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Relationship between Granger non-causality and network graph of state-space representations

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Chapter 3

Granger causality and Kalman representations in coordinated form

In Chapter 2 we have shown that the existence of Kalman representations, whose network graphs are directed graphs with two nodes and one edge, is equivalent to Granger non-causality among the components of the output process of these representations. It has also been shown that Granger non-causality implies the existence of such Kalman representations that can be constructed algorithmically from the covariance sequence of the output process. In this chapter, we step forward to study Kalman representations with more complex network graph. More specifically, we study Kalman representations, whose network graphs are star graphs, called Kalman representations in coordinated form. A star graph is a tree graph which has precisely one root and all its other nodes are leaves. The subsystem of a Kalman representation in coordinated form corresponding to the root node of the network graph will be called coordinator and the other subsystems, corresponding to the leaves of the network graph, will be called agents. In such representations, the coordinator sends information to the agents and the agents do not send information to the coordinator. The existence of Kalman representations in coordinated form is then associated to a collection of conditional and unconditional Granger non-causalities in the output process. We also present algorithms for constructing a Kalman representation in coordinated form in the presence of the appropriate conditional and unconditional Granger non-causalities.

Deterministic LTI-SS representations in coordinated form were already introduced in (Kempker, 2012; Kempker et al., 2014a; Ran and van Schuppen, 2014). In (Kempker, 2012) and (Kempker et al., 2014a), a general method was presented to transform deterministic LTI-SS system into coordinated form. In addition, in (Kempker, 2012) and (Pambakian, 2011), Gaussian coordinated systems were studied in the context of LQG control. The cited papers gave the idea and motivated the name of the representations studied in this chapter. However, the cited papers do not study the relation between the coordinated system structure and causality properties of the observed process. The results in (Caines et al., 2003; Caines et al., 2009) are the closest ones to the results in this chapter. The cited papers provide nec-

essary and sufficient conditions for the existence of LTI–SS representations in the so-called conditional orthogonal form which form a subclass of LTI–SS representations in coordinated form. The conditions of (Caines et al., 2003; Caines et al., 2009) for the existence of such systems are much stronger than the conditions proposed in this chapter. Regarding the proofs of the statements in (Caines et al., 2003; Caines and Wynn, 2007; Caines et al., 2009), only the existence of the LTI–SS representation in conditional orthogonal form is proven in (Caines et al., 2003) which completely differs from the idea behind the proofs of this chapter. Note that (Caines et al., 2003; Caines and Wynn, 2007; Caines et al., 2009) did not provide algorithms to calculate the representations. This chapter, together with Chapter 3 is based on the journal paper (Jozsa et al., 2018b).

The structure of this chapter is as follows: First, we introduce Kalman representations in coordinated form. Then, we characterize their existence in terms of conditional and unconditional Granger causality. This is followed by the presentation of two algorithms for calculating Kalman representations in coordinated form. Finally, we provide an example to illustrate the results. The proofs of the statements can be found in Appendices 3.A and 3.B. If not stated otherwise, we assume throughout this chapter that $\mathbf{y} = [\mathbf{y}_1^T, \dots, \mathbf{y}_n^T]^T$ is a ZMSIR process where $n \geq 2$, $\mathbf{y}_i \in \mathbb{R}^{r_i}$, and $r_i > 0$ for $i = 1, \dots, n$.

3.1 Kalman representation in coordinated form

In this section, we introduce Kalman representations in coordinated form and discuss their properties. Kalman representations in coordinated form have the star graph as their network graph that defines the communication flow among their subsystems. In order to exclude hidden communication that is inconsistent with this network graph, we also introduce a subclass of these representations, called Kalman representations in causal coordinated form. To begin with, we define these classes of representations:

Definition 3.1. A Kalman representation $(A, K, C, I, \mathbf{e} = [\mathbf{e}_1^T, \dots, \mathbf{e}_n^T]^T, \mathbf{y})$, where $\mathbf{e}_i \in \mathbb{R}^{r_i}$, $i = 1, \dots, n$, is called a *Kalman representation in coordinated form*, if

$$A = \left[\begin{array}{cccc|c} A_{11} & 0 & \cdots & 0 & A_{1n} \\ 0 & A_{22} & \cdots & 0 & A_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{(n-1)(n-1)} & A_{(n-1)n} \\ \hline 0 & 0 & \cdots & 0 & A_{nn} \end{array} \right], K = \left[\begin{array}{cccc|c} K_{11} & 0 & \cdots & 0 & K_{1n} \\ 0 & K_{22} & \cdots & 0 & K_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & K_{(n-1)(n-1)} & K_{(n-1)n} \\ \hline 0 & 0 & \cdots & 0 & K_{nn} \end{array} \right], \quad (3.1)$$

$$C = \left[\begin{array}{cccc|c} C_{11} & 0 & \cdots & 0 & C_{1n} \\ 0 & C_{22} & \cdots & 0 & C_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & C_{(n-1)(n-1)} & C_{(n-1)n} \\ \hline 0 & 0 & \cdots & 0 & C_{nn} \end{array} \right], \quad (3.1)$$

where $A_{ij} \in \mathbb{R}^{p_i \times p_j}$, $K_{ij} \in \mathbb{R}^{p_i \times r_j}$, $C_{ij} \in \mathbb{R}^{r_i \times p_j}$ and $p_i \geq 0$ for $i, j = 1, \dots, n$. If, in addition, for each $i = 1, \dots, n-1$

$$\left(\left[\begin{array}{cc} A_{ii} & A_{in} \\ 0 & A_{nn} \end{array} \right], \left[\begin{array}{cc} K_{ii} & K_{in} \\ 0 & K_{nn} \end{array} \right], \left[\begin{array}{cc} C_{ii} & C_{in} \\ 0 & C_{nn} \end{array} \right], I_{r_i+r_n}, \begin{bmatrix} \mathbf{e}_i \\ \mathbf{e}_n \end{bmatrix} \right) \quad (3.2)$$

is a minimal Kalman representation of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ in causal block triangular form, then $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is called a *Kalman representation in causal coordinated form*.

If $n = 2$, then Definition 3.1 coincides with Definition 2.1 on Kalman representations in block triangular form. Furthermore, if (A, K, C, I, \mathbf{e}) is a Kalman representation in causal coordinated form, then the dimensions of the block matrices A_{ij}, K_{ij}, C_{ij} , $i, j = 1, \dots, n$ are uniquely determined by \mathbf{y} . Indeed, since (3.2) is a minimal Kalman representation of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$, $i = 1, \dots, n-1$ in causal block triangular form, the dimensions of A_{ii}, K_{ii}, C_{ii} and A_{in}, K_{in}, C_{in} are uniquely determined by $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$, see Section 2.1 for details. Therefore, all dimensions of the blocks of A, K and C in (3.1) are determined by \mathbf{y} . Definition 3.1 is based on the deterministic terminology (Kempker, 2012; Ran and van Schuppen, 2014) and on the definition of Gaussian coordinated systems (Kempker, 2012; Pambakian, 2011).

The term *coordinated* is used because the LTI-SS representation at hand can be viewed as consisting of several subsystems; one of which plays the role of a coordinator and the others play the role of agents. More precisely, let $(A, K, C, I, \mathbf{e}, \mathbf{y})$ be a Kalman representation in coordinated form as in (3.1) and let $\mathbf{x} = [\mathbf{x}_1^T, \dots, \mathbf{x}_n^T]^T$ be its state such that $\mathbf{x}_i \in \mathbb{R}^{p_i}$, $i = 1, \dots, n$. Then, for $i = 1, \dots, n-1$

$$\mathcal{S}_{a_i} \begin{cases} \mathbf{x}_i(t+1) = \sum_{j=\{i,n\}} A_{ij} \mathbf{x}_j(t) + K_{ij} \mathbf{e}_j(t) \\ \mathbf{y}_i(t) = \sum_{j=\{i,n\}} C_{ij} \mathbf{x}_j(t) + \mathbf{e}_i(t) \end{cases} \quad (3.3)$$

$$\mathcal{S}_c \begin{cases} \mathbf{x}_n(t+1) = A_{nn} \mathbf{x}_n(t) + K_{nn} \mathbf{e}_n(t) \\ \mathbf{y}_n(t) = C_{nn} \mathbf{x}_n(t) + \mathbf{e}_n(t) \end{cases} \quad (3.4)$$

Notice that subsystem \mathcal{S}_{a_i} generates \mathbf{y}_i as output, has $\mathbf{x}_i, \mathbf{e}_i$ as its state and noise process and takes $\mathbf{x}_n, \mathbf{e}_n$ as its inputs, thus takes inputs from subsystem \mathcal{S}_c . In contrast, \mathcal{S}_c is autonomous, generating \mathbf{y}_n as output and having $\mathbf{x}_n, \mathbf{e}_n$ as its state and noise

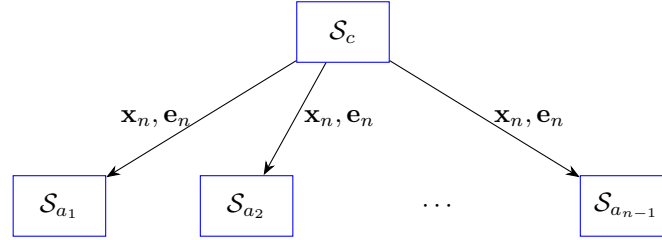


Figure 3.1: Network graph of a Kalman representation in coordinated form: \mathcal{S}_c is the coordinator (3.4) and \mathcal{S}_{a_i} , $i = 1, \dots, n - 1$ are the agents (3.3).

process but not taking input from subsystems \mathcal{S}_{a_i} , $i = 1, \dots, n - 1$ (see Figure 3.1). We call \mathcal{S}_c the *coordinator* and \mathcal{S}_{a_i} with $i = 1, \dots, n - 1$ the *agents*. Intuitively, the agents do not communicate with each other, only the coordinator sends information (\mathbf{x}_n and \mathbf{e}_n) to all agents and does not receive information from them.

Motivation for Kalman representations in causal coordinated form

If we considered a general LTI–SS representation with a network graph like in Figure 3.1, then the noise process \mathbf{e} could be any process. If \mathbf{e} were not the innovation process of \mathbf{y} , then it could happen that the agents communicate with each other in an implicit way through \mathbf{e} . However, if we assume that $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is a Kalman representation in causal coordinated form satisfying (3.1), then $[\mathbf{e}_i^T, \mathbf{e}_n^T]^T$ is the innovation process of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ for $i = 1, \dots, n - 1$ and \mathbf{e}_n is the innovation process of \mathbf{y}_n . Hence, the values of \mathbf{e}_i and \mathbf{e}_n depend only on the past and present values of \mathbf{y}_i and \mathbf{y}_n . Moreover, \mathbf{x}_n depends only on the past values of \mathbf{y}_n and therefore \mathbf{x}_i depends only on the past values of \mathbf{y}_i and \mathbf{y}_n . That is, Kalman representations in causal coordinated form have the property that there is no communication among the agents or from the agents to the coordinator hidden in the noise process. Note that the lack of communication from the agents to the coordinator is ensured by that (3.2) is a Kalman representation in causal block triangular form.

Kalman representations in causal coordinated form have a number of desirable properties, e.g., they have the smallest possible coordinator. By definition, the subsystem (3.2) of a Kalman representation in causal coordinated form (3.1) is a minimal Kalman representation of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ in causal block triangular form. This implies that (3.4) is a minimal Kalman representation of \mathbf{y}_n and thus the coordinator is minimal. It assures observability of the coordinator and enables to estimate the states of Kalman representations in causal coordinated form using distributed filters. That is, in order to estimate the state \mathbf{x}_n of the coordinator using a Kalman filter, only

the output \mathbf{y}_n of the coordinator is necessary. Since (3.3) is also minimal and thus observable, in order to estimate the state \mathbf{x}_i of the i th agent using a Kalman filter, only the output \mathbf{y}_i of this agent and the output \mathbf{y}_n of the coordinator are necessary. Furthermore, from Lemma 3.2 below, Kalman representations in causal coordinated form are isomorphic (see Definition 1.11). Hence, if they represent the same output process, their properties are essentially the same. Note that as a consequence of Lemma 3.2, if a Kalman representation of a process \mathbf{y} in causal coordinated form is not minimal then there does not exist a minimal Kalman representation of \mathbf{y} in causal coordinated form. The **proof** of Lemma 3.2 can be found in Appendix 3.A.

Lemma 3.2. *Any two Kalman representations of \mathbf{y} in causal coordinated form are isomorphic.*

3.2 Conditional Granger causality and coordinated systems

In this section, we show that the existence of a Kalman representation of \mathbf{y} in causal coordinated form can be characterized by conditional Granger non-causalities among the components of \mathbf{y} .

To begin with, we define conditional Granger non-causality: the generalization of Granger non-causality between two components of a process in the presence of a third component. The next definition is a particular case of the concept of causality defined in (Granger, 1963), if the latter is applied to ZMSIR processes, and if, using the terminology of (Granger, 1963), there is one external process.

Definition 3.3. Consider a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{y}_3^T]^T$. We say that \mathbf{y}_1 *conditionally does not Granger cause \mathbf{y}_2 with respect to \mathbf{y}_3* , if for all $t, k \in \mathbb{Z}, k \geq 0$

$$E_l[\mathbf{y}_2(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_2, \mathbf{y}_3}] = E_l[\mathbf{y}_2(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3}].$$

Otherwise, we say that \mathbf{y}_1 *conditionally Granger causes \mathbf{y}_2 with respect to \mathbf{y}_3* .

If $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$, then considering $\mathbf{y}_3 = \mathbf{y}_2$ Definition 3.3 coincides with the unconditional Granger causality defined in Definition 2.3. We will be interested in a particular combination of causal dependencies in a process $\mathbf{y} = [\mathbf{y}_1^T, \dots, \mathbf{y}_n^T]^T$. Namely, when \mathbf{y}_i does not Granger cause \mathbf{y}_n and \mathbf{y}_i does not Granger cause \mathbf{y}_j with respect to \mathbf{y}_n for all $i, j = 1, \dots, n-1, i \neq j$. We will show that these causal relations in $\mathbf{y} = [\mathbf{y}_1^T, \dots, \mathbf{y}_n^T]^T$ hold if and only if \mathbf{y} has a Kalman representation in causal coordinated form whose network graph is as in Figure 3.1. We will also give condition for when this Kalman representation in causal coordinated form is minimal. Kalman

representations in causal coordinated form are observable, therefore we only need to ensure reachability. For the formulation of the reachability condition, we need to introduce the term of conditionally trivial intersection of two subspaces U, V with respect to a third, closed subspace W .

Definition 3.4. Consider the subspaces $U, V, W \subseteq \mathcal{H}$ such that W is closed. Then U, V have a *conditionally trivial intersection* with respect to W if

$$\{u - E_l[u|W] \mid u \in U\} \cap \{v - E_l[v|W] \mid v \in V\} = \{0\},$$

i.e., the intersection of the projections of U and V onto the orthogonal complement of W in \mathcal{H} is the zero subspace. The conditionally trivial intersection of U and V with respect to W is denoted by $U \cap V|W = \{0\}$

Now we are ready to state the main result of this chapter:

Theorem 3.5. Consider the following statements for a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \dots, \mathbf{y}_n^T]^T$:

- (i) \mathbf{y}_i does not Granger cause \mathbf{y}_n , $i = 1, \dots, n-1$;
- (ii) \mathbf{y}_i conditionally does not Granger cause \mathbf{y}_j with respect to \mathbf{y}_n , $i, j = 1, \dots, n-1$, $i \neq j$;
- (iii) (i) and (ii) hold and for $i, j \in \{1, \dots, n-1\}$, $i \neq j$

$$E_l[\mathcal{H}_{t+}^{\mathbf{y}_i} | \mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}] \cap E_l[\mathcal{H}_{t+}^{\mathbf{y}_j} | \mathcal{H}_{t-}^{\mathbf{y}_j, \mathbf{y}_n}] \mid E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}] = \{0\} \quad (3.5)$$

- (iv) there exists a minimal Kalman representation of \mathbf{y} in causal coordinated form;
- (v) there exists a Kalman representation of \mathbf{y} in causal coordinated form;
- (vi) there exists a Kalman representation of \mathbf{y} in coordinated form;

Then, the following hold:

- (a) (iii) \iff (iv);
- (b) (i) and (ii) \iff (v).

If, in addition, \mathbf{y} is coercive, then we have

- (c) (i) and (ii) \iff (v) \iff (vi).

The **proof** can be found in Appendix 3.B. The intuition behind this result is the following. For a coordinator to exist, the outputs of the agents should not influence

the output of the coordinator, i.e., (i) should hold. Moreover, for $i \neq j$ the output of agent i should not influence the output of agent j , except that information which comes from the output of the coordinator, i.e., (ii) should hold. Condition (iii) for minimality can be explained as follows. It can be shown that a Kalman representation in causal coordinated form is observable, so for minimality, we only have to ensure its reachability or equivalently, the linear independence of the components of the state $\mathbf{x} = [\mathbf{x}_1^T, \dots, \mathbf{x}_n^T]^T$ at each time. The spaces generated by the components of $\mathbf{x}_n(t)$ and of $[\mathbf{x}_i^T, \mathbf{x}_n^T]^T(t)$ are $E_t[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}]$ and $E_t[\mathcal{H}_{t+}^{\mathbf{y}_i} | \mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}]$, respectively, where \mathbf{x}_i and \mathbf{x}_n are as in (3.4) and (3.3). As a result, condition (iii) is equivalent to the reachability of a Kalman representation in causal coordinated form.

Our next result helps us in reformulating condition (ii) in Theorem 3.5 by unconditional Granger causality.

Lemma 3.6. Consider a process $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{y}_3^T]^T$ and the following statements

- (i) \mathbf{y}_1 does not Granger cause \mathbf{y}_3
- (ii) \mathbf{y}_2 does not Granger cause \mathbf{y}_3
- (iii) \mathbf{y}_1 conditionally does not Granger cause \mathbf{y}_2 with respect to \mathbf{y}_3
- (iv) \mathbf{y}_1 does not Granger cause $[\mathbf{y}_2^T, \mathbf{y}_3^T]^T$

Then we have that

(i) and (ii) and (iii) \iff (ii) and (iv).

The proof can be found in Appendix 3.A

Remark 3.7 (Alternative formulations of (ii)). From Lemma 3.6 we can reformulate the conditions of Theorem 3.5 as follows: if (i) holds, then condition (ii) is equivalent to saying that \mathbf{y}_i does not Granger cause $[\mathbf{y}_j^T, \mathbf{y}_n^T]^T$, $i, j \in \{1, \dots, n-1\}$, $i \neq j$.

Minimal Kalman representations in causal coordinated form are isomorphic to any other minimal Kalman representation of the same process, see Proposition 1.12. Hence, any property of minimal Kalman representations in causal coordinated form that is invariant under isomorphism remains valid for any other minimal Kalman representation. Theorem 3.5 gives a necessary and sufficient condition for the existence of a minimal Kalman representations in causal coordinated form. From Lemma 3.2, we know that any two Kalman representations in causal coordinated form are isomorphic, thus behave as minimal ones among all Kalman representations in coordinated form. Existence of a minimal Kalman representation in coordinated form (not causal coordinated form) remains a topic of future research.

3.3 Computing Kalman representations in coordinated form

Next, we describe a procedure to calculate a Kalman representation of \mathbf{y} in causal coordinated form. Assume that condition (i) in Theorem 3.5 holds. Consider an LTI-SS representation $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$ of \mathbf{y} and the partitions of \bar{C} and \bar{D}

$$\bar{C} = [\bar{C}_1^T, \dots, \bar{C}_n^T]^T, \quad \bar{D} = [\bar{D}_1^T, \dots, \bar{D}_n^T]^T \quad (3.6)$$

such that \bar{C}_i and \bar{D}_i have $r_i = \dim(\mathbf{y}_i)$ rows for all $i = 1, \dots, n$. Then, notice that the tuple $(\bar{A}, \bar{B}, [\bar{C}_i^T \ \bar{C}_n^T]^T, [\bar{D}_i^T \ \bar{D}_n^T]^T, \mathbf{v})$ is an LTI-SS representation of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ for all $i = 1, \dots, n-1$. Hence, by Corollary 2.6 the latter can be transformed into a minimal Kalman representation $(\hat{A}_i, \hat{K}_i, \hat{C}_i, I_{r_i+r_n}, [\mathbf{e}_i^T, \mathbf{e}_n^T]^T)$ of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ in causal block triangular form, i.e.,

$$\hat{A}_i = \begin{bmatrix} \hat{A}_{ii} & \hat{A}_{in} \\ 0 & \hat{A}_{i,nn} \end{bmatrix}, \quad \hat{K}_i = \begin{bmatrix} \hat{K}_{ii} & \hat{K}_{in} \\ 0 & \hat{K}_{i,nn} \end{bmatrix}, \quad \hat{C}_i = \begin{bmatrix} \hat{C}_{ii} & \hat{C}_{in} \\ 0 & \hat{C}_{i,nn} \end{bmatrix}, \quad (3.7)$$

and the process $[\mathbf{e}_i^T, \mathbf{e}_n^T]^T$ is the innovation process of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$. In addition, $(\hat{A}_{i,nn}, \hat{K}_{i,nn}, \hat{C}_{i,nn}, I_{r_n}, \mathbf{e}_n)$ is a minimal Kalman representation of \mathbf{y}_n . Since all minimal Kalman representations of \mathbf{y}_n are isomorphic (Proposition 1.12), there exist non-singular matrices T_i for $i = 2, \dots, n-1$ such that

$$\hat{A}_{1,nn} = T_i \hat{A}_{i,nn} T_i^{-1}, \quad \hat{K}_{1,nn} = T_i \hat{K}_{i,nn}, \quad \hat{C}_{1,nn} = \hat{C}_{i,nn} T_i^{-1}. \quad (3.8)$$

Let T_1 be the identity matrix and define the matrices A, K and C as in (3.1) such that for $i = 1, \dots, n-1$

$$\begin{aligned} A_{ii} &= \hat{A}_{ii}, & K_{ii} &= \hat{K}_{ii}, & C_{ii} &= \hat{C}_{ii}, \\ A_{in} &= \hat{A}_{in} T_i^{-1}, & K_{in} &= \hat{K}_{in}, & C_{in} &= \hat{C}_{in} T_i^{-1}, \\ A_{nn} &= \hat{A}_{1,nn}, & K_{nn} &= \hat{K}_{1,nn}, & C_{nn} &= \hat{C}_{1,nn}. \end{aligned} \quad (3.9)$$

Then, for the tuple $(A, K, C, I, \mathbf{e}, \mathbf{y})$, where $\mathbf{e} = [\mathbf{e}_1^T, \mathbf{e}_2^T, \dots, \mathbf{e}_n^T]^T$, we can state the following.

Corollary 3.8. *The following statements hold:*

- If \mathbf{y} satisfies conditions (i) and (ii) in Theorem 3.5, then $(A, K, C, I, \mathbf{e}, \mathbf{y})$ defined by (3.7), (3.8), and (3.9) is a Kalman representation in causal coordinated form.
- If \mathbf{y} satisfies (iii), then $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is also minimal.

The **proof** can be found in Appendix 3.B. Note that if condition (ii) in Theorem 3.5 does not hold, then A, K, C and \mathbf{e} can be calculated as above, but the process \mathbf{e} is not necessarily white noise. Hence, if condition (ii) does not hold, then the tuple $(A, K, C, I, \mathbf{e}, \mathbf{y})$ does not necessarily define an LTI-SS representation.

The procedure above is elaborated in Algorithms 6 and 7. Algorithm 6 takes an LTI-SS representation as its input and transforms it into a Kalman representation in causal coordinated form. Algorithm 7 calculates the same representation from covariances of the output. Hence, by using empirical covariances it can be applied to data.

Algorithm 6 Kalman representation in causal coordinated form based on LTI-SS representation

Input $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}, \Lambda_0^{\mathbf{y}}\}$: System matrices of an LTI-SS representation $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$ of \mathbf{y} and variance of \mathbf{v}

Output $\{A, K, C\}$: System matrices of (3.1)
--

for $i = 1 : n - 1$

Step 1 Consider the partition (3.6) and apply Algorithm 4 with input $\{\bar{A}, \bar{B}, \bar{C}_i, \bar{D}_i, \Lambda_0^{\mathbf{y}}\}$. Denote its output by $\{\hat{A}_i, \hat{K}_i, \hat{C}_i\}$, where $(\hat{A}_i, \hat{K}_i, \hat{C}_i, I, \mathbf{e}_{i,n}, [\mathbf{y}_i^T, \mathbf{y}_n^T]^T)$ is a minimal Kalman representation in block triangular form.

Step 2 If $i > 1$, consider the partition (3.7) of $\hat{A}_i, \hat{K}_i, \hat{C}_i$ and define the non-singular matrix T_i as in (3.8).

end for

Step 3 Define A, K and C as in (3.1), such that the submatrices $\{\hat{A}_{ii}, A_{in}C_{ii}, C_{in}K_{ii}, K_{in}\}_{i=1, \dots, n-1}$ satisfy (3.9).

Remark 3.9 (Correctness of Algorithm 6 and Algorithm 7). Consider a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \dots, \mathbf{y}_n^T]^T$ with covariance sequence $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{\infty}$ and an LTI-SS representation $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$ of \mathbf{y} . Let \mathbf{e} be the innovation process of \mathbf{y} and N be any number larger than or equal to the dimension of a minimal LTI-SS representation of \mathbf{y} . Assume that \mathbf{y} satisfies conditions (i) and (ii) in Theorem 3.5 and note that Algorithms 4 and 5 calculate minimal Kalman representations in causal block triangular form (Remark 2.7). Then it follows from Corollary 3.8 that if $\{A, K, C\}$ is the output of Algorithm 6 with input $\{\bar{A}, \bar{B}, \bar{C}, \bar{D}, \Lambda_0^{\mathbf{y}} = E[\mathbf{v}(t)\mathbf{v}^T(t)]\}$, then (A, K, C, I, \mathbf{e}) is a Kalman representation of \mathbf{y} in causal coordinated form. In addition, (A, K, C, I, \mathbf{e}) is minimal if and only if (3.5) holds. Similarly, by Corollary 3.8, if $\{A, K, C\}$ is the output of Algorithm 7 with input $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$, then (A, K, C, I, \mathbf{e}) is a Kalman representation of \mathbf{y} in causal coordinated form and is minimal if and only if (3.5) holds.

Algorithm 7 Kalman representation in causal coordinated form based on output covariances

Input $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$: Covariance sequence of $\mathbf{y} = [\mathbf{y}_1^T, \dots, \mathbf{y}_n^T]^T$

Output $\{A, K, C\}$: System matrices of (3.1)

for $i = 1 : n - 1$

Step 1 Denote the covariance matrix of $\mathbf{y}_{i,n} = [\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ with lag k by $\Lambda_k^{\mathbf{y}_{i,n}} = E[\mathbf{y}_{i,n} \mathbf{y}_{i,n}^T]$.

Step 2 Calculate the rank of the Hankel matrix formed by $\{\Lambda_k^{\mathbf{y}_{i,n}}\}_{k=0}^{2N-1}$ and denote it by N_i . Call Algorithm 5 for $\{\Lambda_k^{\mathbf{y}_{i,n}}\}_{k=1}^{2N_i}$ and denote its output by $\{\hat{A}_i, \hat{K}_i, \hat{C}_i\}$.

Step 3 Step 2 of Algorithm 6.

end for

Step 4 Step 3 of Algorithm 6.

Remark 3.10. In view of Remark 1.9 and Remark 2.8, the computational complexity of Algorithms 7 and 6 are polynomial. Algorithm 6 is polynomial in the dimensions of the state, output, and noise processes of the LTI-SS representation $(\bar{A}, \bar{B}, \bar{C}, \bar{D}, \mathbf{v})$. Algorithm 7 is polynomial in the number and size of the output covariances.

Remark 3.11 (Checking (i)–(ii)). Algorithms 6 and 7 cannot be directly used to check conditions (i) and (ii) in Theorem 3.5. However, (i) consists of unconditional Granger non-causality conditions, and by Remark 3.7, (ii) can also be reformulated as unconditional Granger non-causality conditions. Therefore, by Remark 2.9, Algorithms 4 and 5 can be used to check these conditions.

Note that Algorithms 6 and 7 operate in a distributed manner; they combine subsystems belonging to an agent and the coordinator for which the observation of any other agent is not needed. Furthermore, Algorithm 7 only uses the covariances of the observed process, thus using empirical covariances, it is suitable to estimate Kalman representations in causal coordinated form based on data. Due to its distributed nature, when applied to empirical covariances, Algorithm 7 is possibly advantageous in terms of estimation error compared to non-distributed procedures.

3.4 Example for coordinated representation

In this section, we adopt a case study from (Kempker, 2012, Section 8.1) to illustrate the results of this chapter in a similar manner as we illustrated the results of Chapter 2 in Section 2.4. The focus of this study is the dynamics of three underwater

vehicles that track a reference path in a fixed formation. Among the vehicles there is one acting as a coordinator that tracks a reference path and two others acting as agents that track the coordinator.

In comparison with (Kempker, 2012, Section 8.1) we made the following changes: (1) to ensure stationarity, the coordinator follows the zero position; (2) for convenience, we consider the movements of the vehicles along the first coordinate; (3) besides the position disturbance we include measurement noise.

We will show that the relative positions (concerning the formation) of the vehicles are ZMSIR processes that can be modeled by a minimal Kalman representation in causal coordinated form. In fact, we reverse engineer the coordinated network topology from the observed process in the following way: We verify that conditions (i) and (ii) in Theorem 3.5 hold by calculating Granger non-causal relations based on Remark 2.9. Then, we calculate a minimal Kalman representation in causal coordinated form using Algorithm 7.

Model description Assume that we have three underwater vehicles V_1, V_2 and V_c where V_1, V_2 act as agents and V_c acts as the coordinator. For $j \in \{1, 2, c\}$ denote the first coordinate at time $t \in \mathbb{Z}$ of the position, velocity, acceleration, position disturbance and measurement noise of V_j by $p_j(t), s_j(t), a_j(t), w_j(t)$ and $\tilde{w}_j(t)$, respectively. Also, denote the first coordinate of the reference position and velocity of V_j by $p_j^R(t)$ and $s_j^R(t)$, respectively. Let

$$\begin{aligned} p_c^R(t) &= -(p_c(t) + \tilde{w}_c(t)) \\ p_j^R(t) &= (p_c(t) + \tilde{w}_c(t)) + \Delta_j, \quad j = 1, 2. \end{aligned}$$

That is, V_c follows the zero position based on its own measured position and for $j = 1, 2$, V_j follows V_c in a distance Δ_j based on the same information. To shorten the expressions, we neglect the dependencies on time. That is, for a process $l(t)$ we write l and we use σ to denote the forward time shift operator defined as follows: $\sigma l(t) = l(t+1)$.

The dynamics of $[p_j, s_j]^T, j \in \{1, 2, c\}$ is given by

$$\sigma \begin{bmatrix} p_j \\ s_j \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\tau-1}{\tau} \end{bmatrix} \begin{bmatrix} p_j \\ s_j \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\tau} \end{bmatrix} a_j + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_j, \quad (3.10)$$

where a_j is a control input and a τ is a time constant. The reference signals $[p_j^R, s_j^R]^T$ are estimated by an observer with dynamics

$$\sigma \begin{bmatrix} \hat{p}_j^R \\ \hat{s}_j^R \end{bmatrix} = \begin{bmatrix} 1 - G_j^p & 1 \\ -G_j^s & \frac{\tau-1}{\tau} \end{bmatrix} \begin{bmatrix} \hat{p}_j^R \\ \hat{s}_j^R \end{bmatrix} + \begin{bmatrix} G_j^p \\ G_j^s \end{bmatrix} p_j^R \quad (3.11)$$

where G_j^p, G_j^s are constant gains. The linear feedback control is then given by

$$a_j = \begin{bmatrix} F_j^p & F_j^s \end{bmatrix} \begin{bmatrix} p_j - \hat{p}_j^R \\ s_j - \hat{s}_j^R \end{bmatrix}.$$

Combining (3.10) and (3.11), we obtain the closed loop system

$$\sigma \begin{bmatrix} p_j \\ s_j \\ \hat{p}_j^R \\ \hat{s}_j^R \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ \frac{1}{\tau} F_j^p & \frac{\tau-1}{\tau} + \frac{1}{\tau} F_j^s & -\frac{1}{\tau} F_j^p & -\frac{1}{\tau} F_j^s \\ 0 & 0 & 1 - G_j^p & 1 \\ 0 & 0 & -G_j^s & \frac{\tau-1}{\tau} \end{bmatrix}}_{A_j} \begin{bmatrix} p_j \\ s_j \\ \hat{p}_j^R \\ \hat{s}_j^R \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ G_j^p \\ G_j^s \end{bmatrix}}_{B_j} p_j^R + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_E w_j.$$

Note that $\mathbf{x}_j := [p_j - \Delta_j, s_j, \hat{p}_j^R - \Delta_j, \hat{s}_j^R]^T$ has essentially the same dynamics as $[p_j, s_j, \hat{p}_j^R, \hat{s}_j^R]^T$ for $j = 1, 2$, namely $\sigma \mathbf{x}_j = A_j \mathbf{x}_j + B_j p_c^R + E w_j$.

Assuming that $\mathbf{v} := [w_1, w_2, w_c, \tilde{w}_1, \tilde{w}_2, \tilde{w}_c]^T$ is a white noise process, we can define the following LTI-SS representation (A, B, C, D, \mathbf{v}) of the process $\mathbf{y} = [\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_c]^T$, where $\mathbf{y}_1 = p_1 - \Delta_1$, $\mathbf{y}_2 = p_2 - \Delta_2$ and $\mathbf{y}_3 = p_c$:

$$\sigma \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_c \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & B_2 & 0 \\ 0 & 0 & A_c & -B_c F \end{bmatrix}}_A \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_c \end{bmatrix} + \underbrace{\begin{bmatrix} E & 0 & 0 & 0 & 0 & B_1 \\ 0 & E & 0 & 0 & 0 & B_2 \\ 0 & 0 & E & 0 & 0 & -B_c \end{bmatrix}}_B \mathbf{v}$$

$$\begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \\ \mathbf{y}_c \end{bmatrix} = \underbrace{\begin{bmatrix} E^T & 0 & 0 \\ 0 & E^T & 0 \\ 0 & 0 & E^T \end{bmatrix}}_C \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_c \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_D \mathbf{v}.$$

Parameter settings Following the approach of (Kempker, 2012), we take $F_1^p = F_2^p = F_c^p$ and $F_1^s = F_2^s = F_c^s$ as the solution of the linear quadratic problem $\min_{a_c} \{ \|\mathbf{z}_1\|^2 + \alpha \|a_c^2\|^2 \}$ with respect to the dynamics

$$\sigma \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \frac{\tau-1}{\tau} \end{bmatrix} \begin{bmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\tau} \end{bmatrix} a_c.$$

For $\tau = 2$ and $\alpha = 10$ the optimal solution is $F_1^p = F_2^p = F_c^p = -0.3$ and $F_1^s = F_2^s = F_c^s = -0.5$. The gain constants were chosen to be $G_1^p = 1.5$, $G_1^s = 0.3$, $G_2^p = 1.2$, $G_2^s = 0.1$, $G_{R_c}^p = 0.9$, and $G_{R_c}^s = 0.5$ for which the matrix A is stable. Finally, the noise process \mathbf{v} is chosen to be a normalized Gaussian white noise process.

Reverse engineering of the network graph Assume that the output process \mathbf{y} of the LTI-SS representation $\mathcal{S} := (A, B, C, D, \mathbf{v}, \mathbf{y})$ is observed. Using the result of this chapter, we will calculate a minimal Kalman representation of \mathbf{y} in causal coordinated form. Note that we do not use prior knowledge of the coordinated structure of the network graph. In fact, this representation reconstructs the network graph of \mathcal{S} . Note that the network graph of \mathcal{S} is a star graph with three nodes: one root node and two leaves where there is two directed edges from the root node to the leaves.

First, we check the Granger non-causal relations among the components of \mathbf{y} using the covariance sequence $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$ of \mathbf{y} where N is larger than or equal to the dimension of a minimal LTI-SS representation of \mathbf{y} . For this, we calculate a Kalman representation (A, K, C, I, \mathbf{e}) of \mathbf{y} and verify that \mathbf{y} is coercive by checking that $A - KC$ is invertible (see Section 1.2.2). In view of Corollary 2.6, a Granger non-causal relation can be verified by observing the output matrix K of Algorithm 5: if the left lower block of the matrix K is zero, then an appropriate Granger non-causal relation holds (see Remark 2.9). Following this method, we apply Algorithm 5 choosing the coordinator to be $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_c, [\mathbf{y}_1, \mathbf{y}_2]^T, [\mathbf{y}_1, \mathbf{y}_c]^T$ and $[\mathbf{y}_2, \mathbf{y}_c]^T$, thus trying all the possibilities. We obtain that $[\mathbf{y}_1, \mathbf{y}_2]^T$ does not Granger cause \mathbf{y}_c and \mathbf{y}_j does not Granger cause $[\mathbf{y}_i, \mathbf{y}_c]$ for all $i, j = 1, 2$, where $i \neq j$, thus conditions (i) and (ii) in Theorem 3.5 hold for the partition $\mathbf{y} = [\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_c]^T$.

Second, in order to calculate a Kalman representation of \mathbf{y} in coordinated form, we apply Algorithm 7 with the covariance sequence $\{\Lambda_k^{\mathbf{y}}\}_{k=0}^{2N}$ as its input. Accordingly, first the minimal Kalman representations $(A_{k,1}, K_1, C_{k,1}, I, \mathbf{e}_{1,c}, [\mathbf{y}_1, \mathbf{y}_c]^T)$ and $(A_{k,2}, K_2, C_{k,2}, I, \mathbf{e}_{2,c}, [\mathbf{y}_2, \mathbf{y}_c]^T)$ are calculated in causal block triangular form using Algorithm 4 with the covariances of $[\mathbf{y}_1, \mathbf{y}_c]^T$ and $[\mathbf{y}_2, \mathbf{y}_c]^T$ as its input. With our parameter settings, these matrices are as follows:

$$\begin{aligned}
A_{k,1} &= \left[\begin{array}{c|c} A_{k,11} & A_{k,1c} \\ \hline 0 & A_{k,cc} \end{array} \right] = \left[\begin{array}{cccc|cccc} 0.4 & -0.3 & -0.1 & 0.1 & 0.0 & 0.1 & 0.2 & -0.1 \\ 0.2 & 0.4 & 0.6 & 0.2 & 0.1 & 0.0 & 0.2 & 0.1 \\ 0.0 & -0.3 & 0.4 & -0.1 & 0.0 & 0.0 & 0.0 & -0.4 \\ -0.2 & 0 & -0.2 & 0.7 & 0.0 & -0.1 & 0.0 & 0.0 \\ \hline 0 & 0 & 0 & 0 & 0.2 & -0.9 & 0.1 & 0.0 \\ 0 & 0 & 0 & 0 & 0.6 & 0.3 & 0.3 & 0.1 \\ 0 & 0 & 0 & 0 & -0.1 & 0.2 & -0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0.1 & 0.0 & -0.5 & 0.3 \end{array} \right] \\
K_1^T &= \left[\begin{array}{c|c} K_{11}^T & 0 \\ \hline K_{1c}^T & K_{cc}^T \end{array} \right] = \left[\begin{array}{cccc|cccc} 0.1 & -0.1 & 0.0 & -0.3 & 0 & 0 & 0 & 0 \\ \hline 0.2 & -0.1 & 0.0 & -0.1 & 0.1 & 0.2 & 0.2 & -0.1 \end{array} \right] \\
C_{k,1} &= \left[\begin{array}{c|c} C_{k,11} & C_{k,1c} \\ \hline 0 & C_{cc} \end{array} \right] = \left[\begin{array}{cccc|cccc} -0.4 & -0.3 & 0.4 & -1.8 & 0.2 & -0.2 & 0.0 & 0.2 \\ \hline 0 & 0 & 0 & 0 & 0.5 & 0.0 & -0.2 & 0.0 \end{array} \right]
\end{aligned}$$

$$\begin{aligned}
A_{k,2} &= \left[\begin{array}{c|c} A_{k,22} & \hat{A}_{k,2c} \\ \hline 0 & \hat{A}_{k,cc} \end{array} \right] = \left[\begin{array}{cccc|cccc} 0.5 & 0.2 & 0 & -0.1 & 0 & -0.1 & -0.2 & 0.2 \\ -0.2 & 0.4 & 0.7 & 0.0 & 0.1 & 0.0 & 0.1 & 0.1 \\ 0.0 & -0.1 & 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & -0.3 \\ 0.1 & 0.0 & 0.0 & 0.8 & 0.0 & -0.1 & 0.0 & 0.0 \\ \hline 0 & 0 & 0 & 0 & 0.2 & -0.8 & 0.1 & 0.0 \\ 0 & 0 & 0 & 0 & 0.7 & 0.4 & 0.3 & 0.1 \\ 0 & 0 & 0 & 0 & -0.1 & 0.2 & -0.4 & 0.4 \\ 0 & 0 & 0 & 0 & 0.1 & 0.1 & -0.3 & 0.2 \end{array} \right] \\
K_2^T &= \left[\begin{array}{c|c} K_{22}^T & 0 \\ \hline \hat{K}_{2c}^T & \hat{K}_{cc}^T \end{array} \right] = \left[\begin{array}{cccc|cccc} -0.1 & 0.0 & 0.0 & -0.3 & 0 & 0 & 0 & 0 \\ -0.2 & 0.0 & 0.0 & -0.1 & 0.1 & 0.2 & 0.2 & -0.1 \end{array} \right] \\
C_{k,2} &= \left[\begin{array}{c|c} C_{k,22} & \hat{C}_{k,2c} \\ \hline 0 & \hat{C}_{cc} \end{array} \right] = \left[\begin{array}{cccc|cccc} 0.5 & -0.1 & -0.1 & -1.9 & 0.2 & -0.1 & 0.0 & 0.1 \\ 0 & 0 & 0 & 0 & 0.5 & -0.1 & -0.2 & 0.0 \end{array} \right].
\end{aligned}$$

Next, as in Step 3 of Algorithm 7, we define a transformation matrix

$$T := \left(\left[\begin{array}{c|c} C_{k,23}^T & A_{k,23} C_{k,23}^T \end{array} \right]^T \right)^{-1} \left[\begin{array}{c|c} C_{k,13}^T & A_{k,13} C_{k,13}^T \end{array} \right]^T$$

with which, the output matrices $\{A_k, K, C_k\}$ of Algorithm 7 are calculated as below.

$$\begin{aligned}
A_k &= \left[\begin{array}{c|c|c} A_{k,11} & 0 & A_{k,1c} \\ \hline 0 & A_{k,22} & T \hat{A}_{k,2c} T^{-1} \\ \hline 0 & 0 & A_{k,cc} \end{array} \right] & K &= \left[\begin{array}{c|c|c} K_{11} & 0 & A_{1c} \\ \hline 0 & K_{22} & T \hat{A}_{2c} \\ \hline 0 & 0 & K_{cc} \end{array} \right] \\
C_k &= \left[\begin{array}{c|c|c} C_{k,11} & 0 & C_{k,1c} \\ \hline 0 & C_{k,22} & \hat{C}_{k,2c} T^{-1} \\ \hline 0 & 0 & C_{k,cc} \end{array} \right].
\end{aligned}$$

In view of Remark 3.9, the tuple $\mathcal{S}_k := (A_k, K, C_k, I, \mathbf{e}, \mathbf{y})$, where \mathbf{e} is the innovation process of \mathbf{y} , is a Kalman representation in causal coordinated form. Furthermore, it is easy to check that \mathcal{S}_k is minimal, which implies that condition (iii) in Theorem 3.5 holds. The calculation of \mathcal{S}_k only requires the second order statistics of the output process and does not use prior knowledge of the network topology. Therefore, the construction of \mathcal{S}_k shows that the network graph of \mathcal{S} can be reverse engineered. Moreover, by Theorem 3.5, the reconstructed representation \mathcal{S}_k not only shows the coordinated structure but also characterizes the causal relations that describe the coordinated relationship in the observed process. The procedure can be repeated based on data, using empirical covariances which provide an estimation of \mathcal{S}_k . Note that \mathcal{S}_k is calculated in a distributed way which possibly reduces estimation error.

3.5 Conclusions

This chapter studies the relationship between coordinated state-space representations and (conditional) Granger non-causality. Our results show that a collection of conditional and unconditional Granger non-causalities among the components of a process is equivalent to the existence of an LTI-SS representation with the star graph as its network graph, called Kalman representation in coordinated form. We provided algorithms for calculating this structured representation, in particular, calculating it from the covariance sequence of the observed output process. Hence, the results open up the possibility of calculating this representation from output data by using empirical covariances. In addition, the results deal with the minimality of the representations and the so-called coercive property of the output processes.

3.A Proof of Lemmas 3.2 and 3.6

Proof of Lemma 3.2. Consider a process $\mathbf{y} = [\mathbf{y}_1^T, \dots, \mathbf{y}_n^T]^T$ where $\mathbf{y}_i \in \mathbb{R}^{r_i}$, for $r_i > 0$, $i = 1, \dots, n$. Let (A, K, C, I, \mathbf{e}) and $(\hat{A}, \hat{K}, \hat{C}, I, \mathbf{e})$ be two Kalman representations of \mathbf{y} in causal coordinated form (3.1) with blocks $A_{ij} \in \mathbb{R}^{p_i \times p_j}$, $K_{ij} \in \mathbb{R}^{p_i \times r_j}$, $C_{ij} \in \mathbb{R}^{r_i \times p_j}$ and $\hat{A}_{ij} \in \mathbb{R}^{p_i \times p_j}$, $\hat{K}_{ij} \in \mathbb{R}^{p_i \times r_j}$, $\hat{C}_{ij} \in \mathbb{R}^{r_i \times p_j}$ for $i, j = 1, \dots, n$. Let \mathcal{S}_i be the Kalman representation (3.2), and let $\hat{\mathcal{S}}_i$ be the counterpart of (3.2), obtained by replacing A_{ij}, K_{ij}, C_{ij} by the matrices $\hat{A}_{ij}, \hat{K}_{ij}, \hat{C}_{ij}$ for $j = i, n$, $i = 1, \dots, n$. From Definition 3.1, it follows that \mathcal{S}_i and $\hat{\mathcal{S}}_i$ are minimal Kalman representations of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ in block triangular form, thus there exists an isomorphism T_i from $\hat{\mathcal{S}}_i$ to \mathcal{S}_i , $i = 1, \dots, n-1$. We will show that T_i is of the form

$$T_i = \begin{bmatrix} T_{ii} & T_{in} \\ 0 & T_{nn} \end{bmatrix}, \quad T_{ij} \in \mathbb{R}^{p_i \times p_j} \text{ for } j \in \{i, n\}.$$

This then implies that $(\hat{A}, \hat{K}, \hat{C}, I, \mathbf{e})$ and (A, K, C, I, \mathbf{e}) are isomorphic such as $A = T\hat{A}T^{-1}$, $K = T\hat{K}$ and $C = \hat{C}T^{-1}$ with the matrix T defined by

$$T = \left[\begin{array}{cccc|c} T_{11} & 0 & \cdots & 0 & T_{1n} \\ 0 & T_{22} & \cdots & 0 & T_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & T_{(n-1)(n-1)} & T_{(n-1)n} \\ \hline 0 & 0 & \cdots & 0 & T_{nn} \end{array} \right].$$

Consider the partition

$$T_i = \begin{bmatrix} T_{ii} & T_{in} \\ T_{in} & T_{nn} \end{bmatrix}, \quad T_{kl} \in \mathbb{R}^{p_k \times p_l} \text{ for } k, l \in \{i, n\}.$$

By reordering the rows of the observability matrices \mathcal{O}_i and $\hat{\mathcal{O}}_i$ of \mathcal{S}_i and $\hat{\mathcal{S}}_i$ we obtain that

$$\mathcal{O}_i = \begin{bmatrix} O_i & O_{in} \\ 0 & O_n \end{bmatrix}, \quad \hat{\mathcal{O}}_i = \begin{bmatrix} \hat{O}_i & \hat{O}_{in} \\ 0 & \hat{O}_n \end{bmatrix},$$

where O_n and \hat{O}_n are the observability matrices of (A_{nn}, C_{nn}) and $(\hat{A}_{nn}, \hat{C}_{nn})$, respectively. Since $\mathcal{O}_i T_i = \hat{\mathcal{O}}_i$, it follows that $O_n T_{in} = 0$. Since \mathcal{S}_i is a Kalman representation in causal block triangular form, (A_{nn}, C_{nn}) is observable and therefore, O_n is full row rank. Then $O_n T_{in} = 0$ implies $T_{in} = 0$. \square

Proof of Lemma 3.6. To help the readability, we recall the conditions that the statements were formulated on:

- (i) \mathbf{y}_1 does not Granger cause \mathbf{y}_3
- (ii) \mathbf{y}_2 does not Granger cause \mathbf{y}_3
- (iii) \mathbf{y}_1 conditionally does not Granger cause \mathbf{y}_2 with respect to \mathbf{y}_3
- (iv) \mathbf{y}_1 does not Granger cause $[\mathbf{y}_2^T, \mathbf{y}_3^T]^T$

We first prove that (i)-(ii)-(iii) implies (iv). By definition, (iv) means that

$$E_l \left[\begin{bmatrix} \mathbf{y}_2(t+k) \\ \mathbf{y}_3(t+k) \end{bmatrix} \middle| \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3} \right] = E_l \left[\begin{bmatrix} \mathbf{y}_2(t+k) \\ \mathbf{y}_3(t+k) \end{bmatrix} \middle| \mathcal{H}_{t-}^{\mathbf{y}_2, \mathbf{y}_3} \right]$$

for all $t, k \in \mathbb{Z}, k \geq 0$. Looking at the equation above by the components, by the first component it is equivalent to (iii) and by the second component it is equivalent to $E_l [\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3}] = E_l [\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_2, \mathbf{y}_3}]$. Since (iii) is assumed to hold, we only need to see the latter. Define $\alpha = \mathbf{y}_3(t+k) - E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_3}]$. From (i) and from (ii) we respectively obtain that

$$\alpha = \mathbf{y}_3(t+k) - E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_3}], \quad \alpha = \mathbf{y}_3(t+k) - E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_2, \mathbf{y}_3}].$$

Therefore, α is orthogonal to $\mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_3}$ and to $\mathcal{H}_{t-}^{\mathbf{y}_2, \mathbf{y}_3}$, thus also to their closed union, $\mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3}$. Hence, $E_l[\alpha \mid \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3}] = 0$, which is equivalent to that

$$E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3}] = E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_3}],$$

since $\mathcal{H}_{t-}^{\mathbf{y}_3} \subseteq \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3}$. Then, from (ii) we obtain that $E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_3}] = E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_2, \mathbf{y}_3}]$ and thus $E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3}] = E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_2, \mathbf{y}_3}]$. With this, (iv) follows.

Next, we show that (ii)-(iv) implies (i) and (iii). As we saw above, (iv) is equivalent to the condition (iii) and that $E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3}] = E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_2, \mathbf{y}_3}]$. Therefore, we only have to see that (ii)-(iv) implies (i). From (iii) and from (ii) we know that

$$\begin{aligned} E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3}] &= E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_2, \mathbf{y}_3}] \\ E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_2, \mathbf{y}_3}] &= E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_3}] \end{aligned}$$

and thus $E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3}] = E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_3}]$. Taking the projection of both part of the latter equation onto $\mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_3}$, we obtain that $E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_1, \mathbf{y}_3}] = E_l[\mathbf{y}_3(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_3}]$ which, by definition, gives (i). \square

3.B Proof of Theorem 3.5 and Corollary 3.8

To prove Theorem 3.5, we need an auxiliary result. For the sake of simplicity, a ZM-SIR process $[\mathbf{y}_{j_1}^T, \dots, \mathbf{y}_{j_k}^T]^T$ is shortened by $\mathbf{y}_{j_1, \dots, j_k}$ or by \mathbf{y}_J where $J = \{j_1, \dots, j_k\}$ and the Hilbert spaces generated by the present, past, and future values of \mathbf{y}_J are written by $\mathcal{H}_t^{\mathbf{y}_{j_1, \dots, j_k}}$, $\mathcal{H}_{t-}^{\mathbf{y}_{j_1, \dots, j_k}}$ and $\mathcal{H}_{t+}^{\mathbf{y}_{j_1, \dots, j_k}}$ or by $\mathcal{H}_t^{\mathbf{y}_J}$, $\mathcal{H}_{t-}^{\mathbf{y}_J}$ and $\mathcal{H}_{t+}^{\mathbf{y}_J}$, respectively.

Lemma 3.12. *Consider a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{y}_3^T, \mathbf{y}_4^T]^T$. Then \mathbf{y}_1 and \mathbf{y}_2 conditionally do not Granger cause \mathbf{y}_3 with respect to \mathbf{y}_4 if and only if $[\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ conditionally does not Granger cause \mathbf{y}_3 with respect to \mathbf{y}_4 .*

Proof. *Sufficiency:* By definition, the joint process $[\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ conditionally does not Granger cause \mathbf{y}_3 with respect to \mathbf{y}_4 if $E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}] = E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}}]$ for all $t, k \in \mathbb{Z}, k \geq 0$. By projecting both sides onto $\mathcal{H}_{t-}^{\mathbf{y}_{1,3,4}}$ and to $\mathcal{H}_{t-}^{\mathbf{y}_{2,3,4}}$ we have that

$$\begin{aligned} E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}] &= E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{1,3,4}}] \\ E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}] &= E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{2,3,4}}], \end{aligned} \quad (3.12)$$

which implies that \mathbf{y}_1 and \mathbf{y}_2 conditionally does not Granger cause \mathbf{y}_3 with respect to \mathbf{y}_4 .

Necessity: Define the process $\alpha(t+k) := \mathbf{y}_3(t+k) - E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}]$ for $t, k \in \mathbb{Z}, k \geq 0$. Then, $\alpha(t+k)$ is orthogonal to $\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}$ and from the Granger non-causality conditions we also know that $\alpha(t+k)$ is orthogonal to $\mathcal{H}_{t-}^{\mathbf{y}_{1,3,4}}$ and to $\mathcal{H}_{t-}^{\mathbf{y}_{2,3,4}}$. Therefore, $\alpha(t+k)$ is orthogonal to the sum of the subspaces $\mathcal{H}_{t-}^{\mathbf{y}_{1,3,4}} + \mathcal{H}_{t-}^{\mathbf{y}_{2,3,4}} = \mathcal{H}^{\mathbf{y}_{1,2,3,4}}$, thus to $\mathcal{H}_{t-}^{\mathbf{y}}$. By projecting $\alpha(t+k)$ onto $\mathcal{H}_{t-}^{\mathbf{y}}$ we obtain that $E_l[\alpha(t+k)|\mathcal{H}_{t-}^{\mathbf{y}}] = 0$ thus $E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}}] = E_l[\mathbf{y}_3(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{3,4}}]$, which by definition is that $[\mathbf{y}_1^T, \mathbf{y}_2^T]^T$ conditionally does not Granger cause \mathbf{y}_3 with respect to \mathbf{y}_4 . \square

Proof of Theorem 3.5. To start with, any Kalman representation in causal coordinated form is a Kalman representation in coordinated form, hence (v) \implies (vi) follows. We assume now that $\mathbf{y} = [\mathbf{y}_1^T, \dots, \mathbf{y}_n^T]^T$ is a ZMSIR process where $\mathbf{y}_i \in \mathbb{R}^{r_i}$, $r_i > 0, i = 1, \dots, n$ and we continue with the proof of the remaining implications.

(i) and (ii) \implies (v): Condition (i) and Theorem 2.5 imply the existence of minimal Kalman representations $(\hat{A}_i, \hat{K}_i, \hat{C}_i, I, \mathbf{e}_{i,n})$ of $\mathbf{y}_{i,n} = [\mathbf{y}_i^T, \mathbf{y}_n^T]^T, i = 1, \dots, n-1$ in causal block triangular form. Note that $\mathbf{e}_{i,n} = [\mathbf{e}_i^T, \mathbf{e}_n^T]^T$, where $\mathbf{e}_i \in \mathbb{R}^{r_i}, \mathbf{e}_n \in \mathbb{R}^{r_n}$, is the innovation process of $\mathbf{y}_{i,n}$. Furthermore, \mathbf{e}_n is the innovation process of \mathbf{y}_n . By using (i), (ii) and Lemma 3.12, we obtain that $\mathbf{y}_{\{1,2,\dots,n-1\} \setminus \{i\}}$ does not Granger cause \mathbf{y}_i with respect to \mathbf{y}_n and $\mathbf{y}_{\{1,2,\dots,n-1\}}$ does not Granger cause \mathbf{y}_n , i.e.,

$$\begin{aligned} \mathbf{e}_i(t) &= \mathbf{y}_i(t) - E_l[\mathbf{y}_i(t)|\mathcal{H}_{t-}^{\mathbf{y}_{i,n}}] = \mathbf{y}_i(t) - E_l[\mathbf{y}_i(t)|\mathcal{H}_{t-}^{\mathbf{y}}] \\ \mathbf{e}_n(t) &= \mathbf{y}_n(t) - E_l[\mathbf{y}_n(t)|\mathcal{H}_{t-}^{\mathbf{y}_n}] = \mathbf{y}_n(t) - E_l[\mathbf{y}_n(t)|\mathcal{H}_{t-}^{\mathbf{y}}]. \end{aligned}$$

It then follows that $\mathbf{e} = [\mathbf{e}_1^T, \dots, \mathbf{e}_n^T]^T$ is the innovation process of \mathbf{y} . Consider the partitions of \hat{A}_i for $\hat{K}_i, \hat{C}_i, i = 1, \dots, n-1$

$$\hat{A}_i = \begin{bmatrix} \hat{A}_{ii} & \hat{A}_{in} \\ 0 & \hat{A}_{i,nn} \end{bmatrix} \quad \hat{K}_i = \begin{bmatrix} \hat{K}_{ii} & \hat{K}_{in} \\ 0 & \hat{K}_{i,nn} \end{bmatrix} \quad \hat{C}_i = \begin{bmatrix} \hat{C}_{ii} & \hat{C}_{in} \\ 0 & \hat{C}_{i,nn} \end{bmatrix}$$

as in (3.7). Let $T_i, i = 2, \dots, n-1$ be the matrix as in (3.8) that transforms the Kalman representation $(\hat{A}_{i,nn}, \hat{K}_{i,nn}, \hat{C}_{i,nn}, I, \mathbf{e}_n)$ of \mathbf{y}_n into the isomorphic Kalman representation $(\hat{A}_{1,nn}, \hat{K}_{1,nn}, \hat{C}_{1,nn}, I, \mathbf{e}_n)$ of \mathbf{y}_n and define $T_1 = I$. Then define A_{in}, C_{in}, K_{in} for $i = 1, \dots, n-1$ as in (3.9) and A, K, C as in (3.1). Note that the stability of $\hat{A}_i, i = 1, \dots, n-1$ implies the stability of A . Then, (A, K, C, I, \mathbf{e}) is a Kalman representation of \mathbf{y} in coordinated form. Finally, since $(\hat{A}_i, \hat{K}_i, \hat{C}_i, I, \mathbf{e}_{i,n})$ is isomorphic with

$$\left(\begin{bmatrix} A_{ii} & A_{in} \\ 0 & A_{nn} \end{bmatrix}, \begin{bmatrix} K_{ii} & K_{in} \\ 0 & K_{nn} \end{bmatrix}, \begin{bmatrix} C_{ii} & C_{in} \\ 0 & C_{nn} \end{bmatrix}, I, \mathbf{e}_{i,n} \right) \quad (3.13)$$

by the isomorphism defined by the transformation matrix $\begin{bmatrix} I & 0 \\ 0 & T_i \end{bmatrix}$, it follows that the LTI-SS representation (3.13) is also a minimal Kalman representation of $\mathbf{y}_{i,n}$ in causal block triangular form for all $i = 1, \dots, n-1$. As a result, (A, K, C, I, \mathbf{e}) is a Kalman representation of \mathbf{y} in causal coordinated form.

(iii) \implies (iv): Consider the Kalman representation (A, K, C, I, \mathbf{e}) of \mathbf{y} in causal coordinated form which was constructed in the proof of (i) and (ii) \implies (v). First, we show that (A, C) is an observable pair thus $[C^T (A - \lambda I)^T]^T$ is full column rank for all $\lambda \in \mathbb{C}$. Minimality of (3.13) implies that (A_{ii}, C_{ii}) are observable pairs for $i = 1, \dots, n$ so that the matrices $[C_{ii}^T (A_{ii} - \lambda I)^T]^T$ are full column rank for all $\lambda \in \mathbb{C}$. Notice that $[C^T (A - \lambda I)^T]^T$ can be transformed, by permuting the rows of $[C^T (A - \lambda I)^T]^T$, into an upper block triangular form such that the diagonal blocks are $[C_{ii}^T (A_{ii} - \lambda I)^T]^T$ for $i = 1, \dots, n$. Hence, $[C^T (A - \lambda I)^T]^T$ is full column rank for all $\lambda \in \mathbb{C}$, which implies that (A, C) is observable.

Below, we prove that if condition (iii) holds, then (A, K) is controllable which is equivalent to that the components of the state process $\mathbf{x} = [\mathbf{x}_1^T, \dots, \mathbf{x}_n^T]^T$, which are consistent with (3.4) and (3.3), are linearly independent. Notice that the representations (3.13) are minimal Kalman representations. Hence, the components of $\mathbf{x}_{i,n} = [\mathbf{x}_i^T, \mathbf{x}_n^T]^T$ are linearly independent and $\mathcal{H}_t^{\mathbf{x}_i} \cap \mathcal{H}_t^{\mathbf{x}_n} = \{0\}$ for all $t \in \mathbb{Z}$ and $i = 1, \dots, n-1$. Therefore, for the linear independence of \mathbf{x} it is enough to show that $\dim(\mathcal{H}_t^{\mathbf{x}}) = \sum_{i=1}^n \dim(\mathcal{H}_t^{\mathbf{x}_i})$, where $\dim(\mathcal{H}_t^{\mathbf{x}_i})$ denotes the number of scalar components of a basis in the Hilbert space $\mathcal{H}_t^{\mathbf{x}_i}$ generated by the random variable $\mathbf{z}(t)$. Recall that the orthogonal complement of $B \subseteq \mathcal{H}_{t-}^{\mathbf{x}} + \mathcal{H}_{t+}^{\mathbf{x}}$ in $A \subseteq \mathcal{H}_{t-}^{\mathbf{x}} + \mathcal{H}_{t+}^{\mathbf{x}}$ is denoted

by $A \ominus B$. The linear independence of the components of $\mathbf{x}_{i,n}$ and $\mathcal{H}_t^{\mathbf{x}_i} \cap \mathcal{H}_t^{\mathbf{x}_n} = \{0\}$ imply that $\mathcal{H}_t^{\mathbf{x}_{i,n}} = (\mathcal{H}_t^{\mathbf{x}_i} \ominus \mathcal{H}_t^{\mathbf{x}_n}) \oplus \mathcal{H}_t^{\mathbf{x}_n}$ and $\dim(\mathcal{H}_t^{\mathbf{x}_i} \ominus \mathcal{H}_t^{\mathbf{x}_n}) = \dim(\mathcal{H}_t^{\mathbf{x}_i})$. Next, by using (3.5) in Theorem 3.5, we show that $\mathcal{H}_t^{\mathbf{x}}$ can be decomposed as

$$\mathcal{H}_t^{\mathbf{x}} = \mathcal{H}_t^{\mathbf{x}_n} \oplus ((\mathcal{H}_t^{\mathbf{x}_1} \ominus \mathcal{H}_t^{\mathbf{x}_n}) \dot{+} \dots \dot{+} (\mathcal{H}_t^{\mathbf{x}_{n-1}} \ominus \mathcal{H}_t^{\mathbf{x}_n})), \quad (3.14)$$

from which it follows that $\dim(\mathcal{H}_t^{\mathbf{x}}) = \sum_{i=1}^n \dim(\mathcal{H}_t^{\mathbf{x}_i})$, i.e., the components of $\mathbf{x}(t)$ are linearly independent. Define the process $Y_i(t) := [\mathbf{y}_i^T(t), \dots, \mathbf{y}_i^T(t+N-1)]^T$ for $i = 1, 2, \dots, n$ and notice that $Y_i(t)$ spans $\mathcal{H}_{t+}^{\mathbf{y}_i}$. Consider the observability matrix

$$O_i^N := [C_{ii}^T (C_{ii}A_{ii})^T \dots (C_{ii}A_{ii}^{N-1})^T]^T$$

of (A_{ii}, C_{ii}) , $i = 1, \dots, n$ where $N \geq \dim(\mathbf{x})$ and notice that because of the causal coordinated form of (A, K, C, I, \mathbf{e}) we have the following equations with an appropriate M matrix:

$$E_l[Y_n(t)|\mathcal{H}_{t-}^{\mathbf{y}_n}] = O_n^N \mathbf{x}_n(t), \quad E_l[Y_i(t)|\mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}] = O_i^N \mathbf{x}_i(t) + M \mathbf{x}_n(t).$$

for $i = 1, \dots, n-1$. It implies that $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}] \subseteq \mathcal{H}_t^{\mathbf{x}_n}$ and $E_l[\mathcal{H}_{t+}^{\mathbf{y}_i} | \mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}] \subseteq \mathcal{H}_t^{\mathbf{x}_{i,n}}$. Since (A_{ii}, C_{ii}) is an observable pair, O_i^N has left inverse and we also have that $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}] \supseteq \mathcal{H}_t^{\mathbf{x}_n}$ and $E_l[\mathcal{H}_{t+}^{\mathbf{y}_i} | \mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}] + \mathcal{H}_t^{\mathbf{x}_n} \supseteq \mathcal{H}_t^{\mathbf{x}_i}$, $i = 1, \dots, n$. Hence,

$$\mathcal{H}_t^{\mathbf{x}_n} = E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}], \quad \mathcal{H}_t^{\mathbf{x}_{i,n}} = E_l[\mathcal{H}_{t+}^{\mathbf{y}_i} | \mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}] + E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}]. \quad (3.15)$$

Notice that

$$E_l[\mathcal{H}_{t+}^{\mathbf{y}_i} | \mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}] \cap E_l[\mathcal{H}_{t+}^{\mathbf{y}_j} | \mathcal{H}_{t-}^{\mathbf{y}_j, \mathbf{y}_n}] | E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}] = \{0\} \iff \\ (E_l[\mathcal{H}_{t+}^{\mathbf{y}_i} | \mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}] + E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}]) \cap (E_l[\mathcal{H}_{t+}^{\mathbf{y}_j} | \mathcal{H}_{t-}^{\mathbf{y}_j, \mathbf{y}_n}] + E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}]) | E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}] = \{0\}$$

and similarly, that $\mathcal{H}_t^{\mathbf{x}_i} \cap \mathcal{H}_t^{\mathbf{x}_j} | \mathcal{H}_t^{\mathbf{x}_n} = \{0\} \iff \mathcal{H}_t^{\mathbf{x}_{i,n}} \cap \mathcal{H}_t^{\mathbf{x}_{j,n}} | \mathcal{H}_t^{\mathbf{x}_n} = \{0\}$. By using the equations (3.15), we obtain that the condition (3.5) in Theorem 3.5 is equivalent to $\mathcal{H}_t^{\mathbf{x}_i} \cap \mathcal{H}_t^{\mathbf{x}_j} | \mathcal{H}_t^{\mathbf{x}_n} = \{0\}$ which implies that $\mathcal{H}_t^{\mathbf{x}}$ can be decomposed as in (3.14). Hence, the components of \mathbf{x} are linearly independent and thus (A, K) is controllable. By Proposition 1.10, the observability of (A, C) and the controllability of (A, K) implies the minimality of the Kalman representation $(A, K, C, I, \mathbf{e}, \mathbf{y})$.

(v) \implies (i) and (ii): Assume that (A, K, C, I, \mathbf{e}) is a Kalman representation of \mathbf{y} in causal coordinated form where the matrices A, K, C are as in (3.1). Then, by definition, (3.2) is a minimal Kalman representation of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ in causal block triangular form. Hence, by Theorem 2.5, \mathbf{y}_i does not Granger cause \mathbf{y}_n for $i = 1, \dots, n-1$ and thus condition (i) holds. By Remark 3.7, (ii) is equivalent to saying that \mathbf{y}_j does

not Granger cause $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ for all $i, j = 1, \dots, n-1, i \neq j$. By Lemma 3.12, this is equivalent to saying that $[\mathbf{y}_1^T, \dots, \mathbf{y}_{i-1}^T, \mathbf{y}_{i+1}^T, \dots, \mathbf{y}_{n-1}^T]^T$ does not Granger cause $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$, which is further equivalent to $[\mathbf{e}_i^T, \mathbf{e}_n^T]^T$ being the innovation process of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ due (Dufour and Renault, 1998, Proposition 2.3), where $\mathbf{e} = [\mathbf{e}_1^T, \dots, \mathbf{e}_n^T]^T$ such that $\mathbf{e}_i \in \mathbb{R}^{r_i}, i = 1, \dots, n$. Since (3.2) is a minimal Kalman representation of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$, by Definition 3.1, $[\mathbf{e}_i^T, \mathbf{e}_n^T]^T$ is indeed the innovation process of $[\mathbf{y}_i^T, \mathbf{y}_n^T]^T$ and thus condition (ii) follows.

(iv) \implies (iii): Let (A, K, C, I, \mathbf{e}) be a minimal Kalman representation of \mathbf{y} in causal coordinated form and assume that A, K, C satisfy (3.1). Then for $k \geq 0$

$$\begin{aligned} E_l[\mathbf{y}_n(t+k) | \mathcal{H}_{t-}^{\mathbf{y}_n}] &= C_{nn} A_{nn}^k \mathbf{x}_n(t) \\ E_l[\mathbf{y}_i(t+k) | \mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}] &= C_{ii} A_{ii}^k \mathbf{x}_i(t) + C_{in} A_{nn}^k \mathbf{x}_n(t), \end{aligned} \quad (3.16)$$

from which $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}] \subseteq \mathcal{H}_t^{\mathbf{x}_n}$ and $E_l[\mathcal{H}_{t+}^{\mathbf{y}_i} | \mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}] \subseteq \mathcal{H}_t^{\mathbf{x}_i, \mathbf{x}_n}$. Since the Kalman representation $(A_{nn}, K_{nn}, C_{nn}, I, \mathbf{e}_n, \mathbf{y}_n)$ is minimal, the pair (A_{nn}, C_{nn}) is observable. Define the (finite) observability matrix of (A_{nn}, C_{nn}) by

$$O_n^N := [C_{nn}^T (C_{nn} A_{nn})^T \dots (C_{nn} A_{nn}^{N-1})^T]^T,$$

where $N \geq \dim(\mathbf{x}_n)$. Then from (3.16) we obtain that $E_l[[Y_n(t) | \mathcal{H}_{t-}^{\mathbf{y}_n}] = O_n^N \mathbf{x}_n(t)$, where $Y_n(t) := [\mathbf{y}_n^T(t), \dots, \mathbf{y}_n^T(t+N-1)]^T$. From the observability of (A_{nn}, C_{nn}) we know that O_n^N has left inverse and thus $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}] = \mathcal{H}_t^{\mathbf{x}_n}$. Since the Kalman representation $(A, K, C, I, \mathbf{e}, \mathbf{y})$ is minimal, the components of $\mathbf{x}(t)$ are linearly independent for each $t \in \mathbb{Z}$. In particular, this means that $\mathcal{H}_t^{\mathbf{x}_i, j} \cap \mathcal{H}_t^{\mathbf{x}_n} = \{0\}$ and $\mathcal{H}_t^{\mathbf{x}_i} \cap \mathcal{H}_t^{\mathbf{x}_j} = \{0\}$ for $i, j = 1, \dots, n-1, i \neq j$. In turn, this implies that $\mathcal{H}_t^{\mathbf{x}_i, n} \cap \mathcal{H}_t^{\mathbf{x}_j, n} | \mathcal{H}_t^{\mathbf{x}_n} = \{0\}$. By combining it with $E_l[\mathcal{H}_{t+}^{\mathbf{y}_i} | \mathcal{H}_{t-}^{\mathbf{y}_i, \mathbf{y}_n}] \subseteq \mathcal{H}_t^{\mathbf{x}_i, \mathbf{x}_n}$ and $E_l[\mathcal{H}_{t+}^{\mathbf{y}_n} | \mathcal{H}_{t-}^{\mathbf{y}_n}] = \mathcal{H}_t^{\mathbf{x}_n}$ we can conclude that condition (3.5) in Theorem 3.5 holds.

(vi) \implies (i) and (ii) if \mathbf{y} is coercive: Assume that (A, K, C, I, \mathbf{e}) is a Kalman representation of \mathbf{y} in coordinated form satisfying (3.1). Since \mathbf{y} is coercive,

$$\mathbf{e}(t) = \mathbf{y}(t) - \sum_{k=1}^{\infty} C(A - KC)^{k-1} K \mathbf{y}(t-k).$$

From (3.1) it is easy to see that for any $k \geq 1$,

$$C(A - KC)^{k-1} K = \left[\begin{array}{cccc|c} M_{k,11} & 0 & \cdots & 0 & M_{k,1n} \\ 0 & M_{k,22} & \cdots & 0 & M_{k,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & M_{k,(n-1)(n-1)} & M_{k,(n-1)n} \\ \hline 0 & 0 & \cdots & 0 & M_{k,nn} \end{array} \right],$$

where $M_{k,ii} = C_{ii}(A_{ii} - K_{ii}C_{ii})^{k-1}K_{ii}$, $i = 1, \dots, n$ and $M_{k,in}$ are suitable matrices for $i = 1, \dots, n - 1$. Hence,

$$\mathbf{e}_n(t) = \mathbf{y}_n(t) - \sum_{k=1}^{\infty} M_{k,nn}\mathbf{y}_n(t-k)$$

$$\mathbf{e}_i(t) = \mathbf{y}_i(t) - \sum_{k=1}^{\infty} M_{k,ii}\mathbf{y}_i(t-k) + M_{k,in}\mathbf{y}_n(t-k),$$

where $\mathbf{e}_i(t) = \mathbf{y}_i(t) - E_l[\mathbf{y}_i(t) | \mathcal{H}_{t-}^{\mathbf{y}}]$ for $i = 1, \dots, n$. This implies that $E_l[\mathbf{y}_n(t) | \mathcal{H}_{t-}^{\mathbf{y}}] \subseteq \mathcal{H}_{t-}^{\mathbf{y}_n}$ and $E_l[\mathbf{y}_i(t) | \mathcal{H}_{t-}^{\mathbf{y}}] \subseteq \mathcal{H}_{t-}^{\mathbf{y}_i, n}$. Therefore, $E_l[\mathbf{y}_n(t) | \mathcal{H}_{t-}^{\mathbf{y}}] = E_l[\mathbf{y}_n(t) | \mathcal{H}_{t-}^{\mathbf{y}_n}]$ and $E_l[\mathbf{y}_i(t) | \mathcal{H}_{t-}^{\mathbf{y}}] = E_l[\mathbf{y}_i(t) | \mathcal{H}_{t-}^{\mathbf{y}_i, n}]$. It follows from (Dufour and Renault, 1998, Proposition 2.3) that \mathbf{y}_i does not Granger cause \mathbf{y}_n , and \mathbf{y}_j does not Granger cause $\mathbf{y}_{i,n}$ for all $i, j \in \{1, \dots, n-1\}$, $i \neq j$. In view of Remark 3.7 this is equivalent to (i) and (ii). \square

Proof of Corollary 3.8. Consider the LTI-SS representation (A, K, C, I, \mathbf{e}) defined by (3.7), (3.8), and (3.9) before Corollary 3.8. Then (A, K, C, I, \mathbf{e}) coincides with the Kalman representation defined in the proof of (i) and (ii) \implies (v) of Theorem 3.5. Hence, the first statement of Corollary 3.8 is a consequence of the implication (i) and (ii) \implies (v) of Theorem 3.5. Similarly, the second statement of Corollary 3.8 is a direct consequence of the implication (iii) \implies (iv) of Theorem 3.5. \square