A moment matching-based loop shaping design with closed-loop pole placement

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T. C. Ionescu, O. V. Iftime and R. Ştefan

Abstract— **In this paper, moment matching-based loop shaping design for finite-dimensional linear systems is considered. First, the time-domain moment matching model reduction problem with complex-conjugated pole-zero placement is revisited. We explicitly calculate the free parameters such that the reduced order model, achieving moment-matching at complex-conjugated interpolation points, has pairs of complex-conjugated poles and zeros placed in desired locations. Then, we recast the problem of closed-loop pole placement in terms of moment-matching, with the moments equal to -1 and 0, respectively. Consequently, the free parameters of a model achieving both moment matching at complex-conjugated points and closed-loop pole placement at prescribed locations are determined. Furthermore, we combine the classical loop shaping method with closed-loop pole placement via moment matching to design a robust controller for stable, minimum-phase, proper systems. We illustrate the proposed moment matching-based loop shaping approach on a benchmark example.**

I. Introduction

In model reduction, moment matching-based approximation techniques stand out as computationally efficient and easy to implement [1]. The notion of moment is related to the unique solution of a Sylvester equation ([2], [3]). For a given linear system of high dimensions, families of parametrized reduced order models are computed based on the time-domain approach ([4], [5]).

In [6] for Krylov-based, moment matching techniques, the degrees of freedom left have been computed such that all the poles of the reduced order model are placed at desired locations. In [7], selecting the free parameters in the family of parametrized reduced order models, all matching a set of prescribed moments, a unique model that places poles, zeros and matches higher order moments as constraints, is computed. Therein, for some of the results, the interpolation points as well as the prescribed poles or zeros are restricted to real numbers. However, in the control engineering practice, the sets of poles or the sets of zeros of the rational transfer function of a linear, time-invariant (LTI) system may contain both real numbers as well as pairs of complex-conjugated numbers. Therefore, we revisit the problem of time-domain moment matching model order reduction problem with pole-zero placement in the complex-conjugated case. Thus, in this paper, we derive formulae for the free parameters, such that the reduced order model matching a set of prescribed moments at complex-conjugated interpolation frequencies has pairs of complex-conjugated poles and zeros placed at prescribed locations. Furthermore, we use the results of the moment matching with prescribed poles and zeros to derive a procedure for solving a standard loop shaping controller design problem, see, e.g., [8, Chapter 7]. The method utilizes both the parametrization of the reduced order models achieving moment matching at simple pairs of complex-conjugated frequencies as well as the computation of the free parameters to yield the unique model with the closed-loop poles at prescribed locations. The proposed approach adds closed-loop pole placement (via moment matching) to enhance the classic loop shaping control, where the closed-loop poles are not prescribed, but placed arbitrarily.

The paper is organized as follows. In Section II, we formulate the problem of moment matching for LTI single-input-single-output systems, at complex-conjugated points, with pole-zero placement. In Section III we find the family of parametrized models solving the time-domain moment matching problem for pairs of complex-conjugated points. Subsequently, in Section IV, explicit formulae for the free parameters to place desired poles and/or zeros at desired complex-conjugated locations are derived. Section V is dedicated to the problem of closed-loop pole placement as moment matching, with the moments constrained at -1 and 0. We explicitly compute the free parameters to prescribe the closed-loop poles. Finally, in Section VI, the results are used to solve a robust control problem, imposing prescribed closed-loop poles in a loop shaping procedure.

Notation: A^T denotes the transposed of a matrix A. $\sigma(A)$ represents the spectrum of the square matrix A. $\mathbf{K}(s)$ is the transfer function of an LTI system. For indices from 1 to $N \in \mathbb{N}$, we use the Matlab notation $1:N$.

II. Preliminaries and problem formulation

In this section, we briefly overview the computation of the family of ν order models matching ν moments of a stable LTI system, as in, e.g., [4], [5]. Then, we consider the situation when the interpolation points are complex-valued, arriving in complex-conjugated pairs.

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Consider a single input-single output (SISO) linear time-invariant (LTI) minimal system

$$\Sigma: \ \dot{x} = Ax + Bu, \quad y = Cx, \tag{1}$$

with the state $x \in \mathbb{R}^n$, the input $u \in \mathbb{R}$, the output $y \in \mathbb{R}$ and $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^n, C \in \mathbb{R}^{1\times n}$. The transfer function of (1) is $\mathbf{K}: \mathbb{C} \setminus \sigma(A) \to \mathbb{C}$,

$$\mathbf{K}(s) = C(sI - A)^{-1}B. \tag{2}$$

Throughout the rest of the paper we assume that the system (1) is stable, that is $\sigma(A) \subset \mathbb{C}^-$.

The moments of $\mathbf{K}(s)$ are defined as follows.

Definition 1. [1], [4] The k-moment of $\mathbf{K}(s)$ at $s \in \mathbb{C}$ is $\eta_k(s_1) = (-1)^k/k! \cdot \mathrm{d}^k\,\mathbf{K}(s)/\mathrm{d}s^k \in \mathbb{C}$.

Consider the LTI system

$$\widehat{\Sigma}: \ \dot{\xi} = F\xi + Gu, \quad \psi = H\xi, \tag{3}$$

with $F \in \mathbb{R}^{\nu\times\nu}$, $G \in \mathbb{R}^\nu$ and $H \in \mathbb{R}^{1\times\nu}$, and the corresponding transfer function:

$$\widehat{\mathbf{K}}(s) = H(sI - F)^{-1}G. \tag{4}$$

Let $\{s_i \in \mathbb{C} \setminus \sigma(A) \mid i = 1 : \varsigma\}$, be a symmetric set of complex numbers (including multiplicities). Take $j_i \geq 0$ such that

$$\sum_{i=1}^{\varsigma}(j_i + 1) = \nu. \tag{5}$$

For each i, let $\eta_0(s_i), ..., \eta_{j_i}(s_i)$ denote the ν moments of orders $0 : j_i$ of $\mathbf{K}$ at s_i and let $\widehat{\eta}_0(s_i), ..., \widehat{\eta}_{j_i}(s_i)$ denote the ν moments of orders $0 : j_i$ of $\widehat{\mathbf{K}}$ at s_i. We recall the definition moment matching as follows.

Definition 2 (Moment matching)**.** [1] $\widehat{\mathbf{K}}$ matches ν moments of $\mathbf{K}$ at $\{s_1, ..., s_\varsigma\}$, if $\eta_\kappa(s_i) = \hat{\eta}_\kappa(s_i)$, for all $\kappa = 0 : j_i$, $i = 1 : \varsigma$.

One can parametrize the ν order models (3), with transfer function $\widehat{\mathbf{K}}$, achieving moment matching at a symmetric set of points.

Proposition 1. *[4, Section II-C], [9] Fix $S \in \mathbb{R}^{\nu\times\nu}$, such that $\sigma(S) = \{s_i \in \mathbb{C} \setminus \sigma(A) \mid i = 1 : \varsigma\}$, with $s_i \neq s_j$, $i, j = 1 : \nu$ and $L \in \mathbb{R}^{1\times\nu}$, such that (L, S) is observable. Consider the reduced order model* (3)*, of order ν. Furthermore, assume that $\sigma(F) \cap \sigma(S) = \emptyset$. Then, $\widehat{\mathbf{K}}$ as in* (4) *matches the moments of $\mathbf{K}$, at $\sigma(S)$, if and only if $HP = C\Pi$, where $\Pi \in \mathbb{R}^{n\times\nu}$ is the unique solution of the Sylvester equation $A\Pi + BL = \Pi S$ and $P \in \mathbb{R}^{\nu\times\nu}$ is the unique solution of the Sylvester equation $FP + GL = PS$.*

By [4, Equation (15), Proposition 1], for $P = I$, the system

$$\Sigma_G: \dot{\xi} = (S - GL)\xi + Gu, \quad \psi = C\Pi\xi, \tag{6}$$

with the transfer function

$$\mathbf{K}_G(s) = C\Pi(sI - S + GL)^{-1}G, \tag{7}$$

describes the family of ν order models that match ν moments of $\mathbf{K}$, at $\sigma(S)$, in the sense of Definition 2, for all $G \in \mathbb{R}^\nu$ such that $\sigma(S - GL) \cap \sigma(S) = \emptyset$. Without loss of generality, throughout the rest of the paper, we consider a subfamily of models Σ_G, for all G such that $(C\Pi, S - GL)$ is observable, i.e., minimal realizations (6) of the transfer function (7), see [5], for more details on the minimality of models (6).

In this paper, we restrict to even model orders of the form 2ν and complex-conjugated pairs $\{s_i, \overline{s}_i\}$, $i = 1 : \varsigma$, $s_i = \sigma_i + j\omega_i$, with $\sigma_i, \omega_i \in \mathbb{R}, \omega_i \neq 0$. We now formulate the following particular model moment matching-based model order reduction problem.

Problem 1. Consider a system Σ as in (1) , with the transfer function $\mathbf{K}$, as in (2), and the complex-conjugated pairs $\{s_i, \overline{s}_i\}$, $i = 1 : \varsigma$, $s_i = \sigma_i + j\omega_i$, with $\sigma_i, \omega_i \in \mathbb{R}, \omega_i \neq 0$, such that $\{s_i, \overline{s_i}\} \cap \sigma(A) = \emptyset$, with multiplicity j_i satisfying (5). Compute the 2ν order models $\widehat{\Sigma}$, as in (3), with the transfer function $\widehat{\mathbf{K}}$, as in (4), matching 2ν moments of orders $0 : j_i$ of $\mathbf{K}$ at the complex-conjugated pairs $\{s_i, \overline{s}_i\}$, $i = 1 : \varsigma$, in the sense of Definition 2, such that the following constraints are satisfied

i. $\widehat{\mathbf{K}}$ has 2ℓ poles at $\{\lambda_i, \overline{\lambda}_i\} \in \mathbb{C}\setminus\{\sigma(S) \cup \mathbb{R}\}$, $i = 1 : \ell$,
ii. $\widehat{\mathbf{K}}$ has $2k$ zeros at $\{z_j, \overline{z}_j\} \in \mathbb{C}\setminus\{\sigma(S) \cup \mathbb{R}\}$, $j = 1 : k$.
iii. $2\ell + 2k = 2\nu$.

III. Moment matching at simple pairs of complex-conjugated points

Consider the system (1) with the transfer function $\mathbf{K}(s)$ and a simple pair $\{s_1, \overline{s_1}\}$ in the resolvent of A, where $s_1 = \sigma + j\omega$, with $\sigma, \omega \in \mathbb{R}, \omega \neq 0$. Let the zero order moment at s_1 be $\eta_0(s_1) = \mathbf{K}(s_1) = C(s_1 I - A)^{-1}B \in \mathbb{C}$. Then, the zero order moment at $\overline{s_1}$ is $\eta_0(\overline{s_1}) = \overline{\eta_0(s_1)}$.

Lemma 1. *Consider the system* (1) *with the transfer function* $\mathbf{K}$ *and let* $s_1 = \sigma + j\omega$, *with* $\sigma, \omega \in \mathbb{R}, \omega \neq 0$, *such that* $\{s_1, \overline{s_1}\} \cap \sigma(A) = \emptyset$. *Then the zero order complex moment* $\eta_0(s_1) = \operatorname{Re}\eta_0(s_1) + j\operatorname{Im}\eta_0(s_1)$ *satisfies the relations*

$$\operatorname{Re}\eta_0(s_1) = C\operatorname{Re}\Pi_0, \tag{8a}$$

$$\operatorname{Im}\eta_0(s_1) = C\operatorname{Im}\Pi_0, \tag{8b}$$

where $\Pi_0 = \operatorname{Re}\Pi_0 + j\operatorname{Im}\Pi_0 \in \mathbb{C}^n$ *is the unique solution of the system real-valued equations*

$$A\operatorname{Im}\Pi_0 = \sigma\operatorname{Im}\Pi_0 + \omega\operatorname{Re}\Pi_0, \tag{9a}$$

$$A\operatorname{Re}\Pi_0 + B = \sigma\operatorname{Re}\Pi_0 - \omega\operatorname{Im}\Pi_0. \tag{9b}$$

Proof: Substituting $s = s_1 = \sigma + j\omega$ in (2) yields $\mathbf{K}(\sigma + j\omega) = C(\sigma I + j\omega I - A)^{-1}B = C\Pi_0$, where $\Pi_0 \in \mathbb{C}^n$ is the unique solution of the complex Sylvester equation $A\Pi_0 + B = \Pi_0\sigma + \Pi_0 j\omega$. Similarly, $\overline{\eta_0(s_1)} = C\overline{\Pi_0}$, where $\overline{\Pi_0}$ is the solution of the complex Sylvester equation $A\overline{\Pi_0} + B = \overline{\Pi_0}\sigma - \overline{\Pi_0}j\omega$ Hence, one can write $[\eta_0(s_1)\ \overline{\eta_0(s_1)}] = C[\Pi_0\ \overline{\Pi_0}]$, where $[\Pi_0\ \overline{\Pi_0}] \in \mathbb{C}^{n\times 2}$ is the unique solution of the complex Sylvester equation

$$A[\Pi_0\ \overline{\Pi_0}] + [B\ B] = [\Pi_0\ \overline{\Pi_0}]\Sigma_c, \tag{10}$$

with $\Sigma_c = \mathrm{diag}(\sigma + j\omega, \sigma - j\omega) \in \mathbb{C}^{2\times 2}$. Letting

$$\Sigma_r = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix} \in \mathbb{R}^{2\times 2}, \tag{11}$$

note that $\Sigma_r T = T\Sigma_c$, with

$$T = \frac{1}{\sqrt{2}}\begin{bmatrix} -j & j \\ 1 & 1 \end{bmatrix} \Leftrightarrow T^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} j & 1 \\ -j & 1 \end{bmatrix}. \tag{12}$$

Now substitute $\Sigma_c = T^{-1}\Sigma_r T$ in (10). Since $[\eta_0(s_1)\ \overline{\eta_0(s_1)}]T^{-1} = \sqrt{2}[-\mathrm{Im}\,\eta_0(s_1)\ \ \mathrm{Re}\,\eta_0(s_1)]$ and $[\Pi_0\ \overline{\Pi_0}]T^{-1} = \sqrt{2}[-\mathrm{Im}\,\Pi_0\ \ \mathrm{Re}\,\Pi_0]$, the claim follows. ■

The next result states that a pair of complex-conjugated moments at a pair of complex-conjugated points can be characterized by a real valued vector in $\mathbb{R}^{1\times 2}$.

Proposition 2. *Consider the system* (1) *with the transfer function* $\mathbf{K}$ *as in* (2). *Let* $s_1 = \sigma + j\omega$, *with* $\sigma, \omega \in \mathbb{R}, \omega \neq 0$, *such that* $\{s_1, \overline{s_1}\} \cap \sigma(A) = \emptyset$. *Then the zero order moments* $\eta_0(s_1) \in \mathbb{C}$ *of* $\mathbf{K}$ *at* $\{s_1, \overline{s_1}\}$ *are in a one-to-one relation*[1] *with the matrix* $C\Pi_r \in \mathbb{R}^{1\times 2}$, *where* $\Pi_r \in \mathbb{R}^{n\times 2}$ *is the unique solution of the real valued Sylvester equation*

$$A\Pi_r + BL_r = \Pi_r\Sigma_r, \tag{13}$$

with Σ_r *as in* (11) *and*

$$L_r = [1\ 1]T^{-1} = \sqrt{2}\cdot[0\ 1],$$

where T *is given by* (12).

Note that, by construction, the pair (L_r, Σ_r) is observable. Consider ν distinct pairs of complex-conjugated points $\{s_i, \overline{s_i}, i = 1:\nu\} \subset \{\mathbb{C} \setminus \mathbb{R}\}$. Proposition 2 extends to any matrix S such that $\sigma(S) = \{\{s_i, \overline{s_i}\}, i = 1:\nu\}$.

Corollary 1. *Consider the system* (1) *with the transfer function* $\mathbf{K}$ *as in* (2). *Let* $s_i \in \mathbb{C}\setminus\mathbb{R}$, *such that* $\{s_i, \overline{s_i}\} \cap \sigma(A) = \emptyset$, $i = 1:\nu$. *Then the zero order moments* $\eta_0(s_i) \in \mathbb{C}, i = 1:\nu$ *of* $\mathbf{K}$ *at are in a one-to-one relation (up to a scaling with a coordinate transformation* $U \in \mathbb{R}^{2\nu\times 2\nu}$*) with the matrix* $C\Pi \in \mathbb{R}^{1\times 2\nu}$, *where* $\Pi \in \mathbb{R}^{n\times 2\nu}$ *is the unique solution of the real valued Sylvester equation*

$$A\Pi + BL = \Pi S, \tag{14}$$

where $S \in \mathbb{R}^{2\nu\times 2\nu}$ *is any matrix such that* $\sigma(S) = \sigma(\mathcal{T}\cdot \mathrm{diag}(\{s_i, \overline{s_i}\})\cdot\mathcal{T}^{-1}), i = 1:\nu$, *with* $\mathcal{T} = \mathrm{diag}(\underbrace{T, \ldots, T}_{\nu\ times})$, T *is as in* (12) *and* $LU = \sqrt{2}\cdot[0\ 1\ \ldots\ 0\ 1] \in \mathbb{R}^{1\times 2\nu}$.

Note that, by construction, the pair (L, S) is observable.

A real-valued time-domain interpretation: Following the arguments in [4], [9], consider the signal generator described by the system

$$\begin{aligned} \dot{\omega} &= \Sigma_c\omega,\ \omega \in \mathbb{C}^2,\ \Sigma_c \in \mathbb{C}^{2\times 2}, \\ \theta &= L\omega, \end{aligned} \tag{15}$$

with the matrix $L = [1\ 1]$ and the initial condition $\omega_0 \neq 0$, such that (Σ_c, ω_0, L) is a minimal realization [10]. Note that, even if Σ_c is not a real-valued matrix, but a complex matrix with a spectrum made of a pair of complex-conjugated eigenvalues, the signal $\theta(t)$ is real. Since we would like to model the signal generator (15) with a linear system, with real-valued dynamics, we apply the transformation T, as in (12). Hence, changing the coordinates, (15) becomes

$$\begin{aligned} \dot{\widetilde{\omega}} &= \Sigma_r\widetilde{\omega},\ \widetilde{\omega} \in \mathbb{R}^2,\ \widetilde{\omega}_0 \neq 0,\ \Sigma_r = T\Sigma_c T^{-1} \in \mathbb{R}^{2\times 2}, \\ \theta &= L_r\widetilde{\omega}, \end{aligned} \tag{16}$$

with $L_r = [1\ 1]T^{-1} = \sqrt{2}\cdot[0\ 1]$. Hence θ is a real sinusoidal signal, that can be modeled with a real-valued signal generator. We now present the relation between the moments at a pair of complex-conjugated points and the steady-state of the output real signal of (1) when the input is excited by a harmonic signal generator (with complex-conjugated spectrum).

Proposition 3. *[4], [9] Consider the system* (1) *with the transfer function* $\mathbf{K}$ *as in* (2). *Let* $s_1 \in \mathbb{C}\setminus\mathbb{R}$, *such that* $\{s_1, \overline{s_1}\} \cap \sigma(A) = \emptyset$. *Then the zero order moments* $\eta_0(s_1) \in \mathbb{C}$ *of* $\mathbf{K}$ *at are in a one-to-one relation with the steady-state of the output* $y(t)$ *of* (1) *interconnected with the signal generator* (15)*, or, equivalently,* (16).

The interconnection between a signal generator and a linear system is illustrated in Figure 1.

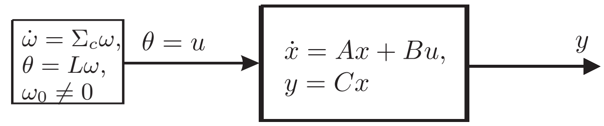

Fig. 1. Diagram representing the output of a linear system subject to an input generated by a signal generator

IV. Moment matching with pole-zero placement

In this section we assume that the matrices A, B, C describing (1) and the transfer function $\mathbf{K}$ as in (2) are not available. However, we assume we have access to the class of 2ν order models Σ_G as in (6) and the transfer functions $\mathbf{K}_G$ as in (7), with the output matrix $C\Pi = H$ obtained through input-output data measurements. The results presented in the sequel are extensions of the statements of [7, Proposition 3, Proposition 5] to the case of distinct pairs of complex-conjugated points.

Consider the system (1) and the class of reduced 2ν order models Σ_G in (6) that match ν pairs of complex-conjugated zero order moments of $\mathbf{K}$. Note that, since ν is even, for the sake of simplicity, we take the model order to be 2ν instead of ν, in the sequel.

[1] By one-to-one relation between a set of moments and the elements of a matrix, we mean that the moments are uniquely determined by the elements of the matrix.

A. Pole placement

Let $\lambda_i \in \mathbb{C} \setminus \mathbb{R}$, $i = 1 : 2\ell$, $\ell \le \nu$ be such that $\lambda_i \notin \sigma(S)$ and $\lambda_i \neq \lambda_j$, $i, j = 1 : 2\ell$. Then λ_i are poles of Σ_G if $\det(\lambda_i I - S + GL) = 0$, $i = 1 : 2\ell$, such that $\{\lambda_1, \dots, \lambda_{2\ell}\}$ is a symmetric set. When $S = \operatorname{diag}(\Sigma_{r_1}, \dots, \Sigma_{r_\nu}) \in \mathbb{R}^{2\nu \times 2\nu}$, with

$$\Sigma_{r_i} = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}, \; \sigma_i, \omega_i \in \mathbb{R}$$

and the zero-order moments are considered, the pole placement problem boils down to solving a linear system in the unknown $G \in \mathbb{R}^{2\nu}$.

Proposition 4. *Let $S = \operatorname{diag}(\Sigma_{r_1}, \dots, \Sigma_{r_\nu}) \in \mathbb{R}^{2\nu \times 2\nu}$, with*

$$\Sigma_{r_i} = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}, \; \sigma_i, \omega_i \in \mathbb{R}$$

and $L = \sqrt{2} \cdot [0 \; 1 \; \dots \; 0 \; 1] \in \mathbb{R}^{1 \times 2\nu}$. Then $\{\lambda_1, \dots, \lambda_{2\ell}\} \subseteq \{\mathbb{C} \setminus \mathbb{R}\}$, with $\ell \le \nu$, are a symmetric set of simple poles of $\mathbf{K}_G(s)$ given by (7) *if and only if $G = \begin{bmatrix} g_1 & \dots & g_{2\nu} \end{bmatrix}^T \in \mathbb{R}^{2\nu}$ is a solution of the linear system*

$$1 + L D_\kappa^{-1} G = 0, \quad \forall \kappa = 1 : 2\ell, \tag{17}$$

with $D_\kappa = \operatorname{diag}(\Theta_{\kappa 1}, \dots, \Theta_{\kappa \nu})$, where

$$\Theta_{\kappa i} = \begin{bmatrix} \lambda_\kappa - \sigma_i & -\omega_i \\ \omega_i & \lambda_\kappa - \sigma_i \end{bmatrix}, \quad i = 1 : \nu, \; \kappa = 1 : 2\ell. \tag{18}$$

B. Zero placement

Consider the system (1) and the family of 2ν order models Σ_G that approximate (1) by matching ν pairs of complex-conjugate zero order moments of $\mathbf{K}$, for all $G \in \mathbb{R}^{2\nu}$. Let $\{z_1, \dots, \; z_{2k}\} \subset \{\mathbb{C} \setminus \mathbb{R}\}$ be a symmetric set of distinct points, with $k < \nu$ and $z_i \neq s_j$, $i = 1 : k, j = 1 : \nu$. By, e.g., [4], [11], [12], there exists a subfamily of models Σ_G, such that the set of zeros of each model contains $z_1, \dots, \; z_{2k}$. Equivalently, there exists G such that

$$\det \begin{bmatrix} S - GL - z_i I & G \\ C\Pi & 0 \end{bmatrix} = \det \begin{bmatrix} S - z_i I & G \\ C\Pi & 0 \end{bmatrix} = 0, i = 1 : 2k. \tag{19}$$

When $S = \operatorname{diag}(\Sigma_{r_1}, \dots, \Sigma_{r_\nu}) \in \mathbb{R}^{2\nu \times 2\nu}$, with

$$\Sigma_{r_i} = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}, \; \sigma_i, \omega_i \in \mathbb{R}$$

and the zero-order moments are considered, the zero placement problem boils down to solving a linear system in the unknown $G \in \mathbb{R}^{2\nu}$.
Let $\nu = 1$. The following result holds.

Lemma 2. *Let $S = \Sigma_r$, as in* (11) *and consider the second order model as in* (6)*, with $C\Pi = [\eta_{0_r} \; \eta_{1_r}] \in \mathbb{R}^{1 \times 2}$. Then $z = \sigma_z + j\omega_z \in \mathbb{C}, z \neq \sigma + j\omega$ is a zero of $\mathbf{K}_G$ if and only if $G = \begin{bmatrix} g_1 & g_2 \end{bmatrix}^T \in \mathbb{R}^2$ is a solution of the linear system*

$$C\Pi(\Sigma_r - zI)^{-1} G = 0. \tag{20}$$

Note that, in practice, (20) is difficult to compute when $k > 2$. The next result is the counterpart of [7, Proposition 5], in the complex-conjugate case.

Proposition 5. *Let $S = \operatorname{diag}(\Sigma_{r_1}, \dots, \Sigma_{r_\nu}) \in \mathbb{R}^{2\nu \times 2\nu}$, with*

$$\Sigma_{r_i} = \begin{bmatrix} \sigma_i & \omega_i \\ -\omega_i & \sigma_i \end{bmatrix}, \; \sigma_i, \omega_i \in \mathbb{R}$$

and let $C\Pi = [\eta_1 \; \dots \; \eta_{2\nu}] \in \mathbb{R}^{1 \times 2\nu}$. Then Σ_G, as in (6)*, is a model with the set $\{z_j, j = 1 : 2k\} \subset \mathbb{C} \setminus \mathbb{R}$, $z_j \neq s_i$, $i = 1 : 2\nu, j = 1 : 2k$, among the zeros of the transfer function $\mathbf{K}_G(s)$, if and only if the elements of $G = [g_1 \; \dots \; g_{2\nu}]^T$ satisfy*

$$C\Pi(S - z_j I)^{-1} G = 0, \; \forall j = 1 : 2k. \tag{21}$$

V. Moment matching-based closed-loop pole placement

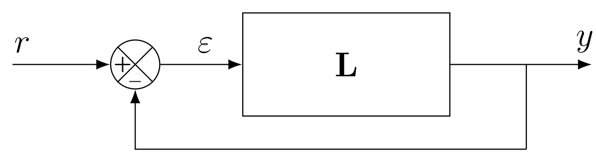

Fig. 2. Closed-loop system

In this section, we consider the standard closed-loop system in Figure 2, where $\mathbf{L}$ is regarded as an open-loop transfer function, of order μ, resulting from a loop shaping control design procedure as in [8, Chapter 7]. We show how to place the closed-loop poles through moment matching constraints satisfied by $\mathbf{L}$. Furthermore, when $\mathbf{L}$ is parametrized by (6), we derive an explicit formula for the free parameters placing all the closed-loop poles of $\mathbf{L}$ in prescribed locations.

A. Closed-loop poles as moments of the open loop system

We now describe the connection between the closed-loop poles of the configuration in Figure 2 and a particular set of moment matching constraints satisfied by $\mathbf{L}$.

Lemma 3. *Consider the system $\mathbf{L}(s)$ as in* (3)*, with real-valued coefficients, of order μ in the closed-loop configuration in Figure 2. Let $\mathcal{P} = \{p_i \in \mathbb{C} \mid i = 1 : \varsigma_p\}$, be a symmetric set of complex numbers (including multiplicities). Take $\kappa_i \ge 0$ such that $\sum_{i=1}^{\varsigma_p} (\kappa_i + 1) = \mu$. For each i, let $\eta_0(p_i), \dots, \eta_{\kappa_i}(p_i)$ denote the μ moments of orders $0 : \kappa_i$ of $\mathbf{L}$ at the given points p_i. Then $p_i, \; i = 1 : \varsigma_p$ are the closed-loop poles of the system in Figure 2, if and only if the μ moments of $\mathbf{L}$ at $\mathcal{P}$ satisfy the constraints*

$$\begin{aligned} \eta_0(p_i) &= -1, \\ \eta_{\kappa_i}(p_i) &= 0. \end{aligned} \tag{22}$$

B. Moment matching with closed-loop pole placement

Consider a system (1), of order n, and the family of μ order models Σ_G that approximate (1) by matching μ moments, for all $G \in \mathbb{R}^\mu$. Assume $\mathcal{P} = \{p_1, \dots, \; p_\mu\} \subset \mathbb{C} \setminus \sigma(A)$ is a symmetric set of distinct points such that $\mathbf{K}(p_i) = -1$, for all $i = 1 : \mu$. That is, assume that $\mathcal{P}$ is a subset of closed-loop poles of $\mathbf{K}$. We consider the case of simple points (without multiplicity). Then, there exists a model $\mathbf{L}(s) = \mathbf{K}_G(s) = C\Pi(sI - S + GL)^{-1} G$, as in (7), in the family Σ_G, with the closed loop poles placed in $\mathcal{P}$. Let

$$\mathbf{E}(s) = 1 + \mathbf{L}(s) = 1 + C\Pi(sI - S + GL)^{-1}G.$$

Hence, we seek $G \in \mathbb{R}^\mu$ such that

$$\mathbf{E}(p_i) = 0 \Leftrightarrow \mathbf{L}(p_i) = -1,$$

for all $i = 1 : \mu$.
Let $Q_\mathbf{C} \in \mathbb{R}^{\mu\times\mu}$ be such that $\sigma(Q_\mathbf{C}) = \mathcal{P} = \{p_1, \dots, p_\mu\}$ such that the pair $(Q_\mathbf{C}, -\mathcal{I})$ is controllable, where $\mathcal{I} = [1\ 1\ \dots\ 1]^T \in \mathbb{R}^\mu$. Let $\Upsilon_\mathbf{C} \in \mathbb{R}^{\mu\times n}$ be the unique solution of the Sylvester equation

$$Q_\mathbf{C}\Upsilon_\mathbf{C} = \Upsilon_\mathbf{C}A - \mathcal{I}C. \tag{23}$$

Note that $\operatorname{rank}\Upsilon_\mathbf{C} = \mu$, see, e.g., [13]. Since p_i are closed-loop poles of $\mathbf{K}$, i.e., $\mathbf{K}(p_i) = -1, i = 1 : \mu$, then we have that $\Upsilon_\mathbf{C}B = \mathcal{I}$. The next result imposes linear constraints on G such that $\mathbf{L}(s)$ has μ closed-loop, simple poles at $\mathcal{P} = \{p_1, \dots, p_\mu\}$, as well.

Proposition 6. *Let $\mathbf{L}(s)$, as in (7), be a μ order, parametrized model, in the family Σ_G, as in (6), that matches μ moments of $\mathbf{K}$ at $\sigma(S)$. Consider a matrix $Q_\mathbf{C}$, with $\sigma(Q_\mathbf{C}) = \mathcal{P} = \{p_1, \dots, p_\mu\} \subset \mathbb{C}\setminus\sigma(A)$, a symmetric set of distinct points, such that $\mathbf{K}(p_i) = -1$, for all $i = 1 : \mu$. Furthermore, let $\Upsilon_\mathbf{C} \in \mathbb{R}^{\mu\times n}$ be the unique solution of (23), such that $\Upsilon_\mathbf{C}B = \mathcal{I}$ and assume that the matrix $\Upsilon_\mathbf{C}\Pi$ is invertible. If $G \in \mathbb{R}^\mu$ is the solution of the equation*

$$\Upsilon_\mathbf{C}\Pi G = \mathcal{I}, \tag{24}$$

then $\mathcal{P} = \sigma(Q_\mathbf{C})$, is the set of closed-loop poles of $\mathbf{L}$.

Remark 1. When $p_i \in \mathcal{P} \cap \{\mathbb{C}\setminus\mathbb{R}\}$ have multiplicity one, using Proposition 4 to $1/(1+\mathbf{L})$, closed-loop pole placement is achieved through solving a corresponding equation (17). Alternatively, applying Proposition 5 to $1 + \mathbf{L}$, closed-loop pole placement is achieved through solving a corresponding equation (21).

VI. Illustrative example

Consider a stable, minimum-phase plant $\mathbf{K}(s)$ as in the preamble of [8, Section 7.1], of relative degree [2]1, with an unknown explicit model, for which one has to design a robust, internally stabilizing controller $\mathbf{C}(s)$, as described in [8, Example 1, Section 7.3]. To obtain a controller through loop shaping synthesis techniques, one must find a rational function $\mathbf{L}(s)$ so that the following hold:

- the closed-loop system in Figure 2 is internally stable[3],
- $|\mathbf{L}(j\omega)| \geq \dfrac{a}{1-|\mathbf{W}_2(j\omega)|}$, $\omega \leq 1$ rad/s,
- $|\mathbf{L}(j\omega)| \leq \dfrac{1}{|\mathbf{W}_2(j\omega)|}$, $\omega \geq 20$ rad/s,

[2]The relative degree of a rational transfer function is the difference between the degree of the denominator polynomial and the nominator polynomial. A transfer function with positive relative degree is strictly proper.

[3]The closed-loop system in Figure 2 is internally stable, as in [8, Example 1, Section 7.3], if and only if all the transfer functions from the exogenous inputs to the internal signals are stable.

with

$$\mathbf{W}_2(s) = \frac{0.01(s+1)}{20(0.01s+1)}$$

and $a = 10$. The constraint a is obtained with a fifth order Butterworth, low-pass filter $\mathbf{W}_1$. To satisfy the constraints, it proves to be sufficient that

$$|\mathbf{L}(j)| = 30,\ \left|\mathbf{L}(10^{-2}j)\right| = 700,\ |\mathbf{L}(20j)| = 0.3.$$

We now design $\mathbf{L}$ to satisfy the moment matching conditions

$$\mathbf{L}(\pm j) = 46.21 \pm 25.44j,\ \mathbf{L}(\pm 10^{-2}j) = 469.52 \pm 698.25j,$$
$$\mathbf{L}(\pm 20j) = -0.5 \pm 0.07j.$$

Furthermore, $\mathbf{L}(s)$ is parametrized as in (7), with a state-space realization (6), where

$$S = \operatorname{diag}\left(\begin{bmatrix} 0 & 10^0 \\ -10^0 & 0\end{bmatrix}, \begin{bmatrix} 0 & 10^{-2} \\ -10^{-2} & 0\end{bmatrix}, \begin{bmatrix} 0 & 2\cdot 10^1 \\ -2\cdot 10^1 & 0\end{bmatrix}\right)),$$
$$L = \sqrt{2}\begin{bmatrix}0 & 1 & 0 & 1 & 0 & 1\end{bmatrix},$$

such that (L, S) is an observable pair and

$$C\Pi = \begin{bmatrix}-25.24 & 16.21 & -698.25 & 49.52 & -0.07 & -0.5\end{bmatrix},$$

for all $G \in \mathbb{R}^6$. For the closed-loop system in Figure 2, we select the stabilizing poles in the set

$$\mathcal{P} = \{-3 \pm 6j, -1 \pm 4j, -3 \pm 2j\}.$$

Applying Lemma 3 along with Proposition 5 yields

$$G = \begin{bmatrix}0.7601 & 0.1842 & -0.2836 & -3.5403 & 13.7379 & 14.3537\end{bmatrix}^T.$$

Then, the resulting open-loop transfer function

$$\mathbf{L}(s) = \frac{-1.55s^5 + 123.6s^4 + 2623s^3 + 3296s^2 + 9089s + 9943}{s^6 + 15.55s^5 + 11.36s^4 - 1883s^3 - 416.9s^2 - 2003s + 1.56}.$$

satisfies the loop shaping constraints, see Figure 3, and stabilizes the closed-loop system in Figure 2, placing the closed-loop poles in the set $\mathcal{P}$, see Figure 4. Indeed $\mathbf{L}$ achieves robust stabilization and performance, since

$$\| |\mathbf{W}_1\mathbf{S}| + |\mathbf{W}_2\mathbf{T}| \|_\infty < 1,$$

where $\mathbf{S} = 1/(1+\mathbf{L})$ and $\mathbf{T} = 1 - \mathbf{S}$ [8], see Figure 5. Note that the procedure yields an even order of $\mathbf{L}$, twice the number of interpolation frequency pairs. Furthermore, since the relative degree of $\mathbf{L}$ is also 1, the resulting controller $\mathbf{C} = \mathbf{L}/\mathbf{K}$ is a biproper transfer function, hence physically implementable. The closed-loop pole placement is a major advantage over the classic loop shaping procedure in [8], where at each iteration, the stability of the closed-loop system must be checked. However, a compromise between a systematic procedure of loop shaping with closed-loop pole placement and increasing the controller dimension must be made. Moreover, the choices of the interpolation frequencies, the values of the moments, the functions $\mathbf{W}_1, \mathbf{W}_2$ and the locations of the closed-loop poles are not yet systematic, but based experience, see also [8].

Remark 2. When an explicit model $\mathbf{K}(s)$ is available, one can apply Propositions 4 or 5 to obtain a stabilizing controller through moment matching-based loop shaping design.

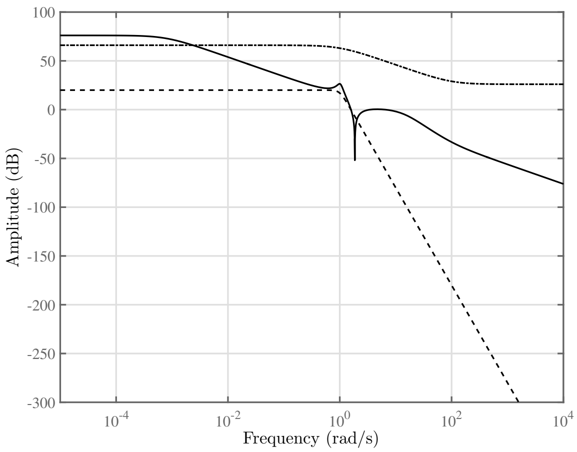

Fig. 3. Magnitude plots of $\mathbf{L}$ (solid), $\mathbf{W}_1$ (dash), $\dfrac{1}{\mathbf{W}_2}$ (dash-dot)

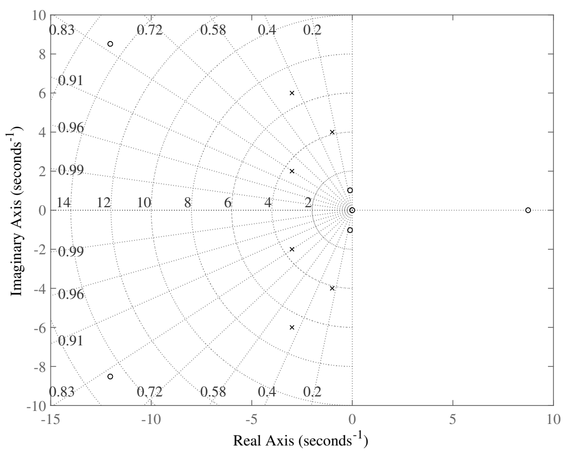

Fig. 4. The closed-loop pole (cross) and zero (circle) map for the closed -loop system $1/(1+\mathbf{L})$, in Figure 2

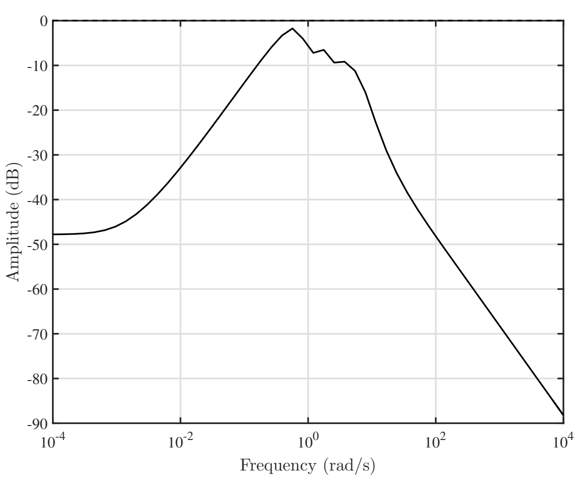

Fig. 5. The loop shaping condition $\||\mathbf{W}_1\mathbf{S}| + |\mathbf{W}_2\mathbf{T}|\|_\infty < 1$, where $\mathbf{S} = 1/(1+\mathbf{L})$ and $\mathbf{T} = 1 - \mathbf{S}$, as in [8]

VII. Conclusions

In this paper, we have revisited the problem of time-domain moment matching model order problem with pole-zero placement in the complex-conjugated framework. We have also solved the problem of finding the free parameters, such that the reduced order model achieving moment matching at complex-conjugated interpolation points, has pairs of complex-conjugated poles and zeros in prescribed locations. We have proven that closed-loop pole placement is also moment matching, with the moments fixed in -1 and 0. We have computed a formula to have (simple) closed-loop poles placed at desired locations. We have combined the time-domain moment matching at complex-conjugated frequencies with loop shaping to design a stabilizing controller with prescribed closed-loop pole placement.

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