Toward controlled ultra-high vacuum chemical vapor deposition processes

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Chapter 6

Shape Control Problem

In this chapter, we introduce the shape control problem (SCP) and solve it for linear systems. In particular, we provide solutions for cases where the initial and the desired probability density function (pdf) of the state are either linearly matching or nonlinearly matching. We show the efficacy of our solution for the linearly matching case through a simulation study. We continue by considering the problem in the general setting which we have also considered for the containment control problem (CCP).

The remainder of this chapter is structured as follows. In Section 6.1 we introduce the dynamical system equations, candidate transient specifications and the SCP formulation. We use Section 6.2 to present a solution to SCP for linear systems where the initial and target pdf are linearly matching. This solution is subsequently expanded in Section 6.3 to initial and target pdfs that are nonlinearly matching. We then use Section 6.4 to numerically evaluate our result for linearly matching pdfs. Lastly, we conclude the chapter with some remarks in Section 6.5.

6.1 Shape control problem definition

We will use this section to formally present the SCP. As in Chapter 5, we start with the dynamical system equations, this is then followed by candidate transient specifications. The selected transient specification is subsequently implemented in the SCP formulation. We do not explicitly show the nontriviality of the SCP, since it is based on the same arguments as the nontriviality for the CCP.

6.1.1 Dynamical system equations

Consider the general dynamical system given by

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0, \tag{6.1}$$

where $x \in X \subseteq \mathbb{R}^n$, $u \in U \subseteq R^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let us assume that x_0 is a known random variable defined on $X_0 \subset X$, satisfying a probability density function (pdf) $\phi_{x_0} : X_0 \to \mathbb{R}_{\geq 0}$. In this case, its forward solution $x(t)$ is random variable for all $t > 0$ and we denote the propagation of ϕ_{x_0} along (6.1) by $\phi_{x_0,t}$. Note that the usual setting where x_0 is deterministic is a particular class of this class of system where ϕ_{x_0} is simply given by a Dirac delta function.

6.1.2 Candidate transient specifications

For defining transient behavior specification corresponding to the evolution of $\phi_{x_0,t}$, there are two possibilities in defining the measure. For the first one, we can relate $\phi_{x_0,t}$ at a terminal time T or in an interval $[0, T]$ to a desired point (or desired set) or to a desired trajectory $x_d(t)$ defined on the time interval $[0, T]$, respectively. For the second one, we can relate $\phi_{x_0,t}$ to a (dynamic or stationary) target distribution. We have focused on the former for the CCP. For the SCP, we focus on the latter.

When considering (dis)similarity between two density functions, we can consider measures that are given by distances or divergences such as the Hellinger distance, Bhattacharyya distance, Kullback-Leibler divergence and Jeffrey's divergence (Ali and Silvey 1966, Kullback 1997, Kailath 1967). A distance deserves preference over a divergence, since it produces the desired scalar-valued output. Note that the Bhattacharyya distance and the Hellinger distance are related to each other through the Bhattacharyya coefficient $\mathcal{B}$ (Buehler et al. 2016, Abou-Moustafa and Ferrie 2012). If ϕ_d denotes the target distribution at terminal time T, the Bhattacharyya coefficient is given by

$$\mathcal{B}(\phi_d, \phi_{x_0,T}) = \int_X \sqrt{\phi_d(\xi)\phi_{x_0,T}(\xi)}\mathrm{d}\xi. \tag{6.2}$$

If both distributions are equal then it will give Bhattacharyya coefficient of 1 and if they are dissimilar then the Bhattacharyya coefficient will be close to 0. Using this coefficient, the Hellinger distance is defined by

$$d_h(\phi_d, \phi_{x_0,T}) = \sqrt{1 - \mathcal{B}(\phi_d, \phi_{x_0,T})}, \tag{6.3}$$

and, correspondingly, the Bhattacharyya distance is given by

$$d_b(\phi_d, \phi_{x_0,T}) = -\ln(\mathcal{B}(\phi_d, \phi_{x_0,T})). \tag{6.4}$$

The two measures are relatively similar. However, the Hellinger distance is a proper metric, as discussed in (Abou-Moustafa and Ferrie 2012), while the Bhattacharyya distance is only a semi-metric because it does not satisfy the triangle inequality. Our distance of choice is therefore the Hellinger distance (6.3).

6.1.3 Shape control problem formulation

We are now ready to define the shape control problem, based on Hellinger distance as given before.

Shape Control Problem (SCP): For the system in (6.1), given a desired pdf ϕ_d, a transient time T, a distance $d(\cdot,\cdot)$, a desired Hellinger distance $\ell \in [0, \infty)$ and a target trajectory x_d, design a control law $u(t) = k(x(t), t)$ such that

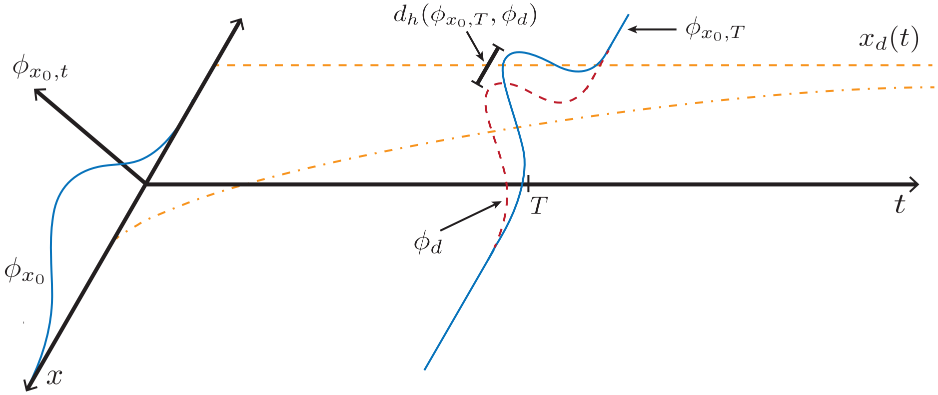

Figure 6.1: Shape control problem illustration

This figure illustrates the control objective of the SCP. The initial distribution ϕ_{x_0} changes with time, such that the distribution at time T, denoted as $\phi_{x_0,T}$, has a Hellinger distance $d_h(\phi_{x_0,T}, \phi_d)$ w.r.t. a desired distribution ϕ_d. Furthermore, all possible initial values should converge to $x_d(t)$, which can result in the indicated trajectory for a specific initial state of ϕ_{x_0}.

SCPa: $d_h(\phi_{x_0,T}, \phi_d) \leq \ell$

SCPb: $\lim_{t\to\infty} d(x(t), x_d(t)) = 0.$ △

Similar to the structure of the CCP in Chapter 5, SCPa is the realization of the transient performance criteria related to the Hellinger distance by requiring it to be smaller than ℓ. This condition is again complemented by the asymptotic convergence criterion as expressed in SCPb. The control objective for the SCP is illustrated in Fig. 6.1.

6.2 SCP for linearly matching pdfs

Before we continue, let us formally define our notion of (linearly) matching pdfs. This definition is based on matching procedures that are used in image processing applications (Inamdar et al. 2008, Shapiro and Stockman 2001), which are typically performed through well-known operations of rotations, scaling, translation, shearing and/or reflections (Shapiro and Stockman 2001).

Definition 3. *(Linearly matching probability density functions) For a given $Y \subset \mathbb{R}^n$, we call two pdfs $\phi : Y \to \mathbb{R}_{\geq 0}$ and $\varphi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ linearly matching with respect to Y if there exist $\eta \in \mathbb{R}^{n\times 1}$, $\beta \in \mathbb{R}^{n\times n}$ and $\varepsilon \in \mathbb{R}$ so that*

$$\phi(x) = \varepsilon\varphi(\beta x + \eta) \tag{6.5}$$

holds for all $x \in Y$.

We note that the linearity refers to the application of a linear affine state transformation for matching both nonlinear maps ϕ and φ. For such linearly matching pdfs, we are now ready to present our controller design that can solve the SCP, with a bounded ℓ. This bound is then dependent on the matching of the pdfs (expressed through ε, β and η) and the system equations.

Theorem 4. *Consider the system as in* (6.1). *Let $T > 0$ be the given transient terminal time and x_d be the desired trajectory. Assume that the pair (A, B) is controllable and there exists a finite $\tau > T$ such that x_d is a solution to* (6.1) *(with an admissible input signal $u_d(t)$) for all $t \geq \tau$. Suppose that: (i) the target distribution at time $T > 0$ is given by ϕ_d and that ϕ_{x_0} and ϕ_d are matching with respect to X_0 for some $\eta \in \mathbb{R}^{n\times 1}$, invertible $\beta \in \mathbb{R}^{n\times n}$ and $\varepsilon \in \mathbb{R}$ and (ii) there exists a finite $\tau > T$, such that $x_d(t)$ is a solution to the system* (6.1), *with an admissible input signal $u_d(t)$, for all $t \geq \tau$. Then, the SCP is solvable for ℓ and $K \in \mathbb{R}^{n\times m}$ satisfying*

$$\ell \geq \min_{\{K|spec(A+BK)\in\mathbb{C}_-\}} \sqrt{1 - \int_X \sqrt{\frac{\phi_{x_0}(\tilde{\beta}^{-1}(\xi - \tilde{\eta}))\phi_{x_0}(\beta^{-1}(\xi - \eta))}{\tilde{\varepsilon}\varepsilon}} \mathrm{d}\xi}, \tag{6.6}$$

where

$$\tilde{\varepsilon} = \int_X \phi_{x_0}(\tilde{\beta}^{-1}(\xi - \tilde{\eta}))\mathrm{d}\xi, \tag{6.7}$$

$$\tilde{\beta} = e^{(A+BK)T}, \tag{6.8}$$

$$\tilde{\eta} = \mu_d - e^{(A+BK)T}\mu, \tag{6.9}$$

with μ the mean value of ϕ_{x_0} and μ_d the mean value of ϕ_d.

PROOF. We will first prove the fulfillment of SCPa. Consider the control law $u(t) = K(x(t) - x_r(t)) + u^*$. We define $x_r(t)$ and $u^*(t)$ with the following properties: (i) $x_r(t) = x_d(t)$, for all $t \geq \tau$, (ii) $x_r(0) = \mu$, (iii) $x_r(T) = \mu_d$ and (iv) $\dot{x}_r(t) = Ax_r(t) + Bu^*(t)$. Since the pair (A, B) is controllable, we can always find a control signal u^* that can bring the state from μ at time 0 to μ_d at time T, and subsequently, to $x_d(\tau)$ at τ. Additionally, since $x_d(t)$ is a solution to (6.1) for $u_d(t)$ and $t \geq \tau$, we can let $u^*(t) = u_d(t)$ for $t \geq \tau$. For this control system, we define an error like signal as $\zeta(t) = x(t) - x_r(t)$. Also, due to ϕ_{x_0} and ϕ_d being matching, we have that $\varepsilon^{-1}\phi_{x_0}(x) = \phi_d(\beta x + \eta)$, for all $x \in X_0$. Let us then write a similar identity for $\phi_{x_0,T}$ as

$$\tilde{\varepsilon}^{-1}\phi_{x_0}(x(0)) = \phi_{x_0,T}(\tilde{\beta}x(0) + \tilde{\eta}). \tag{6.10}$$

Substituting our choices of $\tilde{\beta}$ and $\tilde{\eta}$ as given by (6.8) and (6.9) yields

$$\tilde{\varepsilon}^{-1}\phi_{x_0}(x(0)) = \phi_{x_0,T}\left(e^{(A+BK)T}\zeta(0) + \mu_d\right). \tag{6.11}$$

Notice that we furthermore have $x(T) = x_r(T) + \zeta(T)$ which by design satisfies the solution

$$x(T) = e^{(A+BK)T}\zeta(0) + \mu_d, \tag{6.12}$$

and we hence have

$$\tilde{\varepsilon}^{-1}\phi_{x_0}(x(0)) = \phi_{x_0,T}(x(T)) = \phi_{x_0,T}(\tilde{\beta}x(0) + \tilde{\eta}), \tag{6.13}$$

for all $x(0) \in X_0$. Notice that $\phi_{x_0,T}$ is accordingly the pdf of the state at time T for the closed loop system. We have thus obtained that $\tilde{\varepsilon}^{-1}\phi_{x_0}(x) = \phi_{x_0,T}(\tilde{\beta}x + \tilde{\eta})$ and $\varepsilon^{-1}\phi_{x_0}(x) = \phi_d(\beta x + \eta)$ hold for all $x \in X_0$. Subsequently, we can define two coordinate transformations $\tilde{y} = \tilde{\beta}x + \tilde{\eta}$ and $y = \beta x + \eta$ whose inverses are given by $x = \tilde{\beta}^{-1}(y - \tilde{\eta})$ and $x = \beta^{-1}(z - \eta)$, respectively. Accordingly, we obtain

$$\tilde{\varepsilon}^{-1}\phi_{x_0}(\tilde{\beta}^{-1}(\tilde{y} - \tilde{\eta})) = \phi_{x_0,T}(\tilde{y}), \tag{6.14}$$
$$\varepsilon^{-1}\phi_{x_0}(\beta^{-1}(y - \eta)) = \phi_d(y). \tag{6.15}$$

The inverse of $\tilde{\beta}$ always exists and we assume $\tilde{y}, y \in X$ for all $x \in X_0$. It then follows directly from (6.7) that $\phi_{x_0,T}$ satisfies

$$\int_X \phi_{x_0,T}(\xi)\mathrm{d}\xi = \frac{1}{\tilde{\varepsilon}}\int_X \phi_{x_0}(\tilde{\beta}^{-1}(\xi - \tilde{\eta}))\mathrm{d}\xi = 1. \tag{6.16}$$

Plugging (6.14) and (6.15) in (6.3) yields

$$d_h(\phi_{x_0,T}, \phi_d) = \sqrt{1 - \int_X \sqrt{\frac{\phi_{x_0}(\tilde{\beta}^{-1}(\xi - \tilde{\eta}))\phi_{x_0}(\beta^{-1}(\xi - \eta))}{\tilde{\varepsilon}\varepsilon}}\mathrm{d}\xi}. \tag{6.17}$$

Accordingly, for our simple choices of the control law, reference signal x_r and control signal u^*, we can always find a matrix K such that we satisfy SCPa for a maximum distance ℓ satisfying (6.6).

We are now left to prove SCPb. The proof for the asymptotic convergence to the reference signal follows directly from the design restriction on K that requires $(A + BK)$ to be Hurwitz. Additionally, since we have that $x_r(t) = x_d(t)$ for $t \geq \tau$, the asymptotic property holds:

$$\lim_{t\to\infty} \zeta(t) = 0 \Rightarrow \lim_{t\to\infty} d_E(x(t), x_d(t)) = 0. \tag{6.18}$$

In other words, SCPb holds. This concludes the proof. □

As shown in Theorem 4, for given initial and target distributions ϕ_{x_0} and ϕ_d that are matching, there is a lower bound on achievable ℓ. However, there are cases when the lower bound is equal to zero for some specific combinations of A, B, ϕ_{x_0}, ϕ_d and T. In the following corollary, we show a particular example of such case.

Corollary 2. *Assume that the hypotheses in Theorem 4 holds. Suppose that there exists a $K \in \mathbb{R}^{n\times m}$ such that* (6.8) *and* (6.9) *satisfy $\tilde{\beta} = \beta$ and $\tilde{\eta} = \eta$. Then, the **SCP** is solvable for any $\ell \geq 0$.*

PROOF. The result follows directly from (6.6) and (6.16), when $\tilde{\varepsilon} = \varepsilon$. Rewriting (6.16) for ϕ_d yields

$$\int_X \phi_d(\xi)\mathrm{d}\xi = \frac{1}{\varepsilon}\int_X \phi_{x_0}(\beta^{-1}(\xi - \eta))\mathrm{d}\xi = 1, \tag{6.19}$$

where $\beta = \tilde{\beta}$ and $\eta = \tilde{\eta}$ which also implies that $\varepsilon = \tilde{\varepsilon}$. Hence, (6.6) reduces to

$$\ell \geq \sqrt{1 - \frac{1}{\varepsilon}\int_X \phi_{x_0}(\beta^{-1}(\xi - \eta))\mathrm{d}\xi} = 0. \tag{6.20}$$

Accordingly, SCPa holds for $\ell \geq 0$. The property SCPb follows from Theorem 4. □

6.3 SCP for nonlinearly matching pdfs

In this section, we will propose a solution to the SCP for cases when the initial and the desired pdfs are nonlinearly matching. In this case, the results from the previous subsection can be extended to the situation when there exists nonlinear mappings that gives the relation between both pdfs.

Definition 4. *(Nonlinearly matching probability density functions) For a given $Y, Z \subset \mathbb{R}^n$, we call two pdfs $\phi : Y \to \mathbb{R}_{\geq 0}$ and $\varphi : Z \to \mathbb{R}_{\geq 0}$ nonlinearly matching with respect to the tuple (Y, Z) if there exist a diffeomorphic mapping $\Psi : Y \to Z$, a function $\vartheta : Z \to \mathbb{R}_{>0}$ such that*

$$\phi(x) = \vartheta\big(\Psi(x)\big)\varphi\big(\Psi(x)\big) \tag{6.21}$$

holds for all $x \in Y$.

Since we will later use Ψ in the coordinate transformation, the function ϑ becomes a normalizing function that corrects for the elongation of the pdf in the transformed state space via the mapping Ψ. In this case, when we consider $Y = X$ (with X being the original state space domain) in the above definition, we have

$$\int_W \phi(\xi)\mathrm{d}\xi = \int_W \delta(\Psi(\xi))\varphi(\Psi(\xi))\mathrm{d}\xi, \tag{6.22}$$

holds for all $W \subseteq X$. We are now ready to present our result for nonlinearly matching pdfs.

Theorem 5. *Assume the hypothesis of Theorem 4 holds. Suppose that: (i) the pdfs* ϕ_{x_0} *and* ϕ_d *are nonlinearly matching for a diffeomorphic map* $\Psi : X \to X$, $\vartheta : X \to \mathbb{R}_{\geq 0}$ *and (ii) there exists a finite* $\tau > T$*, such that* $x_d(t)$ *is a solution to the system* (6.1) *for all* $t \geq \tau$ *and for a corresponding admissible input signal* $u_d(t)$*. Then, the SCP is solvable for* ℓ *and* $K \in \mathbb{R}^{n\times m}$ *satisfying*

$$\ell \geq \min_{\{K|spec(A+BK)\in\mathbb{C}_-\}} \sqrt{1 - \int_X \sqrt{\frac{\phi_{x_0}(\tilde{\beta}^{-1}(\xi-\tilde{\eta}))\phi_{x_0}(\Psi^{-1}(\xi))}{\tilde{\varepsilon}\vartheta(\xi)}}\mathrm{d}\xi}, \tag{6.23}$$

where

$$\tilde{\varepsilon} = \int_X \phi_{x_0}(\tilde{\beta}^{-1}(\xi-\tilde{\eta}))\mathrm{d}\xi, \tag{6.24}$$

$$\tilde{\beta} = e^{(A+BK)T}, \tag{6.25}$$

$$\tilde{\eta} = \mu_d - e^{(A+BK)T}\mu, \tag{6.26}$$

with μ *the mean value of* ϕ_{x_0} *and* μ_d *the mean value of* ϕ_d*.*

PROOF. The proof follows the same lines as the proof for Theorem 5. We have a controllable linear system with two nonlinearly matching pdfs ϕ_{x_0} and ϕ_d. Subsequently, we again consider $u(t) = K(x(t) - x_r(t)) + u^*$ and design $x_r(t)$ and u^* as in Theorem 4. Accordingly, (6.14) and (6.15) become

$$\tilde{\varepsilon}^{-1}\phi_{x_0}(\tilde{\beta}^{-1}(\tilde{y}-\tilde{\eta})) = \phi_{x_0,T}(\tilde{y}), \tag{6.27}$$

$$\vartheta(y)^{-1}\phi_{x_0}(\Psi^{-1}(y)) = \phi_d(y). \tag{6.28}$$

Using these substitutions, we obtain

$$d_h(\phi_{x_0,T},\phi_d) = \sqrt{1 - \int_X \sqrt{\frac{\phi_{x_0}(\tilde{\beta}^{-1}(\xi-\tilde{\eta}))\phi_{x_0}(\Psi^{-1}(\xi))}{\tilde{\lambda}\vartheta(\xi)}}\mathrm{d}\xi}. \tag{6.29}$$

And hence, we can always find a K to achieve a maximum distance ℓ as in (6.23), and thus satisfying SCPa. Furthermore, since K is such that $(A + BK)$ is Hurwitz, we obtain

$$\lim_{t\to\infty} \zeta(t) = 0 \Rightarrow \lim_{t\to\infty} d_E(x(t), x_r(t)) = 0. \tag{6.30}$$

Furthermore, since $x_d(t)$ is a solution to (6.1) for input $u_d(t)$, we can let $x_r(t) = x_d(t)$ for all $t \geq \tau$ and we hence also have $\lim_{t\to\infty} d_E(x(t), x_d(t)) = 0$. SCPb thus holds. This concludes the proof. □

6.4 Numerical evaluation of SCP controller for matching pdfs

In this section we will numerically evaluate the result of Theorem 4. The example will show that we can only solve the SCP for $\ell \geq 0$ when the initial pdf ϕ_{x_0} and the desired pdf ϕ_d have very specific (linear) matching properties.

Let us consider the classical second order mass-spring system with unitary parameters given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \qquad x(0) = x_0 \tag{6.31}$$

where x_0 satisfies the pdf $\phi_{x_0} = \mathcal{N}(\mu, \Sigma)$, a normal distribution, with

$$\mu = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} 0.7 & -0.4 \\ -0.4 & 0.3 \end{bmatrix} \tag{6.32}$$

and $X = \mathbb{R}^2$. Let us furthermore define a relevant transient time $T = 5$ and a desired pdf $\phi_d = \mathcal{N}(\mu_d, \Sigma_d)$, with

$$\mu_d = \begin{bmatrix} 0.97 \\ 3.16 \end{bmatrix}, \qquad \Sigma_d = \begin{bmatrix} 0.0004 & 0.0012 \\ 0.0012 & 0.0049 \end{bmatrix}. \tag{6.33}$$

Then, ϕ_{x_0} and ϕ_d satisfy the linear matching property $\phi_{x_0}(x) = \varepsilon\phi_d(\beta x + \eta)$, with

$$\eta = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 0.01 & -0.02 \\ 0.1 & 0.03 \end{bmatrix}, \qquad \lambda = \sqrt{\frac{|\Sigma_d|}{|\Sigma|}}. \tag{6.34}$$

Indeed, let us consider the desired pdf as

$$\phi_d(y) = \frac{1}{2\pi\sqrt{|\Sigma_d|}} \exp(-\frac{1}{2}(z - \mu_d)^\top \Sigma_d^{-1}(y - \mu_d)), \tag{6.35}$$

with $y = \eta + \beta x$. Substituting y for x and letting $\mu_d = \eta + \beta\mu$ yields

$$\phi_d(\beta x + \eta) = \frac{1}{2\pi\sqrt{|\Sigma_d|}} \exp(-\frac{1}{2}(x - \mu)^\top \beta^\top \Sigma_d^{-1} \beta(x - \mu)) \tag{6.36}$$

and we thus have the relations $\mu_d = \eta + \beta\mu$ and $\Sigma_d = \beta\Sigma\beta^\top$. Substituting Σ_d and multiplying with λ then yields the matching property.

Let us furthermore assume that we have a reference signal $x_r(t)$ and a feedforward input u^*, satisfying (i) $x_r(t) = x_d(t)$ for all $t \geq \tau$, with $\tau > T$, (ii) $x_r(0) = \mu$, (iii) $x_r(T) = \mu_d$ and (iv) $\dot{x}_r = Ax_r(t) + Bu^*$. For this system, we then satisfy the conditions of Theorem 4 and we can therefore numerically evaluate the presented insights. Accordingly, we can express the minimal attainable distance ℓ through (6.6). As in

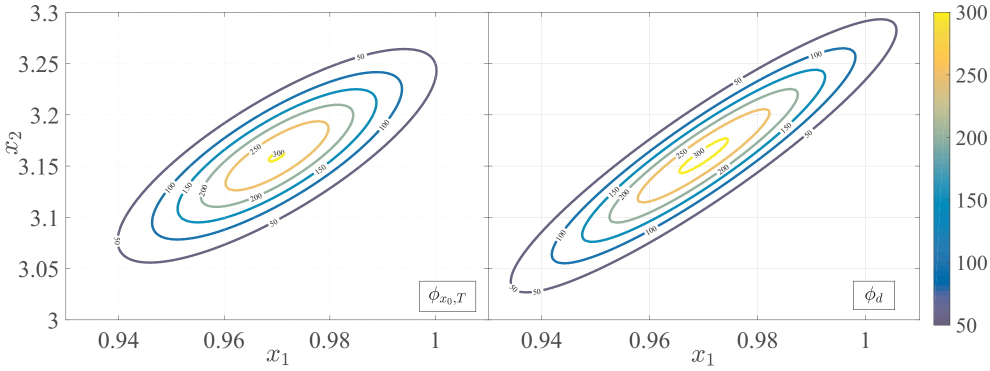

Figure 6.2: Contour plot showing the realized and desired pdfs

This figure shows level sets of the realized pdf $\phi_{x_0,T}$ and the desired pdf ϕ_d for the simulation in Section 6.4. These pdfs have a Hellinger distance $d_h(\phi_{x_0,T}, \phi_d) = 0.1799$.

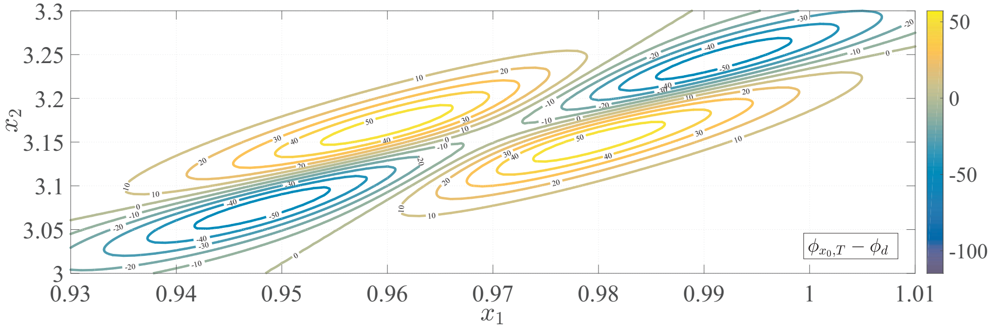

Figure 6.3: Contour plot showing the difference between pdfs

This figure shows the difference between the realized pdf $\phi_{x_0,T}$ and the desired pdf ϕ_d for the simulation in Section 6.4. This differences correspond to hellinger distance $d_h(\phi_{x_0,T}, \phi_d) = 0.1799$. The difference is computed by taking $\phi_{x_0,T}(x) - \phi_d(x)$.

Theorem 4, we take $\tilde{\beta} = e^{(A+BK)T}$ and $\tilde{\eta} = \mu_d - e^{(A+BK)T}\mu$. Subsequently, we can retrace the steps above to find a realization $\phi_{x_0,T} = \mathcal{N}(\mu_T, \Sigma_T)$, where $\mu_T = \tilde{\eta} + \tilde{\beta}\mu$, $\Sigma_T = \tilde{\beta}\Sigma\tilde{\beta}^\top$ and $\tilde{\varepsilon} = \sqrt{\frac{|\Sigma_T|}{|\Sigma|}}$.

We are now ready to find a K that minimizes $d_h(\phi_{x_0}, \phi_{x_0,T})$ through (6.6). For this realization, this is $K = \begin{bmatrix} -2.075 & -0.21 \end{bmatrix}$. We find a corresponding minimum Hellinger distance between pdfs $\ell = 0.1799$. The obtained pdf is shown in Fig. 6.2, the difference between the obtained and the desired pdf is shown in Fig. 6.3. We remark that, in accordance with the above, the shown pdfs are not obtained through a time simulation. The pdfs ϕ_d and $\phi_{x_0,T}$ are instead generated directly, with analytically obtained mean and covariance values.

Let us now consider the interesting case where

$$\mu_d = \begin{bmatrix} 2 \\ 2.5 \end{bmatrix}, \qquad \Sigma_d = 10^{-3} \begin{bmatrix} 0.6546 & 0.3921 \\ 0.3921 & 0.7042 \end{bmatrix}, \tag{6.37}$$

corresponding to

$$\eta = \begin{bmatrix} 2.0691 \\ 2.3396 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 0.0065 & -0.0378 \\ 0.0567 & 0.0518 \end{bmatrix}. \tag{6.38}$$

We selected this example because, by choosing $K = \begin{bmatrix} -0.5 & -0.2 \end{bmatrix}$, we obtain the equalities $\tilde{\beta} = \beta$ and $\tilde{\eta} = \eta$ and a minimum Hellinger distance of $\ell = 0$, in accordance with Corollary 2.

6.5 Concluding remarks

We have presented our SCP formulation and results in this chapter. The SCP poses a more challenging control problem than the CCP, due to the more strict transient specification. We have solved the SCP for linear system dynamics, where the initial and desired pdfs are (nonlinearly) matching.

The presented solutions are simple to implement once the matching property has been identified. However, doing so can prove to be nontrivial in practise. This is especially true for the sets of pdfs that can only be rated through a nonlinear matching.

With this section, we have arrived at the end of the main chapters and Part II of the thesis. In the next chapter, we will present the conclusions and recommendations for further research.