Numerical methods for studying transition probabilities in stochastic ocean-climate models

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INTRODUCTION

You may be reading this thesis on a sunny day in August in your garden surrounded by flowers, or it may be a rainy Sunday afternoon near the fireplace with your feet up, but most likely it's just another day in the office. Fact is that there is a strong variability in the every day weather related to the development of high- and low-pressure fields. These developments do not have anything to do with the external solar forcing, but are due to the internal variability of the atmospheric flow (Dijkstra, 2005). The time scale of this variability of 3-7 days is determined by the nonlinear processes in the atmosphere itself.

Similar processes happen in every part of the climate system on varying time scales. A high-frequency process is the weather as described above; a low-frequency process is the change in land-ice distribution (Mulder et al., 2018). In the ocean, internal variability causes formation of ocean eddies and the meandering of ocean currents such as the Gulf Stream (Schmeits and Dijkstra, 2001). Interaction between all the different processes on different time scales may result in internal variability on other time scales that is not present in decoupled systems. This makes such processes very difficult to study, and difficult to understand. Therefore it is often a good idea to take a step back, and look at a more simplified model to make sure one is at least able to understand the results, and then apply this newly acquired knowledge to the fully coupled system.

In this thesis, we work with such a simplified model of the Meridional Overturning Circulation (MOC). The MOC consists of a global 'conveyor belt' of ocean currents, which are driven by wind stress forces and fluxes of heat and freshwater at the surface. In the Atlantic ocean, it consists of surface currents that transport relatively light water toward high latitudes, deep water currents going in the opposite direction, and sinking and upwelling processes that connect these two. The circulation system contains two overturning cells: one in the north with North Atlantic Deep Water (NADW) and one in the south with Antarctic Bottom Water (AABW). The Atlantic part of the MOC (AMOC) is shown in Figure 1.1.

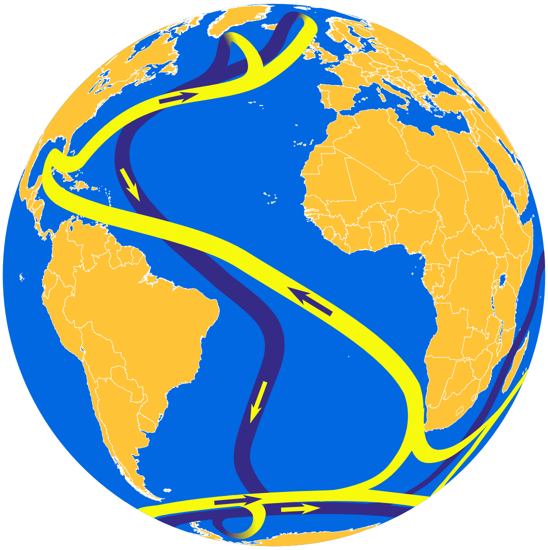

Figure 1.1: Simplified depiction of the Meridional Overturning Circulation in the Atlantic ocean. The yellow paths indicate the circulation at the surface, and the blue paths indicate the deep-water currents. The gradients between the two colors indicate the overturning cells with NADW in the north and AABW cell in the south.

Since the first model proposed by Stommel (1961), many model studies have shown that the MOC may be sensitive to variability in the freshwater forcing. In a global coupled climate model by Vellinga and Wood (2002) the NADW circulation collapses and recovers after 120 years. This is because weakening the MOC by introducing more freshwater in the North Atlantic (melting of the Greenland ice sheet) leads to a reduced northward saltwater transport, which in turn amplifies the freshwater perturbation.

A state of the MOC with a strongly reduced heat transport may have large consequences for the global climate (Rahmstorf, 2000; Lenton et al., 2008). In Vellinga and Wood (2002), a cooling of a few degrees is observed in Europe, which in turn may lead to growing glaciers and then global cooling. Therefore, an estimate of the probability of MOC transitions is crucial for prediction of a collapse of the MOC and with that a rapid climate change.

This collapse may occur due to the existence of a tipping point associated with the salt-advection feedback (Walin, 1985; Lenton, 2011). Mathematically, this tipping point is referred to as a bifurcation point. Tipping points exist due to the presence of multiple stable steady states for the same parameter values. Due to unresolved small-scale variability, however, transitions may even be observed before a tipping point is reached. This unresolved variability is often represented as noise. An example of such a noise induced transition in a simplified 2D model of the MOC is shown in Figure 1.2.

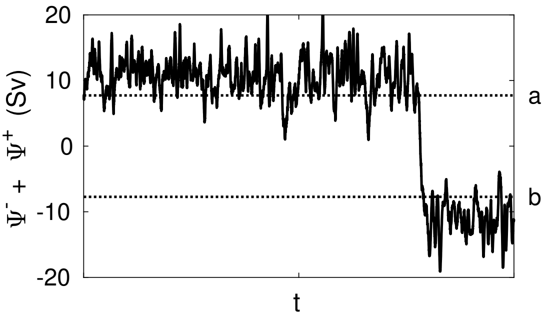

Figure 1.2: Example of a noise induced transition between steady states a and b in a simplified 2D model of the MOC. We are interested in the computation of the probability of such transitions. On the y-axis, the sum of the maximum and minimum of the overturning streamfunction Ψ are shown, meaning that steady state a represents the current state of the MOC as depicted in Figure 1.1, and steady state b rotates in the opposite direction.

The effect of noise on MOC transitions has so far been studied in very idealized models (Cessi, 1994; Monahan et al., 2008; Timmermann and Lohmann, 2000) and more recently in a coupled climate model of intermediate complexity (Sijp et al., 2013). Knowledge of the probability of these transitions is also important to evaluate whether climate variability phenomena in the geological past, such as the Dansgaard–Oeschger events, could be caused by stochastic transitions or by transitions induced by crossing tipping points (Ditlevsen and Johnsen, 2010). The aim of this thesis is to develop methods which can be used for studying transition behavior in the MOC.

To be able to observe these transitions between stable steady states, we first need to be able to compute the steady states themselves. They can be computed by using time integration, which is often expensive, especially if steady states have to be computed for multiple parameter values. Instead one can apply a method called (pseudo-arclength) continuation (Keller, 1977; Crisfield, 1991), which can compute the steady states directly by applying Newton's method. This can speed up the computation of steady states considerably (Dijkstra et al., 2014). Pseudo-arclength continuation is especially useful for computing unstable steady states, i.e. steady states for which a small disturbance causes the state to converge to a different steady state. Time integration methods are usually incapable of computing these unstable steady states.

During the application of Newton's method, many linear systems of equations have to be solved. For the MOC, these are specifically equations that include the Navier–Stokes equations, which are discretized on a staggered grid. In Benzi et al. (2005), the authors gave a survey of methods that are currently used to solve linear systems of this type. Direct (sparse) solvers are not practical since a typical ocean model involves millions of unknowns leading to a huge memory requirement for storing the factorization and an enormous amount of time to compute it. For such problems, iterative methods are preferred, e.g. Krylov subspace methods with suitable preconditioning to ensure robustness, fast convergence and accuracy of the final approximate solution (Benzi et al., 2005; Benzi and Olshanskii, 2006; De Niet et al., 2007; Elman et al., 2002; Kay et al., 2002; Elman et al., 2008). Segal et al. (2010) give an overview of the present state of fast solvers for the incompressible Navier–Stokes equations discretized by the finite element method and linearized by Newton's or Picard's method.

Preconditioners that are often advocated include standard additive Schwarz domain-decomposition, multigrid with aggressive coarsening and strong smoothers (e.g. ILU), and 'block preconditioners' that use an approximate block LU factorization and some approximation of the Schur complement associated with the pressure unknowns, e.g. SIMPLEC, LSC, and PCD (Cyr et al., 2012).

Another class of Schur complement methods is obtained when eliminating the interior of geometric partitions (or subdomains) and constructing a suitable approximation of the reduced system on the separator. In Wubs and Thies (2011); Thies and Wubs (2011) it was shown how to construct a preconditioner for the Navier–Stokes equations, which are discretized on a staggered grid. The resulting operator acts in the divergence-free space, which allows the method to handle the saddle-point structure of the system in a natural way. In Chapter 3 we first present a novel multilevel variant of this method. This method is also described in Baars et al. (2019c), and used in Baars et al. (2019b) for studying bifurcations in a lid-driven cavity.

After computing steady states of the MOC, we are interested in its sensitivity to noise. The sensitivity around a steady state can be determined from the probability density function. It is well known that this probability density function can be computed from the solution of a generalized Lyapunov equation (Gardiner, 1985). Direct methods for solving a generalized Lyapunov equation such as Bartels–Stewart algorithm (Bartels and Stewart, 1972) are based on dense matrix solvers and hence inapplicable for large systems. Other existing methods which use low-rank approximations (Simoncini, 2007; Druskin and Simoncini, 2011; Stykel and Simoncini, 2012; Druskin et al., 2014; Kleinman, 1968; Penzl, 1999) might also become expensive for high-dimensional problems, particularly when trying to use previous initial guesses along a continuation branch. We therefore present a novel method for computing the solution of generalized Lyapunov equations that is particularly well suited for our ocean problem in a continuation context in Chapter 4. Most of this chapter has been published in Baars et al. (2017) and Dijkstra et al. (2016), but some extra work has been done in the context of continuation and extended generalized Lyapunov equations.

A shortcoming to this above approach is that it only describes the sensitivity to noise around a steady state. To study the more global phenomenon of transitions between steady states, stochastic time integration is required. Applying a standard Monte Carlo method, however, is way too expensive, especially for high-dimensional systems and when the probability of a transition is small. Multiple methods exist to work around this problem by applying some form of resampling (Cérou and Guyader, 2007; Rolland and Simonnet, 2015; Lestang et al., 2018; Moral and Garnier, 2005; Moral, 2013; Wouters and Bouchet, 2016). We discuss these methods and show their benefits and shortcomings when applied to transitions between steady states on high-dimensional systems in Chapter 5. In Chapter 6 we propose a method based on the solution of generalized Lyapunov equations that reduces the computational cost and the huge memory requirements that are common for these types of methods. This chapter, and parts of Chapter 5 will appear in Baars et al. (2019a). We also used one of the methods described in Chapter 5 in Mulder et al. (2018) to study transition behavior in marine ice sheet instability problems.