Numerical methods for studying transition probabilities in stochastic ocean-climate models

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Conclusion

The goal of this thesis was to develop methods which can be used for studying transition behavior in the Meridional Overturning Circulation (MOC). The reason why we want to study such behavior is that melting of the polar ice sheets, and thereby increasing the freshwater in the North Atlantic, may cause a weakening of the MOC, which in turn may cause a reduced average temperature in especially Europe. These transitions may happen due to tipping points, or bifurcation points, that are associated with the salt-advection feedback.

State of the art methods are not efficient enough for computing probabilities of transitions in a realistic MOC model, and therefore techniques for computing other indicators of these transitions are being developed. In this thesis, we introduced novel preconditioning and Lyapunov solution methods, and improved the efficiency of methods which can be used for computing actual transition probabilities.

Linear systems

In Chapter 3, a novel multilevel preconditioner was introduced, which is specialized for the Navier–Stokes equations which are discretized on a staggered grid. The preconditioner works by partitioning the domain into parallelepiped shaped subdomains, of which the interior can be eliminated in parallel. On the interfaces, Householder transformations are applied to decouple all but one velocity node from the pressure nodes, after which all decoupled nodes can also be eliminated in parallel. The preconditioner can then be applied recursively to the leftover coupled velocity nodes until a direct solver is used at the last level.

We showed that the preconditioner has grid-independent convergence in case the number of levels is kept constant. Increasing the number of levels with the grid size was actually more efficient in terms of time, but did not show grid-independent convergence. We can again come close to grid-independent convergence by retaining a few more velocity nodes per level.

We also found the preconditioner to be robust for a 3D lid-driven cavity problem with increasing Reynolds numbers, which leads us to believe that it will also perform well when applied in our ocean solver.

Lyapunov equations

The sensitivity of the MOC to noise around a steady state can be expressed in a probability density function. To compute this probability density function, we have to compute the covariance matrix at the steady state. This can be done by solving a generalized Lyapunov equation.

We developed a novel method for computing low-rank solutions of generalized Lyapunov equations in Chapter 4. The method works by applying a Galerkin type projection with the space built from the eigenvectors belonging to the largest eigenvalues of the residual at every iteration of the method. Most important is that the method can be restarted, which allows for less memory usage, faster iterations, and recycling of previous solutions. We showed that for an idealized 2D MOC model, our method is the most efficient method in terms of both memory usage and time.

Due to the recycling properties, the method is also very efficient for solving extended generalized Lyapunov equations, and for solving generalized Lyapunov equations during a continuation process.

Transition probabilities

In Chapter 5 we discussed what a transition probability is, and how to compute them. We defined the transition probability to be the probability of going from a neighborhood A near a deterministic steady state $\bar{\boldsymbol{x}}_A$ to a neighborhood B near a deterministic steady state $\bar{\boldsymbol{x}}_B$ within time T.

Before actually computing transition probabilities, we discussed the possibility to compute covariance ellipsoids, which can give us some idea of the variability around a steady state. We then discussed most probable transition paths, which can give us some idea of the path along which a transition may happen. Both of these ellipsoids and paths, however, do not allow us to compute transition probabilities.

To compute actual transition probabilities, we can use methods like Adaptive Multilevel Splitting (AMS), Trajectory-Adaptive Multilevel Sampling (TAMS) and Genealogical Particle Analysis (GPA). All of these methods work by resampling of the trajectories based on how close they are to the other steady state. We gave a quantitative comparison of these methods, and found that TAMS gave results close to the results obtained with direct sampling, but with a smaller variance.

Transitions in the Meridional Overturning Circulation

To compute transitions in the MOC, we used TAMS in Chapter 6, as this gave us the best results in the previous chapter. To improve on this method, we came up with a projected time stepping method, which reduces the memory usage for our idealized 2D MOC model with 96%, and the time consumption by 30%. We showed that the probability of a transition increases drastically when getting closer to a bifurcation point, and that the projected method is able to obtain the same results as standard TAMS.

Since we only looked at an idealized 2D MOC model, we can not really say anything about the transition probabilities of the MOC in the actual Atlantic ocean, but we did provide methods with which this becomes feasible.

Prospects

In the various summaries that are present in this thesis, we already mentioned several prospects that are applicable to the methods that are presented in the corresponding chapters. In this section we aim to give some prospects for the project as a whole.

Something that we briefly mentioned in Section 5.4 are most probable transition trajectories. These have already been computed for another model that is used in oceanography, which is the two-dimensional barotropic quasi-geostrophic model. Even though we did not go into this direction in this thesis, it would be interesting to also compute these trajectories for the MOC. This would give us insight in the states that exist between the present day MOC and a collapsed MOC.

To compute transition probabilities for the MOC that are more meaningful than the ones we computed with our idealized 2D model, we require a more realistic 3D ocean-climate model. Fortunately, the implementation that we used for the idealized 2D model, the I-EMIC, already allows for this (Mulder, 2019). It would be very interesting to see if we can compute transitions with this 3D model, and if we can, what the probability of such a transition is.

It would also be interesting to see how our preconditioner performs in combination with the I-EMIC. At present, the I-EMIC uses the tailored preconditioner from De Niet et al. (2007); Thies et al. (2009). We expect to achieve more stability and much better scalability with the preconditioner presented in this thesis, which would allow for simulations with a higher resolution.