Practice-inspired contributions to inventory theory

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CHAPTER 3

A general method to address forecasting uncertainty in inventory models

Abstract. *Whereas in practice inventory decisions depend heavily on demand forecasts, the literature typically assumes that demand distributions are known. As a result, estimates are directly substituted for the unknown parameters, which leads to insufficient safety stocks, stock-outs, low service, and high costs. We propose a framework for addressing the estimation uncertainty that is applicable to any inventory model, demand distribution, and parameter estimator. The estimation errors are modeled and a predictive lead time demand distribution is obtained, which is substituted into the inventory model. We illustrate this framework for several demand models. When estimates are based on 10 observations, relative savings are typically between 10% and 30% for mean-stationary demand. Savings are larger when estimates are based on fewer observations, when backorders are costlier, and when the lead time is larger. In the presence of a trend, savings are between 50% and 80% for several scenarios.*

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3.1 Introduction

Inventory control depends heavily on forecasts of future demand. Yet, the inventory control literature exhibits a separation of demand forecasting and inventory decision making. Since the establishment of the Economic Order Quantity model by Harris (1913), a wide range of different models have been developed with varying review structures, cost frameworks, and demand characteristics. Most (medium sized and large) companies nowadays use such inventory models through either specific inventory control software, or more general ERP software. However, inventory models generally rely on complete certainty of the future demand distribution, which never applies in practice. Although much research has been devoted to optimally forecasting various types of demands, the interface of forecasting and decision making remains ill-studied. In this paper we present a general framework for estimating unknown demand parameters and including the estimation uncertainty in the inventory decision. This framework can be applied more generally to any optimization model that depends on some unknown variable that has to be forecasted, but in this paper we focus on inventory control models.

Also in inventory control textbooks the relationship between forecasting and inventory decision making is generally left unaddressed, see e.g. Zipkin (2000), Waters (2012), and Hillier and Lieberman (2014). Hax and Candea (1984) discuss how the distribution of demand forecast errors can be derived empirically, by updating the probabilities with incoming demand observations. Although this yields a consistent estimate of the forecast error distribution, it ignores all uncertainty around the estimate at any point in time. For example, after one observation, the estimated forecast error distribution would have all mass at 0 (unless some prior distribution is used), as the estimated mean of the demand distribution will exactly equal that one observation. By treating point estimates as true parameters one ignores part of the uncertainty about demand, whereas inventories are kept to account for this uncertainty. Inventory calculations that ignore this forecasting uncertainty are flawed, even if the used estimators are unbiased and have minimum variance. This leads to insufficient safety stocks, resulting in frequent stock-outs, high costs (e.g. backorder costs, lost sales and emergency shipment costs), and not achieving the target service level. Moreover, in practice there is often a limited number of demand observations available to estimate demand parameters with, because firstly most companies do not store large histories of demand observations, and secondly - and more importantly - the underlying demand process is subject to frequent changes. The fewer historical demand observations are used, the more volatile are the parameter estimates, and thus the larger is the negative effect of ignoring their uncertainty.

Bayesian inventory modeling provides an exact framework for incorporating unknown demand parameters in inventory decision models. One specifies a class of distributions and a prior parameter distribution, and obtains via Bayes' rule a predictive parameter distribution. The Bayesian forecasting literature is rich and deals with many different types of demand models, such as normally distributed dynamic models, autoregressive models, but also more general models with demand distributions that belong to the exponential family. However, it is noteworthy that applications of Bayesian theory in inventory modeling are very rare. Exact treatments exist only for simple single-period and/or single-parameter demand models. If the demand distribution contains several parameters, then typically some of them (e.g. the variance for normally distributed demand) are considered to be known. This restricts the applicability of Bayesian inventory theory, whereas the negative effects of ignoring parameter uncertainty are typically largest in models with several parameters, multiple periods and/or positive lead times (since the forecast errors over the forecast horizon are correlated). Furthermore, in practice it is difficult to exactly specify the demand distribution, making methods such as exponential smoothing popular. Standard Bayesian methods do not exhibit freedom of the choice of parameter estimator, but rather fix the demand distribution and choose a prior parameter distribution. Finally, the cost consequences of ignoring parameter uncertainty inventory decisions are understudied in the literature, as the focus in the forecasting literature is on the accuracy of demand predictions, whereas the inventory control literature typically ignores forecasting altogether.

This paper serves two goals. First, we present a framework for addressing estimation uncertainty in inventory models that starts from the chosen parameter estimators, rather than from a prior distribution. Via the sample distributions of the parameter estimators, the distributions of the unknown parameters are modeled using Bayes' rule. By taking the expectation of the lead time demand distribution function with respect to the (now stochastic) unknown parameters, a predictive lead time demand distribution is obtained, which can be applied in the inventory decision model. We assume that other than the available data, no information about the unknown parameters is available. Second, we perform an extensive numerical study to show that for many widely-used inventory models and demand distributions, the cost of ignoring parameter uncertainty is substantial. Furthermore, we demonstrate how this cost relates to the cost resulting from misspecification of the demand model or selection of suboptimal parameter estimators.

We demonstrate this method for a discrete-time, continuous-review inventory model with a fixed lead time, and linear holding and backorder costs. In the case of normally distributed demand, the distribution of the estimation errors can be found exactly, but we will also discuss an approximative method, which is useful for two reasons. Firstly, this method is robust to demand distribution misspecification. Secondly, an exact derivation of the parameter estimation errors may not always be available, whereas the approximative method follows straightforwardly for every Maximum Likelihood estimator.

The remainder of this paper is structured as follows. Section 3.2 gives a literature review. In Section 3.3 we present the modeling framework, the inventory model, the demand models, and derive the order levels according to the classical and corrected (both exact and approximative) approaches. In Section 3.4 we perform a numerical study in which we compare the performance of the different approaches. In Section 3.5 we discuss the robustness of the approaches to misspecification of the demand model. Section 3.6 concludes.

3.2 Related literature

The mismatch between forecasting and inventory control has been pointed out occasionally by other researchers. Fildes and Beard (1992) explicitly discuss the correlation between future forecast errors, and Silver et al. (2017) remark that the variability of the forecast error depends in a complicated fashion on both the demand model and the forecast procedure. Strijbosch et al. (2000) discuss the importance of deriving the lead time forecast error rather than only the forecast error per time instant. Toktay and Wein (2001) describe a production model where demand forecast updates are incorporated in the production decision, and also discuss the effect of forecast errors. All authors mentioned above focus on the mean of demand. Beutel and Minner (2012) discuss an integrated framework of least squares demand forecasting and inventory decision making. They find that if demand parameters are replaced by estimates, and the estimates are based on only a few observations, then actual service levels will significantly undershoot their targets. Prak et al. (2017b) provide adjustments to the standard calculations to incorporate uncertainty of both parameters in the case of mean-stationary normally distributed demand, for a service level model. They show that especially in situations where few previous demand observations are available, the service levels achieved by classical approaches significantly undershoot their targets. However, their method holds solely for mean-stationary, normally distributed demand and under a service level constraint.

In the forecasting literature, there is considerable attention for Bayesian approaches to demand forecasting. When applying Bayesian forecasting, one seeks to derive a predictive demand distribution, rather than just a point forecast. This is key to properly addressing estimation uncertainty in the eventual inventory decision. A wide discussion of Bayesian forecasting can be found in West and Harrison (1997), where predictive distributions are presented for e.g. dynamic linear models, autoregressive models, and non-linear models. Key articles in this stream are Harrison and Stevens (1971) and West et al. (1985). Bayesian forecasting is especially convenient for dealing with varying model parameters, whereas traditional methods typically assume that these remain constant for at least some time. Specific applications to demand forecasting are given by Spedding and Chan (2000) and Yelland (2010).

Although already pioneered by Dvoretzky et al. (1952b) and Scarf (1959), applications of Bayesian inventory control remain scarce. Numerical Bayesian applications to inventory control mainly involve one-parameter demand distributions, or, if the distribution has more than one parameter, impose the restriction that one or more parameters are known. Azoury and Miyaoka (2009) and Chen (2010) use a normal distribution of which the standard deviation is assumed to be known, and Rajashree Kamath and Pakkala (2002) use a lognormal distribution with a known standard deviation. This restriction is convenient, because it ensures that the posterior demand distribution is again (log)normal, so that the analysis is similar to that without the Bayesian treatment of the unknown mean. Azoury and Miyaoka (2009) state that uncertainty in the demand standard deviation is of minor importance and cite Gelman et al. (1995). However, Gelman et al. (1995) mainly discusses regression models from economics, where the standard deviation only affects the standard errors of the estimated parameters. Contrarily, in inventory control, the standard deviation of demand relates directly to the safety stock (a key decision variable) and therefore does play a vital role. Another complication of applying Bayesian methods to inventory control is that one has to specify a prior distribution, which is a subjective element. Hill (1997) and Hill (1999) use a non-informative prior.

The inventory control literature contains a few non-Bayesian studies about decision making under parameter uncertainty. Ritchken and Sankar (1984) consistently estimate the quantiles of the normal distribution with unknown mean and variance in order to optimize the reorder level in a single-period inventory model. Janssen et al. (2009) perform a simulation study to add a mark-up to the traditional reorder level. Strijbosch and Moors (2005) numerically search for the optimal batch size in an (r, Q) policy with normally distributed demand and unknown parameters. Strijbosch et al. (2011) find that also for non-stationary demand series, the estimated demand variance should be adjusted to account for the uncertainty in the parameter estimates. They use Simple Moving Average (SMA) and Single Exponential Smoothing (SES) on a large historical dataset of demands for several products, and find that for the majority of the items time horizons of 2 to 8 periods are optimal for SMA, and smoothing parameters between 0.21 and 0.4 are optimal for SES (where it has to be remarked that the search range was restricted to values between 0 and 0.4). This suggests that in practice estimates can often only be based on a few demand observations, which is in accordance with our own experience. The conclusion in all above studies is that ignoring forecasting uncertainty leads to too low inventories and undershooting target service levels.

Fildes (1983) found that a multivariate Bayesian forecasting system typically does not outperform classical methods such as exponential smoothing. Also more recently it has been observed that exponential smoothing performs well in forecasting competitions (Gardner, 2006; Ali and Boylan, 2011, 2012). This may be one of the reasons why practitioners typically resort to classical methods. Bermúdez et al. (2010) discuss that the prediction intervals generated by Maximum Likelihood in the Holt-Winters model are too narrow and show that a Bayesian approach performs better. This points at the key problem when using classical estimation methods for inventory decisions: whereas Bayesian forecasting leads to predictive distributions that naturally include estimation uncertainty, classical methods do not. Ord et al. (1997) discuss several approaches to overcome this issue and find that a simulation method - which draws possible 'true' parameters based on observed estimates - provides the most reliable results in terms prediction intervals for future demand. However, the effects on the eventual inventory decision and resulting costs are not addressed.

In conclusion, although in the forecasting literature Bayesian methods exist to derive predictive forecast distributions, the inventory control literature applies these only in limited contexts. Practitioners typically apply classical methods and ignore estimation uncertainty when making inventory decisions, and there exists little insight into the severity of this problem. Whichever forecasting method is used, the estimation uncertainty should be taken into account in the inventory decision. We present a general approach to facilitate this in any combination of demand model and inventory system. Starting from a chosen parameter estimator, we perform an analysis along classical lines based on the sample distributions of the estimators, but use Bayes' rule to find predictive parameter distributions and eventually a predictive lead time demand distribution. We assume no prior information on the parameters. Furthermore, in contrast with Ord et al. (1997), we perform an extensive numerical study to the resulting inventory costs. We benchmark the performance of the corrected approach against that of the classical approach that ignores estimation uncertainty, thereby seeking to obtain insights into the consequences of ignoring estimation uncertainty, and their relationship to other costs, especially those of misspecifying the demand model or choosing a suboptimal parameter estimator.

3.3 Model

In this section we first describe the general framework that we apply throughout this paper. Then we discuss the specific inventory model and the several demand models and parameter estimators by which we demonstrate the framework. We derive the optimal policy under uncertainty of the demand parameters, and furthermore discuss a robust approximative method based on the asymptotic sample distributions of the parameter estimators.

3.3.1 General framework

The approach to handling unknown demand parameters taken in this paper starts from a cost function that is to be minimized, and that depends in some way on a (lead time) demand distribution with unknown parameters. That demand distribution is transformed into a predictive demand distribution as follows. Denote the vector of parameters by θ and the required distribution function by $F(x;\theta)$. We estimate θ by $\hat{\theta}$, and rewrite θ as a function of $\hat{\theta}$ and the estimation error ϵ_θ. The estimation error ϵ_θ is a random vector of which we either derive or approximate the probability distribution. This is achieved via the sample distribution of the estimator, or its asymptotic distribution (which is for example available for every Maximum Likelihood estimator), and Bayes' rule is applied with an uninformative prior to transform that distribution into the distribution of the unknown parameter. The predictive demand distribution function is now $\mathrm{E}_\theta(F)$, where we notationally omit for brevity that the distribution of θ is conditional on the observed data. Numerically, this boils down to an integration with order equal to the number of unknown parameters. The inventory decision model should use this predictive distribution to arrive at the inventory decision. If the demand model contains many unknown parameters (such as in a multiple regression model), then the lead time demand distribution may be re-parameterized so that the dimension of integration is reduced. We will discuss this further in Section 3.3.3. Finally, we remark that classical approaches arrive at a predictive distribution by simply substituting the obtained point estimates for the true parameters.

3.3.2 Inventory model

To illustrate our approach, we will focus on a discrete-time, single-item, continuous-review inventory model with fixed, linear holding and shortage costs. That is, the time horizon is $t = 1, 2, ...$, holding costs per unit per time unit are $h > 0$, shortage costs per unit per time unit are $p \geq h$, and full back-ordering is assumed. Orders can be placed free of charge at every time instant and arrive after a fixed lead time $L \geq 0$. Because there are no fixed order costs, there is no need to batch orders and so-called base-stock or order-up-to level policies can be applied that keep the inventory position (the current inventory level plus outstanding orders) constant. At every review an optimal order-up-to level is derived, and the order of events at any review time instant is as follows: first outstanding orders arrive, then demand occurs and costs are incurred, and finally new orders can be placed. It is in principle possible that the order-up-to level is smaller than the current inventory position. This is however unlikely, since it can only occur if the difference in the optimal order-up-to level from one period to the next is larger than the demand realization. In these rare cases we allow for negative orders.

The costs arising from the inventory model at any time instant are easily characterized. Denote the inventory level (the inventory on hand) at time t, after the arrival of demand at time t, by $IL(t)$. Then the costs at time t are given by

$$C(t) = h(IL(t))^+ + p(IL(t))^-,$$

where $(x)^+ = \max\{x, 0\}$ and $(x)^- = \max\{-x, 0\}$.

We refer to the current time instant by n. We derive an optimal order-up-to policy that is characterized by an order-up-to level S_n for the current inventory position, which includes outstanding orders. Furthermore, we assume that we are currently at the end of time n, and that we have observed demands at time $1, ..., n$. Orders that are placed at time n will arrive at the beginning of time $n + L$. The next possible order can be placed at time $n + 1$ and will arrive at the beginning of time $n + L + 1$. Therefore, the decision made at time n affects costs at time $n + L$, which are driven by the inventory level directly after the arrival of demand at time $n + L$, but before the next ordering, which we, as above, denote by $IL(n + L)$. The inventory level at time $n + L$ is therefore given by the inventory position (the inventory level plus outstanding orders) after the ordering at the end of time n, minus the demands at times $n + 1, ..., n + L$. Denote the inventory position at the end of time n by $IP(n)$ and denote the sum of demands at times $k, ..., m$ by $D_{[k,m]}$. Then we can write

$$IL(n + L) = IP(n) - D_{[n+1,n+L]}.$$

We find that at time n we face the problem of minimizing

$$TC(S_n) = h\mathrm{E}(S_n - D_{[n+1,n+L]})^+ + p\mathrm{E}(S_n - D_{[n+1,n+L]})^-.$$

Observe that this model is decomposable in the sense that the optimal order-up-to level at time n can be found independently of former and future order-up-to levels. Therefore, in the sequel we omit the subscript n and refer to the optimal order-up-to level at time n as S. We are now back in the framework of Section 3.3.1. The discussed model can easily be extended to e.g. a periodic review framework or a setting where the time horizon is continuous rather than discrete.

Decision making

Denote the (predictive) distribution function of lead time demand $D_{[n+1,n+L]}$ by $F_{D_{[n+1,n+L]}}$. Since

$$\begin{aligned}\mathrm{E}(S - D_{[n+1,n+L]})^{+} &= \int_{-\infty}^{S} (S-s)\mathrm{d}F_{D_{[n+1,n+L]}}(s) = \int_{-\infty}^{S}\int_{s}^{S} \mathrm{d}x\mathrm{d}F_{D_{[n+1,n+L]}}(s)\\ &= \int_{-\infty}^{S}\int_{-\infty}^{x} \mathrm{d}F_{D_{[n+1,n+L]}}(s)\mathrm{d}x = \int_{-\infty}^{S} F_{D_{[n+1,n+L]}}(x)\mathrm{d}x,\end{aligned}$$

and similarly

$$\mathrm{E}(S - D_{[n+1,n+L]})^{-} = \int_{S}^{\infty} \left[1 - F_{D_{[n+1,n+L]}}(x)\right]\mathrm{d}x,$$

we can write

$$TC(S;\mu,\sigma) \;=\; h\int_{-\infty}^{S} F_{D_{[n+1,n+L]}}(x)\mathrm{d}x + p\int_{S}^{\infty}\left[1 - F_{D_{[n+1,n+L]}}(x)\right]\mathrm{d}x.$$

Using

$$\frac{\mathrm{d}}{\mathrm{d}S}\int_{-\infty}^{S} F_{D_{[n+1,n+L]}}(x)\mathrm{d}x = F_{D_{[n+1,n+L]}}(S),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}S}\int_S^\infty \left[1 - F_{D_{[n+1,n+L]}}(x)\right] \mathrm{d}x = -\left[1 - F_{D_{[n+1,n+L]}}(S)\right],$$

we find the first order condition

$$F_{D_{[n+1,n+L]}}(S) = \frac{p}{p+h}, \tag{3.1}$$

which is a newsvendor equation and can be solved for S. Since

$$\frac{\mathrm{d}^2 TC}{\mathrm{d}S^2} = (p+h) f_{D_{[n+1,n+L]}}(S) > 0,$$

it follows that TC is convex and the first order condition leads to the global minimum.

Under the assumption of i.i.d. normally distributed demand with known mean μ and standard deviation σ per time unit, the equation defining the optimal order-up-to level has a closed-form solution which is given by

$$S = L\mu + \sqrt{L}\Phi^{-1}\left(\frac{p}{p+h}\right)\sigma,$$

where Φ denotes the distribution function of the standard normal distribution. Classical inventory control methods directly substitute point estimates for the unknown demand parameters into (3.1) and subsequently calculate the 'optimal' order-up-to level, which we denote by $\tilde{S}$. This order-up-to level is not optimal under parameter uncertainty. The optimal inventory decision that is derived using the predictive distribution is denoted by S^*. In the remainder of this section we are concerned with deriving these predictive distributions that can be used in equation (3.1) to find the optimal order-up-to level under parameter uncertainty.

3.3.3 Demand models and derivation of the predictive lead time demand distribution

The inventory model described in the previous section depends on the distribution of demand in the time interval $[n+1, n+L]$. In this section we present four different demand models, derive the corresponding lead time demand distributions, discuss how the parameters can be estimated and their estimation errors can be modeled, and finally how a predictive lead time demand distribution can be obtained. The choice for a certain demand model is typically made either manually by the decision maker, or automatically using forecasting software by comparing goodness-of-fit measures. We first discuss the often used model of mean-stationary demand, where we compare the sample mean and the SES estimator. Whereas the former is the Minimum Variance Unbiased Estimator for this model, the latter is popular because in practice it is often not known whether, and if so since when demand has been stationary. Next, we consider the trended demand model, which is selected when demand appears to be stationary around a linear trend. This is also a common choice in textbooks and forecasting software, for example for fashion-sensitive products that show an upward trend in the beginning of their sales cycle, and a downward trend at the end of their sales cycle. Finally, we consider the random walk model, where the next demand observation is equal to the previous observation plus a random noise term. This model is realistic for products with demand patterns that are non-stationary and do not follow specific trends.

Mean-stationary, normally distributed demand

Denoting demand at time t by D_t, we assume $D_t \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$, where μ and σ^2 are unknown. Models with normally distributed demand have the drawback that demand can theoretically be negative (although the probability of that occurring can be made arbitrarily small), but are nevertheless widely applied in the literature and especially in textbooks, such as Silver et al. (2017) and Axsäter (2015). It now follows easily that

$$D_{[n+1,n+L]} \overset{\text{i.i.d.}}{\sim} N(L\mu, L\sigma^2).$$

The unknown parameters μ and σ^2 are estimated efficiently by the sample mean and sample variance. That is,

$$\hat{\mu} = \frac{1}{n}\sum_{t=1}^{n} D_t$$

and

$$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{t=1}^{n}(D_t - \hat{\mu})^2.$$

Observe that $\hat{\sigma}^2$ is not equal to the Maximum Likelihood estimator for σ^2, which divides by n rather than $n-1$. As is discussed in e.g. Davidson and MacKinnon (2003), $\hat{\sigma}^2$ is unbiased whereas the Maximum Likelihood estimator is not. Still, $\hat{\sigma}^2$ has the same asymptotic distribution as the Maximum Likelihood estimator, which we use later for our approximate modeling approach.

What remains is to find the exact relationship between the true parameters μ and σ^2, and their estimates $\hat{\mu}$ and $\hat{\sigma}^2$. We start from the following results, that are proven in e.g. Miller and Miller (2004):

$$\frac{\mu - \hat{\mu}}{\sigma/\sqrt{n}} \sim N(0,1),$$

and, independently,

$$\frac{(n-1)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-1},$$

where χ^2_{n-1} denotes a χ^2 distribution with $n-1$ degrees of freedom. We can now write, given σ^2,

$$\mu = \hat{\mu} + \frac{\sigma Z}{\sqrt{n}},$$

where Z is standard normally distributed, and

$$\sigma^2 = \frac{(n-1)\hat{\sigma}^2}{X},$$

where X is χ^2_{n-1}-distributed. This defines the predictive parameter distribution as needed in the framework described in Section 3.3.1. The integral is taken first with respect to σ^2 and then with respect to μ. We denote the resulting order-up-to level by S^*.

This modeling approach fits in the Bayesian framework when an uninformative prior is used. If the normal distribution is assumed for demand, and if an uninformative prior distribution is used, then the posterior distribution of μ, given σ^2, is normal with mean $\hat{\mu}$ and variance σ^2/n. Also, the marginal posterior distribution of σ^2 is scaled inverse χ^2 with scale parameter $\hat{\sigma}^2$ and $n-1$ degrees of freedom, see e.g. Gelman et al. (1995). With Bayesian analysis one can show that using a non-central Student's t-distribution (the posterior distribution that results when a non-informative prior is used on a normal distribution) as the predictive lead time demand distribution, leads to the same results. However, this one-to-one correspondence with a standard Bayesian analysis does not always exist.

If it is not possible or desirable to derive the distribution of the estimation error exactly, then one can resort to the asymptotic approximation of the error distribution. Every Maximum Likelihood estimator is asymptotically normal with a variance that follows directly from the Fisher information matrix. Since in the previous section the estimator for μ is the Maximum Likelihood estimator and the estimator for σ^2 is asymptotically equivalent to the Maximum Likelihood estimator, we can use Maximum Likelihood theory to derive the asymptotic distributions of the estimators. These asymptotic distributions hold even if the demand distribution is not normal, making this approach robust and practically appealing. However, we remark that if the demand distribution is correctly specified and if an exact derivation is available, then this alternative approach is sub-optimal.

As is shown in Greene (2011), in our model,

$$\hat{\mu} - \mu \overset{\text{a}}{\sim} N(0, \sigma^2/n)$$

and

$$\hat{\sigma}^2 - \sigma^2 \overset{\text{a}}{\sim} N(0, 2\sigma^4/n).$$

Replacing σ^2 by $\hat{\sigma}^2$, we find a consistent estimator of these asymptotic distributions. Hence, we find $\mu \approx \hat{\mu} + \sqrt{\frac{\hat{\sigma}^2}{n}} Z_1$ and $\sigma^2 \approx \hat{\sigma}^2 + \sqrt{\frac{2\hat{\sigma}^4}{n}} Z_2$, where Z_1 and Z_2 are independent and standard normally distributed. We are now back in the framework of the previous section.

Note that although σ^2 is restricted to be positive, its normal distribution model allows that the integral is evaluated at negative values of the demand variance. As the distribution function is not defined for negative values of the variance, this will lead to an overestimation of the true demand variance and thereby to safety stocks that are too high. To correct for this, we use a truncated normal distribution with appropriately bounded support to model Z_2. We denote the resulting order-up-to level by S'.

Single exponential smoothing for mean-stationary, normally distributed demand

Consider the same demand model as above, $D_t \overset{\text{i.i.d.}}{\sim} N(\mu, \sigma^2)$, but now estimate μ by single exponential smoothing (SES) with smoothing parameter α. This method is frequently applied in practice, as it is often not known with certainty whether, and if so since when demand has been stationary. The variance σ^2 is again estimated by the sample variance. As above, we are interested in the lead time demand distribution, which is characterized by

$$D_{[n+1,n+L]} \overset{\text{i.i.d.}}{\sim} N(L\mu, L\sigma^2).$$

The SES estimator at time n can be written in expanded form as

$$\hat{\mu} = \alpha \left(D_n + (1-\alpha) D_{n-1} + (1-\alpha)^2 D_{n-2} + ... + (1-\alpha)^{n-2} D_2 \right) + (1-\alpha)^{n-1} D_1$$

Its sample distribution is normal with mean μ and variance

$$\text{Var}(\hat{\mu}) = \sigma^2 \alpha^2 \sum_{i=0}^{n-2} (1-\alpha)^{2i} + \sigma^2 (1-\alpha)^{2(n-1)} = \sigma^2 \frac{\alpha - \alpha^2 + 2(1-\alpha)^{2n}}{(\alpha-2)(\alpha-1)}.$$

We can now write, conditional on the true variance,

$$\mu = \hat{\mu} + \sqrt{\text{Var}(\hat{\mu})}Z,$$

where Z is standard normally distributed. Since we estimate σ^2 by the sample variance, we still have

$$\sigma^2 = \frac{(n-1)\hat{\sigma}^2}{X},$$

where X is χ^2_{n-1}-distributed. Observe that when estimating σ^2, for this sample distribution to hold true, the deviations of the demand observations should be taken with respect to the sample mean, and not with respect to the SES estimator for μ. We have now completely specified the predictive parameter distribution.

The approximate modeling method in the case of the SES estimator uses its asymptotic distribution. By letting $n \to \infty$ in the expression for the sample variance of the SES estimator, we find the familiar result that its asymptotic variance is $\frac{\alpha}{2-\alpha}\sigma^2$. Replacing σ^2 by its estimator, we find that the asymptotic variance can be estimated by $\frac{\alpha}{2-\alpha}\hat{\sigma}^2$. So, we find $\mu \approx \hat{\mu} + \sqrt{\frac{\hat{\sigma}^2\alpha}{2-\alpha}}Z_1$ and $\sigma^2 \approx \hat{\sigma}^2 + \sqrt{\frac{2\hat{\sigma}^4}{n}}Z_2$, where Z_1 and Z_2 are independent and standard normally distributed. For σ^2 we use, as before, a truncated normal distribution.

Normally distributed, trended demand

As a natural extension of the mean-stationary demand model, let us now consider a trended demand model. Denoting demand at time t by D_t, we assume that demand is normally distributed and may have a linear trend, $D_t = \alpha + t\beta + \nu_t$, $\nu_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2)$. The parameters α, β, and σ^2 are unknown and have to be estimated.

In this model we find that $D_{[n+1,n+L]}$ is normal with mean

$$\mu_{\text{lead}} = L\alpha + (n+1+...+n+L)\beta = L\alpha + \frac{1}{2}(L^2 + 2nL + L)\beta$$

and variance $L\sigma^2$. The variance of lead-time demand is again easily dealt with, but the mean requires some more analysis.

It follows from OLS theory (Davidson and MacKinnon, 2003) that for a general regression model $y_t = x_t'\beta + \epsilon_t$, the OLS estimator for the parameter vector β is given by $\hat{\beta} = (X'X)^{-1}X'y$, where the vector y contains all elements y_t and the matrix X contains all row vectors x_t. Furthermore, the error variance σ^2 is estimated by

$$\hat{\sigma}^2 = \frac{(y - X\beta)'(y - X\beta)}{n - k},$$

where k denotes the number of regressors in the model. It can be shown that we can write, conditional on the true variance, $\beta = \hat{\beta} + \epsilon_\beta$, where

$$\epsilon_\beta \sim MVN(0, \sigma^2(X'X)^{-1}),$$

where MVN denotes a multivariate normal distribution. We also have that $\sigma^2 = \frac{\hat{\sigma}^2(n-k)}{\epsilon_{\sigma^2}}$, where $\epsilon_{\sigma^2} \sim \chi^2_{n-k}$.

Applying the OLS theory discussed above to our trend model, we find that at time n the parameters α and β are estimated by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \begin{pmatrix} n & \sum_{t=1}^n t \\ \sum_{t=1}^n t & \sum_{t=1}^n t^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^n D_t \\ \sum_{t=1}^n tD_t \end{pmatrix}. \tag{3.2}$$

Then the OLS predictor of demand at time $t > n$ is $\hat{\mu}_t = \hat{\alpha} + t\hat{\beta}_t$, and that of lead time demand is $\hat{\mu}_{\text{lead}} = L\hat{\alpha} + \frac{1}{2}(L^2 + 2nL + L)\hat{\beta}$.

The covariance matrix of the OLS estimators for α and β is given by

$$\begin{aligned} V &= \sigma^2 \begin{pmatrix} n & \sum_{t=1}^{n} t \\ \sum_{t=1}^{n} t & \sum_{t=1}^{n} t^2 \end{pmatrix}^{-1} = \frac{\sigma^2}{n\sum_t t^2 - \left(\sum_t t\right)^2} \begin{pmatrix} \sum_t t^2 & -\sum_t t \\ -\sum_t t & n \end{pmatrix} \\ &= \sigma^2 \begin{pmatrix} \frac{4n+2}{n(n-1)} & -\frac{6}{n(n-1)} \\ -\frac{6}{n(n-1)} & \frac{12}{n(n^2-1)} \end{pmatrix}, \end{aligned}$$

where we use that

$$\sum_{t=1}^{n} t = \frac{n(n+1)}{2}$$

and

$$\sum_{t=1}^{n} t^2 = \frac{n(n+1)(2n+1)}{6}.$$

Since the predictor for the mean of lead time demand is a linear transformation of $\hat{\alpha}$ and $\hat{\beta}$, it is (conditional on the true demand variance) normally distributed with mean μ_{lead} and variance

$$\text{Var}(\hat{\mu}_{\text{lead}}) = \begin{pmatrix} L & \frac{1}{2}(L^2 + 2nL + L) \end{pmatrix} V \begin{pmatrix} L \\ \frac{1}{2}(L^2 + 2nL + L) \end{pmatrix}.$$

We can now write, conditional on the true variance,

$$\mu_{\text{lead}} = \hat{\mu}_{\text{lead}} + \sqrt{\text{Var}(\hat{\mu}_{\text{lead}})}Z,$$

where Z is standard normally distributed. For σ^2 we find that

$$\sigma^2 = \frac{(n-2)\hat{\sigma}^2}{X},$$

where X is χ^2_{n-2}-distributed. This specifies the predictive parameter distribution. Observe that by directly modeling μ_{lead}, rather than α and β, we have reduced the dimension of the integral leading to the predictive lead time demand distribution from 3 to 2, thereby simplifying its use in practice.

Replacing σ^2 by its estimator in $\text{Var}(\hat{\mu}_{\text{lead}})$ straightforwardly leads to an estimator $\widehat{\text{Var}}(\hat{\mu}_{\text{lead}})$ of the asymptotic variance of $\hat{\mu}_{\text{lead}}$. The asymptotic variance of $\hat{\sigma}^2$ is again estimated by $2\hat{\sigma}^4/n$. Hence, we find $\mu_{\text{lead}} \approx \hat{\mu}_{\text{lead}} + \sqrt{\widehat{\text{Var}}(\hat{\mu}_{\text{lead}})}Z_1$ and $\sigma^2 \approx \hat{\sigma}^2 + \sqrt{\frac{2\hat{\sigma}^4}{n}}Z_2$, where Z_1 and Z_2 are independent and standard normally distributed. For σ^2 we use again a truncated normal distribution.

Demand is a normal random walk

The trend model described in the previous subsection still assumes that demands at different time instants are independent from each other. Here we demonstrate the procedure for a typical model of auto-correlated demand. Denoting demand at time t by D_t, we assume that

$$D_t = D_{t-1} + \nu_t,$$

where $\nu_t \overset{\text{i.i.d.}}{\sim} N(0, \sigma^2)$. That is, demand follows a random walk and the increments are normally distributed. The parameter σ^2 is unknown and has to be estimated based on n previous observations. In this model we find that, conditional on D_n,

$$D_{[n+1,n+L]} = LD_n + L\nu_{n+1} + (L-1)\nu_{n+2} + ... + \nu_{n+L},$$

which is normally distributed with mean LD_n and variance

$$\sigma^2_{\text{lead}} = \sigma^2(L^2 + (L-1)^2 + (L-2)^2 + ... + 1) = \sigma^2 L(L+1)(2L+1)/6.$$

The unknown parameter σ^2 can be estimated by the sample variance of the difference series of the previous n observations. The $n-1$ differences are each i.i.d. normally distributed with mean 0 and variance σ^2, so that the theory for σ^2 as derived for the mean-stationary model, still holds. For the exact approach we can write

$$\sigma^2 = \frac{(n-2)\hat{\sigma}^2}{X},$$

where X is χ^2_{n-2}-distributed, and for the approximate approach we have $\sigma^2 \approx \hat{\sigma}^2 + \sqrt{\frac{2\hat{\sigma}^4}{n}}Z$, where Z is standard normally distributed. Replacing that standard normal distribution by a truncated normal distribution, we are back in the familiar framework and the predictive lead time demand distribution follows directly.

3.4 Numerical study

In this section we numerically evaluate the order-up-to levels and costs that result from the corrected approaches and the classical approach for the inventory model under study and the four demand models. For each set-up, we start from the observed parameter estimates and derive the order-up-to levels S^* corresponding to the exact, corrected approach (based on the exact sample distribution of the estimator), S' corresponding to the approximate, corrected approach (based on the asymptotic sample distribution of the estimator), and $\tilde{S}$ corresponding to the classical approach (which ignores parameter uncertainty). We then perform 1,000,000 replications in which true parameters are drawn according to the earlier derived probability distributions, and the costs are evaluated based on the full information cost function using these true parameters. We report the differences between the three methods in terms of order-up-to levels and resulting average costs, and furthermore report the differences in terms of safety stock, by subtracting the estimated mean of lead time demand from the order-up-to levels.

3.4.1 Mean-stationary, normally distributed demand

We start with the mean-stationary demand model, define a base case of parameter settings, and then vary one parameter at a time. The base case of the comparison has $\hat{\mu} = 10$ and $\hat{\sigma}^2 = 4$. Holding costs per unit per time unit are 1, while shortage costs are 20. The lead time is 5 time units. The other scenarios are constructed by varying the shortage costs to 50 and 100, doubling the lead time to 10 time units, decreasing $\hat{\sigma}^2$ to 1, and doubling $\hat{\mu}$ to 20. Within every scenario, the number of historical observations corresponding to the current demand distribution is varied between 5, 10, 20, and 100. As discussed in Section 3.2, in practice the usable history of demand observations is typically very small, which makes 5 or 10 observations realistic choices.

Observe that as an optimal result, in a model with holding and shortage costs, the fill rate is equal to $\frac{p}{p+h}$. It follows that the base case set-up corresponds to a fill rate of 95%, and the set-ups with $p = 50$ and $b = 100$ correspond to fill rates of 98% and 99%, respectively, which are typical settings in practice. Table 3.1 shows the resulting order-up-to levels as well as the percentage differences between the average safety stocks and costs of the corrected approaches and the classical approach. The safety stock is calculated by subtracting the expected lead time demand from the order-up-to level.

It becomes clear from Table 3.1 that especially when n is small, the calculated order-up-to level resulting from the classical method (which ignores parameter uncertainty and is therefore insensitive to variations of n) is substantially too low, and this results in a significant cost increase. Based on five observations in the base case, the cost benefit of correctly treating the estimation uncertainty in μ and σ^2 is 21%, with a difference in safety stock level of 84%. As n increases, both the safety stock mark-up and the cost benefit decrease, as extra information results in more accurate estimates and hence less severe effects of parameter uncertainty. With 10 observations, the cost difference is still 9%, with 20 observations it is reduced to 3%, and with 100 observations the difference is 0.2%.

The shortage costs have a substantial impact on the actual cost benefit. As the corrected order-up-to levels are larger than the classical order-up-to levels, it is ex-

Table 3.1: *Numerical results: mean-stationary, normally distributed demand*

Parameters						Order-up-to levels			Safety stock diff.		Total expected holding and shortage cost				
$\hat{\mu}$	$\hat{\sigma}^2$	n	p	h	L	Class.	Approx.	Exact	Approx.	Exact	Class.	Approximate		Exact	
10	4	5	20	1	5	57.5	61.0	63.8	+47.9%	+84.4%	26.1	21.3	(-18.2%)	20.5	(-21.2%)
10	4	10	20	1	5	57.5	59.2	60.2	+23.9%	+36.8%	15.0	13.7	(-8.4%)	13.6	(-9.4%)
10	4	20	20	1	5	57.5	58.4	58.8	+12.0%	+17.7%	11.6	11.3	(-2.9%)	11.3	(-3.1%)
10	4	100	20	1	5	57.5	57.6	57.7	+2.4%	+3.5%	9.7	9.7	(-0.1%)	9.7	(-0.2%)
10	4	5	**50**	1	5	59.2	64.2	69.1	+54.2%	+107.1%	42.9	29.7	(-30.7%)	27.1	(-36.6%)
10	4	10	**50**	1	5	59.2	61.8	63.2	+27.6%	+43.2%	20.2	17.0	(-16.1%)	16.6	(-17.8%)
10	4	20	**50**	1	5	59.2	60.5	61.1	+13.9%	+20.1%	14.3	13.5	(-5.6%)	13.4	(-6.1%)
10	4	100	**50**	1	5	59.2	59.5	59.6	+2.8%	+3.9%	11.4	11.3	(-0.3%)	11.3	(-0.3%)
10	4	5	**100**	1	5	60.4	66.5	73.8	+58.3%	+128.3%	65.4	38.9	(-40.5%)	33.1	(-49.4%)
10	4	10	**100**	1	5	60.4	63.6	65.5	+30.1%	+48.7%	25.8	19.7	(-23.6%)	19.0	(-26.3%)
10	4	20	**100**	1	5	60.4	62.0	62.7	+15.3%	+22.1%	16.5	15.1	(-8.6%)	15.0	(-9.3%)
10	4	100	**100**	1	5	60.4	60.7	60.9	+3.2%	+4.2%	12.5	12.5	(-0.4%)	12.5	(-0.4%)
10	4	5	20	1	**10**	110.6	119.1	123.8	+81.2%	+125.9%	51.4	37.0	(-28.1%)	35.6	(-30.7%)
10	4	10	20	1	**10**	110.6	115.1	116.7	+43.1%	+58.0%	26.9	22.5	(-16.6%)	22.2	(-17.4%)
10	4	20	20	1	**10**	110.6	112.9	113.6	+22.7%	+28.9%	18.8	17.5	(-7.0%)	17.5	(-7.3%)
10	4	100	20	1	**10**	110.6	111.1	111.2	+4.9%	+5.9%	14.1	14.0	(-0.4%)	14.0	(-0.4%)
10	**1**	5	20	1	5	53.7	55.5	56.9	+47.9%	+84.4%	13.0	10.7	(-18.1%)	10.3	(-21.1%)
10	**1**	10	20	1	5	53.7	54.6	55.1	+23.9%	+36.8%	7.5	6.9	(-8.5%)	6.8	(-9.5%)
10	**1**	20	20	1	5	53.7	54.2	54.4	+12.0%	+17.7%	5.8	5.7	(-2.9%)	5.6	(-3.2%)
10	**1**	100	20	1	5	53.7	53.8	53.9	+2.4%	+3.5%	4.9	4.8	(-0.1%)	4.8	(-0.2%)
20	4	5	20	1	5	107.5	111.0	113.8	+47.9%	+84.4%	26.1	21.4	(-18.2%)	20.6	(-21.2%)
20	4	10	20	1	5	107.5	109.2	110.2	+23.9%	+36.8%	15.0	13.8	(-8.4%)	13.6	(-9.4%)
20	4	20	20	1	5	107.5	108.4	108.8	+12.0%	+17.7%	11.6	11.3	(-2.9%)	11.3	(-3.1%)
20	4	100	20	1	5	107.5	107.6	107.7	+2.4%	+3.5%	9.7	9.7	(-0.1%)	9.7	(-0.2%)

pected and underpinned by the results that the costlier backorders are, the larger the cost benefit of applying the correction is. If $p = 100$, then even with $n = 20$ historical observations, the exact approach leads to a cost advantage of 9% compared to the classical approach, which increases to 49% for $n = 5$. Also the lead time has a large effect. If the lead time is doubled to 10, then the cost benefits increase, especially in the cases where estimates are not based on very few observations. This can be explained by the fact that if the lead time increases, then the auto-correlative effect of the estimation errors magnifies. For example, if μ is underestimated, then demand in all time instants during the lead time is underestimated.

If the demand variance is decreased, then this decreases the fluctuation in lead time demand and hence the safety stock mark-up decreases. However, the percentage safety stock difference and cost benefit of the exact approach remain the same. So, the demand variance can be seen as a scaling parameter of the uncertainty in demand. A similar observation holds for the mean.

Finally, we compare the approximate approach to the exact approach. As one would expect from asymptotic theory, the exact approach outperforms the approximate approach, but the difference decreases if n increases. For $n = 10$ and larger, the exact and asymptotic approach typically yield results that are very close to each other. The shape of the distribution that the asymptotic approach assumes for the demand variance error term, only holds asymptotically. Therefore, two effects come into play that dictate the benefit of the asymptotic approach as compared to the classical approach. Firstly, the smaller n is, the larger is the effect of the estimation uncertainty and hence the larger is the possible gain. However, secondly, the smaller n is, the more severe is the misfit of the shape of the asymptotic normal distribution for the demand variance error term, which has a negative effect on the gain. In conclusion, if very few data points are available, then it is worthwhile to derive the exact error term distribution, if possible, but for 10 or more data points the approximate method (which is easy to derive and always available) suffices.

We now inspect the cost differences between the different approaches in closer detail for the base case scenario. We do so by means of the violin plots in Figure 3.1, which show a combination of a box plot and a kernel density plot. Typically, the exact approach sets the largest order-up-to level, followed by the approximate ap-

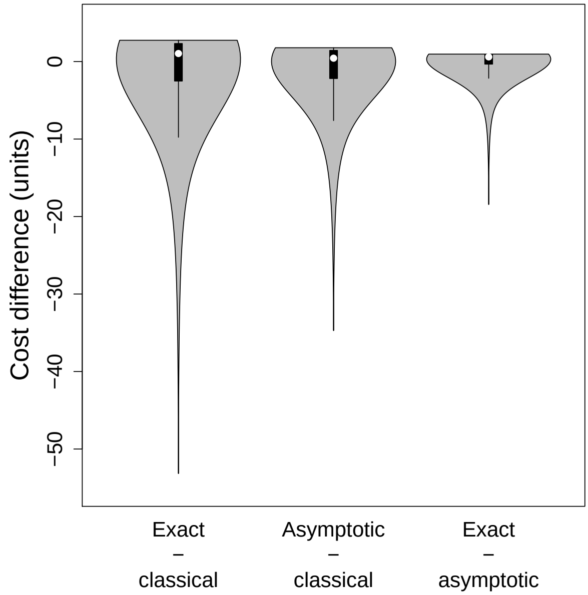

Figure 3.1: *Violin plots of cost differences between the classical approach and the improved approaches, for* $\hat{\mu} = 10$, $\hat{\sigma}^2 = 4$, $p = 20$, $h = 1$, $L = 5$, *and* $n = 10$

proach and the classical approach. This implies that if μ and σ are underestimated, then the exact approach yields the largest benefits, followed by the asymptotic approach, as backorder costs are much larger than holding costs. This is reflected in the tails of the plots. Observe that there are also cases where the exact and asymptotic approaches lead to slightly larger costs than the classical approach. This typically occurs when μ and/or σ are severely overestimated. In those situations, the safety stocks of the classical approach were already too high and adding an extra mark-up leads to larger holding costs. However, overall the cost consequences of underestimating the demand parameters are more severe, which is underpinned by the fact that the corrected approaches do on average lead to a substantial cost benefit compared to the classical approach. The average cost of the classical approach is 15.0, whereas that of the approximate approach is 13.7 and that of the exact approach is 13.6. So, by setting an extra safety-stock mark-up, slightly larger costs are incurred in situations where the demand parameters are overestimated, but simultaneously the decision-maker is protected against very large costs when the demand parameters are underestimated, which on average has a larger effect. These effects behave similarly for the demand models that we discuss next.

3.4.2 Single exponential smoothing for mean-stationary, normally distributed demand

Table 3.2 shows the results of the numerical comparisons for the mean-stationary normal demand model, where SES is used to estimate the mean. We focus on those results that provide new insights compared to the previous section. It is of particular interest to observe that all costs in Table 3.2 are higher than those in Table 3.1. This reflects the fact that the SES estimator has a larger sample variance than the sample mean, which is the minimum variance unbiased estimator for the mean-stationary

Table 3.2: *Numerical results: single exponential smoothing for mean-stationary, normally distributed demand*

Parameters							Order-up-to levels			Safety stock diff.		Total expected holding and shortage cost				
$\hat{\mu}$	$\hat{\sigma}^2$	α	n	p	h	L	Class.	Approx.	Exact	Approx.	Exact	Class.	Approximate		Exact	
10	4	0.2	5	20	1	5	57.5	59.7	64.8	+30.4%	+97.8%	29.3	25.0	(-14.7%)	22.1	(-24.7%)
10	4	0.2	10	20	1	5	57.5	59.4	60.7	+26.2%	+42.9%	16.1	14.5	(-10.3%)	14.2	(-11.8%)
10	4	0.2	20	20	1	5	57.5	59.3	59.8	+24.9%	+31.3%	13.7	12.6	(-7.9%)	12.6	(-8.2%)
10	4	0.2	100	20	1	5	57.5	59.3	59.4	+24.7%	+25.9%	12.6	11.8	(-6.5%)	11.8	(-6.5%)
10	4	0.2	5	**100**	1	5	60.4	64.6	75.5	+39.6%	+144.9%	76.6	50.9	(-33.6%)	35.5	(-53.6%)
10	4	0.2	10	**100**	1	5	60.4	63.8	66.2	+32.5%	+55.3%	28.6	20.8	(-27.3%)	19.8	(-30.8%)
10	4	0.2	20	**100**	1	5	60.4	63.4	64.2	+28.7%	+36.3%	21.1	16.9	(-20.1%)	16.7	(-20.7%)
10	4	0.2	100	**100**	1	5	60.4	63.1	63.2	+25.5%	+26.8%	17.8	15.2	(-14.8%)	15.2	(-14.8%)
10	4	0.2	5	20	1	**10**	110.6	116.0	126.1	+52.0%	+147.5%	59.7	46.0	(-23.0%)	39.1	(-34.6%)
10	4	0.2	10	20	1	**10**	110.6	115.5	117.8	+47.0%	+68.4%	30.0	24.1	(-19.7%)	23.6	(-21.3%)
10	4	0.2	20	20	1	**10**	110.6	115.4	116.1	+45.5%	+53.0%	25.0	20.8	(-16.9%)	20.7	(-17.2%)
10	4	0.2	100	20	1	**10**	110.6	115.3	115.5	+45.3%	+46.7%	23.0	19.4	(-15.7%)	19.4	(-15.7%)
10	**1**	0.2	5	20	1	5	53.7	54.9	57.4	+30.4%	+97.8%	14.6	12.5	(-14.7%)	11.0	(-24.6%)
10	**1**	0.2	10	20	1	5	53.7	54.7	55.3	+26.2%	+42.9%	8.0	7.2	(-10.3%)	7.1	(-11.7%)
10	**1**	0.2	20	20	1	5	53.7	54.7	54.9	+24.9%	+31.3%	6.8	6.3	(-7.9%)	6.3	(-8.2%)
10	**1**	0.2	100	20	1	5	53.7	54.7	54.7	+24.7%	+25.9%	6.3	5.9	(-6.5%)	5.9	(-6.6%)
20	4	0.2	5	20	1	5	107.5	109.7	114.8	+30.4%	+97.8%	29.2	24.9	(-14.7%)	22.1	(-24.6%)
20	4	0.2	10	20	1	5	107.5	109.4	110.7	+26.2%	+42.9%	16.1	14.5	(-10.3%)	14.2	(-11.8%)
20	4	0.2	20	20	1	5	107.5	109.3	109.8	+24.9%	+31.3%	13.7	12.6	(-7.9%)	12.6	(-8.2%)
20	4	0.2	100	20	1	5	107.5	109.3	109.4	+24.7%	+25.9%	12.6	11.8	(-6.5%)	11.8	(-6.5%)
10	4	**0.5**	5	20	1	5	57.5	62.7	65.9	+70.8%	+113.5%	33.1	24.7	(-25.4%)	23.8	(-28.2%)
10	4	**0.5**	10	20	1	5	57.5	62.3	63.6	+65.2%	+82.5%	24.5	18.4	(-25.1%)	18.2	(-25.9%)
10	4	**0.5**	20	20	1	5	57.5	62.2	62.8	+63.5%	+71.9%	21.8	16.5	(-24.2%)	16.5	(-24.5%)
10	4	**0.5**	100	20	1	5	57.5	62.2	62.3	+63.3%	+64.9%	20.1	15.4	(-23.1%)	15.4	(-23.1%)
10	4	0.5	5	**100**	1	5	60.4	69.0	77.5	+82.8%	+164.2%	90.3	45.0	(-50.2%)	38.2	(-57.6%)
10	4	0.5	10	**100**	1	5	60.4	68.1	70.7	+73.4%	+98.3%	54.3	26.2	(-51.7%)	25.3	(-53.4%)
10	4	0.5	20	**100**	1	5	60.4	67.6	68.6	+68.5%	+78.4%	43.6	22.1	(-49.4%)	21.9	(-49.8%)
10	4	0.5	100	**100**	1	5	60.4	67.1	67.3	+64.4%	+66.1%	37.2	19.9	(-46.6%)	19.9	(-46.6%)
10	4	**0.8**	5	20	1	5	57.5	66.2	70.3	+117.7%	+171.5%	49.1	31.6	(-35.7%)	30.4	(-38.0%)
10	4	**0.8**	10	20	1	5	57.5	65.7	67.4	+110.6%	+132.6%	37.5	23.4	(-37.6%)	23.1	(-38.2%)
10	4	**0.8**	20	20	1	5	57.5	65.6	66.3	+108.5%	+119.1%	33.8	21.1	(-37.8%)	21.0	(-38.0%)
10	4	**0.8**	100	20	1	5	57.5	65.5	65.7	+108.1%	+110.2%	31.4	19.7	(-37.3%)	19.7	(-37.3%)
10	4	0.8	5	**100**	1	5	60.4	74.3	85.0	+133.0%	+236.0%	151.7	57.2	(-62.3%)	48.7	(-67.9%)
10	4	0.8	10	**100**	1	5	60.4	73.0	76.3	+121.1%	+152.7%	100.6	33.5	(-66.7%)	32.3	(-67.9%)
10	4	0.8	20	**100**	1	5	60.4	72.4	73.7	+114.7%	+127.4%	84.7	28.1	(-66.8%)	27.9	(-67.1%)
10	4	0.8	100	**100**	1	5	60.4	71.8	72.1	+109.5%	+111.7%	74.8	25.4	(-66.1%)	25.4	(-66.1%)

normal demand model. For given estimators $\hat{\mu}$ and $\hat{\sigma}^2$, the classical order-up-to levels in this section are the same as those in the previous section. However, given these estimates, the true values for μ and σ^2 can vary more under the SES estimator than under the sample mean. Therefore, even when applying the exact correction which leads to the optimal decision under the mean-stationary normal demand model, the costs using the SES estimator are larger than those using the sample mean.

If the smoothing parameter α is chosen larger, then relatively more weight is given to recent observations. The SES estimator gives exponentially decreasing weights to older observations, and therefore increasing α increases the sample variance of the estimator, and thus also the necessary safety stock mark-up and the cost difference between the corrected approaches and the classical approach. Furthermore, especially for large values of α, the cost benefit of applying the corrected approaches remains very large (up to 66% even for $n = 100$). The discussion in Section 3.2 suggests that values between 0.21 and 0.4, and possibly larger, are realistic. Similar to the previous section, we find also here that if there are 10 or more observations, then the approximate approach performs very similarly to the exact approach. Furthermore, a larger lead time increases the possible cost differences, and the percentage cost differences are invariant to the demand parameterization. In conclusion, although using the SES estimator in the mean-stationary normal demand model leads to a cost disadvantage compared to using the sample mean, the cost benefit that can be realized by correcting for the parameter uncertainty is significantly larger for SES and also substantial in proportion to the extra cost associated with using SES instead of the sample average.

3.4.3 Normally distributed, trended demand

Table 3.3 shows the comparative results for the trended demand model. As many effects have already been already interpreted for the previous two demand models, we focus again on the insights that are specific to the trended demand model. If we compare Table 3.3 to Table 3.1, then we notice that both the percentage safety stock mark-ups and the percentage cost savings are significantly larger for trended demand than for mean-stationary demand. This reflects the auto-correlative effect of future forecast errors. In the trend model, not only forecast errors of the mean parameter accumulate over the lead time, but also those of the trend parameter. If a trend is underestimated, then this has a substantially larger effect on the error in the lead time demand forecast, than if only a mean is underestimated. For example, if the trend parameter is underestimated by 0.1, then the (one-period) demand forecast for 5 periods later is off by 0.5. Contrarily, if the mean parameter is underestimated by 0.1, then the demand forecast for 5 periods later is still off by 0.1. Therefore, in the trend model the consequences of ignoring parameter uncertainty are also substantially larger than in the mean-stationary model.

In the base case, the safety stocks resulting from the classical approach have to be adjusted by 11% for 100 observations, to up to 450% when only 5 observations are available. This mark-up increases to up to more than 1000% (so with a multiplication factor of 11) if the lead time is doubled to 10. For this longer lead time, even with 100 observations the necessary mark-up is still 22% and the resulting cost saving is 5%, which increases to up to 59% for $n = 5$. When backorder costs are set at 100 (and the lead time is 5), then savings of 4% for $n = 100$ to up to 69% for $n = 5$ can be achieved.

Table 3.3: *Numerical results: normally distributed, trended demand*

Parameters							Order-up-to levels			Safety stock diff.		Total expected holding and shortage cost				
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}^2$	n	p	h	L	Class.	Approx.	Exact	Approx.	Exact	Class.	Approximate		Exact	
10	1	4	5	20	1	5	97.5	119.7	131.0	+298.3%	+449.6%	138.7	71.7	(-48.3%)	67.4	(-51.4%)
10	1	4	10	20	1	5	122.5	131.7	133.7	+124.2%	+151.2%	42.8	25.6	(-40.3%)	25.2	(-41.1%)
10	1	4	20	20	1	5	172.5	176.6	177.3	+55.9%	+64.4%	20.2	15.8	(-21.4%)	15.8	(-21.7%)
10	1	4	100	20	1	5	572.5	573.2	573.3	+10.2%	+11.3%	10.6	10.4	(-1.5%)	10.4	(-1.5%)
10	1	4	5	**50**	1	5	99.2	128.3	149.8	+315.1%	+548.3%	298.2	109.0	(-63.4%)	93.8	(-68.6%)
10	1	4	10	**50**	1	5	124.2	136.3	139.4	+130.8%	+164.6%	75.7	31.8	(-58.0%)	30.9	(-59.1%)
10	1	4	20	**50**	1	5	174.2	179.6	180.5	+58.7%	+68.0%	28.9	18.9	(-34.5%)	18.8	(-35.0%)
10	1	4	100	**50**	1	5	574.2	575.2	575.3	+10.6%	+11.8%	12.5	12.2	(-2.6%)	12.2	(-2.6%)
10	1	4	5	**100**	1	5	100.4	134.4	167.7	+326.3%	+645.3%	547.9	156.1	(-71.5%)	119.8	(-78.1%)
10	1	4	10	**100**	1	5	125.4	139.5	143.8	+135.3%	+176.2%	121.6	37.6	(-69.1%)	35.8	(-70.6%)
10	1	4	20	**100**	1	5	175.4	181.7	182.8	+60.6%	+71.0%	38.6	21.2	(-45.1%)	21.0	(-45.6%)
10	1	4	100	**100**	1	5	575.4	576.6	576.7	+11.0%	+12.1%	13.9	13.4	(-3.5%)	13.4	(-3.6%)
10	1	4	5	20	1	**10**	215.6	289.9	322.2	+705.1%	+1010.9%	472.1	204.8	(-56.6%)	192.2	(-59.3%)
10	1	4	10	20	1	**10**	265.6	295.1	300.0	+280.3%	+326.1%	139.0	61.3	(-55.9%)	60.5	(-56.5%)
10	1	4	20	20	1	**10**	365.6	378.4	379.6	+121.3%	+133.3%	53.4	31.7	(-40.5%)	31.6	(-40.7%)
10	1	4	100	20	1	**10**	1165.6	1167.8	1167.9	+20.9%	+22.2%	17.0	16.2	(-5.0%)	16.2	(-5.0%)
10	1	**1**	5	20	1	5	93.7	104.9	110.5	+298.3%	+449.6%	69.2	35.7	(-48.4%)	33.5	(-51.5%)
10	1	**1**	10	20	1	5	118.7	123.4	124.4	+124.2%	+151.2%	21.5	12.8	(-40.2%)	12.6	(-41.0%)
10	1	**1**	20	20	1	5	168.7	170.8	171.1	+56.0%	+64.4%	10.1	7.9	(-21.3%)	7.9	(-21.6%)
10	1	**1**	100	20	1	5	568.7	569.1	569.2	+10.2%	+11.3%	5.3	5.2	(-1.5%)	5.2	(-1.5%)
10	**0.5**	4	5	20	1	5	77.5	99.7	111.0	+298.3%	+449.6%	138.7	71.6	(-48.3%)	67.3	(-51.5%)
10	**0.5**	4	10	20	1	5	90.0	99.2	101.2	+124.2%	+151.2%	42.8	25.6	(-40.1%)	25.3	(-40.9%)
10	**0.5**	4	20	20	1	5	115.0	119.1	119.8	+55.9%	+64.4%	20.1	15.8	(-21.2%)	15.8	(-21.5%)
10	**0.5**	4	100	20	1	5	315.0	315.7	315.8	+10.2%	+11.3%	10.6	10.4	(-1.5%)	10.4	(-1.5%)
20	1	4	5	20	1	5	147.5	169.7	181.0	+298.3%	+449.6%	139.0	71.9	(-48.3%)	67.4	(-51.5%)
20	1	4	10	20	1	5	172.5	181.7	183.7	+124.2%	+151.2%	42.7	25.5	(-40.2%)	25.2	(-41.0%)
20	1	4	20	20	1	5	222.5	226.6	227.3	+55.9%	+64.4%	20.1	15.8	(-21.4%)	15.8	(-21.6%)
20	1	4	100	20	1	5	622.5	623.2	623.3	+10.2%	+11.3%	10.6	10.4	(-1.5%)	10.4	(-1.5%)

The percentage cost differences are invariant to the demand parameterization, both to $\hat{\alpha}$ and $\hat{\beta}$, and we again observe that for 10 observations or more, the approximate approach and exact approach perform nearly equally well.

3.4.4 Demand is a normal random walk

When demand follows a random walk with normally distributed increments, and this model is specified correctly, then there is only one parameter left to estimate: the variance of the increments. Therefore, Table 3.4 does not show safety stock markups or cost savings that are as dramatic as those in previous sections. Still, if the backorder cost is set to 100, then savings of 9% for $n = 10$ or 31% for $n = 5$ are still possible. However, typically, if there are more than 10 observations available, then the possible savings become very small, and for $n > 100$ they are negligible for many settings. It is interesting to note that for this model a longer lead time does not lead to larger percentage cost savings of the corrected approaches compared to the classical approach. This is easily explained, as other than with a mean or trend parameter, underestimation of the variance of the increments does not have a multiplicative ef-

Table 3.4: *Numerical results: normal random walk*

Parameters						Order-up-to levels			Safety stock diff.		Total expected holding and shortage cost				
D_n	$\hat{\sigma}^2$	n	p	h	L	Class.	Approx.	Exact	Approx.	Exact	Class.	Approximate		Exact	
10	4	5	20	1	5	74.7	76.3	85.7	+6.4%	+44.3%	63.3	61.8	(-2.3%)	58.4	(-7.7%)
10	4	10	20	1	5	74.7	75.1	78.1	+1.5%	+13.4%	38.4	38.2	(-0.4%)	37.8	(-1.6%)
10	4	20	20	1	5	74.7	74.8	76.1	+0.2%	+5.6%	33.7	33.7	(-0.0%)	33.6	(-0.4%)
10	4	100	20	1	5	74.7	74.7	75.0	+0.0%	+1.0%	31.4	31.4	(-0.0%)	31.4	(-0.0%)
10	4	5	**50**	1	5	80.6	84.0	102.1	+11.3%	+70.2%	100.8	93.9	(-6.9%)	81.7	(-19.0%)
10	4	10	**50**	1	5	80.6	82.0	86.5	+4.7%	+19.4%	48.7	47.7	(-2.2%)	46.4	(-4.7%)
10	4	20	**50**	1	5	80.6	81.2	83.0	+2.0%	+7.9%	40.5	40.3	(-0.5%)	40.0	(-1.1%)
10	4	100	**50**	1	5	80.6	80.7	81.0	+0.3%	+1.4%	36.7	36.7	(-0.0%)	36.7	(-0.0%)
10	4	5	**100**	1	5	84.6	89.6	117.6	+14.5%	+95.6%	150.8	133.5	(-11.5%)	104.8	(-30.5%)
10	4	10	**100**	1	5	84.6	86.9	93.1	+6.9%	+24.6%	58.5	55.8	(-4.7%)	53.3	(-8.8%)
10	4	20	**100**	1	5	84.6	85.7	87.9	+3.3%	+9.8%	45.7	45.1	(-1.2%)	44.8	(-2.0%)
10	4	100	**100**	1	5	84.6	84.8	85.1	+0.6%	+1.7%	40.4	40.4	(-0.0%)	40.4	(-0.1%)
10	4	5	20	1	**10**	165.5	169.7	194.5	+6.4%	+44.3%	168.2	164.3	(-2.3%)	155.2	(-7.7%)
10	4	10	20	1	**10**	165.5	166.5	174.2	+1.5%	+13.4%	101.5	101.2	(-0.4%)	99.9	(-1.6%)
10	4	20	20	1	**10**	165.5	165.6	169.1	+0.2%	+5.6%	89.3	89.2	(-0.0%)	88.9	(-0.4%)
10	4	100	20	1	**10**	165.5	165.4	166.1	+0.0%	+1.0%	83.0	83.0	(-0.0%)	83.0	(-0.0%)
10	**1**	5	20	1	5	62.4	63.2	67.9	+6.4%	+44.3%	31.8	31.0	(-2.3%)	29.3	(-7.7%)
10	**1**	10	20	1	5	62.4	62.6	64.0	+1.5%	+13.4%	19.2	19.1	(-0.4%)	18.9	(-1.6%)
10	**1**	20	20	1	5	62.4	62.4	63.1	+0.2%	+5.6%	16.9	16.9	(-0.0%)	16.8	(-0.4%)
10	**1**	100	20	1	5	62.4	62.4	62.5	+0.0%	+1.0%	15.7	15.7	(-0.0%)	15.7	(-0.0%)
20	4	5	20	1	5	124.7	126.3	135.7	+6.4%	+44.3%	63.6	62.1	(-2.3%)	58.6	(-7.7%)
20	4	10	20	1	5	124.7	125.1	128.1	+1.5%	+13.4%	38.4	38.2	(-0.4%)	37.7	(-1.6%)
20	4	20	20	1	5	124.7	124.8	126.1	+0.2%	+5.6%	33.7	33.7	(-0.0%)	33.6	(-0.4%)
20	4	100	20	1	5	124.7	124.7	125.0	+0.0%	+1.0%	31.4	31.4	(-0.0%)	31.4	(-0.0%)

fect when future demands are aggregated. Contrarily, the proportional error remains constant over the lead time. Furthermore, in the random walk model there is no estimation error around the level of future demands. We conclude that the random walk demand model with normal increments does not lead to cost consequences of ignoring parameter uncertainty that are as severe as for the other demand models.

3.5 Demand misspecification

The numerical results in the previous section hold under the assumption that the demand model was specified correctly. However, in practice it may be difficult to specify whether, and if so since when demand is stationary, has a trend, or is autocorrelated. In Section 3.4.2 we already considered using the SES estimator instead of the sample mean in the mean-stationary demand model. The underlying demand model - that of mean-stationary, normally distributed demand - was specified correctly and used in the analysis. Contrarily, in this section we study a scenario where the demand model is not specified correctly and the analysis is based on that incorrect specification. We study the negative cost consequences and the performance of the classical and corrected approaches in the presence of such misspecification. Specifically, we consider the scenario where a trend is fitted on a data series that is actually mean-stationary. We focus on the base case scenarios of the previous section and compare the order-up-to levels and costs of the classical and corrected approaches, in order to analyze how the cost consequences of demand misspecification compare to the cost consequences of ignoring parameter uncertainty.

So, assume that the decision maker suspects that demand is normally distributed and stationary around a linear trend and acts accordingly, whereas actually demand is mean-stationary. Table 3.5 shows the numerical results for this scenario. We consider the case where the point estimate for the trend parameter is 0, which is the most likely scenario in absence of a trend. Note that the uncertainty around the trend parameter is still taken into account by the proposed methods; so the misspecification affects the results even if the point estimate for the trend parameter is 0. The costs are uniformly higher compared to those in Table 3.1. For 5 observations, the classical approach overall results in costs that are approximately 3 times as large as those without misspecification. The exact approach reduced this to a factor approximately

2 for cases with a lead time of 5 time instants.

It is noteworthy that the asymptotic approach outperforms the exact approach for this case of demand misspecification. The difference is especially large if few observations are used, and for $n = 10$ or more the two approaches yield similar results. So, in this scenario the asymptotic approach proves to be valuable, suggesting that if the decision maker is unsure of the demand specification, then the asymptotic approach can yield better results than the exact approach.

Comparing Table 3.1 with Table 3.5, we find that for the base case with $n = 5$, the total cost (in absolute units) without misspecification is 26 for the classical approach, 21 for the approximate approach, and 21 for the exact approach. Under demand misspecification, these figures are 67, 38, and 45, respectively. This implies that the cost effect of mis-specifying demand is 41 for the classical approach, 17 for the approximate approach, and 23 for the exact approach. The gain of applying the approximate correction factor is 29 compared to the classical approach with misspecification. For the exact correction factor the gain is 22. If the demand model is misspecified, then

Table 3.5: *Numerical results: misspecification, trend fitted to mean-stationary demand*

Parameters							Order-up-to levels			Safety stock diff.		Total expected holding and shortage cost				
$\hat{\alpha}$	$\hat{\beta}$	$\hat{\sigma}^2$	n	p	h	L	Class.	Approx.	Exact	Approx.	Exact	Class.	Approximate		Exact	
10	0	4	5	20	1	5	57.5	79.7	91.0	+38.7%	+58.4%	67.1	38.0	(-43.4%)	44.8	(-33.2%)
10	0	4	10	20	1	5	57.5	66.7	68.7	+16.1%	+19.6%	29.8	20.5	(-31.2%)	21.3	(-28.6%)
10	0	4	20	20	1	5	57.5	61.6	62.3	+7.3%	+8.4%	17.4	14.5	(-16.6%)	14.6	(-16.0%)
10	0	4	100	20	1	5	57.5	58.2	58.3	+1.3%	+1.5%	10.5	10.4	(-1.4%)	10.4	(-1.4%)
10	0	4	5	**50**	1	5	59.2	88.3	109.8	+49.1%	+85.4%	131.5	49.4	(-62.4%)	63.2	(-51.9%)
10	0	4	10	**50**	1	5	59.2	71.3	74.4	+20.4%	+25.6%	48.6	25.0	(-48.5%)	26.5	(-45.6%)
10	0	4	20	**50**	1	5	59.2	64.6	65.5	+9.1%	+10.6%	24.0	17.3	(-27.7%)	17.5	(-26.9%)
10	0	4	100	**50**	1	5	59.2	60.2	60.3	+1.7%	+1.8%	12.4	12.1	(-2.3%)	12.1	(-2.3%)
10	0	4	5	**100**	1	5	60.4	94.4	127.7	+56.3%	+111.3%	226.1	59.5	(-73.7%)	80.8	(-64.3%)
10	0	4	10	**100**	1	5	60.4	74.5	78.8	+23.3%	+30.4%	72.6	28.6	(-60.6%)	30.6	(-57.8%)
10	0	4	20	**100**	1	5	60.4	66.7	67.8	+10.5%	+12.2%	30.8	19.3	(-37.2%)	19.6	(-36.4%)
10	0	4	100	**100**	1	5	60.4	61.6	61.7	+1.9%	+2.1%	13.8	13.4	(-3.3%)	13.4	(-3.3%)
10	0	4	5	20	1	**10**	110.6	184.9	217.2	+67.3%	+96.5%	149.3	91.2	(-38.9%)	119.9	(-19.7%)
10	0	4	10	20	1	**10**	110.6	140.1	145.0	+26.7%	+31.1%	66.7	42.7	(-35.9%)	46.5	(-30.2%)
10	0	4	20	20	1	**10**	110.6	123.4	124.6	+11.6%	+12.7%	36.0	26.2	(-27.0%)	26.9	(-25.2%)
10	0	4	100	20	1	**10**	110.6	112.8	112.9	+2.0%	+2.1%	16.5	15.9	(-4.1%)	15.9	(-4.1%)
10	0	**1**	5	20	1	5	53.7	64.9	70.5	+20.7%	+31.2%	33.6	19.0	(-43.3%)	22.4	(-33.2%)
10	0	**1**	10	20	1	5	53.7	58.4	59.4	+8.6%	+10.5%	14.9	10.2	(-31.3%)	10.6	(-28.6%)
10	0	**1**	20	20	1	5	53.7	55.8	56.1	+3.9%	+4.5%	8.7	7.3	(-16.7%)	7.3	(-16.1%)
10	0	**1**	100	20	1	5	53.7	54.1	54.2	+0.7%	+0.8%	5.3	5.2	(-1.4%)	5.2	(-1.4%)
20	0	4	5	20	1	5	107.5	129.7	141.0	+20.7%	+31.2%	67.1	38.1	(-43.3%)	44.9	(-33.1%)
20	0	4	10	20	1	5	107.5	116.7	118.7	+8.6%	+10.5%	29.9	20.5	(-31.4%)	21.3	(-28.8%)
20	0	4	20	20	1	5	107.5	111.6	112.3	+3.9%	+4.5%	17.5	14.6	(-16.6%)	14.7	(-16.0%)
20	0	4	100	20	1	5	107.5	108.2	108.3	+0.7%	+0.8%	10.5	10.4	(-1.4%)	10.4	(-1.4%)

the approximate method (which performs best in that scenario) does not yield lower costs than the classical method under a correctly specified demand model. Therefore, we conclude that the cost effect of mis-specifying the demand model is larger than the cost reduction that can be achieved by applying the correction. However, the figures above show that the latter is still substantial, as the gain of 29 units corrects 71% of the loss due to demand misspecification. It is thus still worthwhile to apply a corrected method (especially the approximate method) also when demand is misspecified. However, the largest gain can be achieved by correctly specifying the demand model. As another example, in the base case with $n = 20$, the classical method under demand misspecification has a 6 units higher cost than that under a correct specification of the demand model, but applying any of both corrected methods under misspecification leads to a gain of 3 units compared to using the classical method. So, we find that 50% of the loss due to demand misspecification can be corrected for by applying the corrected methods.

3.6 Conclusion

An adaptation of inventory models to incorporate parameter estimation uncertainty for future demand was proposed. The approach has four steps:

1. Formulate the decision model in terms of the lead time demand distribution function

2. Estimate the parameters efficiently and estimate the distributions of their estimation errors

3. Replace the true parameters by an appropriate function of the point estimates and estimation errors in the lead time demand distribution function

4. Use the expectation of the lead time demand distribution function with respect to the estimation errors as the new predictive lead time demand distribution in the decision model.

We demonstrated the approach in a model with holding and shortage costs, using various demand models. The method can be applied to any inventory model using any demand process and parameter estimator, as long as the distribution of its estimation error can be derived or approximated. Typical results of this approach are safety stock mark-ups that increase in the degree of uncertainty in the model. The degree of uncertainty increases if more parameters have to be estimated, if estimates are based on a small number of observations, if the lead time is large, or if the variance of demand is large with respect to the mean. The method allows for freedom in the choice of approximation method of the estimators' error distributions. We discussed the exact error distribution and the asymptotic approach which is typically easier to derive and furthermore robust to demand model misspecification.

The cost benefits of properly treating estimation uncertainty in inventory models are twofold: the expected costs are lower and the cost variance is reduced. The point estimate of a parameter in the model is sometimes close to the true parameter, leading to lower costs, but it can also be relatively far away from the true parameter, leading to very high costs if no correction is applied. By incorporating a distribution of the estimation error, one acknowledges that the actual parameter value may be smaller than the point estimate, but also larger. A safety stock mark-up is established, and since shortage costs are relatively higher than holding costs, this reduces the chance of extraordinary high future costs. This two-sided benefit is even more beneficial for risk-averse decision makers. The asymptotic approach that is generally applicable performed almost as well as the exact approach when estimates were based on 10 or more observations. This implies that also for demand distributions where the exact error distribution cannot be derived, the correction framework as discussed in this paper can be applied.

A particular case where very large cost savings can be achieved is the trend model, because if a trend is mis-estimated and lead time demands are aggregated, then this has a much larger effect then when for example only a mean is mis-estimated. Contrarily, the random walk model showed relatively low possible cost savings. In that model, if specified correctly and if the variance of lead time demand is derived correctly, parameter uncertainty (in only the variance of the increments) plays a minor role.

We also studied the severity of ignoring parameter uncertainty in relationship with choosing a suboptimal estimator or mis-specifying the demand model. Although the latter has a larger effect on costs, the savings that can be achieved by applying the correction are still substantial. Also, in this case the approximate method showed favorable results due to its robustness. In conclusion, we find that the current standard of separating forecasting and decision making has severe cost consequences, which could be easily prevented by applying the derived, simple, four-step method in any inventory model where forecasts are used.

The message that we aim to transmit to inventory decision makers in practice is that inventories are held as a protection against the uncertainty around future demand, but that a major part of this uncertainty is ignored if parameter estimates are treated as the true parameters of the demand distribution. Especially since in practice estimates are typically based on only a few observations, and therefore have a large error, this uncertainty comprises a large part of the total uncertainty around future demand. Ignoring this leads to too low safety stocks, and therefore too frequent stock-outs. This in turn implies that target service levels are not achieved, that customers are dissatisfied, and that high backorder costs are incurred. This paper proposes a framework that can correct this flaw for general combinations of demand models and parameter estimators.

There are ample opportunities for further research, mainly on three strands. The first strand is studying the four-step method in different inventory models, under different demand distributions, and for different processes. The second strand is studying different approaches to modeling the estimation error distribution, such as bootstrapping. The final strand should serve to find efficient parameterizations and model formulations so that even for models with many parameters, the four-step approach can be implemented and the optimal solution can be calculated efficiently. A related issue is that of the choice of estimator. The generally used parameter estimators are based on loss functions that may be inappropriate for some inventory models. For example, least squares considers an underestimation just as damaging as an overestimation, whereas if $b > h$ an underestimation of e.g. a mean has a larger cost effect than an overestimation. Therefore, an interesting future research topic is to make a connection between the actual loss in the inventory model and the loss function of the estimator.