined Chabauty–Kim loci are automatically empty. We have $b = 2$ and $r_p = r = g = \rho_f = h_{\mathrm{BK}} = 0$ since the compactification $\mathbb{P}^1$ has trivial Jacobian, so Theorems A–D apply whenever $\#S < 2$, that is, for $S = \{2\}$. Theorem A(2) (or C) yields the finiteness of $\mathcal{Y}(\mathbb{Z}_p)_{\{2\},2}$, and Theorem B (or D) shows that for each of the three reduction types Σ (corresponding to the three cusps 0, 1, ∞) there exists a nonzero Coleman algebraic function of weight at most 2 vanishing on $\mathcal{Y}(\mathbb{Z}_p)_{\{2\},2,\Sigma}$, and we have the bound

$$\#\mathcal{Y}(\mathbb{Z}[1/2]) \leq \#\mathcal{Y}(\mathbb{Z}_p)_{\{2\},2} \leq 48(p-2)\kappa_p = 48\Big(p-2+\frac{p-1}{\log(p)}\Big).$$

We actually know explicit equations cutting out the refined Chabauty–Kim loci $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ for $\#S \leq 2$ by the work of Best–Betts–Kumpitsch–Lüdtke–McAndrew–Qian–Studnia–Xu [11]. Namely, $\mathcal{Y}(\mathbb{Z}_p)_{\{2\},2}$ consists of the common solutions in $\mathcal{Y}(\mathbb{Z}_p)$ of the two equations

$$\log(z) = 0, \quad \mathrm{Li}_2(z) = 0,$$

along with their translates under the natural S_3-action [11, Theorem A]. The p-adic logarithm $\log(z)$, which is Coleman algebraic of weight 2, is the function whose existence is predicted by Theorem B for one of the three reduction types. Indeed, it vanishes on $\{-1\}$, the set of $\{2\}$-integral points reducing to the cusp 1 modulo 2. The dilogarithm $\mathrm{Li}_2(z)$ on the other hand is Coleman algebraic of weight 4, so its vanishing on $\{-1\}$ is not predicted by Theorem B. The reason that the results of this paper do not capture all information coming from the depth 2 Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ is that $U_{Y,2}$ has a subgroup isomorphic to $\mathbb{Q}_p(2)$, which is of weight -4, whereas the proofs of our theorems make use only of the weight ≥ -2 quotient of U_Y.

For the same reason, the results of this paper do not show the finiteness of $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ for $\#S = 2$, although we know by [11, Theorem B] that the locus is finite also for such S and defined by explicit Coleman algebraic functions of weight 4.

3 General Strategy

For the proofs of Theorems A–D, we follow Betts' strategy of exploiting weight filtrations on Selmer schemes. Specifically, Theorem A follows from a calculation of the dimensions of the local and global Selmer scheme, and Theorem B will follow from [12, Theorem 6.2.1 A)+B)], for suitable quotients $U_Y \twoheadrightarrow U$ of the fundamental group.

We briefly explain the strategy and sketch the arguments by which results in Chabauty–Kim theory follow from (abelian) Galois cohomology calculations. The reader who is willing to apply Betts' machinery as a black box may skip this section.

We write $G_v := G_{\mathbb{Q}_v}$ for a place v of $\mathbb{Q}$. Let U be a $G_{\mathbb{Q}}$-equivariant quotient of U_Y. The local Bloch–Kato Selmer scheme $\mathrm{H}^1_f(G_p, U)$ consists of the crystalline classes in $\mathrm{H}^1(G_p, U)$. Let $\mathrm{H}^1_f(G_{\mathbb{Q}}, U)$ denote the subspace of $\mathrm{H}^1(G_{\mathbb{Q}}, U)$ containing those classes that are crystalline at p and unramified at all other places. Recall from [12, §6.2] that

$\mathcal{Y}(\mathbb{Z}_S)$ can be partitioned according to reduction types. Two S-integral points have the same reduction type if and only if for all primes ℓ, their mod-ℓ reductions are either two non-cuspidal points on the same irreducible component or are the same cuspidal point. Here, mod-ℓ reduction refers to the special fibre of $\mathcal{X}$ (if $\ell \notin S$), respectively of the minimal regular normal crossings model of (X, D) over $\mathbb{Z}_\ell$ (if $\ell \in S$; see [15, Appendix B]).

For every reduction type Σ, Betts–Dogra define the global refined Selmer scheme $\mathrm{Sel}_{\Sigma,U} \subset \mathrm{H}^1(G_\mathbb{Q}, U)$ in [14, Definition 1.2.2] (see also [12, §3.2, §6.2]). Then we have a commutative diagram as follows [12, §6.2]:

$$\begin{array}{ccc} \mathcal{Y}(\mathbb{Z}_S)_\Sigma & \hookrightarrow & \mathcal{Y}(\mathbb{Z}_p) \\ \downarrow{\scriptstyle j_S} & & \downarrow{\scriptstyle j_p} \\ \mathrm{Sel}_{\Sigma,U}(\mathbb{Q}_p) & \xrightarrow{\mathrm{loc}_p} & \mathrm{H}^1_f(G_p, U)(\mathbb{Q}_p). \end{array}$$

The map loc_p in the bottom row is an algebraic map of affine $\mathbb{Q}_p$-schemes.

Definition 3.1. The *refined Chabauty–Kim locus* $\mathcal{Y}(\mathbb{Z}_p)_{S,U,\Sigma}$ for the reduction type Σ is the subset of those points of $\mathcal{Y}(\mathbb{Z}_p)$ whose image in $\mathrm{H}^1_f(G_p, U)$ is contained in the scheme-theoretic image of $\mathrm{Sel}_{\Sigma,U}$ under the localisation map. The total *refined Chabauty–Kim locus* $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$ is defined as the union over all reduction types

$$\mathcal{Y}(\mathbb{Z}_p)_{S,U} := \bigcup_{\Sigma} \mathcal{Y}(\mathbb{Z}_p)_{S,U,\Sigma}.$$

We have the inclusion $\mathcal{Y}(\mathbb{Z}_S)_\Sigma \subseteq \mathcal{Y}(\mathbb{Z}_p)_{S,U,\Sigma}$ for all reduction types Σ, and hence

$$\mathcal{Y}(\mathbb{Z}_S) \subseteq \mathcal{Y}(\mathbb{Z}_p)_{S,U}.$$

In particular, finiteness results and size bounds for the Chabauty–Kim loci imply the same for the set of S-integral points.

The set $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$ is finite whenever the localisation map loc_p is not dominant, for example when

$$\dim \mathrm{Sel}_{\Sigma,U} < \dim \mathrm{H}^1_f(G_p, U)$$

for all Σ. The dimensions of these non-abelian cohomology groups can be controlled by calculating dimensions of abelian cohomology groups arising as graded pieces of the weight filtration as follows.

The pro-unipotent group U carries a weight filtration by subgroups [12, Lemma 2.1.8]

$$\ldots \subseteq W_{-2}U \subseteq W_{-1}U = U,$$

such that $[W_{-i}U, W_{-j}U] \subseteq W_{-(i+j)}U$ for all $i,j \geq 1$. The graded pieces $\mathrm{gr}^W_{-k} U = W_{-k}U/W_{-k-1}U$ are representations of $G_{\mathbb{Q}}$ on finite-dimensional $\mathbb{Q}_p$-vector spaces.

Lemma 3.2. Let $U_Y \twoheadrightarrow U$ be a finite-dimensional $G_{\mathbb{Q}}$-equivariant quotient. Then the dimensions of the local and global Selmer scheme satisfy

$$\dim \mathrm{Sel}_{\Sigma,U} \leq s + \sum_{k=1}^{\infty} \dim \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-k} U),$$
$$\dim \mathrm{H}^1_f(G_p, U) = \sum_{k=1}^{\infty} \dim \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-k} U).$$

In particular, if

$$\sum_{k=1}^{\infty}\big(\dim \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-k} U) - \dim \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-k} U)\big) - s > 0, \tag{3.1}$$

then $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$ is finite.

Proof. By [12, Lemma 3.2.6], each of the spaces $\mathrm{Sel}_{\Sigma,U}$ is (non-canonically) a closed subscheme of $\prod_{k=1}^{\infty} \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-k} U) \times \prod_{\ell \neq p} \mathfrak{S}_{\Sigma_\ell}$, where each $\mathfrak{S}_{\Sigma_\ell}$ is empty, a single point, or a curve of genus 0 [12, Lemma 6.1.4]. (Here, the vector spaces $\mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-k} U)$ are viewed as affine schemes over $\mathbb{Q}_p$.) The latter can happen only for $\ell \in S$ by [12, Lemma 6.1.1]. This implies the upper bound on $\dim \mathrm{Sel}_{\Sigma,U}$. By [12, Corollary 3.1.11], the local Selmer scheme $\mathrm{H}^1_f(G_p, U)$ is (non-canonically) isomorphic to $\prod_{k=1}^{\infty} \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-k} U)$, which implies the claim on its dimension.

If (3.1) is satisfied, then $\mathrm{Sel}_{\Sigma,U}$ has strictly smaller dimension than $\mathrm{H}^1_f(G_p, U)$ for every reduction type Σ. The localisation map is thus not dominant, which implies the finiteness of $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$. ■

Part A(1) of Theorem A will follow from Lemma 3.2 applied to the quotient U_Y^{ab} of U_Y. For part (2)A, we will construct an intermediate quotient of $U_{Y,2} \twoheadrightarrow U_{Y,1} = U_Y^{\mathrm{ab}}$.

In order to guarantee the existence of a Coleman algebraic function of a certain weight vanishing on S-integral points, as in Theorem B, Betts defines the following Hilbert series in $\mathbb{N}_0[\![t]\!]$ associated to U:

$$\mathrm{HS}_{\mathrm{glob}}(t) := (1-t^2)^{-s} \prod_{k=1}^{\infty} (1-t^k)^{-\dim \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-k} U)},$$

$$\mathrm{HS}_{\mathrm{loc}}(t) := \prod_{k=1}^{\infty} (1-t^k)^{-\dim \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-k} U)}.$$

Denote their coefficients by c_i^{glob} and c_i^{loc}, respectively. The weight filtration on U by subgroups induces a weight filtration on the $\mathbb{Q}_p$-algebra $\mathcal{O}(U)$ [12, Example 2.1.5], which in turn induces weight filtrations on the rings of functions of the global and local Selmer scheme. The coefficients c_i^{glob} and c_i^{loc} are upper bounds for, respectively are equal to, the dimensions of the weight-graded pieces of the latter:

Lemma 3.3. For all $i \geq 0$, we have

$$\dim \mathrm{gr}_i^W \mathcal{O}(\mathrm{Sel}_{\Sigma,U}) \leq c_i^{\mathrm{glob}},$$

$$\dim \mathrm{gr}_i^W \mathcal{O}(\mathrm{H}^1_f(G_p, U)) = c_i^{\mathrm{loc}}.$$

Proof (sketch). For every affine scheme $X/\mathbb{Q}_p$ whose ring of functions is equipped with a weight filtration $W_\bullet$, we can define its Hilbert series as the generating function of the dimensions of the weight-graded pieces:

$$\mathrm{HS}_X(t) := \sum_{i=0}^{\infty} \dim(\mathrm{gr}_i^W \mathcal{O}(X))\, t^i \in \mathbb{N}_0^{\infty}[\![t]\!].$$

The claim is thus equivalent to

$$\mathrm{HS}_{\mathrm{Sel}_{\Sigma,U}}(t) \preceq \mathrm{HS}_{\mathrm{glob}}(t),$$

$$\mathrm{HS}_{\mathrm{H}^1_f(G_p,U)}(t) = \mathrm{HS}_{\mathrm{loc}}(t),$$

where $\preceq$ denotes coefficient-wise inequality.

In the case where $X = V = \prod_{k=1}^{\infty} V_{-k}$ is a weight-graded $\mathbb{Q}_p$-vector space, viewed as an affine $\mathbb{Q}_p$-scheme, there is an induced weight filtration on $\mathcal{O}(V) = \mathrm{Sym}(V)^\vee$, and the Hilbert series is given by

$$\mathrm{HS}_V(t) = \prod_{k>0} \mathrm{HS}_{V_{-k}}(t) = \prod_{k>0} (1-t^k)^{-\dim V_{-k}}.$$

As in the proof of Lemma 3.2, the local Selmer scheme $\mathrm{H}^1_f(G_p, U)$ is non-canonically isomorphic to $\prod_{k=1}^{\infty} \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-k} U)$, compatibly with the weight filtrations on rings of functions if the k-th factor is placed in weight $-k$. This implies the equality of their Hilbert series, the latter being precisely $\mathrm{HS}_{\mathrm{loc}}(t)$.

The global Selmer scheme $\mathrm{Sel}_{\Sigma,U}$ is non-canonically a closed subscheme of $\prod_{k=1}^{\infty} \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-k} U) \times \prod_{\ell \neq p} \mathfrak{S}_{\Sigma_\ell}$ as above. This implies the inequality of Hilbert series

$$\mathrm{HS}_{\mathrm{Sel}_{\Sigma,U}} \preceq \prod_{k=1}^{\infty} (1-t^k)^{-\dim \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-k} U)} \cdot \prod_{\ell \neq p} \mathrm{HS}_{\mathfrak{S}_{\Sigma_\ell}}(t).$$

For $\ell \notin S$, the scheme $\mathfrak{S}_{\Sigma_\ell}$ is either empty or a single point [12, Lemma 6.1.1]. For $\ell \in S$, we have the coefficient-wise bound [12, Lemma 6.1.5]

$$\mathrm{HS}_{\mathfrak{S}_{\Sigma_\ell}}(t) \preceq (1-t^2)^{-1}.$$

The bound $\mathrm{HS}_{\mathrm{Sel}_{\Sigma,U}}(t) \preceq \mathrm{HS}_{\mathrm{glob}}(t)$ follows. ■

Lemma 3.4 ([12, Theorem 6.2.1 A)]). Assume that $\sum_{i=0}^{m} c_i^{\mathrm{glob}} < \sum_{i=0}^{m} c_i^{\mathrm{loc}}$ for some positive integer m. Then for each reduction type Σ, there exists a nonzero Coleman algebraic function of weight at most m that vanishes on $\mathcal{Y}(\mathbb{Z}_p)_{S,U,\Sigma}$.

Proof (sketch) For each reduction type Σ, the pullback map along the localisation map

$$\mathrm{loc}_p^\sharp \colon \mathcal{O}(\mathrm{H}^1_f(G_p, U)) \to \mathcal{O}(\mathrm{Sel}_{\Sigma,U})$$

is filtered with respect to the weight filtrations on both rings, that is, maps functions of weight at most m to functions of weight at most m. The spaces of functions of bounded weight are finite-dimensional. If we have the inequality

$$\dim W_m \mathcal{O}(\mathrm{Sel}_{\Sigma,U}) < \dim W_m \mathcal{O}(\mathrm{H}^1_f(G_p, U)), \tag{3.2}$$

there is a nonzero element f of $W_m\mathcal{O}(\mathrm{H}^1_f(G_p, U))$ such that $f \circ \mathrm{loc}_p = 0$, and then $f \circ j_p$ is a nonzero Coleman algebraic function on $\mathcal{Y}(\mathbb{Z}_p)$ of weight at most m that vanishes on $\mathcal{Y}(\mathbb{Z}_p)_{S,U,\Sigma}$.

By Lemma 3.3, we have

$$\dim \mathrm{gr}_i^W \mathcal{O}(\mathrm{Sel}_{\Sigma,U}) \le c_i^{\mathrm{glob}},$$
$$\dim \mathrm{gr}_i^W \mathcal{O}(\mathrm{H}^1_f(G_p, U)) = c_i^{\mathrm{loc}}$$

for all $i \ge 0$. Hence, the inequality (3.2) is satisfied whenever $\sum_{i=0}^m c_i^{\mathrm{glob}} < \sum_{i=0}^m c_i^{\mathrm{loc}}$. ■

4 Weight-Graded Pieces of U_Y^{ab}

The inclusion $Y \hookrightarrow X$ induces a surjection of $\mathbb{Q}_p$-pro-unipotent fundamental groups $U_Y \twoheadrightarrow U_X$ and hence a surjection on their abelianisations: $U_Y^{\mathrm{ab}} \twoheadrightarrow U_X^{\mathrm{ab}}$. The latter group is the rational p-adic Tate module of the Jacobian of X:

$$U_X^{\mathrm{ab}} = V_p \,\mathrm{Jac}_X = \big(\varprojlim_k \mathrm{Jac}_X[p^k](\overline{\mathbb{Q}})\big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

The inclusions of the cusps induce a map

$$I := \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1) \to U_Y^{\mathrm{ab}},$$

where $\mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}$ denotes the Galois module of maps $D(\overline{\mathbb{Q}}) \to \mathbb{Q}_p(1)$, and $\mathbb{Q}_p(1)$ is embedded diagonally, that is, as the constant maps. This yields a short exact sequence

$$1 \longrightarrow \underbrace{I}_{\mathrm{gr}_{-2}^W U_Y^{\mathrm{ab}}} \longrightarrow U_Y^{\mathrm{ab}} \longrightarrow \underbrace{V_p \,\mathrm{Jac}_X}_{\mathrm{gr}_{-1}^W U_Y^{\mathrm{ab}}} \longrightarrow 1,$$

which exhibits U_Y^{ab} as an extension of $V_p \,\mathrm{Jac}_X$ in weight -1 by I in weight -2. We now calculate the dimensions of the global and local Galois cohomology groups of these weight-graded pieces.

First, the Galois cohomology of the Tate module follows from the work of Bloch–Kato [16, Section 3], see [19, Theorem 3.11, Corollary 3.12]:

$$\dim \mathrm{H}^1_f(G_{\mathbb{Q}}, V_p \,\mathrm{Jac}_X) = r_p, \tag{4.1}$$
$$\dim \mathrm{H}^1_f(G_p, V_p \,\mathrm{Jac}_X) = g. \tag{4.2}$$

We now compute the Galois cohomology of the cuspidal inertia I.

Lemma 4.1 (Galois cohomology of cuspidal inertia).

(a) $\dim \mathrm{H}^1_f(G_{\mathbb{Q}}, I) = n_1 + n_2 - \#|D|$,
(b) $\dim \mathrm{H}^1_f(G_p, I) = n - 1$.

Proof. The sequence of Galois representations

$$0 \longrightarrow \mathbb{Q}_p(1) \longrightarrow \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})} \longrightarrow I \longrightarrow 0$$

is split exact: a retraction $\mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})} \to \mathbb{Q}_p(1)$ is given by the averaging map $(a_x)_{x\in D(\overline{\mathbb{Q}})} \mapsto \frac{1}{n}\sum_x a_x$. Hence, it remains exact after applying $\mathrm{H}^1_f(G_{\mathbb{Q}}, -)$ or $\mathrm{H}^1_f(G_p, -)$ and we have

$$\dim \mathrm{H}^1_f(G_{\mathbb{Q}}, I) = \dim \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}) - \dim \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathbb{Q}_p(1)),$$
$$\dim \mathrm{H}^1_f(G_p, I) = \dim \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}) - \dim \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)).$$

To calculate the cohomology of $\mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}$, note that $\mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})} = (\pi_D)_*\mathbb{Q}_p(1)$, where $\pi_D\colon D \to \mathrm{Spec}(\mathbb{Q})$ is the structural morphism. We get

$$\mathrm{H}^1(G_{\mathbb{Q}}, \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}) = \mathrm{H}^1(D, \mathbb{Q}_p(1)) = \bigoplus_{x\in|D|} \mathrm{H}^1(\kappa(x), \mathbb{Q}_p(1)),$$

where $\kappa(x)$ is the residue field of the cusp x. This implies

$$\mathrm{H}^1_f(G_{\mathbb{Q}}, \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}) = \bigoplus_{x\in|D|} \mathrm{H}^1_f(\kappa(x), \mathbb{Q}_p(1)),$$

where $\mathrm{H}^1_f(\kappa(x), \mathbb{Q}_p(1))$ is the subspace of cohomology classes that are crystalline at all places of $\kappa(x)$ dividing p, and unramified at all other places. These are precisely the classes in the image of

$$\widehat{\mathcal{O}^\times_{\kappa(x)}} \otimes \mathbb{Q}_p \hookrightarrow \widehat{\kappa(x)^\times} \otimes \mathbb{Q}_p = \mathrm{H}^1(\kappa(x), \mathbb{Q}_p(1)),$$

where $\widehat{M} := \varprojlim M/p^k M$ denotes the p-adic completion. By the Dirichlet Unit Theorem, we have

$$\dim_{\mathbb{Q}_p}(\widehat{\mathcal{O}^\times_{\kappa(x)}} \otimes \mathbb{Q}_p) = r_1(x) + r_2(x) - 1$$

with $r_1(x)$ and $r_2(x)$ denoting the number of real embeddings, respectively pairs of complex embeddings of $\kappa(x)$. Taking everything together, we find

$$\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}) = \sum_{x \in |D|} (r_1(x) + r_2(x) - 1) = n_1 + n_2 - \#|D|.$$

Together with $\mathrm{H}^1_f(G_{\mathbb{Q}}, \mathbb{Q}_p(1)) = 0$, this implies (a).

For the local cohomology group, we calculate

$$\mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}) = \bigoplus_{x \in |D_{\mathbb{Q}_p}|} \mathrm{H}^1_f(\kappa(x), \mathbb{Q}_p(1)) = \bigoplus_{x \in |D_{\mathbb{Q}_p}|} \widehat{\mathcal{O}^{\times}_{\kappa(x)}} \otimes \mathbb{Q}_p,$$

whose dimension is

$$\begin{aligned}\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}) &= \sum_{x \in |D_{\mathbb{Q}_p}|} \dim_{\mathbb{Q}_p} \big(\widehat{\mathcal{O}^{\times}_{\kappa(x)}} \otimes \mathbb{Q}_p\big) \\ &= \sum_{x \in |D_{\mathbb{Q}_p}|} [\kappa(x) : \mathbb{Q}_p] \\ &= n.\end{aligned}$$

Since $\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)) = 1$, this implies (b). ■

Remark 4.2. The proof of Lemma 4.1 simplifies if we assume that all cusps are rational: in this case $\mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}$ is a direct sum of copies of $\mathbb{Q}_p(1)$ rather than a twisted form of it.

Having calculated the dimensions of Galois cohomology of the weight-graded pieces of U_Y^{ab}, we can now prove part A(1) of Theorem A.

Proof of Theorem A(1). By Lemma 3.2, $\mathcal{Y}(\mathbb{Z}_p)_{S,1}$ is finite whenever

$$\begin{aligned}0 &< \sum_{k=1}^{2} \big(\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-k} U_Y^{\mathrm{ab}}) - \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-k} U_Y^{\mathrm{ab}})\big) - s \\ &= (g - r_p) + ((n-1) - (n_1 + n_2 - \#|D|)) - s \qquad \text{((4.1), (4.2), and Lemma 4.1)} \\ &= g - r_p + \#|D| + n_2 - 1 - s.\end{aligned}$$

■

Remark 4.3. Instead of working with the full abelianisation U_Y^{ab}, we can form the pushout along the $G_{\mathbb{Q}}$-equivariant map

$$\mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1) = \Big(\bigoplus_{x\in|D|} \mathbb{Q}_p(1)^{x(\overline{\mathbb{Q}})}\Big)/\mathbb{Q}_p(1) \twoheadrightarrow \Big(\bigoplus_{x\in|D|} \mathbb{Q}_p(1)\Big)/\mathbb{Q}_p(1)$$

that takes the average on each Galois orbit of cusps. This produces an extension U of $V_p\,\mathrm{Jac}_X$ by $\mathbb{Q}_p(1)^{|D|}/\mathbb{Q}_p(1)$. The cohomology of the latter is easier to calculate:

$$\dim \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathbb{Q}_p(1)^{|D|}/\mathbb{Q}_p(1)) = 0,$$
$$\dim \mathrm{H}^1_f(G_p, \mathbb{Q}_p(1)^{|D|}/\mathbb{Q}_p(1)) = \#|D| - 1.$$

The refined Chabauty–Kim locus $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$ associated to this quotient U is finite whenever $0 < g - r_p + \#|D| - 1 - s$. This is more restrictive compared to the full abelianisation U_Y^{ab}, where the right-hand side contains an additional summand of n_2. But if all cusps are real, that is, if $n_2 = 0$, the conditions ensuring finiteness actually agree.

It would be interesting to study in more detail what kind of Coleman functions appear in the equations defining the locus $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$ (and similarly for the full abelianisation U_Y^{ab}). In addition to the weight 1 functions arising from U_X, these are of weight 2 by the above. For instance, for the thrice-punctured line $\mathcal{Y} := \mathbb{P}^1_{\mathbb{Z}[1/2]} \setminus \{0, 1, \infty\}$, we have that $\mathcal{Y}(\mathbb{Z}_p)_{\{2\},1,\Sigma}$ is cut out by the weight 2 function $\log(z)$ for some reduction type Σ by [11, Proposition 3.1].

5 The Abelian-by-Artin–Tate Quotient

The key step in the proofs of Theorem A(2) and of Theorem B is the construction of a suitable quotient $U_Y \twoheadrightarrow U$ of the fundamental group of Y, which is inspired by the proof of [6, Proposition 2.2]. It lies between the depth 1 and depth 2 quotient,

$$U_{Y,2} \twoheadrightarrow U \twoheadrightarrow U_{Y,1} = U_Y^{\mathrm{ab}},$$

and can be described as the largest quotient of U_Y of weight ≥ -2, which is a central extension of an abelian group by an Artin–Tate representation. (We call a $G_{\mathbb{Q}}$-representation *Artin–Tate* if it is a Tate twist of an Artin (i.e., finite image) representation.)

Lemma 5.1. There exists a $G_{\mathbb{Q}}$-equivariant quotient $U_Y \twoheadrightarrow U$ that is a central extension of $V_p\,\mathrm{Jac}_X$ by $(\mathbb{Q}_p \otimes \mathrm{NS}(\mathrm{Jac}_{X_{\overline{\mathbb{Q}}}}))^\vee(1) \oplus \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1)$.

Proof. We construct U as a quotient of the weight ≥ -2 quotient $U_Y/W_{-3}U_Y$, which is a central extension as follows:

$$1 \longrightarrow \underbrace{\bigwedge^2 V_p \operatorname{Jac}_X \oplus \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1)}_{\mathrm{gr}^W_{-2} U_Y} \longrightarrow U_Y/W_{-3}U_Y \longrightarrow \underbrace{V_p \operatorname{Jac}_X}_{\mathrm{gr}^W_{-1} U_Y} \longrightarrow 1. \tag{5.1}$$

The proof of [13, Lemma 2.10] identifies $\mathbb{Q}_p \otimes \mathrm{NS}(\operatorname{Jac}_{X_L})$ with $\operatorname{Hom}_{G_L}(\bigwedge^2 V_p \operatorname{Jac}_X, \mathbb{Q}_p(1))$ for any number field L and thus

$$\mathbb{Q}_p \otimes \mathrm{NS}(\operatorname{Jac}_{X_{\overline{\mathbb{Q}}}}) \cong \operatorname{colim}_L \operatorname{Hom}_{G_L}(\bigwedge^2 V_p \operatorname{Jac}_X, \mathbb{Q}_p(1)) \subset \operatorname{Hom}(\bigwedge^2 V_p \operatorname{Jac}_X, \mathbb{Q}_p(1)). \tag{5.2}$$

Therefore, we have a $G_{\mathbb{Q}}$-equivariant surjection $\bigwedge^2 V_p \operatorname{Jac}_X \twoheadrightarrow (\mathbb{Q}_p \otimes \mathrm{NS}(\operatorname{Jac}_{X_{\overline{\mathbb{Q}}}}))^\vee(1)$. Since the extension (5.1) is central, the kernel is normal in $U_Y/W_{-3}U_Y$ and we can form a pushout:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{gr}^W_{-2} U_Y & \longrightarrow & U_Y/W_{-3}U_Y & \longrightarrow & \mathrm{gr}^W_{-1} U_Y & \longrightarrow & 1 \\
 & & \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow & & \| & & \\
1 & \longrightarrow & (\mathbb{Q}_p \otimes \mathrm{NS}(\operatorname{Jac}_{X_{\overline{\mathbb{Q}}}}))^\vee(1) \oplus \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1) & \longrightarrow & U & \longrightarrow & V_p \operatorname{Jac}_X & \longrightarrow & 1.
\end{array} \tag{5.3}$$

The resulting quotient U of U_Y is the desired extension. ■

Remark 5.2. As pointed out to us by Alex Betts, the quotient U constructed in Lemma 5.1 is of geometric origin, in that there exists a smooth connected variety $E/\mathbb{Q}$ whose $\mathbb{Q}_p$-pro-unipotent fundamental group is equal to U, and there is a morphism $f\colon Y \to E$, which induces the quotient map $f_*\colon U_Y \twoheadrightarrow U_E = U$. This variety can be constructed as a torsor $E \to J_X$ under a torus T. The pullback of this torsor along the Abel–Jacobi map is trivial, giving rise to the morphism $f\colon Y \to E$. This construction generalises the $\mathbb{G}_m^{\rho-1}$-torsor over J_X of Edixhoven–Lido [20] in their geometric quadratic Chabauty method. The generalisation is two-fold: firstly, T may be a non-split torus (so the fundamental group of T is Artin–Tate rather than Tate); secondly, our curve Y, in contrast with the setting of Edixhoven–Lido, is affine rather than projective (so T contains the toric part of the generalised Jacobian of Y).

The group U from Lemma 5.1 sits between $U_{Y,2}$ and $U_{Y,1}$ as follows:

$$U_{Y,2} \twoheadrightarrow U_Y/W_{-3}U_Y \twoheadrightarrow U \twoheadrightarrow U_{Y,1}.$$

In particular, we have inclusions of Chabauty–Kim loci

$$\mathcal{Y}(\mathbb{Z}_p)_{S,2} \subseteq \mathcal{Y}(\mathbb{Z}_p)_{S,U} \subseteq \mathcal{Y}(\mathbb{Z}_p)_{S,1}.$$

In order to prove Theorem A(2) and Theorem B, we need two preparatory lemmas that allow us to calculate the Selmer dimensions in weight -2.

Lemma 5.3.

(a) Let K be a finite extension of $\mathbb{Q}_\ell$ and L/K a finite Galois extension with Galois group G. Let V be a representation of G_K on a finite dimensional $\mathbb{Q}_p$-vector space ($\ell = p$ is allowed). Then the restriction is an isomorphism

$$\mathrm{H}^1_f(G_K, V) \cong \mathrm{H}^1_f(G_L, V)^G.$$

(b) Let K be a number field and L/K a finite Galois extension with Galois group G. Let V be a representation of G_K on a finite dimensional $\mathbb{Q}_p$-vector space. Then the restriction is an isomorphism

$$\mathrm{H}^1_f(G_K, V) \cong \mathrm{H}^1_f(G_L, V)^G.$$

Proof. We use inflation–restriction, which also works for continuous group cohomology by [25, Theorem 10.26]. Let us start with (a). Then we have the exact sequence

$$0 \longrightarrow \mathrm{H}^1(G, V^{G_L}) \longrightarrow \mathrm{H}^1(G_K, V) \longrightarrow \mathrm{H}^1(G_L, V)^G \longrightarrow \mathrm{H}^2(G, V^{G_L}).$$

Multiplication by $\#G$ is the zero map on $\mathrm{H}^i(G, V^{G_L})$ for all $i \geq 1$, but it is also an isomorphism, since V^{G_L} is a $\mathbb{Q}_p$-vector space. Thus, restriction is an isomorphism

$$\mathrm{H}^1(G_K, V) \cong \mathrm{H}^1(G_L, V)^G.$$

To see that this isomorphism restricts to an isomorphism on H^1_f, we distinguish the cases $\ell \neq p$ and $\ell = p$. If $\ell \neq p$, an analogous inflation–restriction argument yields an isomorphism

$$\mathrm{H}^1(I_K, V) \cong \mathrm{H}^1(I_L, V)^{I_{L/K}},$$

where $I_K \subset G_K, I_L = I_K \cap G_L \subset G_L$, and $I_{L/K} \subset G$ are the inertia subgroups. By the definition of H^1_f for $\ell \neq p$, the rows in the following commutative diagram are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^1_f(G_K, V) & \longrightarrow & \mathrm{H}^1(G_K, V) & \longrightarrow & \mathrm{H}^1(I_K, V) \\ \Big\| & & \downarrow & & \sim\downarrow \mathrm{res} & & \downarrow \\ 0 & \longrightarrow & \mathrm{H}^1_f(G_L, V)^G & \longrightarrow & \mathrm{H}^1(G_L, V)^G & \longrightarrow & \mathrm{H}^1(I_L, V). \end{array} \tag{5.4}$$

We conclude by using the Four Lemma. The case $\ell = p$ is similar, replacing the right vertical arrow in (5.4) by $\mathrm{H}^1(G_K, V \otimes_{\mathbb{Q}_p} \mathrm{B}_{cris}) \to \mathrm{H}^1(G_L, V \otimes_{\mathbb{Q}_p} \mathrm{B}_{cris})$ and arguing as before.

For (b), inflation–restriction yields an isomorphism

$$\mathrm{H}^1(G_K, V) \cong \mathrm{H}^1(G_L, V)^G$$

in the same manner as above. Using the definition of the global Selmer groups, the rows in the following commutative diagram are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{H}^1_f(G_K, V) & \longrightarrow & \mathrm{H}^1(G_K, V) & \longrightarrow & \prod_v \mathrm{H}^1(G_{K_v}, V)/\mathrm{H}^1_f(G_{K_v}, V) \\ \Big\| & & \downarrow & & \sim\downarrow \mathrm{res} & & \downarrow \prod_v \prod_{w|v} \mathrm{res}^{L_w}_{K_v} \\ 0 & \longrightarrow & \mathrm{H}^1_f(G_L, V)^G & \longrightarrow & \mathrm{H}^1(G_L, V)^G & \longrightarrow & \prod_v \prod_{w|v} \mathrm{H}^1(G_{L_w}, V)/\mathrm{H}^1_f(G_{L_w}, V), \end{array} \tag{5.5}$$

where v runs over the finite places of K, and the horizontal arrows on the right are given by restricting to all decomposition groups. Using part (a) for every local Galois extension L_w/K_v, with $\mathrm{Gal}(L_w/K_v) \subset G$ as the decomposition group at w, we see that the rightmost vertical arrow in (5.5) is injective. So we apply the Four Lemma again to finish the proof. ■

Lemma 5.4. Let W be a finite-dimensional $\mathbb{Q}_p$-representation of $G_{\mathbb{Q}}$ such that $W^\vee(1)$ is an Artin representation, that is, has finite image. Then

(a) $\dim \mathrm{H}^1_f(G_p, W) = \dim W$,

(b) $\dim \mathrm{H}^1_f(G_{\mathbb{Q}}, W) = -\dim \mathrm{H}^0(G_{\mathbb{Q}}, W^\vee(1)) + \dim W - \dim W^{\sigma=1}$.

Proof. By assumption, there is a finite Galois extension $L/\mathbb{Q}$ with Galois group G such that $N := W^\vee(1)$ restricted to G_L is the trivial representation $\mathbb{Q}_p^d$, where $d = \dim N = \dim W$. Note that $W = N^\vee(1)$ is equal to $\mathbb{Q}_p(1)^d$ when restricted to G_L.

For (a), let $L_p = L\mathbb{Q}_p$ and $D_p = \mathrm{Gal}(L_p/\mathbb{Q}_p) \subset G$. Then part (a) of Lemma 5.3 yields $\mathrm{H}^1_f(G_p, W) \cong \mathrm{H}^1_f(G_{L_p}, \mathbb{Q}_p(1)^d)^{D_p}$. By [8, Prop. 2.9], the Kummer map gives an identification of $\mathrm{H}^1_f(G_{L_p}, \mathbb{Q}_p(1))$ with $\widehat{\mathcal{O}_{L_p}^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ in a D_p-equivariant way, so (a) follows from

$$\mathrm{H}^1_f(G_{L_p}, \mathbb{Q}_p(1)^d)^{D_p} = (\widehat{\mathbb{Z}_p^\times} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)^d = \mathbb{Q}_p^d.$$

For (b), we use the following consequence of Poitou–Tate duality [13, Fact 2.9]: Let W be a geometric Galois representation, that is, W is unramified at almost all places and de Rham at p. Then

$$\begin{aligned} \dim \mathrm{H}^1_f(G_\mathbb{Q}, W) = \dim \mathrm{H}^0(G_\mathbb{Q}, W) + \dim \mathrm{H}^1_f(G_\mathbb{Q}, W^\vee(1)) \\ - \dim \mathrm{H}^0(G_\mathbb{Q}, W^\vee(1)) + \dim \mathrm{H}^1_f(G_p, W) - \dim W^{\sigma=1}, \end{aligned} \tag{5.6}$$

where σ is complex conjugation (for some embedding $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$). The given W is geometric as $W^\vee(1)$ is Artin. The first summand $\dim \mathrm{H}^0(G_\mathbb{Q}, W)$ vanishes as W is pure of weight -2. The representation $W^\vee(1)$ is trivial when restricted to G_L, thus part (b) of Lemma 5.3 yields $\mathrm{H}^1_f(G_\mathbb{Q}, W^\vee(1)) \cong \mathrm{H}^1_f(G_L, \mathbb{Q}_p^d)^G$. But $\mathrm{H}^1_f(G_L, \mathbb{Q}_p) = 0$ by [8, Exercise 2.24.a]. Thus, (5.6) and (a) imply (b). ■

Proof of Theorem A(2). Let $U_Y \twoheadrightarrow U$ be the quotient from Lemma 5.1. By construction, its weight-graded pieces are given by

$$\begin{aligned} \mathrm{gr}^W_{-1} U &= V_p \operatorname{Jac}_X, \\ \mathrm{gr}^W_{-2} U &= (\mathbb{Q}_p \otimes \mathrm{NS}(\operatorname{Jac}_{X_{\overline{\mathbb{Q}}}}))^\vee(1) \oplus \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1). \end{aligned}$$

We calculate the dimensions of their Selmer groups. In weight -1, (4.1) and (4.2) yield

$$\dim \mathrm{H}^1_f(G_\mathbb{Q}, \mathrm{gr}^W_{-1} U) = r_p, \tag{5.7}$$

$$\dim \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-1} U) = g. \tag{5.8}$$

In weight -2, we use Lemma 5.4 with $W = (\mathbb{Q}_p \otimes \mathrm{NS}(\operatorname{Jac}_{X_{\overline{\mathbb{Q}}}}))^\vee(1)$. Then $W^\vee(1) = \mathbb{Q}_p \otimes \mathrm{NS}(\operatorname{Jac}_{X_{\overline{\mathbb{Q}}}})$ is an Artin representation because $\mathrm{NS}(\operatorname{Jac}_{X_{\overline{\mathbb{Q}}}})$ is a finitely generated abelian group. Note that $W^\vee(1)^{G_\mathbb{Q}} = \mathbb{Q}_p \otimes \mathrm{NS}(\mathrm{Jac}_X)$ and $W^{\sigma=1} = (\mathbb{Q}_p \otimes \mathrm{NS}(\mathrm{Jac}_{X_{\overline{\mathbb{Q}}}})^{\sigma=-1})^\vee(1)$. Together with Lemma 4.1, this yields

$$\dim \mathrm{H}^1_f(G_\mathbb{Q}, \mathrm{gr}^W_{-2} U) = n_1 + n_2 - \#|D| + \dim W - \rho_f, \tag{5.9}$$

$$\dim \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-2} U) = n - 1 + \dim W. \tag{5.10}$$

By Lemma 3.2, $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$ and thus $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ are finite whenever

$$\begin{aligned} 0 &< \sum_{k=1}^{2} \big(\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-k} U) - \mathrm{H}^1_f(G_\mathbb{Q}, \mathrm{gr}^W_{-k} U)\big) - s \\ &= (g - r_p) + ((n - 1 + \dim W) - (n_1 + n_2 - \#|D| + \dim W - \rho_f)) - s \\ &= g - r_p + \rho_f + \#|D| + n_2 - 1 - s. \end{aligned}$$

■

Proof of Theorem B. Again let $U_Y \twoheadrightarrow U$ be the quotient from Lemma 5.1. Having calculated the Galois cohomology dimensions of its weight-graded pieces in Eqs. (5.7)–(5.10), the global and local Hilbert series associated to U are given by

$$\begin{aligned} \mathrm{HS}_{\mathrm{glob}}(t) &= (1-t)^{-r_p}(1-t^2)^{-(s+n_1+n_2-\#|D|+\dim W-\rho_f)} \\ &= 1 + r_p t + (s + n_1 + n_2 - \#|D| + \dim W - \rho_f + \tfrac{1}{2} r_p(r_p+1))t^2 + \dots, \\ \mathrm{HS}_{\mathrm{loc}}(t) &= (1-t)^{-g}(1-t^2)^{-(\dim W + n - 1)} \\ &= 1 + gt + (\dim W + n - 1 + \tfrac{1}{2} g(g+1))t^2 + \dots \end{aligned}$$

If the coefficients satisfy the inequality $\sum_{i=0}^2 c_i^{\mathrm{glob}} < \sum_{i=0}^2 c_i^{\mathrm{loc}}$, then Lemma 3.4 applies and yields the existence, for every reduction type Σ, of a nonzero Coleman algebraic function of weight at most 2 vanishing on $\mathcal{Y}(\mathbb{Z}_p)_{S,U,\Sigma}$ and thus on $\mathcal{Y}(\mathbb{Z}_p)_{S,2,\Sigma}$. Abbreviating $d = \dim W$, we have:

$$\begin{aligned} &\sum_{i=0}^{2} c_i^{\mathrm{glob}} < \sum_{i=0}^{2} c_i^{\mathrm{loc}} \\ \Leftrightarrow\ & 1 + r_p + (s + n_1 + n_2 - \#|D| + d - \rho_f + \tfrac{1}{2} r_p(r_p+1)) < 1 + g + (d + n - 1 + \tfrac{1}{2} g(g+1)) \\ \Leftrightarrow\ & 0 < \tfrac{1}{2} g(g+3) - \tfrac{1}{2} r_p(r_p+3) + \rho_f + \#|D| + n_2 - 1 - s. \end{aligned}$$

This proves the existence statement of Theorem B.

Finally, the claimed bound on $\#\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ is the one obtained from [12, Theorem 6.2.1 B)] for $m = 2$, noting that the term c_1^{loc} appearing in the general formula is equal to

g by the calculation of the local Hilbert series above. Note that Betts' result is actually stated as a bound on $\#\mathcal{Y}(\mathbb{Z}_S)$ but the proof, which goes via bounding the number of zeros of a Coleman algebraic function of bounded weight, applies in fact to the superset $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$. ■

Remark 5.5. As mentioned in Remark 1.2, the same method can be used to show the depth 1 analogue of Theorem B. For this, one simply replaces the quotient $U_Y \twoheadrightarrow U$ constructed in Lemma 5.1 with the depth 1 quotient $U_{Y,1} = U_Y^{\mathrm{ab}}$.

Remark 5.6. One can prove weaker versions of Theorems A(2) and B, with ρ in place of ρ_f in the definition of $\alpha_2(Y, s, p)$ and $\beta(Y, s, p)$, by constructing a coarser quotient U' of $U_Y/W_{-3}U_Y$. Namely, the irreducible representation $\mathbb{Q}_p(1)$ occurs as a direct summand of the semisimple [12, Lemma 6.0.1] Galois representation $\bigwedge^2 V_p \operatorname{Jac}_X$ with multiplicity given by

$$\dim_{\mathbb{Q}_p} \operatorname{Hom}_{G_{\mathbb{Q}}}(\bigwedge^2 V_p \operatorname{Jac}_X, \mathbb{Q}_p(1)).$$

This dimension is equal to the Picard number ρ [13, Proof of Lemma 2.10]. Forming the pushout as in (5.3), we obtain the quotient U' of $U_Y/W_{-3}U_Y$ with $\mathrm{gr}^W_{-2}(U') = \mathbb{Q}_p(1)^{\rho} \oplus \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1)$. Its Selmer dimensions in weight -2 are therefore

$$\dim \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-2} U') = n_1 + n_2 - \#|D|,$$

$$\dim \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-2} U') = \rho + n - 1.$$

Now one proceeds as before to prove the analogues of Theorem A(2) and Theorem B.

Remark 5.7. Note that [6, Proposition 2.2], which we used as our starting point for the results of the present section, strengthens [4, Lemma 3.2], which has ρ in place of ρ_f. In fact, Balakrishnan–Dogra state in [4, Remark 3.3] that one can use the same method as in the proof of [4, Lemma 3.2] to show finiteness of $\mathcal{Y}(\mathbb{Z}_p)^{\mathrm{BD}}_{\varnothing,2}$ when (in our notation) $n = \#D(\mathbb{Q}) = 1$ and $g - r + \rho > 0$, but give no further details. One may view Theorem A(2) as a generalisation of this.

6 The Full Weight ≥ -2 Quotient

Let us consider the case where we choose for U the full weight ≥ -2 quotient of the fundamental group:

$$U := U_Y/W_{-3}U_Y.$$

As in (5.1) above, the graded piece of weight −2 is semisimple and isomorphic to a direct sum

$$\mathrm{gr}^W_{-2} U = \mathrm{gr}^W_{-2} U_Y \cong \bigwedge^2 V_p \,\mathrm{Jac}_X \oplus \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1).$$

The dimensions of the local and global Galois cohomology of $\mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1)$ have been calculated in Lemma 4.1, so we focus on the first summand. The dimension of the global cohomology group $\mathrm{H}^1_f(G_{\mathbb{Q}}, \bigwedge^2 V_p \,\mathrm{Jac}_X)$ involves the term h_{BK}, defined in (1.1).

Lemma 6.1 (Galois cohomology of wedge-squared Tate module).

1. $\dim \mathrm{H}^1_f(G_{\mathbb{Q}}, \bigwedge^2 V_p \,\mathrm{Jac}_X) = \frac{1}{2}g(g+1) - \rho + h_{\mathrm{BK}}$,
2. $\dim \mathrm{H}^1_f(G_p, \bigwedge^2 V_p \,\mathrm{Jac}_X) = \frac{1}{2}g(3g-1)$.

Proof. We start with the local dimension. By [13, Lemma 2.6], the local Hilbert series of Y is given by

$$\prod_{k=1}^{\infty}(1-t^k)^{-\dim \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-k} U_Y)} = \frac{1-gt}{1-2gt-(n-1)t^2}.$$

Expanding the power series up to the quadratic terms yields

$$1 + d_1 t + (\tfrac{1}{2}d_1(d_1+1) + d_2)t^2 + \ldots = 1 + gt + (2g^2 + n - 1)t^2 + \ldots,$$

where $d_k := \dim \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-k} U_Y)$. Comparing coefficients yields

$$d_1 = g, \quad d_2 = \tfrac{1}{2}g(3g-1) + n - 1.$$

Since $\mathrm{gr}^W_{-2} U_Y$ is a direct sum of $\bigwedge^2 V_p \,\mathrm{Jac}_X$ and $\mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1)$, and the dimension of the local cohomology of the latter is $n-1$ by Lemma 4.1, (b) follows.

For the global cohomology, let $W := \bigwedge^2 V_p \,\mathrm{Jac}_X$. We use (5.6) and go through the summands one by one. The first summand $\mathrm{H}^0(G_{\mathbb{Q}}, W)$ vanishes since W is pure of weight −2. The second summand is precisely h_{BK} since $W^\vee(1) = \mathrm{Hom}(W, \mathbb{Q}_p(1))$. The third is

$$\dim \mathrm{H}^0(G_{\mathbb{Q}}, W^\vee(1)) = \dim \mathrm{Hom}(\bigwedge^2 V_p \,\mathrm{Jac}_X, \mathbb{Q}_p(1))^{G_{\mathbb{Q}}} = \rho,$$

which we already used in the proof of Lemma 5.1. The local Galois cohomology $\mathrm{H}^1_f(G_p, W)$ has dimension $\frac{1}{2}g(3g-1)$, as we just proved. Finally, consider W as a representation of $G_\infty = \langle\sigma\rangle$. The two irreducible representations of G_∞ are the trivial representation

$\mathbf{1}$ and the sign representation, which we denote by ξ. The isomorphism $V_p \operatorname{Jac}_X \cong (V_p \operatorname{Jac}_X)^\vee(1)$ given by the Weil pairing implies that the trivial representation and the sign representation appear in $V_p \operatorname{Jac}_X$ with equal multiplicity, so we have

$$V_p \operatorname{Jac}_X \cong g \cdot \mathbf{1} \oplus g \cdot \xi$$

as a G_∞-representation, which implies

$$\bigwedge^2 V_p \operatorname{Jac}_X = g(g-1) \cdot \mathbf{1} \oplus g^2 \cdot \xi.$$

In particular,

$$\dim W^\sigma = g(g-1). \tag{6.1}$$

Putting everything together in (5.6) yields

$$\begin{aligned}\dim \mathrm{H}^1_f(G_\mathbb{Q}, \bigwedge^2 V_p \operatorname{Jac}_X) &= 0 + h_{\mathrm{BK}} - \rho + \tfrac{1}{2}g(3g-1) - g(g-1)\\ &= \tfrac{1}{2}g(g+1) - \rho + h_{\mathrm{BK}},\end{aligned}$$

as claimed. ■

Using these calculations, we can apply the general theory of Betts to the quotient $U = U_Y/W_{-3}U_Y$ to obtain a finiteness criterion and a criterion for the existence of weight 2 Coleman functions vanishing on depth 2 Chabauty–Kim loci.

Proof of Theorem C The weight ≥ -2 quotient $U := U_Y/W_{-3}U_Y$ is a quotient of $U_{Y,2}$, so we have $\mathcal{Y}(\mathbb{Z}_p)_{S,2} \subseteq \mathcal{Y}(\mathbb{Z}_p)_{S,U}$ and it suffices to show finiteness of the latter. The weight-graded pieces of U are given by

$$\begin{aligned}\mathrm{gr}^W_{-1} U &= \mathrm{gr}^W_{-1} U_Y = V_p \operatorname{Jac}_X,\\ \mathrm{gr}^W_{-2} U &= \mathrm{gr}^W_{-2} U_Y = \bigwedge^2 V_p \operatorname{Jac}_X \oplus \mathbb{Q}_p(1)^{D(\overline{\mathbb{Q}})}/\mathbb{Q}_p(1).\end{aligned}$$

The dimensions of their local and global Galois cohomology follow from (4.1), (4.2), Lemma 4.1, and Lemma 6.1. In weight -1, they are given by r_p (global) and g (local), as above. In weight -2, they are given by

$$\dim \mathrm{H}^1_f(G_\mathbb{Q}, \mathrm{gr}^W_{-2} U) = \tfrac{1}{2}g(g+1) - \rho + h_{\mathrm{BK}} + n_1 + n_2 - \#|D|, \tag{6.2}$$

$$\dim \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-2} U) = \tfrac{1}{2}g(3g-1) + n - 1. \tag{6.3}$$

By Lemma 3.2, the set $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$ is finite if

$$\begin{aligned}
0 &< \sum_{k=1}^{2}\Big(\dim_{\mathbb{Q}_p} \mathrm{H}^1_f(G_p, \mathrm{gr}^W_{-k} U) - \mathrm{H}^1_f(G_{\mathbb{Q}}, \mathrm{gr}^W_{-k} U)\Big) - s \\
&= (g - r_p) + ((\tfrac{1}{2}g(3g-1) + n - 1) - (\tfrac{1}{2}g(g+1) - \rho + h_{\mathrm{BK}} + n_1 + n_2 - \#|D|)) - s \\
&= g^2 - r_p + \rho + \#|D| + n_2 - 1 - h_{\mathrm{BK}} - s. \qquad \blacksquare
\end{aligned}$$

Proof of Theorem D. Choose $U := U_Y/W_{-3}U_Y$ as above. Having calculated the dimensions of global and local cohomology of its weight-graded pieces in Eqs. (5.7)–(5.8) and Eqs. (6.2)–(6.3), the associated Hilbert series can be calculated as follows:

$$\begin{aligned}
\mathrm{HS}_{\mathrm{glob}}(t) &= (1-t)^{-r_p}(1-t^2)^{-(s+\frac{1}{2}g(g+1)-\rho+h_{\mathrm{BK}}+n_1+n_2-\#|D|)} \\
&= 1 + r_p t + \big(\tfrac{1}{2}r_p(r_p+1) + s + \tfrac{1}{2}g(g+1) - \rho \\
&\qquad + h_{\mathrm{BK}} + n_1 + n_2 - \#|D|\big)t^2 + \ldots, \\
\mathrm{HS}_{\mathrm{loc}}(t) &= (1-t)^{-g}(1-t^2)^{-(\frac{1}{2}g(3g-1)+n-1)} \\
&= 1 + gt + (\tfrac{1}{2}g(g+1) + \tfrac{1}{2}g(3g-1) + n - 1)t^2 + \ldots \\
&= 1 + gt + (2g^2 + n - 1)t^2 + \ldots
\end{aligned}$$

Let c_i^{glob} and c_i^{loc} be the respective coefficients. By Lemma 3.4, for every reduction type Σ, there exists a Coleman algebraic function of weight at most 2 that vanishes on $\mathcal{Y}(\mathbb{Z}_p)_{S,U,\Sigma}$, whenever the following inequality holds:

$$\begin{aligned}
&\sum_{i=0}^{2} c_i^{\mathrm{glob}} < \sum_{i=0}^{2} c_i^{\mathrm{loc}} \\
\Leftrightarrow\ & 1 + r_p + \tfrac{1}{2}r_p(r_p+1) + s + \tfrac{1}{2}g(g+1) - \rho + h_{\mathrm{BK}} + n_1 + n_2 - \#|D| \\
&\qquad < 1 + g + 2g^2 + n - 1 \\
\Leftrightarrow\ & 0 < \tfrac{1}{2}g(3g+1) - \tfrac{1}{2}r_p(r_p+3) + \rho + \#|D| + n_2 - 1 - s - h_{\mathrm{BK}}.
\end{aligned}$$

Finally, the bound on $\#\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ from [12, Theorem 6.2.1 B)] with $m = 2$ depends on U only through the Hilbert series coefficient $c_1^{\mathrm{loc}} = g$, so we get the same bound as in Theorem B. Here, as in the proof of Theorem B above, we are using the fact that Betts' bound on $\#\mathcal{Y}(\mathbb{Z}_S)$ does in fact apply to the superset $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$. $\blacksquare$

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