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Impulse Controllability: From Descriptor Systems to Higher Order DAEs

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Abstract—Impulsive solutions in LTI dynamical systems have received ample attention, but primarily for descriptor systems, i.e., first order Differential Algebraic Equations (DAEs). This paper focuses on the impulsive behavior of higher order dynamical systems and analyzes the causes of impulses in the context of interconnection of one or more dynamical systems. We extend the definition of impulse-controllability to the higher order case. Amongst the various nonequivalent notions of impulse-controllability for first order systems available in the literature, which mostly rely on the input/output structure of the system, our definition, based on a so-called state-map obtained directly from the system equations, generalizes many key first order results to the higher order case. In particular, we show that our higher-order-extension of the definition of impulse controllability generalizes the equivalence between impulse controllability and the ability to eliminate impulses in the closed loop by interconnecting with a suitable controller. This requires an extension of the definition of regularity of interconnection from behaviors involving only smooth trajectories to behaviors on the positive half line involving impulsive-smooth trajectories.

Index Terms—impulsive solutions, initial conditions, state maps, zeros at infinity.

I. INTRODUCTION

IN this paper we consider linear systems described by the following set of differential equations:

$$R_0 w + R_1 \frac{d}{dt} w + \cdots + R_N \frac{d^N}{dt^N} w = 0 \quad (1)$$

for constant matrices $R_i \in \mathbb{R}^{n \times m}$, with $R_N \neq 0$. The set of all w that satisfy the above equations in an appropriate solution space, \mathcal{L} , is called the behavior of the system. (1) can be written as $R(d/dt)w = 0$, where $R(\xi) := R_0 + R_1 \xi + \cdots + R_N \xi^N$, which is called a kernel representation of the system. We are particularly interested in solutions on the closed half line $\mathbb{R}^+ := [0, \infty)$, from the point of view of interconnected systems: we consider (1) as the result of interconnection of two or more systems at time 0, the original systems described by disjoint sets of rows of R . In this set up the initial conditions

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for the solutions when $t \geq 0$ arise from the final conditions of the composing systems. This may lead to initial conditions that have *no smooth* (i.e., trajectories that are infinitely often differentiable) solutions on \mathbb{R}^+ . In order to deal with this, we will consider (1) with impulsive-smooth distributions [7], [29] as the underlying function class: linear combinations of distributions with support at zero and smooth functions on \mathbb{R}^+ . The “impulsive-smooth” behaviors are of interest in switched or multimode systems: see [5], [14], [15], for example.

Interconnection of systems has been studied by [24], [28], and conditions for regular interconnection and regular feedback have been derived in the context of smooth behaviors. In this paper we extend these results to a larger space of solutions, now allowing impulses and their derivatives too. Regular feedback interconnection and smooth/impulsive behaviors have been studied respectively in [12] and [13] using techniques from homological¹ algebra.

Impulsive solutions in singular descriptor systems have been well investigated [3], [4], [8], [27]. In that setting impulse controllability aids in the elimination of impulses in the dynamics by feedback control. It must be noted that for the first order case, there are many nonequivalent but inter-related notions of impulse-controllability (see [1] for a recent exposition of the inter-relations). Both time domain and frequency domain approaches have been used in the context of defining impulse controllability. [2] develops a unifying theory to account for the slightly varying definitions of impulse controllability. Our contribution is in light of these variety of differences, and identifying/formulating in such a way that the results extend to higher order dynamical systems and in obtaining the first order results as a special case.

Next, we are interested in the initial conditions which, along with free variables (input), determine the behavior uniquely. We obtain these from the so called “canonical state map” (see Section VIII-B). This has been investigated for smooth behaviors in [23] and we focus on impulsive-smooth behaviors [21].

Summarizing the contributions in this paper:

- We extend results in the literature about regular implementability of smooth sub-behaviors to regular implementability of impulsive-smooth sub-behaviors (of a given plant behavior).
- We extend the concept of impulse controllability defined for descriptor singular system to systems described by higher order differential equations.

¹The third author expresses hearty thanks to Manoj Keshari for help towards understanding the “Ext” operation on the modules built from the plant and controlled system equations as pursued in [13].

- We relate the initial conditions from which there exists an impulsive-smooth solution to a natural map constructed from the system equations: the canonical state map.
- We provide conditions such that the interconnection of two systems does not result in impulses in the dynamics.

Note that all these aspects together help in generalizing the classical first order dynamical systems' equivalence result proved three decades ago in [2] to the higher order case: the equivalence between impulse controllability and the existence of a feedback controller that eliminates impulses in the closed loop. The behavior of unimodular systems (i.e., when R defined after (1) is square and $\det R$ is a nonzero constant), also studied in [25], belong to the class of trajectories we consider in this paper.

The paper is organized as follows: the following section formulates the problems considered in this paper. Section III considers the interconnection of systems where the impulsive-smooth behavior is of interest. Section IV explicitly brings out the relation between state maps and initial conditions. Section V deals with impulse controllability of higher order systems. Impulsive solutions that arise out of interconnections are considered in Section VI and concluding remarks are given in Section VII. Preliminaries required for this paper can be found in the Appendix: Section VIII. The reader is encouraged to read the Appendix for preliminaries about polynomial matrices, poles/zeros at infinity of polynomial/rational matrices, the state-map in behaviors, and essential definitions and results on interconnection of behaviors.

Notation: \mathbb{R} denotes the set of real numbers. $\mathbb{R}[\xi]$ represents the ring of polynomials in one indeterminate ξ , and $\mathbb{R}(\xi)$ stands for the field of rationals over this ring. For a polynomial matrix $R \in \mathbb{R}^{n \times m}[\xi]$, $\text{rank}_{\mathbb{R}[\xi]}(R)$ refers to the rank of R over $\mathbb{R}[\xi]$ and $\text{rank}_{\mathbb{R}}(R)$ denotes the rank over \mathbb{R} . For two matrices R_1 and R_2 with same number of columns $\text{col}(R_1, R_2)$ denotes $[R_1^T \ R_2^T]^T$. We denote the behavior of a system in (1) as follows:

$$\mathfrak{B}(R, \mathcal{L}) := \left\{ w \in \mathcal{L} \mid R \left(\frac{d}{dt} \right) w = 0 \right\}$$

\mathcal{L} denotes the function space of the behavior. $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^m)$ denotes the space of infinitely differentiable functions from \mathbb{R} to \mathbb{R}^m , which we sometimes denote as $\mathcal{L}_{\mathbb{R}}^C$, when the domain and co-domain are unambiguous. \mathcal{D}' is the space of distributions on \mathbb{R} . $\mathcal{L}_{\mathbb{R}^+}^C := \mathcal{C}^\infty(\mathbb{R}^+, \mathbb{R}^m)$ which is the space of smooth functions on \mathbb{R}^+ , such that for any $w \in \mathcal{L}_{\mathbb{R}^+}^C$, the limit $\lim_{t \downarrow 0} w^{(i)}(t)$ exists and is finite for all $i \geq 0$.

II. PROBLEM FORMULATION

We briefly describe the space of “impulsive-smooth” distribution, $\mathcal{L}_{\mathbb{R}^+}^I$ introduced in [7] and [29]. Consider $\mathcal{D}'_0 := \{u \in \mathcal{D}'(\mathbb{R}, \mathbb{R}^m) \mid u = \sum_{i=0}^N a_i \delta^{(i)}, a_i \in \mathbb{R}^m\}$, where δ is the Dirac delta distribution and $\delta^{(i)}$ denotes its i th distributional derivative. The set \mathcal{D}'_0 consists of all distributions supported at zero. $\mathcal{L}_{\mathbb{R}^+}^I$ consists of distributions that are linear combinations of elements from $\mathcal{L}_{\mathbb{R}^+}^C$ and \mathcal{D}'_0 . The behavior depends on the solution space. For example, for a nonconstant unimodular matrix U , $\mathfrak{B}(U, \mathcal{D}'_0)$ is just zero (also when \mathcal{L} is $\mathcal{L}_{\mathbb{R}}^C$, but $\mathfrak{B}(U, \mathcal{L}_{\mathbb{R}^+}^I)$ contains nonzero elements. The following example demonstrates this in a second order system.

Example 2.1: Consider a system described by the following differential equations:

$$\begin{aligned} w_1 + \ddot{w}_2 &= 0 \\ w_2 &= 0 \end{aligned}$$

when $\mathcal{L} = \mathcal{L}_{\mathbb{R}^+}^I$, the solutions of the above differential equations are parametrized by $w_2(0^-)$ and $\dot{w}_2(0^-)$

$$w_2 = 0, \quad w_1 = w_2(0^-)\delta^{(1)} + \dot{w}_2(0^-)\delta.$$

The above equation implies that the solutions of the given differential equations are impulsive for any nonzero initial condition of w_2 and \dot{w}_2 . Next if $\mathcal{L} = \mathcal{L}_{\mathbb{R}}^C$, then $\mathfrak{B} = \{0\}$.

Note that here we consider initial conditions that results in jumps at time $t=0$. Also we consider only those jumps that will lead to impulses in the solutions. For instance, in the above example jumps in w_1 do not cause impulses. As explained in the introduction these jumps might occur as the result of interconnecting two independent systems at $t=0$, that have too many final conditions compared to the initial conditions of the inter-connected system.

The first problem we address is about the interconnection at $t=0$ of two behaviors in $\mathcal{L}_{\mathbb{R}^+}^I$. An interconnection is said to be a *regular interconnection* (RI) if it can be implemented by a (possibly singular) feedback. Refer Section VIII-C (of the Appendix) for the precise definition.

Problem 2.2: For two behaviors such that $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I) \subset \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$, when is $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$ regularly implementable with respect to $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$?

An interconnection is said to be a *regular feedback interconnection* (RFI) if it can be implemented by a regular feedback. The precise definition is in Section VIII-C of the Appendix. Depending on the context, “RI” would refer to regular interconnection or regularly implementable. The same is followed for regular feedback interconnection/implementable, both abbreviated as RFI. We consider the regular feedback interconnection for smooth behaviors. In [26] an algorithm is provided which checks if a given regularly implementable sub-behavior is also regular feedback implementable and provides a controller that implements it, when one exists. The following problem is about obtaining conditions to check if a sub-behavior is regular feedback implementable from a given smooth behavior. The difference in our solution and the one obtained in [26] is elaborated after Theorem 3.3.

Problem 2.3: Assume $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C) \subset \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$. Find necessary and sufficient conditions such that $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$ is regular feedback implementable with respect to $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$.

Next we are interested in the initial condition space for the behavior of a system when the function space is $\mathcal{L}_{\mathbb{R}^+}^I$. A so-called state map acts on the variables of the system and provides the state variables. One can obtain the canonical state map by a shift and cut procedure which is defined and worked out in Section VIII-B of the Appendix. Using the canonical state map we define a subspace, denoted by S , as follows.

Definition 2.4: For a matrix $R(\xi) \in \mathbb{R}^{n \times m}[\xi]$, let $X(\xi) \in \mathbb{R}^{n \times m}[\xi]$ be such that $X(d/dt)$ is the state map obtained by the shift and cut operation on $R(\xi)$. Consider the map $X_\tau : \mathcal{L}_{\mathbb{R}}^C \rightarrow \mathbb{R}^{n \times m}$, for $\tau \in \mathbb{R}$, defined as $X_\tau(f) := (X(d/dt)f)|_{t=\tau^-}$. The set $S \subseteq \mathbb{R}^{n \times m}$ is defined as $S := X_0(\mathcal{L}_{\mathbb{R}}^C)$.

It is well-known (see [23], for example) that if R is row-reduced, then for every $a \in S$, there exists a trajectory $w \in \mathfrak{B}(R, \mathcal{L}_{\mathbb{R}}^C)$ such that $X_0(w) = a$. In this sense, S also serves the purpose of “space of initial conditions.” [23] also considers the problem of $X_0(w)$ giving rise to too many initial conditions, which is the case when R is not row-reduced. An equivalence relation on the row space of $X(\xi)$ is introduced there to get rid of the excess states thereby resulting in a state map that gives exactly the initial conditions for the smooth behavior: see Section VI for our treatment to this problem. We shall term the smooth behavior of a system as the slow behavior. We will show that these additional initial conditions, i.e., the excess states, lead to jumps and impulses. For instance in Example 2.1, the state map is $X(\xi) = \begin{bmatrix} 0 & \xi \\ 0 & 1 \end{bmatrix}$ from which we get $w_2(0)$ and $\dot{w}_2(0)$ as the initial conditions that leads to jump in w_2 and hence impulses in w_1 . In this context we define a consistent initial condition as in [4].

Definition 2.5: Consider $R \in \mathbb{R}^{n \times m}[\xi]$ with $\text{rank}_{\mathbb{R}[\xi]}(R) = n$ and the associated canonical state map $X(d/dt)$ with $X(\xi) \in \mathbb{R}^{n \times n}[\xi]$. A vector $v \in \mathbb{R}^{n \times 1}$ is said to be a *consistent initial condition* for $\mathfrak{B}(R, \mathcal{L}_{\mathbb{R}^+}^I)$ if there exists a trajectory $w \in \mathfrak{B}(R, \mathcal{L}_{\mathbb{R}^+}^C)$ such that $X_0(w) = v$.

An initial condition is *inconsistent* if there does not exist a $w \in \mathfrak{B}(R, \mathcal{L}_{\mathbb{R}^+}^C)$ from that initial condition. Clearly, impulses occur in a system when there are inconsistent initial conditions in the behavior. Inconsistent initial conditions in a behavior are due to the presence of “zeros at infinity” (refer Section VIII-A of the Appendix) in the polynomial matrix which induces a kernel representation for the behavior [11]. It is known that an interconnection being regular feedback is a sufficient condition for zeros at infinity not being introduced as a result of the interconnection of two systems. However, this is not necessary as is shown in the example in [26, Section 3].

In the following problem we look for conditions such that a regular interconnection does not introduce new zeros at infinity. For a polynomial matrix $P(\xi)$, let $z_\infty(P)$ denote the number of zeros at infinity of $P(\xi)$.

Problem 2.6: Suppose the plant, $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$ and controller, $\mathfrak{B}(C, \mathcal{L}_{\mathbb{R}^+}^I)$ are interconnected to get $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$. Find conditions such that $z_\infty(P) + z_\infty(C) = z_\infty(K)$.

In the course of solving the above problems, we generalize the notion of impulse controllability for higher order dynamical systems. We give a definition for impulse controllability of a system in terms of the associated canonical state map (i.e., the shift and cut map: see Section VIII-B of the Appendix). We relate impulse controllability to the existence of a controller that rules out impulses in the closed loop autonomous system. This generalization requires the extension of the notion of regularity of interconnection to behaviors with solutions defined over the half-line (Problem 2.2).

III. INTERCONNECTION OF BEHAVIORS

In this section we deal with regular implementability of an impulsive-smooth sub-behavior with respect to a given $\mathcal{L}_{\mathbb{R}^+}^I$ -behavior.

A. Regular Interconnection

The following proposition is about a necessary and sufficient condition for the equality of two impulsive-smooth behaviors termed as the so-called *fundamental equivalence* in [15]. For a polynomial matrix $P(\xi)$, let $\delta_M(P)$ denote the number of poles at infinity of $P(\xi)$.

Proposition 3.1: [15, Theorem 19] Let $R_1, R_2 \in \mathbb{R}^{n \times m}[\xi]$ be such that $\text{rank}_{\mathbb{R}[\xi]}(R_1) = \text{rank}_{\mathbb{R}[\xi]}(R_2) = n$. Two behaviors $\mathfrak{B}(R_1, \mathcal{L}_{\mathbb{R}^+}^I)$ and $\mathfrak{B}(R_2, \mathcal{L}_{\mathbb{R}^+}^I)$ are equal if and only if there exists a unimodular matrix $U(\xi)$ satisfying

- 1) $U(\xi)R_1(\xi) = R_2(\xi)$,
- 2) $Y(\xi) := [U(\xi) R_2(\xi)]; \delta_M(Y) = \delta_M(R_2)$,
- 3) $Y(\xi)$ has no zeros at infinity.

The following theorem directly addresses Problem 2.2. This is the first main result of this paper: a necessary and sufficient condition for regular implementability of an impulsive-smooth desired sub-behavior, and a characterization of all regular controllers yielding such a desired sub-behavior. This result generalizes [19, Theorem 9] (included later as Proposition 8.8) to the case of impulsive-smooth behaviors. (The notions of left-primeness, zeros at infinity of a polynomial matrix and regular implementability are defined in the Appendix.)

Theorem 3.2: Let $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$ denote the plant behavior and consider a sub-behavior $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$. Then $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$ is regularly implementable with respect to $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$ if and only if there exist polynomial matrices F and G satisfying the following properties:

- (a) $P = FK$ with F left-prime.
- (b) $U := \text{col}(F, G)$ is unimodular.
- (c) $\delta_M([U UK]) = \delta_M(UK)$.
- (d) $[U UK]$ has no zeros at infinity.

Further, any G as above results in a regular controller $\mathfrak{B}(GK, \mathcal{L}_{\mathbb{R}^+}^I)$ that implements $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$.

Proof: Only If: Assume $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$ is RI from $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$. Then there exists $\mathfrak{B}(C, \mathcal{L}_{\mathbb{R}^+}^I)$ such that $\mathfrak{B}(\text{col}(P, C), \mathcal{L}_{\mathbb{R}^+}^I) = \mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$. From Proposition 3.1 it follows that there exists a unimodular matrix U such that $\text{col}(P, C) = UK$ and U satisfies the conditions in Proposition 3.1. Let F be the first few rows (depending on the dimension of P) of U and G the remaining rows of U . This pair, F and G , satisfy all the stated conditions.

If: We assume that there exists an F and G satisfying the stated conditions. From Proposition 3.1, we get $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I) = \mathfrak{B}(\text{col}(F, G)K, \mathcal{L}_{\mathbb{R}^+}^I)$. We have $P = FK$ and let $C = GK$. Then a regular controller that implements $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$ is given by $\mathfrak{B}(C, \mathcal{L}_{\mathbb{R}^+}^I)$. \square

Note that \mathcal{K} being regularly implementable w.r.t. $\mathfrak{B}(P, \mathcal{L})$ depends crucially on the space of which \mathcal{P} and \mathcal{K} are subsets, though the definition of regular interconnection depended only on the polynomial matrices (and their row ranks). For example, suppose $P \in \mathbb{R}^{n \times m}[\xi]$, with $\text{rank}_{\mathbb{R}[\xi]}(P) = n$, can be completed to a unimodular matrix K by adding more rows. Then $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^C) = \{0\} \subset \mathcal{L}_{\mathbb{R}^+}^C$ is always regularly implementable with respect to $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^C)$. However, $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I) = \{0\} \subset \mathcal{L}_{\mathbb{R}^+}^I$ is not regularly implementable with respect to $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$, unless K and P are constant matrices: Corollary 6.6 makes this concrete.

B. Regular Feedback Interconnection

In this section we investigate when a sub-behavior is regular feedback implementable. Here the behavior is assumed to be a subspace of $\mathcal{L}_{\mathbb{R}}^C$. The following result addresses Problem 2.3.

Theorem 3.3: Let $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$ denote the plant behavior and consider a sub-behavior, $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$ of the plant behavior. Suppose both P and K are row-reduced. Then $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$ is regular feedback implementable with respect to $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$ if and only if there exist polynomial matrices F and G such that

- $P = FK$,
- $\text{col}(F, G)$ is unimodular,
- $\text{col}(F, G)K$ is row-reduced.

Further, any G as above results in an RFI controller $\mathfrak{B}(GK, \mathcal{L}_{\mathbb{R}}^C)$ that implements $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$.

Proof: **Only If:** We assume that $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$ is RFI with respect to $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$. Therefore $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$ is RI with respect to $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$ and hence from Proposition 8.8, there exists a controller, $\mathfrak{B}(C, \mathcal{L}_{\mathbb{R}}^C)$ such that $\mathfrak{B}(\text{col}(P, C), \mathcal{L}_{\mathbb{R}}^C) = \mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$, with $C \in \mathbb{R}^{(m-n) \times n}[\xi]$. A polynomial matrix can be made row-reduced by premultiplication by a suitable unimodular matrix ([9, Section 6.3.3]). From Proposition 8.7 in the Appendix below, when $\mathcal{L} = \mathcal{L}_{\mathbb{R}}^C$, two behaviors are equal if the corresponding minimal kernel representation matrices are related by a unimodular matrix. Therefore we may assume without loss of generality that C is row-reduced. Since P and C are row-reduced, $\sum_{i=1}^n d_i(P) = n_{\text{slow}}(P)$ and $\sum_{i=1}^{(m-n)} d_i(C) = n_{\text{slow}}(C)$. $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$ is assumed to be RFI with respect to $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$ and hence from Definition 8.6, $n_{\text{slow}}(\text{col}(P, C)) = n_{\text{slow}}(P) + n_{\text{slow}}(C) = \sum_{i=1}^n d_i(P) + \sum_{i=1}^{(m-n)} d_i(C) = \sum_{i=1}^m d_i(\text{col}(P, C))$. Therefore $\text{col}(P, C)$ is row-reduced. From Proposition 8.7, $UK = \text{col}(P, C)$, for some unimodular matrix U . Let $U = \text{col}(F, G)$. Then $P = FK$. Also $\text{col}(F, G)K = \text{col}(P, C)$ and since $\text{col}(P, C)$ is row-reduced, $\text{col}(F, G)K$ is also row-reduced.

If: Assume there exist matrices F and G satisfying the three properties mentioned in the theorem statement. Since $\text{col}(F, G)$ is unimodular, from Proposition 8.7, $\mathfrak{B}(\text{col}(F, G)K, \mathcal{L}_{\mathbb{R}}^C) = \mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$. Also since $\text{col}(F, G)K$ is row-reduced, $\text{col}(F, G)K$ does not have any zeros at infinity. Hence $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$ is RFI from $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$. An RFI controller that implements $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$ is $\mathfrak{B}(GK, \mathcal{L}_{\mathbb{R}}^C)$. \square

The following points may be noted about Theorems 3.2 and 3.3. When dealing with *slow* behaviors, it has been shown in [19, Theorem 9] (reproduced below as Proposition 8.8) that left-primeness of the matrix F is necessary and sufficient for regular implementability of the desired sub-behavior. Since regular *feedback* implementability imposes conditions on fast “modes” too, it is expected that the F needs to satisfy suitable conditions related to the pole/zero structure at infinity. Theorem 3.3 states that just row-reducedness retention is enough as far as RFI of *slow*-behaviors is concerned. On the other hand, equality of fast-behaviors for regular interconnection requires, loosely speaking, “relative left-primeness at infinity” conditions (last two conditions in Proposition 3.1 and Theorem 3.2) on the unimodular matrix that relates the desired behavior and the interconnection of the plant and the controller

behaviors. Formalizing this loose interpretation and extending this to impulsive-smooth *RFI* implementability requires further investigation.

IV. INITIAL CONDITIONS FROM STATE MAPS

In this section the focus is to show that the initial conditions for a system can be obtained from the state map.

Consider a system with behavior $\mathfrak{B}(R, \mathcal{L}_{\mathbb{R}^+}^I)$, where $R(\xi) = R_N \xi^N + \dots + R_0$ and $R(\xi) \in \mathbb{R}^{n \times m}[\xi]$. We use the distributional framework as in [4], [15] and write $R(d/dt)w = 0$ as

$$R(p)w = \mathfrak{S}_{N-1}(p)Z_R v \quad (2)$$

where p represents² $\delta^{(1)}$, $\mathfrak{S}_{N-1}(p) := [Ip^{N-1}, Ip^{N-2}, \dots, I]$

$$v := \begin{bmatrix} w(0^-) \\ \dot{w}(0^-) \\ \vdots \\ w^{(N-1)}(0^-) \end{bmatrix} \quad \text{and} \quad Z_R := \begin{bmatrix} R_N & 0 & \dots & 0 \\ R_{N-1} & R_N & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_1 & R_2 & \dots & R_N \end{bmatrix}. \quad (3)$$

The initial condition for the system is given by $Z_R v$ and the initial condition space is the image of Z_R denoted as $\text{Im } Z_R$. We assume $\text{rank}_{\mathbb{R}[\xi]}(R(\xi)) = n$ and hence the system in (2) is solvable for all initial conditions: see [15, Theorem 5 and Corollary 6].

Consider the subspace $S \subseteq \mathbb{R}^{n_X}$, where n_X is the number of rows in the state map X , defined in Definition 2.4 as $S := X_0(\mathcal{L}_{\mathbb{R}}^C)$. The following result illustrates how the initial conditions of the system are obtained from the state map.

Lemma 4.1: Consider the system $R(d/dt)w = 0$ where $R(\xi) = R_N \xi^N + \dots + R_0$. Let X be the state map obtained from R by the shift and cut method. Define $S := X_0(\mathcal{L}_{\mathbb{R}}^C)$. Then $S = \text{Im}(Z_R)$ with Z_R defined in (3). Further, $\delta_M(R) = \text{rank}_{\mathbb{R}}(X)$.

Proof: The state map obtained by the shift and cut procedure (elaborated in the Appendix) is given by

$$X(\xi) = \begin{bmatrix} R_N \\ R_N \xi + R_{N-1} \\ \vdots \\ R_N \xi^{N-2} + R_{N-1} \xi^{N-3} \dots R_2 \\ R_N \xi^{N-1} + R_{N-1} \xi^{N-2} \dots R_1 \end{bmatrix} = \begin{bmatrix} R_N & 0 & \dots & 0 \\ R_{N-1} & R_N & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_2 & R_3 & \dots & 0 \\ R_1 & R_2 & \dots & R_N \end{bmatrix} \begin{bmatrix} I \\ \vdots \\ I \xi^{N-2} \\ I \xi^{N-1} \end{bmatrix}$$

²More precisely, similar to [4, equation (1.3)] and [15, Equation (2)], in (2) above, p^i is $\delta^{(i)}$, the LHS $R(p)w$, is interpreted as *convolution* of $R(p)$ and w , while in the RHS of (2), for convenience of matrix equations, the “coefficients” of $\delta^{(i)}$ are on the right of $\delta^{(i)}$. It is important to note that, when dealing with initial conditions and half-line solutions, $R(d/dt)w = 0$ is not a “homogeneous” system of differential equations: this is central to this paper.

which equals $Z_R \mathfrak{G}_{N-1}^T(\xi)$. Hence

$$X_0(\mathcal{L}_{\mathbb{R}}^C) = \left\{ \left(Z_R \mathfrak{G}_{N-1}^T \left(\frac{d}{dt} \right) \right) w \Big|_{0^-} \text{ with } w \in \mathcal{L}_{\mathbb{R}}^C \right\}$$

which implies $X_0(\mathcal{L}_{\mathbb{R}}^C) = S = \text{Im } Z_R$. Also the $\text{rank}_{\mathbb{R}}(X)$ is the same as the rank of Z_R . It is known (see [25, Page 139]) that $\text{rank}_{\mathbb{R}}(Z_R) = \delta_M(R)$. Hence $\text{rank}_{\mathbb{R}}(X) = \delta_M(R)$. This completes the proof. \square

From the above lemma we have that the space S is, in fact, the space of all initial conditions for the behavior, $\mathfrak{B}(R, \mathcal{L}_{\mathbb{R}^+}^I)$. Using the above lemma we have the following result.

Lemma 4.2: For a plant $\mathcal{P} := \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$ with state map X , $\text{rank}_{\mathbb{R}}(X) = n_{\text{slow}}(P) + z_{\infty}(P)$.

Proof: Denote the number of finite zeros of $P(\xi)$ by n and the sum of the minimal indices³ of the right nullspace of $P(\xi)$ by ϵ . From [15, equation (19)] (see also [27]), we have $n + z_{\infty} + \epsilon = \delta_M(P)$. Factorize $P = P_1 P_2$, where P_1 is square and nonsingular and P_2 is left-prime. Since P_2 is left-prime it has no finite zeros. Hence the number of finite zeros in P_1 is n . Since P_1 is square, $n_{\text{slow}}(P_1) = n$. Choose a Q of maximum rank which is right-prime and such that $P_2 Q = 0$. Since $P_2 Q = 0$, $n_{\text{slow}}(P_2) = n_{\text{slow}}(Q) = \epsilon$. This follows from [9, Lemmas 6.3-8 and 6.5-6] though the context and notation are different.⁴ Then $n_{\text{slow}}(P) = n_{\text{slow}}(P_1) + n_{\text{slow}}(P_2)$ ([25, Proposition 1.5]) which implies $n + \epsilon = n_{\text{slow}}(P)$. From Lemma 4.1, $\delta_M(P) = \text{rank}_{\mathbb{R}}(X)$. Combining all the equalities gives the result: $\text{rank}_{\mathbb{R}}(X) = n_{\text{slow}}(P) + z_{\infty}(P)$. \square

Of course, the above result is a close reformulation of the well-known ‘‘index-sum’’ theorem: see [17], for example. The following result is for an autonomous⁵ system.

Lemma 4.3: Let $\mathcal{P} := \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$ denote an autonomous behavior and let $X(d/dt)$ denote the state map obtained from P . Then for any $v \in X_0(\mathcal{L}_{\mathbb{R}}^C)$ there exists a unique impulsive-smooth solution $w \in \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$ such that (2) is satisfied. Further there exist

- $n_{\text{slow}}(P)$ linearly independent smooth solutions, and
- $z_{\infty}(P)$ linearly independent impulsive solutions.

Proof: Since \mathcal{P} is autonomous, $\dim(\mathcal{P}) = \delta_M$, which is equal to the dimension of the space of initial conditions, $\text{rank}_{\mathbb{R}}(X)$ (Lemma 4.1). Therefore, there exists a unique trajectory $w \in \mathcal{P}$ for each initial condition, $v \in X_0(\mathcal{L}_{\mathbb{R}}^C)$. We also have $\dim(\mathcal{P}) = \delta_M(P) = n_{\text{slow}}(P) + z_{\infty}(P)$. Hence there are $n_{\text{slow}}(P)$ linearly independent smooth solutions and $z_{\infty}(P)$ linearly independent impulsive solutions. \square

³For a matrix $R(\xi) \in \mathbb{R}^{n \times m}[\xi]$, $n \leq m$, let \mathcal{N} denote the right nullspace of $R(\xi)$. Let $B = \{t_1, \dots, t_k\}$, $t_i \in \mathbb{R}^m[\xi]$ be a basis for \mathcal{N} such that the degrees d_i of respective t_i satisfy $d_1 \leq \dots \leq d_k$. Then B is said to be a minimal polynomial basis for \mathcal{N} if every other basis of \mathcal{N} , that is ordered in the same way, with degrees r_1, \dots, r_k satisfies $d_i \leq r_i$ for $i = 1, \dots, k$. The numbers d_1, \dots, d_k are the right minimal indices of $R(\xi)$.

⁴Note that left/right irreducible matrix fraction descriptions are exactly controllable-minimal-kernel/observable-image representations, respectively.

⁵A behavior \mathfrak{B} is called *autonomous* if the following implication holds: If w_1 and $w_2 \in \mathfrak{B}$ and $T > 0$ satisfy $w_1(t) = w_2(t)$ for all $t \leq T$ then $w_1 = w_2$. The equalities of trajectories above are also to be understood in a distributional sense. A behavior is autonomous if and only if the polynomial matrix R defining the behavior above has full column rank over $\mathbb{R}[\xi]$.

V. IMPULSE CONTROLLABILITY

Impulse controllability of descriptor systems has been well studied in the literature. One of the classic interpretations of impulse controllability for a descriptor state space system is ‘‘the ability to generate a maximal set of impulses by the input,’’ see [2, Page 1078]. We extend this concept to higher order systems. For this, we first define impulse controllability without an input/output partition. Definition 2.5 about consistency of an initial condition plays a key role.

Definition 5.1: Consider $R(\xi) \in \mathbb{R}^{n \times m}[\xi]$ and define $S \subseteq \mathbb{R}^{n \times n}$ by $S := X_0(\mathcal{L}_{\mathbb{R}}^C)$. A system $\mathfrak{B}(R, \mathcal{L}_{\mathbb{R}^+}^I)$ is said to be impulse controllable if every initial condition $v \in S$ is consistent.

Note that the initial condition vector v is obtained by the state map operating on an arbitrary element of $\mathcal{L}_{\mathbb{R}}^C$ and not just those satisfying $R(d/dt)w = 0$. In general for a system, for every $v \in X_0(\mathcal{L}_{\mathbb{R}}^C)$, there exists an element $w \in \mathfrak{B}(R, \mathcal{L}_{\mathbb{R}^+}^I)$ such that (2) is satisfied whereas impulse controllability ensures that for every v there exists an element $w \in \mathfrak{B}(R, \mathcal{L}_{\mathbb{R}^+}^C)$ such that (2) is satisfied. We develop other equivalent statements for impulse controllability. A major motivation for impulse controllability for regular⁶ descriptor systems is the ability to remove impulses in the system by means of feedback. In this paper we relate the impulsive modes of the system to zeros at infinity of the underlying polynomial matrix representing the system. This brings us to the following result which is a slight modification of [2, Theorem 4].

Lemma 5.2: For a system, $E\dot{x} = Ax + Bu$, with $E, A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$, assume $\det(E\xi - A) \neq 0$. The following are equivalent.

- 1) The system is impulse controllable.
- 2) The matrix $[E\xi - A \ B]$ has no zeros at infinity.
- 3) There exists a linear feedback law $u = Kx$, with $K \in \mathbb{R}^{m \times n}$ such that the closed loop system has only consistent initial conditions.
- 4) There exists $K \in \mathbb{R}^{m \times n}$ such that $\deg \det(E\xi - (A + BK)) = \text{rank } E$.

Proof:

1 \Leftrightarrow 4: Follows directly from [2, Theorem 4].

2 \Leftrightarrow 3: Consider the 3rd statement. Using the feedback law $u = Kx$, the closed loop system is $E\dot{x} = (A + BK)x$. Since the closed loop system has only consistent initial conditions all solutions are nonimpulsive. This is equivalent to $z_{\infty}(E\xi - (A + BK)) = 0$ (Lemma 4.3). Consider the matrix $\begin{bmatrix} E\xi - A & B \\ K & I \end{bmatrix}$.

Then premultiplying by a constant matrix results in

$$\begin{bmatrix} I_n & -B \\ 0 & I_m \end{bmatrix} \begin{bmatrix} E\xi - A & B \\ K & I \end{bmatrix} = \begin{bmatrix} E\xi - (A + BK) & 0 \\ K & I \end{bmatrix}. \quad (4)$$

Since the two polynomial matrices in (4) are related by a nonsingular constant matrix, the zeros at infinity of both are same. This in turn implies that $z_{\infty}(E\xi - (A + BK)) = z_{\infty}([E\xi - A \ B])$ thus proving the equivalence between Statements 2 and 3.

3 \Leftrightarrow 4: Using the arguments in the proof of 2 \Leftrightarrow 3, we have that Statement 3 is equivalent to $z_{\infty}(E\xi - (A +$

⁶We call the descriptor system $E\dot{x} = Ax + Bu$ (with E and A both square and possibly singular) a *regular* descriptor system if $\det(sE - A) \neq 0$. If the determinant is zero, we call the descriptor system nonregular.

$BK)) = 0$. Moreover $z_\infty(E\xi - (A + BK)) = 0$ if and only if $\deg \det(E\xi - (A + BK)) = \text{rank } E$ (see [25, Page 139]). Using this we get the equivalence between Statements 3 and 4. \square

The equivalence in the above lemma is applicable to higher order systems also which will be explained in the next theorem. The following remark addresses the differences between our definition of impulse controllability and the existing (and non-equivalent) definitions in literature especially for nonregular (see Footnote 6) first order systems.

Remark 5.3: For the case of a regular descriptor system (see Footnote 6), [2] links elimination of impulses with impulse controllability. In [8], it has been shown that impulse controllability and ability to eliminate impulses by feedback are not equivalent for nonregular descriptor systems. The concept of impulsive mode controllability was introduced in [6] and this was shown to be equivalent to the ability to eliminate impulses by feedback. Implicit in these descriptor system studies is the input/output partition between variables: the input is always u and the output, the state x . There are also differences in the formulation whether the state needs to be nonimpulsive for piece-wise continuous inputs or in the resulting closed loop autonomous system. In our case, first, we do not have any input/output partition and we seek absence of impulsive behavior only in the autonomous closed loop system. Secondly, we seek “elimination of impulses” under a suitable generalization of the notion of interconnection from the traditional first order descriptor case to the higher order case. It turns out that “absence/elimination of impulsive behavior” needs to be defined differently⁷ for autonomous and nonautonomous systems: thus bringing in the notion of impulse controllability.

Remark 5.4: Traditionally (see [2], and the references within there, for example), impulse controllability has been linked (to different extents) to controllability at ∞ . In the behavioral approach, one speaks of controllability, but without an input/output partition: just like we developed impulse controllability without such a partition. This remark links controllability at a finite complex number λ to impulse controllability as proposed and developed above. For a full row rank polynomial matrix $P(\xi) \in \mathbb{R}^{n \times m}[\xi]$ and the system $\mathcal{P} := \mathfrak{B}(P, \mathcal{L})$, define $\lambda \in \mathbb{C}$ to be a controllable mode if $\text{rank}_{\mathbb{R}}(P(\lambda)) = \text{rank}_{\mathbb{R}[\xi]}(P(\xi))$. It is possible to prove that controllability of \mathcal{P} at a given *finite* $\lambda \in \mathbb{C}$ is equivalent⁸ to existence of a regularly implementable $\mathcal{K} \subset \mathcal{P}$ such that \mathcal{K} has no trajectories of the

⁷For an autonomous system, absence of impulsive behavior is, by definition, just absence of initial conditions that cause impulses. However, for nonautonomous systems, impulsive behavior needs to be distinguished depending on the cause: impulsive inputs or intrinsic/natural response. This paper pursues the latter, since the impulsive inputs expectedly cause impulsive behaviors.

⁸The definition of λ -controllability and the stated equivalence allows one to interpret controllability at a finite complex number λ_0 as ability to steer from exponential system trajectories of the form $ve^{\lambda_0 t}$ to the zero trajectory. Though the equivalence result is significant in its own right, we do not delve into the precise proof since our paper focuses on *fast* behavior and not on controllability at a finite λ : a proof outline is as follows. Suppose \mathcal{P} is uncontrollable at $\lambda \in \mathbb{C}$. Then every completion of P to a square nonsingular polynomial matrix by appending further rows (i.e., regularly interconnecting with a controller that restricts \mathcal{P} to an autonomous sub-behavior \mathcal{K}) results in *loss* of rank of the completion-nonsingular matrix also at λ , thus causing \mathcal{K} to have trajectories of the form $ve^{\lambda t}$ for some $v \in \mathbb{C}^m \setminus 0$. The converse follows in the same fashion using the full row rank assumption on $\text{rank}_{\mathbb{R}}(P(\lambda))$ and constructing a completion that ensures nonsingularity of the completion.

form $ve^{\lambda t}$ for any $v \in \mathbb{C}^m \setminus 0$. Regular interconnection plays a crucial role here. Our paper extends the notion of regular implementability to $\mathcal{L}_{\mathbb{R}^+}^I$ (Theorem 3.2) so that Lemma 5.2 (which deals with controllability “at ∞ ” for first order systems) extends to higher order systems as stated in Theorem 5.5 below.

Theorem 5.5: Consider a behavior $\mathcal{P} := \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$, where $P \in \mathbb{R}^{n \times m}[\xi]$ and $\text{rank}_{\mathbb{R}[\xi]}(P) = n$. Let X be the state map obtained by the shift and cut method. Then the following are equivalent.

- 1) \mathcal{P} is impulse controllable.
- 2) $\text{rank}_{\mathbb{R}}(X) = n_{\text{slow}}(P)$.
- 3) $P(\xi)$ has no zeros at infinity.
- 4) There exists an autonomous sub-behavior $\mathcal{K} \subset \mathcal{P}$ with no inconsistent initial conditions, which is regularly implementable with respect to \mathcal{P} .

Proof: We prove $1 \Leftrightarrow 2$, $2 \Leftrightarrow 3$, $3 \Rightarrow 4$ and then $4 \Rightarrow 3$.

$1 \Leftrightarrow 2$: Using the definition of impulse controllability, for every initial condition v obtained from the state map ($v \in X_0(\mathcal{L}_{\mathbb{R}}^C)$) there exists a nonimpulsive solution. From Lemma 4.1 we have that the dimension of initial condition space is given by $\text{rank}_{\mathbb{R}}(X)$. The dimension of initial condition space that results in slow solutions is n_{slow} . Hence impulse controllability is equivalent to the condition $\text{rank}_{\mathbb{R}}(X) = n_{\text{slow}}(P)$.

$2 \Leftrightarrow 3$: Using Lemma 4.2, $\text{rank}_{\mathbb{R}}(X) = n_{\text{slow}}(P) + z_\infty(P)$. Hence $\text{rank}_{\mathbb{R}}(X) = n_{\text{slow}}(P)$ is equivalent to P having no zeros at infinity.

$3 \Rightarrow 4$: We assume $P(\xi)$ has no zeros at infinity. Then P can be completed⁹ to a square nonsingular polynomial matrix, K which has no zeros at infinity. Using the notations from Theorem 3.2, $F = [I \ 0]$ and $G = [0 \ I]$ with appropriate dimensions. Therefore $U = I$ and conditions 2(c) and 2(d) are also met as K does not have zeros at infinity. Hence from Theorem 3.2 the behavior $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$ is regularly implementable and the behavior does not have inconsistent initial conditions, i.e., $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I) = \mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^C)$.

$4 \Rightarrow 3$: As \mathcal{K} is regularly implementable there exists a C such that $\text{col}(P, C) = K$, where K is the matrix which induces a kernel representation for \mathcal{K} . Since \mathcal{K} is autonomous with no inconsistent initial conditions, $\mathcal{K} = \mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^C)$. Therefore K has no zeros at infinity which implies P also has no zeros at infinity. \square

Example 5.6: Consider a system described by the following differential equations:

$$\begin{aligned} w_1 + \ddot{w}_2 + w_3 &= 0 \\ w_2 &= 0. \end{aligned}$$

The system is not impulse controllable because for any nonzero initial condition of w_2 and \dot{w}_2 the system has only impulsive solutions. We now arrive at this conclusion based on the associated polynomial matrix $R(\xi)$ which is $\begin{bmatrix} 1 & \xi^2 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

⁹A procedure to obtain a K is as follows. For $P \in \mathbb{R}^{n \times m}[\xi]$ satisfying $\text{rank}_{\mathbb{R}[\xi]}(P) = n$, assume without loss of generality that $P = [P_1 \ P_2]$, with $P_1 \in \mathbb{R}^{n \times n}[\xi]$ nonsingular and such that $P_1(\xi)$ is a minor with maximum degree in P . Then use the matrix $[0 \ I_{(m-n)}]$ to complete P to K . This matrix K is square, nonsingular and does not have zeros at infinity.

Since this matrix has zeros at infinity, we conclude from Theorem 5.5 that the system is not impulse controllable. From the same theorem we also conclude that the impulses in the system cannot be eliminated by feedback.

VI. IMPULSES FROM INTERCONNECTION

Observing that the polynomial matrix associated to a system described by only one equation can have no zeros at infinity, and has $\delta_M = n_{\text{slow}}$, one may interpret that the zeros at infinity arise only from interconnecting systems. Hence in this section we consider RI interconnection of two systems and study the consequences on the zeros at infinity. Since the zeros at infinity are closely related to the rank of the state map, and the dimension of the slow space: $z_\infty(R) = \text{rank}_{\mathbb{R}}(X) - n_{\text{slow}}(R)$, we will study at the same time the effects of the interconnection on the state map and the slow space.

The procedure to construct a state map for the slow behavior as given in Appendix, Section VIII-B will produce a state map for the slow behavior if R is row-reduced. If R is not row-reduced, then a state map for the *slow* behavior can be obtained by the following “modulo operation” as in [23]. This leads to a state space that is not a subspace of the original full state space. For the canonical state map $X(\xi)$ obtained from a polynomial matrix $R(\xi)$, \mathcal{X}_R denotes the vector space over \mathbb{R} generated by the rows of X . We now consider the quotient space, $\mathcal{X}(\text{mod } R)$ denoted as \mathcal{X}^s . Elements $a, b \in \mathcal{X}$ can be identified to elements in \mathcal{X}^s as follows: a and b are said to be equivalent in \mathcal{X} if $a = b + pR$ for some $p \in \mathbb{R}^{1 \times n}[\xi]$. Arrange the elements of a basis of \mathcal{X}_R^s as rows of a matrix $X^s(\xi)$. A minimal state map for the slow behavior is given by $X^s(d/dt)$ and we shall term these states as the *slow* states. In this case for every $a \in X_0^s(\mathcal{L}_{\mathbb{R}}^C)$ there exists a w in the slow behavior, $\mathfrak{B}(R, \mathcal{L}_{\mathbb{R}}^C)$ such that $X_0(w) = a$.

Consider the following situation: we have a plant behavior described by $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$, a regular interconnection with a controller behavior given by $\mathfrak{B}(C, \mathcal{L}_{\mathbb{R}^+}^I)$. The behavior of the interconnected system is described by $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$, where $K = \text{col}(P, C)$. Obviously $\mathcal{X}_K = \mathcal{X}_P + \mathcal{X}_C$, and since $\delta_M(\cdot) = \dim(\mathcal{X}(\cdot))$ we have $\delta_M(K) \leq \delta_M(P) + \delta_M(C)$. It is clear that $\mathcal{X}^s(K) \subseteq \mathcal{X}^s(P) + \mathcal{X}^s(C)$, so $n_{\text{slow}}(K) \leq n_{\text{slow}}(P) + n_{\text{slow}}(C)$ and that $z_\infty(K) \geq z_\infty(P) + z_\infty(C)$.

After interconnection, the number of zeros at infinity can increase, if the dimension of the slow states decreases. If the number of zeros at infinity remains the same, it is still possible that the dimension of the slow states decrease. This happens when the dimension of the total state space decreases. The following examples illustrates the above two cases.

Example 6.1: Consider the following two cases with $K = \text{col}(P; C)$:

1) Let

$$P = [s \quad 0 \quad -1] \text{ and } C = \begin{bmatrix} 1 & -1 & 0 \\ 0 & s & 0 \end{bmatrix}.$$

Then $\mathfrak{B}(P, \mathcal{L}) = \{(w_1, w_2, w_3) | w_1(t) = a + \int_0^t w_3(\tau) d\tau\}$ and $\mathfrak{B}(C, \mathcal{L}) = \{(w_1, w_2, w_3) | w_1 = w_2 = b\}$ for a and $b \in \mathbb{R}$. We see that $\mathfrak{B}(K, \mathcal{L}) = \{(w_1, w_2, w_3) | w_1 = w_2 = b, w_3 = 0\}$, so $n_{\text{slow}}(K) = n_{\text{slow}}(P) = n_{\text{slow}}(C) = 1$.

Since $X_P = [1 \ 0 \ 0]$ and $X_C = [0 \ 1 \ 0]$, we see that $\delta_M(K) = \delta_M(P) + \delta_M(C)$, and hence $z_\infty(K) = 1 > z_\infty(P) + z_\infty(C) = 0$.

2) Let

$$P = [s \quad 0] \text{ and } C = [s \quad -1].$$

Then $\mathfrak{B}(P, \mathcal{L}) = \{(w_1, w_2) | w_1(t) = a\}$, $\mathfrak{B}(C, \mathcal{L}) = \{(w_1, w_2) | w_1(t) = b + \int_0^t w_2(\tau) d\tau\}$. It is easily seen that $\mathfrak{B}(K, \mathcal{L}) = \{(w_1, w_2) | w_1(t) = a, w_2(t) = 0\}$. Hence $n_{\text{slow}}(K) = n_{\text{slow}}(P) = n_{\text{slow}}(C) = 1$. In this case $X_P = X_C = X_K = [1 \ 0]$, and there are no zeros at infinity in the polynomial matrix K .

Note that in the first case we have $\mathcal{X}^s(K) \subsetneq \mathcal{X}^s(P) + \mathcal{X}^s(C)$, while in the second case $\mathcal{X}^s(K) = \mathcal{X}^s(P) + \mathcal{X}^s(C)$. We will prove that this equality in general is a sufficient condition for no additional zeros at infinity. We start with a preliminary lemma.

Lemma 6.2: Let $\mathcal{P} := \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$, $\mathcal{C} := \mathfrak{B}(C, \mathcal{L}_{\mathbb{R}^+}^I)$ and $\mathcal{K} := \mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$ denote the plant, controller and controlled behaviors respectively. Then

$$\dim(\mathcal{X}_P^s + \mathcal{X}_C^s) - n_{\text{slow}}(K) = z_\infty(K) - z_\infty(P) - z_\infty(C) + \dim(\mathcal{X}_P \cap \mathcal{X}_C) - \dim(\mathcal{X}_P^s \cap \mathcal{X}_C^s). \quad (5)$$

Proof: For a polynomial matrix $R(\xi) \in \mathbb{R}^{n \times m}[\xi]$ with $\text{rank}_{\mathbb{R}[\xi]}(R) = n$, $\dim(\mathcal{X}_R) = \delta_M(R)$, (by Lemma 4.1), and $\dim(\mathcal{X}_R^s) = n_{\text{slow}}(R)$. Since $\delta_M(R) = n_{\text{slow}}(R) + z_\infty(R)$, we have

$$\dim(\mathcal{X}_R) - \dim(\mathcal{X}_R^s) = z_\infty(R). \quad (6)$$

Using these equalities, we now prove the lemma as follows.

$$\dim(\mathcal{X}_P + \mathcal{X}_C) = \dim(\mathcal{X}_P) + \dim(\mathcal{X}_C) - \dim(\mathcal{X}_P \cap \mathcal{X}_C) \quad (7)$$

$$\dim(\mathcal{X}_P^s + \mathcal{X}_C^s) = \dim(\mathcal{X}_P^s) + \dim(\mathcal{X}_C^s) - \dim(\mathcal{X}_P^s \cap \mathcal{X}_C^s). \quad (8)$$

Subtracting (7) from (8) and using $\mathcal{X}_K = \mathcal{X}_P + \mathcal{X}_C$, we get

$$\begin{aligned} & \dim(\mathcal{X}_P^s + \mathcal{X}_C^s) - \dim(\mathcal{X}_K) \\ &= -(\dim(\mathcal{X}_P) - \dim(\mathcal{X}_P^s)) - (\dim(\mathcal{X}_C) - \dim(\mathcal{X}_C^s)) \\ & \quad + \dim(\mathcal{X}_P \cap \mathcal{X}_C) - \dim(\mathcal{X}_P^s \cap \mathcal{X}_C^s). \end{aligned}$$

Using (6) we get the required equality

$$\begin{aligned} \dim(\mathcal{X}_P^s + \mathcal{X}_C^s) - n_{\text{slow}}(K) &= z_\infty(K) - z_\infty(P) - z_\infty(C) \\ & \quad + \dim(\mathcal{X}_P \cap \mathcal{X}_C) - \dim(\mathcal{X}_P^s \cap \mathcal{X}_C^s). \end{aligned}$$

□

In (5), the LHS is non-negative. Also the expression $z_\infty(K) - (z_\infty(P) + z_\infty(C))$ on that equation's RHS is non-negative. Moreover, the term $\dim(\mathcal{X}_P \cap \mathcal{X}_C) - \dim(\mathcal{X}_P^s \cap \mathcal{X}_C^s)$ from RHS of (5) is also non-negative. This leads immediately to the following theorem.

Theorem 6.3: Let $\mathcal{P} := \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$ and $\mathcal{C} := \mathfrak{B}(C, \mathcal{L}_{\mathbb{R}^+}^I)$ denote the plant and the controller behaviors respectively. Define the slow state spaces $\mathcal{X}_P^s := \mathcal{X}_P(\text{mod } P)$ and $\mathcal{X}_C^s := \mathcal{X}_C(\text{mod } C)$. Assume the interconnected system behavior is $\mathcal{K} := \mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$ and $\mathcal{X}_K^s := \mathcal{X}_K(\text{mod } K)$. If $\mathcal{X}_K^s = \mathcal{X}_P^s + \mathcal{X}_C^s$, then $z_\infty(K) = z_\infty(P) + z_\infty(C)$.

Proof: If $\mathcal{X}_K^s = \mathcal{X}_P^s + \mathcal{X}_C^s$, then the left hand side of (5) is zero. Since both $z_\infty(K) - (z_\infty(P) + z_\infty(C))$ and $\dim(\mathcal{X}_P \cap \mathcal{X}_C) - \dim(\mathcal{X}_P^s \cap \mathcal{X}_C^s)$ are non-negative terms, we infer that each of these terms are zero. \square

It is well known that RFI is sufficient for no impulsive initial conditions in an interconnection, i.e., no additional zeros at infinity are introduced by the interconnection. RFI requires that $n_{\text{slow}}(K) = n_{\text{slow}}(P) + n_{\text{slow}}(C)$, or equivalently that $\mathcal{X}_P^s \cap \mathcal{X}_C^s = 0$. Theorem 6.3 provides a more general condition than RFI. A special case is when the polynomial matrices P and C are row-reduced; this is addressed in the following corollary. See also [22] for a link with so-called ‘‘canonical’’ state spaces.

Corollary 6.4: Assume plant, $\mathcal{P} := \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$ and controller, $\mathcal{C} := \mathfrak{B}(C, \mathcal{L}_{\mathbb{R}^+}^I)$ such that $P(\xi)$ and $C(\xi)$ are row-reduced. Then, all the initial conditions of the controlled system $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$ are consistent if and only if $\mathcal{X}_K^s = \mathcal{X}_P^s + \mathcal{X}_C^s$.¹⁰

Proof: Since P and C are row-reduced the slow spaces and the total spaces are the same. Hence in (5), $\dim(\mathcal{X}_P \cap \mathcal{X}_C) = \dim(\mathcal{X}_P^s \cap \mathcal{X}_C^s)$. This results in the condition

$$\dim(\mathcal{X}_P^s + \mathcal{X}_C^s) - n_{\text{slow}}(K) = z_\infty(K) - (z_\infty(P) + z_\infty(C)).$$

Replacing $n_{\text{slow}}(K)$ with $\dim(\mathcal{X}_K^s)$ we get

$$\dim(\mathcal{X}_P^s + \mathcal{X}_C^s) - \dim(\mathcal{X}_K^s) = z_\infty(K) - (z_\infty(P) + z_\infty(C)).$$

From the above equation we have that there are no inconsistent initial conditions in $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$ if and only if $\dim(\mathcal{X}_P^s + \mathcal{X}_C^s) = \dim(\mathcal{X}_K^s)$. Since $\mathcal{X}^s(K) \subseteq \mathcal{X}^s(P) + \mathcal{X}^s(C)$, the condition $\dim(\mathcal{X}_P^s + \mathcal{X}_C^s) = \dim(\mathcal{X}_K^s)$ is same as $\mathcal{X}^s(K) = \mathcal{X}^s(P) + \mathcal{X}^s(C)$. Hence we have the result. \square

The following example shows that the condition, $\mathcal{X}_K^s = \mathcal{X}_P^s + \mathcal{X}_C^s$ of Theorem 6.3 is not necessary for an interconnection to not result in additional impulses.

Example 6.5: Let

$$P = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \end{bmatrix} \quad \text{and} \quad C = [s \quad 0 \quad 1].$$

Clearly $\dim(\mathcal{X}_P) = \dim(\mathcal{X}_C) = \dim(\mathcal{X}_K) = 1$, $z_\infty(K) = z_\infty(P) = 1$, while $z_\infty(C) = 0$. We have $\dim(\mathcal{X}^s(K)) = 0$, while $\mathcal{X}^s(P) + \mathcal{X}^s(C) = \mathcal{X}^s(C) \neq 0$.

Corollary 6.6: Consider a plant, $\mathcal{P} := \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$, where P is a nonconstant polynomial matrix. Then $\{0\}$ is not a regularly implementable sub-behavior of $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}^+}^I)$.

Continuing the discussion at the end of Section III, the above corollary highlights the role of the structure at infinity when dealing with regular implementability of behaviors in $\mathcal{L}_{\mathbb{R}^+}^I$. More precisely, for a behavior $\mathcal{P} = \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$ to be controllable at all finite complex numbers it is necessary and sufficient

that $\{0\}$ is a regularly implementable sub-behavior. In contrast, any nonconstant polynomial matrix always has poles at infinity due to which there would have to be zeros at infinity in K for obtaining $\{0\}$ by regular interconnection, thus ruling out $\{0\}$ as a regularly implementable $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}^+}^I)$. This reinstates the central role that the signal space \mathcal{L} plays in all the results of this paper.

VII. CONCLUSION

We extended notions in behavioral theory like regular interconnection from smooth function spaces to impulsive-smooth spaces and provided necessary and sufficient conditions for a sub-behavior to be regularly implementable in the impulsive-smooth space (Theorem 3.2). While there is ample work on behaviors in the distributional setting, this has been restricted primarily to the *full-line* where the issue of initial conditions causing impulses is not relevant. Our work builds on results from literature (for example [2], [15]) to relate impulsive behavior to interconnections, the state-map and impulse controllability.

The set of initial conditions that admit an impulsive-smooth solution gets related in a natural manner to the state map. The various dimension counts of the slow and fast subspaces get linked in this paper through the ranks (over \mathbb{R} and $\mathbb{R}[\xi]$) of the matrix X obtained using the shift and cut procedure and the number of zeros at infinity of the underlying polynomial matrix (Lemmas 4.2 and 4.3). We generalized the notion of impulse controllability from first order systems to higher order systems. Further, we showed that impulse controllability of a behavior is equivalent to the absence of zeros at infinity of the polynomial matrix in any minimal kernel representation. Finally we showed that for a behavior there exists a regularly implementable sub-behavior with all initial conditions being consistent if and only if the given behavior is impulse controllable (Theorem 5.5). Our extension of study of regular interconnection to impulsive-smooth spaces of solutions played a key role in the results.

VIII. APPENDIX

In this appendix we include preliminaries about polynomial matrices, the state map and interconnection of behaviors. We also include precise definitions of various notions we used in the paper.

A. Polynomial Matrices

This subsection explains about the various properties of polynomial matrices that are relevant for this paper. The leading row coefficient matrix of R is the constant matrix whose i -th row is the coefficient of the leading degree of ξ in the i -th row of R .

Definition 8.1: A polynomial matrix R is called *row-reduced*¹¹ if its leading row coefficient matrix has full row rank.

¹⁰Since P and C are assumed to be row-reduced, the result is also true for the condition $\mathcal{X}_K = \mathcal{X}_P + \mathcal{X}_C$.

¹¹The term row proper, often used in the literature, means the same.

Column-reduced is defined similarly. A matrix $R \in \mathbb{R}^{q \times q}[\xi]$ is said to be unimodular if $\det(R)$ is a nonzero constant. In this paper we require the concept of poles and zeros at infinity of a polynomial matrix $R(s)$, denoted respectively as $\delta_M(R)$ and $z_\infty(R)$. A direct count of the zeros and poles at infinity can be obtained by counting the valuations at ∞ for a rational matrix as elaborated in [9, Section 6.5] and as explained briefly below. For a rational $p(s) \in \mathbb{R}(s)$, with $p = a/b$ where a and b are polynomials, $b \neq 0$, define $\nu(p)$ the valuation at ∞ of p by $\nu(p) := \text{degree } b - \text{degree } a$ and $\nu(0) := \infty$. For any rational matrix $R \in \mathbb{R}^{n \times m}(s)$ with $n \leq m$, define $\sigma_i(R)$ as the minimum of the valuations of all $i \times i$ minors of R for each $1 \leq i \leq n$. The structural indices at infinity of the rational matrix R are defined as $\nu_1(R) := \sigma_1(R)$, $\nu_j(R) := \sigma_j(R) - \sigma_{j-1}(R)$ for $j = 2, \dots, n$. Typically, some ν_i are negative, some zero, and the rest positive. The absolute values of the negative ones are summed to give the number of *poles* at infinity of R counted with multiplicity, while the positive ones are summed to give the *zeros* at ∞ of R counted with multiplicity

$$z_\infty(R) := \sum_{\nu_i > 0} \nu_i \quad \text{and} \quad \delta_M(R) := - \sum_{\nu_i < 0} \nu_i.$$

The polynomial matrix R is said to have no zeros at infinity if all $\nu_i \leq 0$. A row-reduced matrix does not have zeros at infinity. For a polynomial matrix R , $\delta_M(R)$ is also called the *full* McMillan degree of R (see [25, Proposition 1.75]). This is different from the *slow* McMillan degree which we define as follows.

Definition 8.2: The *slow* McMillan degree, denoted as n_{slow} , of a polynomial matrix $R \in \mathbb{R}^{n \times m}[\xi]$ is defined as the maximum degree among all maximal minors of R .

It is well known that n_{slow} is the dimension of a minimal state space representation when we are interested in just smooth solutions. This paper deals with both fast (impulsive) and slow (smooth) solutions; δ_M and n_{slow} play a key role in this paper. It is possible to characterize row-reducedness of a matrix using n_{slow} as in the following proposition.

Proposition 8.3: [9, Section 6.3] For a polynomial matrix $R \in \mathbb{R}^{n \times m}[\xi]$ with $n \leq m$, let $d_i(R)$ for $i = 1$ to n denote the highest degree among the entries of i th row. Then R is row-reduced if and only if $\sum_{i=1}^n d_i(R) = n_{\text{slow}}(R)$.

The following definition is about a polynomial matrix being left-prime.

Definition 8.4: A polynomial matrix $R(\xi) \in \mathbb{R}^{n \times m}[\xi]$ with $n \leq m$ is said to be *left-prime* if $\text{rank } R(\lambda) = n$ for all $\lambda \in \mathbb{C}$. Right-primeness is defined in the obvious way: requiring full *column* rank for every complex number.

B. State Map

The construction of a state variable for a behavior can be done through the “state map” $X(d/dt)$: a map that acts on the variable w and gives a state variable x , i.e., $x := X(d/dt)w$. We focus in this paper on the state map constructed using the “shift-and-cut-procedure” on a polynomial matrix R defining the behavior [23]. See also [16], [18].

The shift and cut operator $\sigma : \mathbb{R}^{n \times m}[\xi] \rightarrow \mathbb{R}^{n \times m}[\xi]$ for a polynomial matrix R is defined by

$$\sigma(R) := \xi^{-1} (R(\xi) - R(0)).$$

Higher order actions of σ are defined in the obvious way: $\sigma^2(R) = \sigma(\sigma(R))$, etc. Let N be the highest degree amongst the entries in R . Then a state map $X(\xi) \in \mathbb{R}^{Nn \times m}[\xi]$ is constructed by $X(\xi) := \text{col}(\sigma(R), \sigma^2(R), \dots, \sigma^N(R))$. One might as well remove the zero rows from this matrix X and redefine X as only its nonzero rows. Let the number of nonzero rows of X be denoted by n_X . The state map X obtained by the above procedure (shift and cut, followed by removing zero rows) on a polynomial matrix R is denoted as $X_R \in \mathbb{R}^{n_X \times m}[\xi]$ and is called the *canonical* state map.

C. Interconnection

In the behavioral approach we view control as *restriction* of the plant behavior $\mathcal{P} := \mathfrak{B}(P, \mathcal{L})$ to a desired sub-behavior. This restriction is achieved by designing new laws that the system variables have to satisfy in addition to the existing plant equations. These additional laws themselves constitute a dynamical system: the controller, whose behavior we denote by $\mathcal{C} := \mathfrak{B}(C, \mathcal{L})$. The interconnection of \mathcal{P} and \mathcal{C} is defined as the system with behavior $\mathcal{K} := \mathcal{P} \cap \mathcal{C}$. So $\mathcal{K} = \mathfrak{B}(\text{col}(P, C), \mathcal{L})$. The *output cardinality* of a behavior, denoted as $\text{p}(\mathcal{P})$, is given by the $\text{rank}_{\mathbb{R}[\xi]}(P(\xi))$ and $n_{\text{slow}}(\mathcal{P})$ is given by $n_{\text{slow}}(P(\xi))$. The output cardinality, as the name suggests, is the number of components in the output in any of the (in general, nonunique) input/output partitions of the system variables (see [28]).

Definition 8.5: The interconnection of \mathcal{P} and \mathcal{C} is said to be a regular interconnection (RI) if $\text{p}(\mathcal{P} \cap \mathcal{C}) = \text{p}(\mathcal{P}) + \text{p}(\mathcal{C})$. The controller \mathcal{C} is then said to be a regular controller, and it is said to implement $\mathcal{P} \cap \mathcal{C}$ regularly. Given $\mathcal{K} \subseteq \mathcal{P}$, the behavior \mathcal{K} is said to be regularly implementable if there exists a regular controller \mathcal{C} such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$.

Definition 8.6: A regular interconnection is said to be a *regular feedback interconnection* (RFI) if $\text{p}(\mathcal{P} \cap \mathcal{C}) = \text{p}(\mathcal{P}) + \text{p}(\mathcal{C})$ and $n_{\text{slow}}(\mathcal{P} \cap \mathcal{C}) = n_{\text{slow}}(\mathcal{P}) + n_{\text{slow}}(\mathcal{C})$. The controller \mathcal{C} is then said to be an RFI controller, and it is said to implement $\mathcal{P} \cap \mathcal{C}$ by RFI. The behavior \mathcal{K} is said to be *regular feedback implementable* (RFI) if there exists an RFI controller \mathcal{C} such that $\mathcal{K} = \mathcal{P} \cap \mathcal{C}$.

It has been noted in [28] how regular interconnection is equivalent to the practical implementation of the ‘closed loop’ system (the interconnected system) in the familiar feedback configuration. The transfer matrices of the plant/controller could however be improper, thus possibly resulting in impulsive solutions upon interconnection. This is due to the so-called ill-posedness of interconnection. See also [10]. The following proposition gives conditions under which one has equality of two $\mathcal{L}_{\mathbb{R}}^C$ -behaviors, i.e., only the slow behavior is of interest.

Proposition 8.7: [20, Theorem 2.5.4] Let $R_1, R_2 \in \mathbb{R}^{n \times m}[\xi]$ be such that $\text{rank}_{\mathbb{R}[\xi]}(R_1) = \text{rank}_{\mathbb{R}[\xi]}(R_2) = n$. The two behaviors $\mathfrak{B}(R_1, \mathcal{L}_{\mathbb{R}}^C)$ and $\mathfrak{B}(R_2, \mathcal{L}_{\mathbb{R}}^C)$ are equal if and only if there exists a unimodular matrix $U(\xi)$ such that $R_1(\xi) = U(\xi)R_2(\xi)$.

The following result from [19] answers the question of when a smooth sub-behavior is regularly implementable with respect to a given behavior.

Proposition 8.8: [19, Theorem 9] For two behaviors $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$ and $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$ such that $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C) \subset \mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$ the following are equivalent.

- 1) $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$ is regularly implementable with respect to $\mathfrak{B}(P, \mathcal{L}_{\mathbb{R}}^C)$.
- 2) There exists a polynomial matrix F with $F(\lambda)$ full row rank for all $\lambda \in \mathbb{C}$, such that $P = FK$.

Further, if G is such that $\text{col}(F, G)$ is unimodular, then $\mathfrak{B}(GK, \mathcal{L}_{\mathbb{R}}^C)$ is a controller that regularly implements $\mathfrak{B}(K, \mathcal{L}_{\mathbb{R}}^C)$.

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