Toward controlled ultra-high vacuum chemical vapor deposition processes

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Chapter 5

Containment Control problem

In this chapter, we introduce the containment control problem (CCP) and solve it for both linear and nonlinear systems. The nonlinear solution relies on results in contraction-based control methods. We show the efficacy of our solution to the nonlinear case through a numerical simulation of a robot manipulator. We consider the problem is a general setting and therefore move away from explicit consideration of the ultra-high vacuum chemical vapor deposition (UHVCVD) process. Arguments on how these contributions can be useful for UHVCVD are provided in Sections 1.4.2 & 4.6.

For background reading on stochastic processes, we refer the reader to (Arnold 1990, Grimmett and Stirzaker 2001), for background reading on non-linear systems control and analysis to (Khalil 1996, Isidori 2013, Nijmeijer and Van der Schaft 1990, Vidyasagar 2002) and for background reading on contraction (and the closely related notion of incremental stability) to (Lohmiller and Slotine 1998, Jouffroy and Fossen 2010, Andrieu et al. 2016, Angeli 2002).

The remainder of the chapter is structured as follows. In Section 5.1 we present the system dynamics, candidate transient specifications and the CCP formulation. We furthermore show the non-triviality of the control problem through a simple example. We use Section 5.2 to present our solution to the CCP for linear systems. Subsequently, we present the nonlinear result in Section 5.3. We show the efficacy of our solution to the CCP for nonlinear systems through a numerical robot manipulator simulation in Section 5.4. Lastly, we end this chapter with some concluding remarks in Section 5.5.

5.1 Containment control problem definition

We will use this section to formally present the CCP. We start with the dynamical system equations, candidate transient specifications and the CCP formulation. We then provide a simple example to show the nontriviality of the problem and some controller design considerations.

5.1.1 Dynamical system equations

Consider the general dynamical system given by

$$\dot{x} = f(x, u, t), \qquad x(0) = x_0, \tag{5.1}$$

where $x \in X \subseteq \mathbb{R}^n$, $u \in U \subseteq R^m$, $t \in \mathbb{R}_{\geq 0}$ the time and $f : X \times U \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is a continuously differentiable vector field. Let us assume that x_0 is a known random variable defined on $X_0 \subset X$, satisfying a probability density function (pdf) $\phi_{x_0} : X_0 \to \mathbb{R}_{\geq 0}$. In this case, its forward solution $x(t)$ is random variable for all $t > 0$ and we denote the propagation of ϕ_{x_0} along (5.1) by $\phi_{x_0,t}$. Note that the usual setting where x_0 is deterministic is a particular class of this class of system where ϕ_{x_0} is simply given by a Dirac delta function.

5.1.2 Candidate transient specifications

For defining a transient behavior specification corresponding to the evolution of $\phi_{x_0,t}$, there are two possibilities in defining the measure. For the first one, we can relate $\phi_{x_0,t}$ at a terminal time T or in an interval $[0, T]$ to, respectively, a desired point (can also be desired set) or a desired trajectory $x_d(t)$ defined on the time interval $[0, T]$. For the second one, we can relate $\phi_{x_0,t}$ to a (dynamic or stationary) target distribution. For the CCP, we are interested in the former.

We will discuss two measures that can be used to define transient specification based on $\phi_{x_0,t}$ and a given desired point or set. The first measure is given by the cumulative density of $\phi_{x_0,T}$ at the terminal time T over a prescribed set Ξ. More precisely, we can define the following measure

$$\Phi_{\Xi,T} := \int_{\Xi} \phi_{x_0,T}(\xi)\mathrm{d}\xi, \tag{5.2}$$

where $\Xi \subset X$ and T is the relevant terminal transient time. This transient specification is straightforward and it yields a scalar value. The second candidate measure is the second moment with respect to a point, which (for a single dimension) can be expressed as

$$\sigma(\phi_{x_0,T}, \mu) := \int_X (\xi - \mu)^2 \phi_{x_0,T}(\xi)\mathrm{d}\xi, \tag{5.3}$$

where μ is a desired point corresponding to the transient time of interest T. Notice that this expression is equal to the computation of the variance if $\phi_{x_0,T}$ is normally distributed with μ the mean value.

Both of the specifications above can be interpreted in a similar manner as the classical specifications of rise time or settling time and are thus highly relevant for any control problem. When comparing the two specifications, it is furthermore easy to see that the computation of the cumulative density is relatively simple. This transient specification has the advantage that we obtain a scalar valued output which again simplifies interpretation and implementation in the control design.

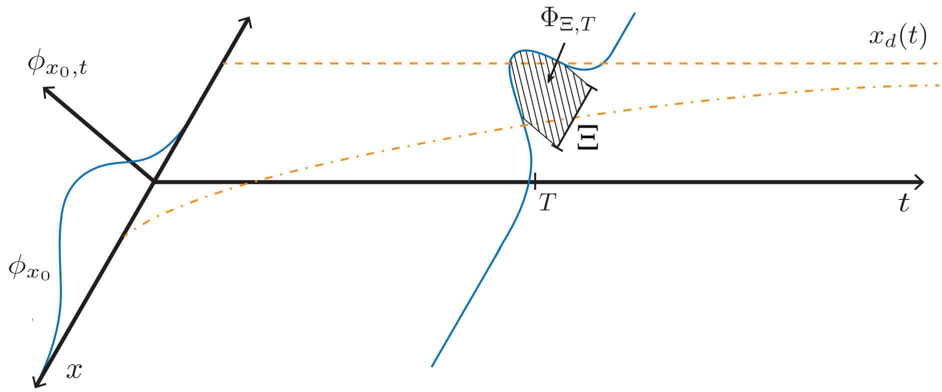

Figure 5.1: Containment control problem illustration

This figure illustrates the control objective of the CCP. The initial distribution ϕ_{x_0} changes with time such that it has a certain cumulative density over the set Ξ at time T, denoted as $\Phi_{\Xi,T}$. In the CCPa, we then require $\Phi_{\Xi,T} \geq p^*$, where $p^* \in (0,1)$ is a desired containment level. Furthermore, for the CCPb, all possible initial values should converge to $x_d(t)$. This in turn could result in the indicated trajectory for a specific realization of x_0.

5.1.3 Containment control problem formulation

We are now ready to define the containment control problem, based on cumulative density transient specifications as given before.

Containment Control Problem (CCP): For a system as in (5.1), given a desired containment set Ξ, a desired containment level $p^* \in (0,1)$, a transient time $T > 0$, a distance $d(\cdot,\cdot)$ and a target trajectory $x_d(t)$, design a control law $u(t) = k(x(t),t)$ such that

CCPa: $\Phi_{\Xi,T} \geq p^*$

CCPb: $\lim_{t\to\infty} d(x(t), x_d(t)) = 0.$ △

In the formulation as above, CCPa is the realization of a minimum containment criteria during the transient. The control problem hence incorporates the cumulative density transient specification (as in (5.2)). This condition is complemented by CCPb, which requires convergence to a desired trajectory in the asymptote. The control objective for the CCP is illustrated in Fig. 5.1.

5.1.4 Containment control problem example

Before moving on to our contributions, we will show the nontriviality of our control problem by considering the following simple example for the CCP with $n = 1$. The example furthermore highlights some of the considerations relevant for solving the CCP (and the shape control problem (SCP) in Chapter 6). We try to solve the CCP with a standard control law, namely a state feedback law, for a first-order LTI system. For this first-order LTI system, we have the system (5.1) with $f(x, u, t) = Ax(t) + Bu(t)$, with $A \in \mathbb{R}$ and $B \in \mathbb{R}$. Furthermore, let $x_d(t) = x^*$. Applying the linear feedback

$$u(t) = K(x(t) - x^*) - \frac{A}{B}x^*, \tag{5.4}$$

with $K \in \mathbb{R}$, to (5.1) will lead us to the following simple expression of the closed-loop system

$$\dot{\tilde{x}}(t) = (A + BK)\tilde{x}(t), \quad \tilde{x}(0) = x_0 - x^*, \tag{5.5}$$

where $\tilde{x}(t) = x(t) - x^*$ is the error state and the gain K can be chosen arbitrarily to ensure that $(A + BK) < 0$. To simplify things further, we will assume a normal distribution for the initial state, e.g. $\phi_{\tilde{x}_0} = \mathcal{N}(\mu - x^*, \sigma)$. Since we are interested in a non-trivial solution of the CCP, we assume that $\mu \neq 0$. For defining the first transient behavior specification of the closed-loop system (5.5), we take $\tilde{\Xi} = [x_{T,low}, x_{T,up}] - x^*$ where $x_{T,low}$ and $x_{T,up}$ are the lower and upper bound of the containment interval Ξ.

Since we are dealing with a simple first-order linear system, we can use the bounds of $\tilde{\Xi}$ and the explicit solution of (5.5) to construct the image of this containment interval at time $t = 0$, which we denote as $\tilde{\Xi}_0$. In this way, the value $\Phi_{\tilde{\Xi},T}$ will be equivalent to cumulative distribution of $\tilde{x}_0$ on $\tilde{\Xi}_0$.

Based on the solution of (5.5), we have

$$\tilde{x}_{0,low} = e^{-(A+BK)T}\tilde{x}_{T,low} \tag{5.6}$$

and

$$\tilde{x}_{0,up} = e^{-(A+BK)T}\tilde{x}_{T,up}, \tag{5.7}$$

where, understandably, $\tilde{x}_{0,low}$ and $\tilde{x}_{0,up}$ are the lower and upper bound of $\tilde{\Xi}_0$.

Since $\phi_{\tilde{x}_0} = \mathcal{N}(\mu - x^*, \sigma)$, we can determine the maximum containment level p_{max} by solving

$$p_{\max} = \max_k \frac{1}{2}\left[\operatorname{erf}\left(\frac{e^{-(A+BK)T}\tilde{x}_{T,up} - \mu + x^*}{\sigma\sqrt{2}}\right) - \operatorname{erf}\left(\frac{e^{-(A+BK)T}\tilde{x}_{T,low} - \mu + x^*}{\sigma\sqrt{2}}\right)\right], \tag{5.8}$$

where erf is the error function. This quantity tells us that we will always have $\Phi_{\tilde{\Xi},T} \leq p_{\max}$. This implies that if $p_{\max} < 1$, we cannot solve CCP for arbitrary containment level $p^* \in (0, 1)$.

In the following numerical example, we will demonstrate a case where a simple linear state-feedback control law without feedforward control cannot solve the CCP for an arbitrary containment level.

Example 1: Consider a system that satisfies

$$\dot{x} = u, \qquad x(0) = x_0, \tag{5.9}$$

where we assume that $\phi_{x_0} = \mathcal{N}(10, 1)$. Furthermore, consider the desired containment set $\Xi = [4, 5]$ with a relevant transient time $T = 5$.

If we consider a non-zero desired equilibrium point of $x^* = 4$. Using the linear feedback controller as given before, we can obtain the gain $K < 0$ for any desired containment level $p^* \in (0, 1)$. For instance, by taking $K = -3.6776$, we get p^* very close to 1. Since $K < 0$, the closed-loop system is stable which implies that $x(t)$ converges to x^* as $t \to \infty$. Hence we achieve both CCPa and CCPb.

On the other hand, if we change the desired steady-state to $x^* = 0$ then the aforementioned feedback control will no longer solve the CCP for arbitrary p^*. The main reason for this is that we can no longer design K such that CCPa is met for some desired containment level p^*. Indeed, solving (5.8) results in $p_{\max} = 0.7359 < 1$ which occurs for $K = -0.1617$. Hence, we can no longer find a feasible solution that satisfies both CCPa and CCPb for any $p^* > p_{\max}$. △

In Example 1, we have shown that the previous simple linear feedback control law only allows us to solve the CCP for specific cases. Particularly, achieving a desired containment level p^* close to 1 may not be possible at all, even for the case of a simple integrator.

5.2 CCP for linear systems

We will start our exposition by considering the CCP in a linear time-invariant setting. The system (5.1) becomes

$$\dot{x} = Ax + Bu, \qquad x(0) = x_0, \tag{5.10}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are the system matrices and x_0 is a random variable defined on $\mathbb{R}^n$. We are now ready to present our first result.

Theorem 2. *Consider the system as in* (5.10). *Let $T > 0$ be the given transient terminal time and x_d be the desired trajectory. Assume that the pair (A, B) is controllable and there exists a finite $\tau > T$ such that x_d is a solution to* (5.10) *(with an admissible input signal $u_d(t)$) for all $t \geq \tau$. Then the CCP is solvable for any p^* where $d(\cdot,\cdot) = d_E(\cdot,\cdot)$ is the Euclidean distance and the set $\Xi \subset \mathbb{R}^n$ is compact, connected and non-empty.*

PROOF. Consider the control law

$$u(t) = K(x(t) - x_r(t)) + u^*(t), \tag{5.11}$$

where x_r and u^* are the tracking reference signal and additional feedforward input signal to be designed.

Let us first define two closed balls. The first one is centered in ϵ_1 and has a radius κ_1 (which we will denote by $\mathbb{B}_{\kappa_1}(\epsilon_1)$). For this ball, ϵ_1 and κ_1 are such that

$$\int_{\mathbb{B}_{\kappa_1}(\epsilon_1)} \phi_{x_0}(\xi)\mathrm{d}\xi \geq p^*. \tag{5.12}$$

We will denote the second ball by $\mathbb{B}_{\kappa_2}(\epsilon_2)$. Since Ξ is compact, connected and non-empty, we can choose ϵ_2 and κ_2 such that $\mathbb{B}_{\kappa_2}(\epsilon_2) \subseteq \Xi$. Furthermore, we require $\kappa_1 > \kappa_2$. Define $x_r(t)$ and $u^*(t)$ with the following properties: (i) $x_r(t) = x_d(t)$, for all $t \geq \tau$, (ii) $x_r(0) = \epsilon_1$, (iii) $x_r(T) = \epsilon_2$ and (iv) $\dot{x}_r(t) = Ax_r(t) + Bu^*(t)$. Note that since the pair (A, B) is controllable, we can always find a control signal u^* that can bring the state from ϵ_1 at time 0 to ϵ_2 at time T, and subsequently, to $x_d(\tau)$ at τ. Furthermore, since $x_d(t)$ is a solution to (5.10) for $u_d(t)$ and $t \geq \tau$, we can let $u^*(t) = u_d(t)$ for $t \geq \tau$. Using such u^*, the tracking reference signal x_r is then given by the solution z of

$$\dot{z} = Az + Bu^*, \qquad z(0) = \epsilon_1. \tag{5.13}$$

Define now $\zeta = x - x_r$ as the error signal between the state and the tracking reference signal. Note that with such coordinate transformation, if $\zeta(T) \in \mathbb{B}_{\kappa_2}(0)$ then, since $x_r(T) = \epsilon_2$, it implies that $x(T) \in \mathbb{B}_{\kappa_2}(\epsilon_2)$, which is a subset of Ξ. Also, it follows that $\zeta_0 \in \mathbb{B}_{\kappa_1}(0)$ implies that $x_0 \in \mathbb{B}_{\kappa_1}(\epsilon_1)$. Accordingly, the dynamics of the error signal where we have applied the control law (5.11) are given by

$$\dot{\zeta} = (A + BK)\zeta, \qquad \zeta(0) = \zeta_0. \tag{5.14}$$

Let us now define a contraction exponential rate constant $\lambda = -\frac{1}{T}\ln(\kappa_2/\kappa_1)$. In the following, we will design K so that $\mathbb{B}_{\kappa_1}(0)$ under the closed-loop dynamics (5.14) will be contracted with an exponential rate of λ, to $\mathbb{B}_{\kappa_2}(0)$ at time T.

From the pair (A, B) being controllable, it follows that we can design K such that $A + BK$ has eigenvalues whose real part is less than $-\lambda$ (for example, by the well-known pole-placement method). This implies that

$$\|\zeta(t)\| \leq e^{-\lambda t}\|\zeta(0)\| \tag{5.15}$$

holds for all initial condition $\zeta(0)$. By our choice of λ as given before and by considering initial conditions along the boundary of $\mathbb{B}_{\kappa_1}(0)$ (having a Euclidean norm of κ_1),

$$\|\zeta(T)\| \leq \frac{\kappa_2}{\kappa_1}\|\zeta(0)\| = \kappa_2. \tag{5.16}$$

Hence,

$$\zeta(0) \in \mathbb{B}_{\kappa_1}(0) \Rightarrow \zeta(T) \in \mathbb{B}_{\kappa_2}(0) \Rightarrow x(T) \in \mathbb{B}_{\kappa_2}(\epsilon_2) \subseteq \Xi. \tag{5.17}$$

And, since (5.12) holds, we obtain

$$\Phi_{\Xi,T} \geq \int_{\mathbb{B}_{\kappa_1}(0)} \phi_{\zeta_0}(\xi)\mathrm{d}\xi \geq p^*. \tag{5.18}$$

In other words, CCPa is satisfied. Additionally, since we have $x_r(t) = x_d(t)$ for $t \geq \tau$, the following asymptotic property holds:

$$\lim_{t\to\infty} \zeta(t) = 0 \Rightarrow \lim_{t\to\infty} d_E(x(t), x_d(t)) = 0. \tag{5.19}$$

In other words, CCPb holds. This concludes the proof. □

In the special case where $x_d(t)$ is a constant point, the result above becomes more simple. This is presented in the following corollary.

Corollary 1. *Consider the system as in* (5.10)*. Assume that the pair (A, B) is controllable and that $x_d(t) = x^*$. Then, the CCP is solvable for any T and p^*, where $d(\cdot,\cdot) = d_E(\cdot,\cdot)$ is the Euclidean distance, and Ξ is compact, connected and non-empty.*

PROOF. The proof is almost identical to the proof of Theorem 2. Here, we let the reference signal satisfy the following: (i) $\lim_{t\to\infty} x_r(t) = x^*$, (ii) $x_r(0) = \epsilon_1$, (iii) $x_r(T) = \epsilon_2$ and (iv) $\dot{x}_r(t) = Ax_r(t) + Bu^*(t)$. Notice that the new characteristic (i) is always achievable for a controllable system. The proof of CCPa remains unchanged. For CCPb we now obtain

$$\lim_{t\to\infty} \zeta(t) = 0 \Rightarrow \lim_{t\to\infty} d_E(x(t), x^*) = 0. \tag{5.20}$$

Hence, CCPb is satisfied. This concludes the proof. □

5.3 CCP for nonlinear systems

In this section, we will extend the result of Section 5.2 to the nonlinear case using results in contraction-based control design. For interested readers, we present relevant contraction results in Appendix 5.A. The overall main idea is that we use contraction results for quantifying the rate of decay among all trajectories which include the target trajectory. For further background reading on this subject, we refer the interested reader to (Lohmiller and Slotine 1998, Jouffroy and Fossen 2010, Andrieu et al. 2016, Andrieu et al. 2015).

Let us now proceed by presenting our solution to CCP for the nonlinear systems case. We can apply the contraction-based control design by implementing a control law for the system (5.1), such that a partial contraction with a desired contraction rate w.r.t. a desired reference trajectory $x_r(t)$ is achieved. The design procedure is accordingly an iterative process which should yield three components: (i) a reference trajectory x_r that starts close to the initial conditions and satisfies $x_r(T) \in \Xi$ and $\lim_{t\to\infty} d_F(x_r(t), x_d(t)) = 0$, with d_F a Finsler distance as in Definition 2 in Appendix 5.A, (ii) a control law that yields a closed-loop system having x_r as a solution and (iii) a Finsler-Lyapunov function that yields a desired contraction rate λ for the closed-loop system.

We apply an adaption of Lemma 2 in Appendix 5.A as presented in (Jouffroy and Fossen 2010, Wang and Slotine 2005) to facilitate control law design. Accordingly, for a given system (5.1), we assume the existence of a control law $u(t) = k(x, x_r, t)$ that causes the closed-loop system, given by

$$\dot{x} = f_c(x, x_r, t), \tag{5.21}$$

to be partially contracting with contraction region $C \subseteq X$. We require C to be so that $X_0 \subset C$. Furthermore, we require a minimum rate λ for all initial conditions belonging to a specific set that satisfies a condition similar to (5.12). For such a control law, we obtain λ, the choice of a set of initial conditions and the reference trajectory x_r as control design parameters for achieving the two control objectives in the CCP.

Before presenting a particular design for λ and x_r in the following theorem, we will define an open ball induced by the Finsler distance d_F. For a given $\kappa > 0$ and a Finsler structure F satisfying 1) to 4) of Definition 1 in Appendix 5.A, we define the ball with radius κ_1 centered at x_1 by

$$\mathbb{D}_{\kappa_1}(x_1) = \{x_2 \in X \mid d_F(x_1, x_2) < \kappa\}. \tag{5.22}$$

We are now ready to present our result.

Theorem 3. *Consider the system* (5.1) *with the control law* $u(t) = k(x, x_r, t)$*. Suppose that the closed-loop system defined by* (5.21) *is contracting w.r.t.* x *and a contraction region* $C \supseteq X_0$*, and that there exist points* $\epsilon_1 \in C$*,* $\epsilon_2 \in \Xi$*, a reference trajectory* x_r *and constants* $\kappa_1, \kappa_2 > 0$ *such that the following conditions hold.*

1. *The reference signal* x_r *satisfies*

$$\dot{x}_r = f_c(x_r, x_r, t) \tag{5.23}$$

for all $t \geq 0$*, with* $x_r(0) = \epsilon_1$*,* $x_r(T) = \epsilon_2$ *and* $\lim_{t\to\infty} d_F(x_r(t), x_d(t)) = 0$*.*

2. *There are two sets* $\mathbb{D}_{\kappa_1}(\epsilon_1)$ *and* $\mathbb{D}_{\kappa_2}(\epsilon_2)$ *satisfying*

$$\int_{\mathbb{D}_{\kappa_1}(\epsilon_1)} \phi_{x_0}(\xi)\mathrm{d}\xi \geq p^* \tag{5.24}$$

and $\mathbb{D}_{\kappa_2}(\epsilon_2) \subseteq \Xi$*.*

3. *The contraction rate* λ *satisfies*

$$\lambda \geq -\frac{\ln\left(\frac{\kappa_2}{\kappa_1}\right)}{T}, \tag{5.25}$$

for all $x_0 \in \mathbb{D}_{\kappa_1}(\epsilon_1)$*.*

Then, the control law $u(t) = k(x, x_r, t)$ *solves the CCP.*

PROOF. The proof is similar to the proof for the linear system. We will show that the initial ball $\mathbb{D}_{\kappa_1}(x_r(0))$, which has the desired minimum cumulative distribution, will contract to $\mathbb{D}_{\kappa_2}(x_r(T))$, which is contained in the desired containment set Ξ, at the transient time T.

By the hypothesis of the proposition, the closed-loop system is contracting w.r.t x. Furthermore, since x_r is an admissible solution to the system, by Lemma 2 in Appendix 5.A, this implies that all trajectories starting in C converge to x_r. Notice that x_r is reachable since it is a solution to the closed loop system which has a contraction property. We accordingly have that all trajectories starting in $\mathbb{D}_{\kappa_1}(\epsilon_1) \subset C$ converge with an exponential rate λ. Hence,

$$d_F(x_r(t), x(t)) \leq d_F(x_r(0), x_0)e^{-\lambda t}, \tag{5.26}$$

for all $x_0 \in \mathbb{D}_{\kappa_1}(\epsilon_1)$. For all initial conditions $x_0 \in \mathbb{D}_{\kappa_1}(\epsilon_1)$, as $x_r(0) = \epsilon_1$, we obtain that $d_F(x_r(0), x_0) \leq \kappa_1$. Hence

$$d_F(x_r(t), x(t)) \leq \kappa_1 e^{-\lambda t}. \tag{5.27}$$

Thus at time T, by the hypothesis on λ as in (5.25),

$$d_F(x_r(T), x(T)) \leq \frac{\kappa_2}{\kappa_1}\kappa_1 \Rightarrow d_F(x_r(T), x(T)) \leq \kappa_2. \tag{5.28}$$

Hence, for this λ we have

$$x_0 \in \mathbb{D}_{\kappa_1}(x_r(0)) \Rightarrow x(T) \in \mathbb{D}_{\kappa_2}(x_r(T)) \subseteq \Xi. \tag{5.29}$$

And since we have (5.24), it follows that

$$\int_{\Xi} \phi_{x_0,T}(\xi)\mathrm{d}\xi \geq \int_{\mathbb{D}_{\kappa_2}(x_r(T))} \phi_{x_0,T}(\xi)\mathrm{d}\xi \geq p^*, \tag{5.30}$$

which implies that the first control objective CCPa is satisfied.

It remains to show that all trajectories converge to x_d as $t \to \infty$. Firstly, since C is such that it contains X_0, we have the contraction property for all initial conditions. Secondly, since the reference signal x_r is such that $\lim_{t\to\infty} d_F(x_r(t), x_d(t)) = 0$, it is sufficient to show that all contracting trajectories converge to x_r. From the partial contraction property, we have

$$\lim_{t\to\infty} d_F(x(t), x_r(t)) = 0 \Rightarrow \lim_{t\to\infty} d_F(x(t), x_d(t)) = 0, \tag{5.31}$$

for all $x_0 \in C$. Hence, CCPb is satisfied. This concludes the proof. $\square$

5.4 CCP controller simulation for a nonlinear robotic manipulator

In this section, we will evaluate the nonlinear contraction based controller design for the CCP, applied to a well-known 2nd order mechanical system operating with 3-DOF, which is a SCARA robot as presented in (Reyes-Báez et al. 2017). The controller design that we implement for this simulation is in accordance with Theorem 3.

5.4.1 Dynamics and controller design

The robot operates on the manifold $\mathcal{X} = \mathcal{Q} \times \mathbb{R}^3$, with states $q \in \mathcal{Q} = \mathcal{S} \times \mathcal{S} \times \mathbb{R}$, where $\mathcal{S}$ the unitary circumference, and $p \in \mathbb{R}^3$. Here, $q^\top = \begin{bmatrix} \theta_1 & \theta_2 & s \end{bmatrix}$ is the generalized position, and $p^\top = \begin{bmatrix} p_{\theta_1} & p_{\theta_2} & p_s \end{bmatrix} = M(q)\dot{q}$ is the generalized momentum, with $M(q) = M^\top(q)$ the inertia matrix, and $u^\top = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}$ the generalized force. The system satisfies the port-Hamiltonian form

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_3 & I_3 \\ -I_3 & -D(q) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_3 \\ G(q) \end{bmatrix} u, \tag{5.32}$$

where $\mathbf{0}$ is a matrix of zeros, $H(q,p)$ is the Hamiltonian function, $D(q) = D^\top(q) : \mathcal{Q} \to \mathbb{R}^{3\times 3}_{\geq 0}$ is the damping matrix and $G(q) : \mathcal{Q} \to \mathbb{R}^{3\times 3}$ is the input matrix. For the Hamiltonian function we have the total energy as

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q) \tag{5.33}$$

with $V(q) = (m_1 + m_2 + m_3)gz$ the potential energy, where m_1, m_2 and m_3 are the masses of the robot manipulator links. For the mass matrix we have

$$M(q) = \begin{bmatrix} M_1 & M_2 & 0 \\ M_2 & m_3 l_2^2 & 0 \\ 0 & 0 & (m_1, m_2, m_3)g \end{bmatrix}, \tag{5.34}$$

where

$$M_1 = (m_2 + m_3)l_1^2 + m_3 l_2^2 + 2m_3 l_1 l_2 \cos\theta_2, \tag{5.35}$$

$$M_2 = m_3 l_2^2 + m_3 l_1 l_2 \cos\theta_2, \tag{5.36}$$

with l_1, l_2 the lengths of the robot manipulator links.

We assume stochastic initial conditions for the two rotational joints, satisfying a normal distribution. Hence, we have $q_0 \sim \mathcal{N}(\mu_q, \Sigma_q)$, where

$$\mu_q = \begin{bmatrix} \mu_{q,1} \\ \mu_{q,2} \\ \mu_{q,3} \end{bmatrix}, \quad \Sigma_q = \begin{bmatrix} \sigma_{q,1} & 0 & 0 \\ 0 & \sigma_{q,2} & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5.37}$$

We take the system to be idle upon initialization, e.g. $p_0^\top = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$. The initial conditions $x_0 = \begin{bmatrix} q_0^\top & p_0^\top \end{bmatrix}^\top \sim \mathcal{N}(\mu, \Sigma)$ then satisfy

$$\mu = \begin{bmatrix} \mu_q \\ \mathbf{0}_{3\times 1} \end{bmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma_q & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 \end{bmatrix}. \tag{5.38}$$

In this example, we will take as constants $D = \texttt{diag}\begin{bmatrix} 0.2 & 0.2 & 0.2 \end{bmatrix}$, $G = I_3$, $\begin{bmatrix} m_1 & m_2 & m_3 \end{bmatrix} = \begin{bmatrix} 1.5 & 1 & 0.5 \end{bmatrix}$, $\begin{bmatrix} l_1 & l_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \end{bmatrix}$, $g_c = 9.81$, $\mu_q = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\top$ and $\begin{bmatrix} \sigma_{q,1} & \sigma_{q,2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$. Let us furthermore consider the desired trajectory as $q_d = \begin{bmatrix} \sin(t)+1 & \sin(t) & \sin(t) \end{bmatrix}^\top$ and $p_d(t) = M(q_d(t))\dot{q}_d(t)$. For the reference signal, we design q_r s.t. $q_r(0) = \mu_q$, $q_r(T) = \epsilon_q \in \Xi$ and $\lim_{t\to\infty} q_r(t) = q_d(t)$. We will discuss characteristics of Ξ shortly. For now, assume that Ξ is so that we can take $\epsilon_q = \begin{bmatrix} \sin(T)+1 & \sin(T) & \sin(T) \end{bmatrix}^\top$ and since we also have that $q_d(0) = \mu$, we can conveniently let $q_r(t) = q_d(t)$.

Subsequently, the error system is given by

$$\zeta := \begin{bmatrix} \tilde{q} \\ \omega \end{bmatrix} = \begin{bmatrix} q - q_d \\ p - p_r \end{bmatrix}, \tag{5.39}$$

where p_r is a momentum reference signal, to be defined. The dynamics of $\tilde{q}$ are given by

$$\dot{\tilde{q}} = M^{-1}(\omega + q_d)p - M^{-1}(q_d)p_d. \tag{5.40}$$

We define $p_r = p_{d\omega} - \Lambda\tilde{q}$, with $p_{d\omega} = M(\tilde{q} + q_d)\dot{q}_d$ and $-\Lambda = -\Lambda^\top$ Hurwitz. Hence, we obtain the properties $\lim_{t\to\infty} q(t) = q_d(t)$ and $\lim_{t\to\infty} p_r(t) = p_d(t)$. We then take $\epsilon_p = p_r(T)$ and accordingly obtain, in reference to condition 1) in Theorem 3, that the reference signal satisfies $x_r(0) = \begin{bmatrix} q_r(0)^\top & p_r(0)^\top \end{bmatrix}^\top = \mu$, $x_r(T) = \begin{bmatrix} \epsilon_q^\top, \epsilon_p^\top \end{bmatrix}^\top$ and $\lim_{t\to\infty} d_F(x_r(t), x_d(t)) = 0$. The full error dynamics are given by

$$\dot{\tilde{q}} = M^{-1}(\tilde{q} + q_d)(\omega - \Lambda\tilde{q}) \tag{5.41}$$

$$\dot{\omega} = -\left[\frac{\partial H}{\partial q}(q,p) + D\frac{\partial H}{\partial p}(q,p) - u + \dot{p}_r\right] \tag{5.42}$$

Accordingly, we obtain the closed-loop error system by applying the control law

$$u = u_{\text{eq}} + u_{\text{at}}, \tag{5.43}$$

$$u_{\text{eq}} = \dot{p}_r + \frac{\partial H}{\partial q}(q, p_r) + D\frac{\partial H}{\partial p}(q, p_r), \tag{5.44}$$

$$u_{\text{at}} = -K_d\frac{\partial H}{\partial p}(q,\omega) - M^{-1}(q)\Lambda\tilde{q} + \frac{\partial}{\partial q}(p_r^\top M^{-1}(q)\omega), \tag{5.45}$$

where K_d is so that

$$D + K_d + \frac{1}{2}I_3 - \frac{1}{4}(M^{-1} + M) > 0. \tag{5.46}$$

Our system then specifies the contraction properties proven in, and for a virtual system as provided in, (Reyes-Báez et al. 2017). This virtual system admits x and $\begin{bmatrix} q_d^\top & p_r^\top \end{bmatrix}^\top$ as solutions and we have therefore satisfied condition 1) in Theorem 3. The candidate Finsler-Lyapunov function for this virtual system, as in Definition 1 in Appendix 5.A, is given by

$$V_F(x_v, \delta x_v) = \frac{1}{2}\delta x_v^\top \Theta^\top P(\zeta)\Theta \delta x_v, \tag{5.47}$$

with x_v and $\delta x_v = \begin{bmatrix} \delta q_v^\top & \delta p_v^\top \end{bmatrix}^\top$ the virtual system states (with reference to (5.60)) and where

$$\Theta = \begin{bmatrix} I_3 & \mathbf{0}_3 \\ \Lambda & I_3 \end{bmatrix}, \qquad P(\zeta) = \begin{bmatrix} \Lambda & \mathbf{0}_3 \\ \mathbf{0}_3 & M^{-1}(\tilde{q} + q_d) \end{bmatrix}. \tag{5.48}$$

It follows that the distance as in Definition 2 in Appendix 5.A satisfies

$$d_F(x, x_r) = \inf_{\Gamma(x,x_r)} \int_{\mathcal{I}} \sqrt{\left(V_F(\gamma(s))\frac{\partial \gamma(s)}{\partial s}\right)} \mathrm{d}s. \tag{5.49}$$

Lastly, we obtain the property

$$d_F(x(t), x_r(t)) < d_F(x(0), x_r(0))e^{-\lambda t}, \tag{5.50}$$

related to a rate λ as

$$\lambda(\zeta) = \min \mathtt{eig}(P^{1/2}(\zeta)\Upsilon(\zeta)P^{1/2}(\zeta)), \tag{5.51}$$

with

$$\Upsilon(\zeta) = \begin{bmatrix} 2M^{-1}(\tilde{q} + q_d) & (M^{-1}(\tilde{q} + q_d) - I_3) \\ (M^{-1}(\tilde{q} + q_d) - I_3) & 2(D + K_d) \end{bmatrix}. \tag{5.52}$$

5.4.2 Control design parameters

In this section, we will discuss the relevant control design parameters T, p^*, Ξ, Λ and K_d. Firstly, let us apply our notion of the distance as in (5.62), with respect to the reference signal $x_r = [q_d^T, p_r^T]^T$ as $d(x, x_r)$, with $\gamma(s) = [q_d^T, p_r^T]^T + \zeta s$, e.g. a line, which is a valid infimum for our coordinates. Hence, we also have $\frac{\partial \gamma(s)}{\partial s} = \zeta$. Notice that the distance is furthermore dependent on Λ, which simultaneously influences the contraction rate. Λ should be chosen so that the minimal contraction rate λ is achievable, for our purpose, we can use $\Lambda = \text{diag}\{2, 2, 2\}$. Furthermore, we take $T = 10$, $p^* = 0.7$ and Ξ such that

$$\Xi = \{x \mid d(x, x_r(T)) \leq \kappa_2\} = \mathbb{D}_{\kappa_2}(x_r(T)), \tag{5.53}$$

for $\kappa_2 = 6$. The first part of condition 2 of Theorem 3 is accordingly satisfied.

Remark 1. *For a higher order, multidimensional system such as this, it is not trivial to relate the containment set to the original coordinates. The main reason is that the map $d_F^{-1} : \mathbb{R}_{\geq 0} \mapsto \mathcal{X} \times \mathcal{X}$ is not unique. However, desired restrictions on the distances in the transient can still be obtained by conducting a study of the distances for different coordinates or empirically. Both options being often time-consuming. Here, we will suffice with the choice of Ξ as given above.*

For the initial conditions, we can specify a distance set as follows. For the pdf of our initial condition around μ, we can use the bivariate standard normal pdf for θ_1 and θ_2, since the other states are deterministic. This is expressed in polar coordinates as

$$\phi_{x_0}(r, \theta) = \frac{r}{2\pi} e^{-0.5r^2}. \tag{5.54}$$

We then find that

$$\int_0^{1.5517} \frac{r}{2\pi} e^{-0.5r^2} \mathrm{d}r = 0.7. \tag{5.55}$$

By mapping the contour of this radius through d_F, we obtain the shape in Fig. 5.2. These points can hence be captured by a distance set as in (5.22), $\mathbb{D}_{\kappa_1}(\mu)$, with $\kappa_1 = 15.73$. We accordingly have

$$\int_{\mathbb{D}_{\kappa_1}(\mu)} \phi_{x_0}(\xi)\mathrm{d}\xi \geq p^*, \tag{5.56}$$

and we therefore satisfy condition 2) in Theorem 3 completely.

Remark 2. *In contrast with $\Xi = \mathbb{D}_{\kappa_2}(x_r(T))$, we are now able to easily relate the distance set $\mathbb{D}_{\kappa_1}(\mu)$ to the original coordinates. This is due to the knowledge that we assume on the initial conditions, and specifically due to the variations being only present in two dimensions upon initialization.*

Let us now determine the minimal contraction rate λ and the corresponding K_d so that we achieve this rate. We find λ as

$$\lambda \geq -\frac{\ln\left(\frac{\kappa_2}{\kappa_1}\right)}{T} = 0.0964. \tag{5.57}$$

Accordingly, we can choose $K_d = \text{diag}\{3, 1, 25\}$ which is so that this minimal rate is always satisfied in accordance with (5.51). In fact, we find that that (5.51) yields $\lambda = 0.0965$ for all trajectories in our example. We therefore satisfy condition 3) in Theorem 3. We are now ready to move on to the simulation results, as we have satisfied all criteria of Theorem 3.

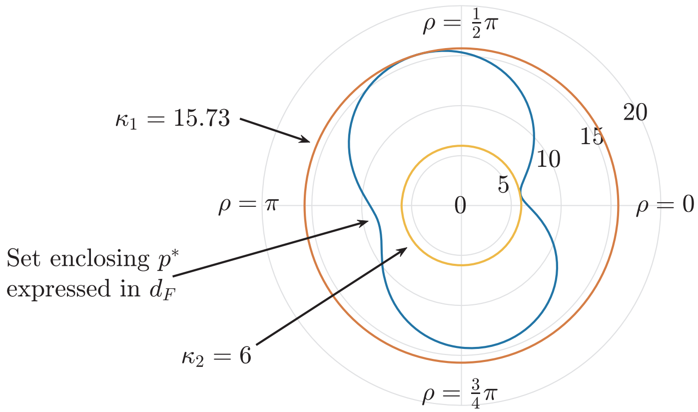

Figure 5.2: Distance sets for CCP applied in the robot manipulator simulation
In this figure, we show the circumference of the two distance sets that are relevant for our simulation in Section 5.4; $\mathbb{D}_{\kappa_1}(x_r(0))$ and $\mathbb{D}_{\kappa_2}(x_r(T))$, and the contour of a set of initial conditions whose cumulative density is p^*. The angle (ρ) of this polar plot is interpretable with respect to θ_1 and θ_2, where $\rho = -\pi\mathbf{1}(-\theta_1 + \mu_1) + \tan^{-1}\left(\frac{\theta_1-\mu_1}{\theta_2-\mu_2}\right)$, $\mathbf{1}(\cdot)$ being the step function.

5.4.3 Simulation results

The simulation results are shown in Fig. 5.3, and 5.4. From Fig. 5.3, it is easy to see that we have achieved

$$\Phi_{\Xi,T} = 1 \geq p^* \tag{5.58}$$

and that we thus satisfy CCPa. The performance of our system is strong, due to the following:

1. The distance set $\mathbb{D}_{\kappa_1}(\mu)$ is greater than a marginal set covering p^* fraction of initial conditions.
2. The rate λ is a minimal rate, but the other eigenvalues in (5.51) generally cause the convergence to be faster than this minimum.

In Fig. 5.4, we show the convergence of an initial condition that satisfies $d_F(x_0, x_r) = \kappa_1$. The asymptotic convergence, satisfying CCPb, to the reference signal can clearly be seen, as well as the fast decay of the initial distance $d_F(x_0, x_r)$.

5.5 Concluding remarks

We have used this chapter to present our CCP formulation and to provide solutions for both linear and nonlinear systems. We have furthermore applied our nonlinear solution to a nontrivial robot manipulator through a numerical simulation.

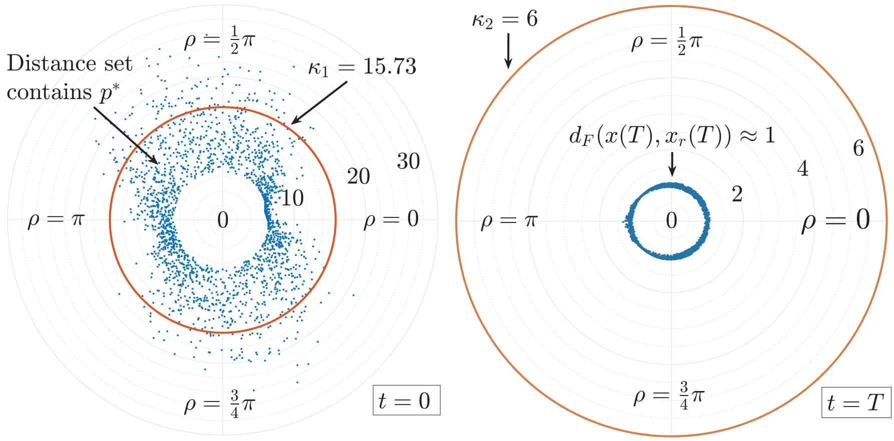

Figure 5.3: States and distance sets for robotic manipulator simulation

In this figure, we show the distances $d_F(x(t), x_r(t))$ of both the initial distribution at time $t = 0$ (left) and the distribution at time $t = T$ (right) for the simulation in Section 5.4. At time T, all trajectories are in the set $\Xi = \mathbb{D}_{\kappa_2}(x_r(T))$, hence we have achieved our desired performance $\Phi_{\Xi,T} \geq p^*$. The interpretation of the plot angle ρ is as in Fig. 5.2.

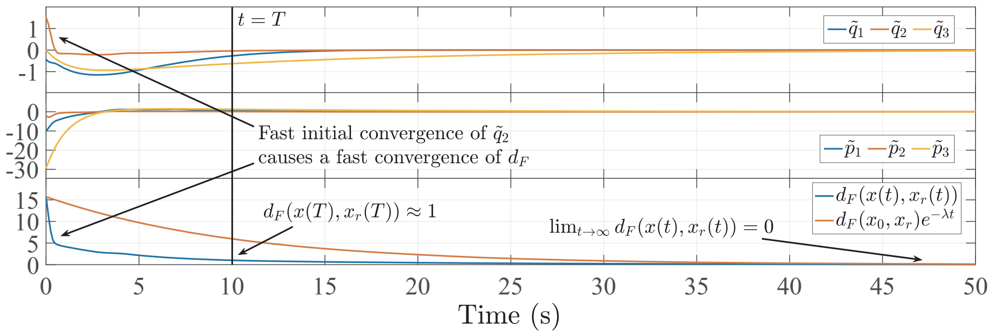

Figure 5.4: Time evolution of robot manipulator states

This figure depicts, for the simulation in Section 5.4, from top to bottom: (i) the time evolution of $\tilde{q}(t) = q(t) - q_d(t)$, (ii) $\tilde{p}(t) = p(t) - p_d(t)$ and (iii) the time evolution of the distance, for a trajectory satisfying $d_F(x(0), x_r(0)) = \kappa_1$. Furthermore, in the bottom plot we show the difference with the nominal decay, which is an effective upper bound. Lastly, the bottom plot shows convergence to a distance $d_F = 0$.

This chapter contains the first results on analytic controller design for deterministic with stochastic initial conditions. Although this class of system is restrictive, the results are very general. Our linear solution to the CCP is furthermore easy to implement. The nonlinear result on the other hand, leaves the user with a dual design problem of the control law and the Finsler-Lyapunov function. Such a result is normal for nonlinear systems but finding a control law accordingly remains nontrivial.

For the CCP, we have considered a transient behaviour specification that relates the attained pdf to a set. However, as argued before, we can also relate the attained pdf to a desired pdf. We will do so in the next chapter.

5.A Contraction preliminaries

Consider the system (5.1). The contraction-based control method is applied through a control law $u(t) = k(x, x_r, t)$, such that the resulting closed-loop system is contracting. Accordingly, we obtain a closed loop system

$$\dot{x} = f(x, x_r, t), \tag{5.59}$$

where f is still a continuously differentiable, e.g. $\mathbb{C}^1$, vector field. For each point $x \in X$ we denote its tangent space as $T_x X$. Furthermore, let $TX = \bigcup_{x \in X} \{x\} \times T_x X$ be the tangent bundle of X.

The contraction analysis is performed on the prolonged system (Crouch and Van der Schaft 1987), which is obtained by combining the system (5.59) with its variational system. The prolonged system is then given by

$$\begin{aligned} \dot{x} &= f(x, x_r, t), \\ \dot{\delta x} &= \frac{\partial f}{\partial x}(x, x_r, t)\delta x, \end{aligned} \tag{5.60}$$

where δx is the tangent vector and $(x, \delta x, t) \in TX \times \mathbb{R}_{\geq 0}$. Accordingly, we can consider a system to be contracting if relevant vector lengths δx (defined by a distance) are uniformly decreasing for all trajectories that start in a certain set. The natural choice for a distance is the Finsler distance, which is related to a Finsler-Lyapunov function. We adopt the corresponding definitions from (Forni and Sepulchre 2014).

Definition 1. *(Finsler-Lyapunov function) A $\mathbb{C}^1$ function $V_F : TX \to \mathbb{R}_{\geq 0}$, that maps every $(x, \delta x) \in TX$ to $V_F(x, \delta x) \in \mathbb{R}_{\geq 0}$, is a candidate Finsler-Lyapunov function for* (5.60)*, if there exist $c_1, c_2 \in \mathbb{R}_{\geq 0}$, $c_3 \in \mathbb{R}_{\geq 1}$, and a Finsler structure $F : TX \to \mathbb{R}_{\geq 0}$ such that, $\forall (x, \delta x) \in TX$,*

$$c_1 F(x, \delta x)^{c_3} \leq V_F(x, \delta x, t) \leq c_2 F(x, \delta x)^{c_3}, \tag{5.61}$$

where the Finsler structure F satisfies the following conditions:

1. *F is a $\mathbb{C}^1$ function for each $(x, \delta x) \in TX$ such that $\delta x \neq 0$;*
2. *$F(x, \delta x) > 0$ for each $(x, \delta x) \in TX$ such that $\delta x \neq 0$;*
3. *$F(x, \lambda \delta x) = \lambda F(x, \delta x)$ for each $\lambda \geq 0$ and each $(x, \delta x) \in TX$;*
4. *$F(x, \delta x_1 + \delta x_2) < F(x, \delta x_1) + F(x, \delta x_2)$ for each $(x, \delta x_1), (x, \delta x_2) \in TX$ such that $\delta x_1 \neq \lambda \delta x_2$ for any given $\lambda \in \mathbb{R}$.*

It follows that the Finsler-Lyapunov function is a measure of length of the tangent vector, the corresponding Finsler distance is then obtained through integration.

Definition 2. *(Finsler distance) Consider a candidate Finsler-Lyapunov function V_F on X and the associated Finsler structure F as in Definition 1. For any two points $(x_1, x_2) \in X \times X$, let $\Gamma(x_1, x_2)$ be the collection of piecewise $\mathbb{C}^1$ curves $\gamma : \mathcal{I} \to X$, $\mathcal{I} := \{s \in \mathbb{R} \mid 0 \leq s \leq 1\}$, $\gamma(0) = x_1$ and $\gamma(1) = x_2$. The distance $d_F : X \times X \to \mathbb{R}_{\geq 0}$ induced by F satisfies*

$$d_F(x_1, x_2) := \inf_{\Gamma(x_1,x_2)} \int_{\mathcal{I}} F(\gamma(s), \dot{\gamma}(s))\mathrm{d}s. \tag{5.62}$$

We are now ready to present the existing results on contraction for nonlinear systems.

Lemma 1. *(Contraction) Consider the system* (5.60) *on the smooth manifold X with F a $\mathbb{C}^2$ function, a connected and forward invariant set $C \subseteq X$ and a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Let V_F be a candidate Finsler-Lyapunov function such that,*

$$\frac{\partial V_F(x, \delta x)}{\partial x} f(x, x_r, t) + \frac{\partial V_F(x, \delta x)}{\partial \delta x} \frac{\partial f(x, x_r, t)}{\partial x} \delta x \leq \alpha(V_F(x, \delta x)) \tag{5.63}$$

for each $t \in \mathbb{R}_{\geq 0}$, $x \in C \subseteq X$, and $\delta x \in T_x X$. Then, (5.60) *is*

- *incrementally stable on C if $\alpha(s) = 0$, for each $s \geq 0$;*
- *incrementally asymptotically stable on C if α is a class $\mathcal{K}$ function;*
- *incrementally exponentially stable on C if $\alpha(s) = \lambda s$.*

We refer to (Forni and Sepulchre 2014) for the proof of this lemma.

The above can then be interpreted as follows. A system (5.60) is contracting if for a Finsler distance d_F, there exists a Finsler-Lyapunov function V_F as in Lemma 1 and $\alpha \in \mathcal{K}$ such that (5.63) holds. The system is said to be exponentially contracting in the case $\alpha(s) = \lambda s$. Here, C is the contraction region and V_F the contraction measure.

For a system that is contracting, all trajectory starting in the contraction region will converge to a single trajectory. However, which trajectory is not specified by the contraction property. In order to specify this, we can use the result on partial contraction as in (Reyes-Báez et al. 2017, Slotine and Wang 2005).

Lemma 2. *(Partial contraction) Consider the nonlinear system as in* (5.59) *with an admissible target trajectory $x_r(t)$, i.e., x_r satisfies $\dot{x}_r(t) = f(x_r(t), x_r(t), t)$ for all $t \geq 0$. If* (5.59) *is contracting w.r.t. x, then x converges to $x_r(t)$.*

The proof of the lemma follows the result from (Slotine and Wang 2005) and (Forni and Sepulchre 2014). The system (5.59) is called partially contracting if it satisfies the hypothesis in Lemma 2 for a given admissible target trajectory x_r.