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Two-product storage-capacitated inventory systems: A technical note

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ABSTRACT

This paper considers a two-product inventory model with limited storage capacity and constant demand rates. We aim at finding an ordering policy that minimizes the cost per time unit. In the literature, several solution methods have been developed for this problem, but these are limited to very restrictive classes of policies. We consider a much more general class where the order quantity of one of the products is allowed to vary. These policies are still cyclic and easy to implement. Closed-form expressions are derived for determining the optimal order quantities. It is shown that savings of up to 25% are possible compared to existing approaches.

1. Introduction

The majority of inventory control literature is based on cost minimization and ignores capacity constraints. In real life, such constraints often do exist, either storage or production related. This paper focuses on storage related capacity restrictions, which may result from physical limitations or internal budget restrictions.

Many textbooks, including Hadley and Whitin (1963), suggest coping with storage capacity limitations through partitioning of the storage capacity. In this approach, each product gets its own share of the storage capacity. This approach is mainly referred to as the *Lagrange multiplier approach*, but the terms *independent cycle*, *independent solutions* and *fixed storage* approach are also used. The advantage of this approach is that the negative effects of the capacity restriction can be spread evenly over the products. An important disadvantage is that the capacity is not used efficiently. Indeed, for deterministic demand, the average capacity usage is always 50%.

Another approach is to use the same cycle time for all products. This approach is referred to as the *fixed cycle*, *pure cycle*, *rotation cycle* or *common cycle* approach. In this approach, a common cycle time is determined and all orders are phased within this cycle, such that the storage capacity is used more efficiently. The main problem is to determine this phasing, i.e. the sequencing of the products, usually called the *staggering* of the products. Homer (1966) was the first who solved this staggering problem to optimality. His result was rediscovered, independently, by Page and Paul (1976), Zoller (1977), Hall (1988). The main advantage of this approach is that capacity is used more efficiently. An important disadvantage, however, is that forcing ordering cycles to become equal can be very costly.

Combining the two main approaches has also been suggested. Page and Paul (1976) provide a method that clusters the products. All products within a cluster have the same order interval, leading to an efficient use of the capacity, while the order intervals vary across the clusters. Anily (1991) provides another method that builds on the same rationale but determines the clusters in a different way. She shows that the performance of her clustering method does not exceed some lower bound on the costs by a factor larger than $\sqrt{2}$.

Besides combinations of the two main approaches, several authors generalized the methods. Gallego et al. (1996) generalized the fixed cycle approach to a powers-of-two policy. Order quantities remain fixed. The authors provide heuristics that determine ordering policies. An even more general approach is used by Hartley and Thomas (1982). They consider a two-product model in which they allow each product to be ordered several times per cycle, though still in fixed amounts. They provide an optimal solution procedure in a companion paper (Thomas and Hartley, 1983). Murthy et al. (2003) and Boctor (2010) consider the problem of offsetting the replenishment cycles by integer multiples of some base period, and use the result that, if the integer multiples of two items are not relatively prime, it is possible to offset their cycles such that the peaks of their inventory cycles never coincide over an infinite time horizon. Murthy et al. propose a heuristic for this framework, which is improved by Boctor.

Several authors provide structural insights. Anily (1991) provides a lower bound on the costs of inventory policies with constant order quantities and Gallego et al. (1996) prove that this bound holds even for varying quantities. Finally, Gallego et al. (1992) prove that the problem of determining an optimal ordering policy subject to a storage capacity restriction is strongly NP-complete.

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All the above discussed contributions consider policies with constant order quantities. Some other contributions do focus on varying order quantities. Hariga and Jackson (1995) formulate a nonlinear program that minimizes holding and ordering costs by selecting order quantities and the overall cycle length. They provide a heuristic solution procedure and conditions under which fixed order quantities are optimal. However, throughout the entire paper, they assume that the sequence of orders is given, which, in fact, constitutes the most challenging part of the determination of an optimal ordering policy.

Güder et al. (1995) also allow varying order quantities. They provide a myopic procedure that determines order quantities one by one, which in general results in a non-cyclic ordering policy. The order quantities that the procedure selects are based on the Economic Order Quantity as well as the available storage space. An upper bound on the optimality gap is not provided.

In this paper, we present an exact approach for the two product inventory model with a capacity constraint and varying order quantities. We provide closed-form expressions for optimizing order quantities and ordering moments for a very general class of ordering policies. We further derive theoretical and numerical results on the suboptimality of existing approaches, which turns out to be considerable in many situations.

This paper proceeds as follows. Section 2 describes the inventory system and introduces the concepts of simple and general cycles. In Section 3, structural properties of the optimal simple cycle policy are derived and used to determine the optimal timing and quantity of orders. Section 4 compares the cost performance of the optimal simple cycle policy to that of existing approaches. Section 5 shortly discusses general cycle policies. We end in Section 6 with a summary of the key findings and insights, and a discussion of research opportunities.

2. System description

Consider a two product, infinite time horizon inventory system. The products share a common storage resource with limited capacity. The products are numbered 1 and 2. Demand rates are deterministic and denoted by d_1 and d_2. Demand is measured in *capacity units per time unit*, i.e. as the rate with which capacity decreases, to allow for normalization of the capacity to unity. Subscripts refer to the product number. Lead times are constant and backorders are not allowed.

The objective is minimizing costs per time unit by determining ordering *moments* and *quantities*. This average cost per time unit is denoted by C and based on a cost per order for each of the two products, which are denoted by A_1 and A_2, respectively. We will focus purely on the ordering costs as we consider situations where ordering quantities are restricted by the limited available capacity rather than the need to avoid excessive holding costs.

Our attention is restricted to cyclic solutions. A *cyclic solution* is a solution in which ordering moments and quantities are described for a finite time interval, the *cycle*, and the inventory levels at the beginning and at the end of this cycle are equal. As a result, the ordering policy within the cycle can be repeated infinitely many times. The length of the cycle is denoted by T, which is a decision variable that will vary for different configurations. Cycles in which at least one of the products is ordered only once, which implies that its order quantity does not vary, will be called *simple cycles*. Cycles without any further restrictions will be referred to as *general cycles*.

3. Simple cycles

Let the *base product* refer to the product that is ordered once per cycle. If both products are ordered once, an arbitrary product can be selected as the base product. We will present the analysis for the case where product 1 serves as the basis, leading to the optimal policy of that type, and than 'copy' the result for the other case where product 2 is the base product.

The replenishment order of product 1 is referred to as the *product 1 order* and its order quantity is denoted by Q_1. Recall from Section 1 that the capacity is normalized to one and hence $Q_1 \le 1$. The number of orders of product 2 during the cycle is denoted by m, which is restricted to be integer. The product 2 orders are numbered according to their position in the cycle, starting from the product 1 order. Hence the i-th product 2 order after the product 1 order will be named *product 2 order i* and its quantity will be denoted by $Q_{2,i}$, for $i = 1, \ldots, m$, where $0 < Q_{2,i} \le 1$.

The above properties and notations lead to the following expression for the cycle time T of simple cycles:

$$T = \frac{Q_1}{d_1} = \frac{\sum_{i=1}^{m} Q_{2,i}}{d_2}. \tag{1}$$

The cost per time unit, which should be minimized, is

$$C = \frac{A_1 + mA_2}{T}. \tag{2}$$

It is easy to see that, for fixed m, maximization of T implies minimization of C. In turn, (1) shows that maximizing the order quantities maximizes T. This observation suggests that (a) the products should be ordered only when their stock is empty, and (b) the order quantities fill all remaining capacity, i.e. order quantities are of maximum size. In the next subsection, we show that optimal solutions indeed always satisfy properties (a) and (b) and we derive expressions for optimal order quantities by applying these.

3.1. Deriving optimal order quantities

Let us assume that we have decided on the base product (numbered 1) and the number, m, of orders per cycle for product 2. In the next subsection, we will show how to select the base product and the value of m optimally, in order to find the overall best simple cycle policy.

The following theorem states that all orders must be of maximum size. We remark that Hariga and Jackson (1995) obtain the same results under the objective of minimizing the storage capacity, but not under cost minimization as considered here.

Theorem 1. *Under an optimal simple cycle policy, every order is of maximum size, i.e. uses up all spare storage capacity when it arrives.*

A proof by contradiction is provided in Appendix A.

The following corollary shows that next to being of maximum size, orders should always arrive just in time. A formal proof is provided in Appendix A, but the logic is as follows. An order that arrives early (when the inventory level is still positive) can be postponed, which leads to an alternative policy with an order of non-maximum size. As the alternative policy cannot be optimal (using Theorem 1), neither can the original policy. We remark that Anily (1991) proves the same, but along different lines.

Corollary 1. *Under an optimal policy, a product arrives exactly when its inventory level reaches zero.*

Now, we derive optimal order quantities as follows. Since the product 1 order is of maximum size, by Theorem 1, and arrives when the product 1 stock level is zero, by Corollary 1, the stock level of product 2 at that time is $(1 - Q_1)$. So, product 2 order 1 will be placed $\frac{1-Q_1}{d_2}$ time units later and, by Theorem 1, its quantity is equal to the total demand over this period, i.e.

$$Q_{2,1} = \frac{d_1 + d_2}{d_2}(1 - Q_1). \tag{3}$$

In the same way, we find that

$$Q_{2,j} = \frac{d_1 + d_2}{d_2} Q_{2,j-1}, \quad \text{for } j = 2, \ldots, m. \tag{4}$$

Combining (3) and (4) gives

$$Q_{2,j} = \left(\frac{d_1 + d_2}{d_2}\right)^j (1 - Q_1), \quad \text{for } j = 1, \ldots, m. \tag{5}$$

Using the same reasoning, but now applied to the relation between product 2 order m and the next product 1 order that follows, we get

$$Q_1 = \frac{d_1 + d_2}{d_1}(1 - Q_{2,m}). \tag{6}$$

Combining (5), applied to product 2 order m, and (6) yields a system of two linear equations with two unknowns. After some algebra this results in

$$Q_1 = 1 - \frac{1}{\left(\frac{d_1 + d_2}{d_2}\right)^{m+1} - \frac{d_1}{d_2}}. \tag{7}$$

Inserting (7) into (5) gives

$$Q_{2,j} = \frac{\left(\frac{d_1 + d_2}{d_2}\right)^j}{\left(\frac{d_1 + d_2}{d_2}\right)^{m+1} - \frac{d_1}{d_2}}, \quad \text{for } j = 1, \ldots, m. \tag{8}$$

Letting $C_1(m)$ denote the cost per time unit in case product 1 is the base product and product 2 is ordered m times, combining (1), (2) and (7) gives

$$C_1(m) = \frac{d_1(A_1 + mA_2)}{1 - \frac{1}{\left(\frac{d_1 + d_2}{d_2}\right)^{m+1} - \frac{d_1}{d_2}}}, \quad \text{for } m \in \mathbb{N}. \tag{9}$$

As mentioned before, the analysis of the case where product 2 is the base product is completely similar. Letting $C_2(n)$ denote the cost per time unit in case product 2 is the base product and product 1 is ordered n times, we get

$$C_2(n) = \frac{d_2(nA_1 + A_2)}{1 - \frac{1}{\left(\frac{d_1 + d_2}{d_1}\right)^{n+1} - \frac{d_2}{d_1}}}, \quad \text{for } n \in \mathbb{N}. \tag{10}$$

3.2. Selecting the base product and optimizing m or n

We can obtain the optimal value of m and n, given as $m^* = \arg\min\{C_1(m) | m \in \mathbb{N}\}$ and $n^* = \arg\min\{C_2(n) | n \in \mathbb{N}\}$, respectively, by a straightforward numerical search procedure. Furthermore, we next develop a graphical aid for determining the optimal base product and the number of order per cycle for the other product.

It is straightforward to rewrite $C_1(m+1) \le C_1(m)$ and $C_2(n+1) \le C_2(n)$ for general m and n, respectively, in terms of $\frac{A_1}{A_2}$ and $\frac{d_1}{d_2}$ rather than the individual cost parameters. Therefore, when determining m^* and n^*, only the two cost dimensions $\frac{A_1}{A_2}$ and $\frac{d_1}{d_2}$ matter. So we can display m^* and n^* graphically in Fig. 1.

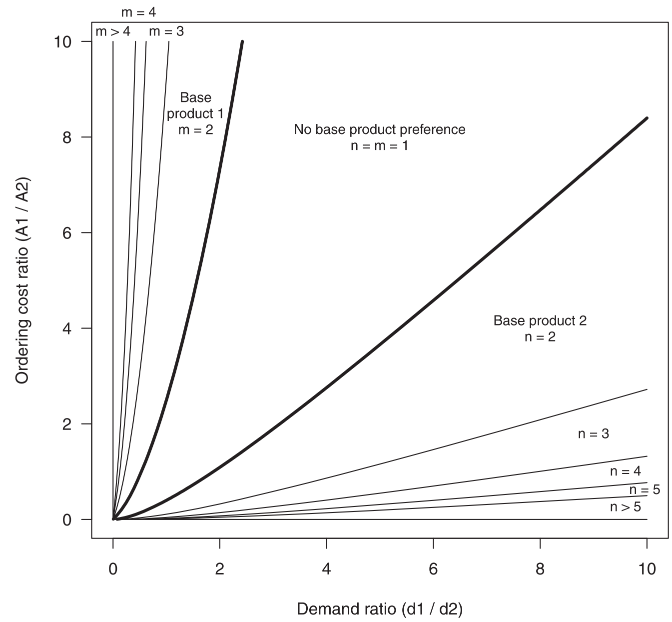

Fig. 1. Optimal choices of the base product and of m or n depending on $\frac{A_1}{A_2}$ and $\frac{d_1}{d_2}$.

4. Comparison to other methods

In this section, the simple cycle optimization procedure of the previous section will be compared to methods that are used in the literature. We restrict to solution methods generating cyclic ordering policies, as these are comparable (in complexity) to ours.

4.1. Capacity partitioning

Although in general an optimal partitioning cannot easily be obtained, as several authors show, a capacity partitioning variant of the model considered in this paper can be solved to optimality algebraically. This solution will be derived here in order to compare it with the solution of Section 3.

Consider the model as introduced in Section 2. Introduce a capacity partitioning P, where $0 < P < 1$. Assume that product 1 uses at most P capacity units and product 2 at most $1 - P$ capacity units. In this model, the cost per time unit, denoted by C^{CP}, becomes,

$$C^{CP} = \frac{A_1}{T_1} + \frac{A_2}{T_2}, \tag{11}$$

where T_1 and T_2 denote the cycle lengths of product 1 and 2, respectively. Obviously, given P, the optimal cycle lengths are $T_1 = \frac{P}{d_1}$ and $T_2 = \frac{1-P}{d_2}$. Omitting the straightforward algebra, the optimal partitioning P^* and corresponding costs C^{CP*} are

$$P^* = \begin{cases} \dfrac{d_1A_1 - \sqrt{d_1A_1d_2A_2}}{d_1A_1 - d_2A_2} & \text{if } d_1A_1 \neq d_2A_2, \\ \dfrac{1}{2} & \text{if } d_1A_1 = d_2A_2, \end{cases} \tag{12}$$

$$C^{CP*} = d_1A_1 + d_2A_2 + 2\sqrt{d_1A_1d_2A_2}. \tag{13}$$

The latter expression can be used to compare the costs with the optimal costs in case of simple cycles. In our numerical tests, we observed the largest relative difference for cases with $d_1 = d_2$ and $A_1 = A_2$. Using the results of Section 3, it follows that an optimal simple cycle policy for this case with product 1 as the base product, $m = 1$ and $Q_1 = Q_{2,1} = T = \frac{2}{3}$ with cost $C = 3d_1A_1$ per time unit.

Using (12) and (13), we find that the best partitioning is $P = \frac{1}{2}$ with cost $C^{CP} = 4d_1A_1$ per time unit. Hence, capacity partitioning can lead to 33% extra cost compared to simple cycle policies.

4.2. Other approaches

The other basic approach, besides capacity partitioning, is the fixed cycle approach. Anily (1991) has shown that costs resulting from this approach may be infinitely many times larger than her lower bound. Our numerical results indicate that this result carries over to policies with varying order sizes, i.e. that fixed cycles can be arbitrarily worse than simple cycles.

Güder et al. (1995) provide a myopic procedure that determines an ordering policy that is always at least as good as the optimal capacity partitioning policy. However, if the products have the same cycle time in the optimal capacity partitioning solution, which is clearly the case if $P = \frac{1}{2}$ and $d_1 = d_2$, the procedure of Güder et al. (1995) yields exactly the same policy. So ordering policies generated by the procedure of Güder et al. (1995) can also lead to 33% extra cost compared to simple cycle policies.

5. General cycles

In this section, the model of Section 3 will be extended to general cycles and we present an example where a general cycle policy outperforms the best simple cycle policy. The cost saving is small though, a finding that was replicated for many other settings that we considered. A cost gap of 1.74% was the largest that we found in our numerical testing.

If both n (the number of product 1 orders in a cycle) and m (the number of product 2 orders in the same cycle) are allowed to be larger than 1, then (1) and (2) become, respectively,

$$T = \frac{\sum_{i=1}^{n} Q_{1,i}}{d_1} = \frac{\sum_{i=1}^{m} Q_{2,i}}{d_2}, \tag{14}$$

$$C = \frac{nA_1 + mA_2}{T}. \tag{15}$$

Consider the following example. Let $d_1 = d_2 = A_2 = 1$ and $A_1 = 3$. Using the optimization procedure of Section 3, the following optimal simple cycle is determined. Product 1 is the base product, m=2, $Q_1 = \frac{6}{7}$, $Q_{2,1} = \frac{2}{7}$ and $Q_{2,2} = \frac{4}{7}$. The cost per time unit of this solution is $C = \frac{35}{6}$.

Now, consider a general cycle with n=2 and m=3. Note that these values do not allow an equivalent simple cycle policy. Assume that product 2 is ordered twice between product 1 order 1 and product 1 order 2, and once between product 1 order 2 and the beginning of the next cycle. Using similar reasoning as in Section 3.1, we get: $Q_{2,1} = 2(1 - Q_{1,1})$, $Q_{2,2} = 2Q_{2,1} = 4(1 - Q_{1,1})$, $Q_{1,2} = 2(1 - Q_{2,2})$, $Q_{2,3} = 2(1 - Q_{1,2})$, $Q_{1,1} = 2(1 - Q_{2,3})$.

This gives $Q_{1,1} = \frac{26}{31}$, $Q_{2,1} = \frac{10}{31}$, $Q_{2,2} = \frac{20}{31}$, $Q_{1,2} = \frac{22}{31}$ and $Q_{2,3} = \frac{18}{31}$. Using (14) and (15), it turns out that $T = \frac{48}{31}$ and $C = \frac{93}{16} < \frac{35}{6}$. Hence the costs are smaller than the costs of the optimal simple cycle, though only by 0.4%.

6. Conclusion and further research

We proposed a new class of the so-called simple cycle policies for two-product capacitated inventory systems with constant demand. These policies differ from those proposed in the literature by allowing varying order sizes for one of the products, where the other product serves as the base product with a constant order quantity. Insightful structural properties on the optimal policy of this type were derived for situations where holding costs are negligible, i.e. when order quantities are capped because of the limited storage capacity available rather than to limit storage costs. These properties state that orders should only arrive if stock drops to zero, and should always be of maximum size, i.e. use up all available spare capacity upon arrival. Using these properties, we were able to derive closed form expressions for determining the optimal simple cycle policy. We showed that simple cycle policies can lead to considerable savings compared to existing approaches with fixed order quantities. Capacity partitioning can lead to 33% extra cost. The same holds for the approach proposed by Güder et al. (1995).

An important extension of our research would be to consider non-negligible holding costs. In fact, our cost and order quantity expressions can easily be generalized for such situations. However, it is important to realize that it may no longer be optimal to always maximize the size of orders when holding costs are positive. Intuitively, this remains logical when storage availability and not storage costs drive ordering decisions, i.e. in situations with relatively small holding costs and low storage availability (as for the examples that we considered). In opposite situations with high holding costs and ample storage capacity, as usually assumed in the inventory literature, products can always be ordered in constant economic order quantities. Future research could address when to best switch from one type of policy to another, and also develop and evaluate hybrid policies.

Another important extension is to consider more than two products. Our analysis of deriving optimal order quantities can be extended to such situations for given order frequencies and sequences, but the number of possible order frequencies and sequences grows exponentially with the number of products. With respect to order frequencies, one possibility is to only allow power-of-2 policies that have been shown to perform very well for other multi-product problems such as the joint replenishment problem and the economic lot scheduling problem (Axsäter, 2006).

Situations with varying demand rates also deserve attention. The traditional approach (for single product models) of using a fixed safety stock per product will no longer be effective if this uses up a large fraction of the limited available storage capacity (in periods with peak demand). Policies should be developed that dynamically optimize order quantities, safety stocks and storage buffer space for future orders simultaneously.

Finally, it would be interesting to consider pooling of stocks and storage space between multiple inventory locations of a company to deal with storage restrictions. It is well known from the literature that pooling reduces total stocks needed. Sharing storage space may help avoid the need for renting additional space.

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Appendix A. Proofs of Theorem 1 and Corollary 1

We introduce the following additional notation. Let t_1 and $t_{2,k}$, for $k = 1, \ldots, m$, denote the moments at which the product 1 order and product 2 order k arrive, respectively. As all orders are placed within the cycle, we have that $0 \le t_1 < T$ and $0 \le t_{2,k} < T$ for $k = 1, \ldots, m$. Let IL_1^- and IL_1^+ denote the inventory level of product 1 right before and right after the arrival of the product 1 order, respectively, and let

TL_1^- and TL_1^+ denote the total inventory level at these moments. Equivalently, we denote $IL_{2,k}^-$ and $IL_{2,k}^+$ as the product 2 inventory level right before and after the arrival of its k-th order and we denote $TL_{2,k}^-$ and $TL_{2,k}^+$ as the total inventory level at these moments. The inventory levels at time 0 ('starting level') will be denoted by SL_1 and SL_2, respectively. If one or more orders arrive at time 0, the 'starting level' refers to the inventory level right before these arrival(s).

We next introduce feasibility criteria for simple cycle policies, which follow from the assumptions in Section 2 and will be used in the proofs. It easily follows that the conditions (a) $IL_1^- \geq 0$ and $IL_{2,k}^- \geq 0$, for $k = 1, \ldots, m$, ensure that backorders are prevented. Equivalently, (b) $TL_1^+ \leq 1$ and $TL_{2,k}^+ \leq 1$, for $k = 1, \ldots, m$, ensure that the capacity restriction is met. In addition, the condition (c) $\frac{Q_1}{d_1} = \frac{\sum_{i=1}^{m} Q_{2,i}}{d_2}$ ensures that the orders satisfy demand for equal periods of time, which implies cyclicality. Altogether, these three conditions imply feasibility of a simple cycle ordering policy.

Lemma 1. *If, under some feasible ordering policy, two or more order moments coincide, the policy can be modified such that all orders arrive at different moments, without increasing the costs per time unit.*

Proof. Suppose that product 2 order k and product 2 order $k+1$ coincide, i.e. $t_{2,k} = t_{2,k+1}$, under some feasible ordering policy. Now, let $t_{2,k}' = t_{2,k} - \Delta$, i.e. product 2 order k arrives a small but positive number of Δ time units earlier. If we select Δ such that $\Delta < \frac{Q2,k+1}{d_1+d_2}$ and $t_{2,k} - \Delta > \max\{t_1, t_{2,k-1}\}$, we ensure that the stock used to satisfy demand between $t_{2,k}'$ and $t_{2,k+1}$ is less than the amount of stock arriving at time $t_{2,k+1}$, which implies that $TL_{2,k}^{+'} < TL_{2,k+1}^{+} \leq 1$, and we ensure that no other orders arrive in between. Hence, this adaptation does not violate condition (b). Clearly, conditions (a) and (c) remain satisfied as well, and also the cycle time T and costs per time unit do not change. Hence, an alternative feasible policy exists, which yields the same costs per time unit as the original one.

An equivalent argument can be made if the product 1 order coincides with a product 2 order. As this argument holds for any two orders, iterative application proves the claim.□

Theorem 1. *Under an optimal simple cycle policy, every order is of maximum size, i.e. uses up all spare storage capacity when it arrives.*

Proof. We provide a proof by contradiction, i.e. we show how any feasible policy with an order of non-maximum size can be modified into an alternative feasible policy with lower average cost per time unit. In fact, by application of Lemma 1, it is sufficient to do so for policies where all orders arrive at different moments. For the considered feasible simple cycle ordering policy, let Q_1, t_1, m, and $Q_{2,k}$ and $t_{2,k}$, for $k = 1, \ldots, m$, be given. We can assume, without loss of generality, that $t_1 = 0$. Finally, let SL_1 and SL_2 be given. To distinguish the alternative from the original policy, we add a superscript $'$ to all relevant notation. We will separately consider the product 1 order (case 1), product 2 order l, for $l = 1, \ldots, m-1$ (case 2a), and product 2 order m (case 2b).

Case 1. *Product* 1

Assume that product 1 is not of maximum size, i.e. $TL_1^+ < 1$. Consider the following alternative ordering policy. Let $\Delta > 0$ and let $Q_1' = Q_1 + d_1\Delta$; $Q_{2,1}' = Q_{2,1} + d_2\Delta$; $Q_{2,k}' = Q_{2,k}$ for $k = 2, \ldots, m$; $t_1' = t_1 = 0$; $t_{2,k}' = t_{2,k} + \left(\frac{d_1+d_2}{d_1}\right)\Delta$ for $k = 1, \ldots, m$; $SL_1' = SL_1$; $SL_2' = SL_2 + d_2\left(\frac{d_1+d_2}{d_1}\right)\Delta$; $T' = T + \Delta$.

We will show that this alternative policy is a feasible simple cycle policy for a sufficiently small, but positive value of Δ. Since the alternative has a larger cycle time, but the same number of orders per cycle, it has a lower average cost per time unit. Note first of all that $t_{2,k}' < T'$ for each $k = 2, \ldots, m$ for Δ sufficiently small. So, we can let all orders 'stay in the same cycle'. In order to prove feasibility of the alternative policy formally, we verify the above conditions (a) and (b) together, for each order, and we then verify condition (c).

Product 1 *order*: IL_1^- equals SL_1 which does not change, so (a) holds. TL_1^+ increases by the combined increase in the product 1 order size and SL_2, i.e. by $d_1\Delta + d_2\frac{d_1+d_2}{d_1}\Delta$. So (b) holds for sufficiently small values of Δ, because $TL_1^+ < 1$ for the original policy.

Product 2 *order* 1: $IL_{2,1}^-$ changes by $SL_2' - SL_2 = d_2\frac{d_1+d_2}{d_1}\Delta$ and by the stock used to satisfy demand before $t_{2,1}$, which increases by $d_2(t_{2,1}' - t_{2,1}) = d_2\frac{d_1+d_2}{d_1}\Delta$. Hence, $IL_{2,1}^-$ is unchanged, so (a) holds. $TL_{2,1}^+$ increases by $d_2\frac{d_1+d_2}{d_1}\Delta + (Q_{2,1}' - Q_{2,1}) + (Q_1' - Q_1) - (d_1+d_2)(t_{2,1}' - t_{2,1}) = d_2\frac{d_1+d_2}{d_1}\Delta + d_2\Delta + d_1\Delta - (d_1+d_2)\frac{d_1+d_2}{d_1}\Delta = 0$, so (b) holds.

Product 2 *order* k, *for* $k = 2, \ldots, m$: $IL_{2,k}^-$ is changed, analogously to product 2 order 1, by the increasing starting stock and the increase of the stock used to satisfy demand, which cancel out again, and by the increase of $Q_{2,1}$. This leads to a total increase of $d_2\Delta$ units, which validates (a). Verification of (b) is completely analogous to product 2 order 1.

Cyclicality: Note that $Q_1' - Q_1 = d_1\Delta$ and that $\sum_{i=1}^{m}(Q_{2,i}' - Q_{2,i}) = d_2\Delta$, which both cover the demand in the additional $T' - T = \Delta$ time units. Because the original ordering policy was cyclic, the alternative policy is cyclic as well, which proves (c).

It follows that the alternative policy is feasible for sufficiently small values of Δ. This completes the proof by contradiction that the product 1 order must be of maximum size.

Case 2a: Product 2 order l, for l < m

Assume that product 2 order l, with $l < m$, is not of maximum size, i.e. $TL_{2,l}^+ < 1$. Consider the following alternative ordering policy. Let

$\Delta > 0$; $Q_1' = Q_1 + d_1\Delta$; $Q_{2,k}' = Q_{2,k} - d_1\left(\frac{d_1+d_2}{d_2}\right)^k\Delta$ for $k = 1, \ldots, l-1$; $Q_{2,l}' = Q_{2,l} + d_2\Delta + d_1\ \Delta + \sum_{i=1}^{l-1} d_1(\frac{d_1+d_2}{d_2})^i\Delta$; $Q_{2,k}' = Q_{2,k}$ for $k = l+1, \ldots, m-1$; $Q_{2,m}' = Q_{2,m} - d_1\Delta$; $t_1' = t_1 = 0$; $t_{2,k}' = t_{2,k} - \sum_{i=0}^{k-1}\left(\frac{d_1}{d_2}\right)\left(\frac{d_1+d_2}{d_2}\right)^i\Delta$ for $k = 1, \ldots, l$; $t_{2,k}' = t_{2,k} + \Delta$ for $k = l+1, \ldots, m$; $SL_1' = SL_1$; $SL_2' = SL_2 - d_1\Delta$; $T' = T + \Delta$.

Note that all order quantities and SL_2' remain positive for sufficiently small values of Δ. Equivalently, for sufficiently small Δ, the ordering of the product 2 orders is preserved, i.e. $0 < t_{2,1}' < \ldots < t_{2,m}' < T'$. As for case 1, we will show that properties (a) and (b) hold for all orders, and that property (c) holds as well, if Δ is sufficiently small. Again, this implies that the alternative policy is a feasible simple cycle policy with lower cost per time unit.

Product 1 *order*: IL_1^- does not change again, so (a) holds. The decrease in SL_2 cancels out against the increase Q_1, leaving TL_1^+ unchanged. Hence, (b) holds as well.

Product 2 *order* k, *for* $k = 1, \ldots l-1$: SL_2 is $d_1\Delta$ units lower and each order i, $i < k$, is reduced in size by $d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta$ units, and so the total decrease in $IL_{2,k}^-$ caused by these reductions equals $d_1\Delta + \sum_{i=1}^{k-1} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta = \sum_{i=0}^{k-1} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta$ units. Product 2 order k arrives $(t_{2,k}' - t_{2,k})$ time units earlier, and so the demanded amount of product 2 before the arrival decreases by $d_2(t_{2,k}' - t_{2,k}) = d_2\sum_{i=0}^{k-1}\frac{d_1}{d_2}\left(\frac{d_1+d_2}{d_2}\right)^i\Delta = \sum_{i=0}^{k-1} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta$ units. As these amounts are equal, $IL_{2,k}^-$ does not change, so (a) holds. $TL_{2,k}^+$ changes by $d_1\Delta - d_1\Delta - \sum_{i=1}^{k} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta$. Using the change in the arrival moment again, demand until that time is reduced by $(d_1+d_2)(t_{2,k}' - t_{2,k}) = (d_1+d_2)\sum_{i=0}^{k-1}\frac{d_1}{d_2}\left(\frac{d_1+d_2}{d_2}\right)^i\Delta = \sum_{i=1}^{k} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta$

units. These amounts are equal again, which shows validity of (b).

Product 2 *order* l: Validity of (a) can be shown in exactly the same way as for product 2 orders 1 up to $l-1$. Moreover, this reasoning shows that $IL_{2,l}^{-}$ is unchanged. Therefore, the change in $TL_{2,l}^{+}$ is $(Q_1' - Q_1) - d_1(t_{2,l}' - t_{2,l}) + (Q_{2,l}' - Q_{2,l}) = d_1\Delta + d_1\left(\sum_{i=0}^{k-1}\frac{d_1}{d_2}\left(\frac{d_1+d_2}{d_2}\right)^i\Delta\right) + d_2\Delta + d_1\Delta + \sum_{i=1}^{l-1} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta = 2d_1\Delta + d_2\Delta + 2d_1\sum_{i=0}^{l-1}\left(\frac{d_1+d_2}{d_2}\right)^i\Delta$.

Since this increase is proportional to Δ and $TL_{2,l}^{+} < 1$ by assumption, (b) holds for sufficiently small Δ.

Product 2 order k, for $k = l+1, \ldots, m$: The change in $IL_{2,k}^{-}$ is equal to the change in the starting stock level, $-d_1\Delta$, plus the change in the number of arriving units until that time, $\sum_{i=1}^{l-1}(Q_{2,i}' - Q_{2,i}) + Q_{2,l}' - Q_{2,l}$, minus the additional demand due to postponement, $d_2(t_{2,k}' - t_{2,k}) = d_2\Delta$, which gives a total change of $-d_1\Delta - \sum_{i=1}^{l-1} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta + d_2\Delta + d_1\Delta + \sum_{i=1}^{l-1} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta - d_2\Delta = 0$. This shows validity of (a). The product 1 inventory level at the arrival moment of product 2 order k changes by $Q_1' - Q_1 - d_1(t_{2,k}' - t_{2,k}) = d_1\Delta - d_1\Delta = 0$, hence $TL_{2,k}^{-}$ is unchanged. Because also $Q_{2,k}' \le Q_{2,k}$ for $k = l+1, \ldots, m$, it follows that $TL_{2,k}^{+'} \le TL_{2,k}^{+}$ and hence (b) is valid.

Cyclicality: Clearly, the total increase of the product 1 order quantity satisfies the demand for an additional Δ time units. The total amount ordered for product 2 changes by $-\sum_{i=1}^{l-1} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta + d_2\Delta + d_1\Delta + \sum_{i=1}^{l-1} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta - d_1\Delta = d_2\Delta$ capacity units, which also satisfies demand for an additional Δ time units. As the order quantities of both products satisfy demand for exactly the same amount of time, the policy remains cyclic.

It follows again that the alternative policy is feasible for sufficiently small values of Δ. Because it is better than the original policy, the original policy cannot be optimal. Hence, product 2 order l must be of maximum size, for $l = 1, \ldots, m-1$.

Case 2b: Product 2 order m

Assume that product 2 order m is not of maximum size, i.e. $TL_{2,m}^{+} < 1$. Consider the following alternative ordering policy. Let $\Delta > 0$; $Q_1' = Q_1 + d_1\Delta$; $Q_{2,k}' = Q_{2,k} - d_1\left(\frac{d_1+d_2}{d_2}\right)^k\Delta$ for $k = 1, \ldots, m-1$; $Q_{2,m}' = Q_{2,m} + d_2\Delta + \sum_{i=1}^{m-1} d_1\left(\frac{d_1+d_2}{d_2}\right)^i\Delta$; $t_1' = t_1 = 0$; $t_{2,k}' = t_{2,k} - \sum_{i=0}^{k-1}\frac{d_1}{d_2}\left(\frac{d_1+d_2}{d_2}\right)^i\Delta$ for $k = 1, \ldots, m$; $SL_1' = SL_1$; $SL_2' = SL_2 - d_1\Delta$; $T' = T + \Delta$.

Since this alternative is almost the same as in case 2a, a detailed proof will be omitted. The only difference is that $Q_{2,m}' - Q_{2,m}$ in case 2b is equal to $Q_{2,m}' - Q_{2,m} + Q_{2,l}' - Q_{2,l}$ in case 2a, i.e. the change made to product 2 order m in the current case is equal to the sum of the changes made to product 2 orders l and m in the previous case. The reasoning applied to all orders before product 2 order m is equivalent to the reasoning applied to all orders before product 2 order l in case 2a. Also the validity of (a) for product 2 order m is completely analogous. Examining the validity of (b) for product 2 order m results in an expression similar to that of product 2 order l in the previous case, which is again proportional to Δ. Validation of (c) is analogous again. It follows again that the alternative policy outperforms the original one for sufficiently small values of Δ, i.e. the original policy is not optimal.

Combining the results of cases 1, 2a and 2b proves the theorem. □

Corollary 1. *Under an optimal policy, a product arrives exactly when its inventory level reaches zero.*

Proof. Suppose product 2 order k does not satisfy this condition, i.e. $IL_{2,k}^{-} > 0$, under some feasible ordering policy. Now, let $t_{2,k}' = t_{2,k} + \Delta$, i.e. product 2 order k is postponed Δ time units. If we select Δ such that $\Delta < \frac{IL_{2,k}^{-}}{d_2}$ and $t_{2,k} + \Delta < \min\{t_1, t_{2,k+1}, T\}$, we ensure that $IL_{2,k}^{-'}$ remains positive, i.e. condition (a) remains satisfied, and we ensure that no other orders arrive in between, which, in turn, implies that $TL_{2,k}^{+'} < TL_{2,k}^{+} \le 1$, i.e. condition (b) remains satisfied as well. Clearly, also condition (c) remains satisfied, hence the alternative policy is feasible. However, $TL_{2,k}^{+'} < 1$ also implies, by application of Theorem 1, that the alternative policy is not optimal. As the original and the alternative policies yield equal costs per time unit, the original policy is not optimal. If we suppose that $IL_1^{-} > 0$, i.e. the product 1 order arrives while its inventory level is positive, we can apply the same argument. Hence, the claim holds for any order in the cycle. □

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