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Conduction and entropy analysis of a mixed memristor-resistor model for neuromorphic networks

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Abstract
To build neuromorphic hardware with self-assembled memristive networks, it is necessary to determine how the functional connectivity between electrodes can be adjusted, under the application of external signals. In this work, we analyse a model of a disordered memristor-resistor network, within the framework of graph theory. Such a model is well suited for the simulation of physical self-assembled neuromorphic materials where impurities are likely to be present. Two primary mechanisms that modulate the collective dynamics are investigated: the strength of interaction, i.e. the ratio of the two limiting conductance states of the memristive components, and the role of disorder in the form of density of Ohmic conductors (OCs) diluting the network. We consider the case where a fraction of the network edges has memristive properties, while the remaining part shows pure Ohmic behaviour. We consider both the case of poor and good OCs. Both the role of the interaction strength and the presence of OCs are investigated in relation to the trace formation between electrodes at the fixed point of the dynamics. The latter is analysed through an ideal observer approach. Thus, network entropy is used to understand the self-reinforcing and cooperative inhibition of other memristive elements resulting in the formation of a winner-take-all path. Both the low interaction strength and the dilution of the memristive fraction in a network provide a reduction of the steep non-linearity in the network conductance under the application of a steady input voltage. Entropy analysis shows enhanced robustness in selective trace formation to the applied voltage for heterogeneous networks of memristors diluted by poor OCs in the vicinity of the percolation threshold. The input voltage controls the diversity in trace formation.

1. Introduction
As modern life demand for computational power increases, new types of hardware solutions are needed and neuromorphic computing has gained increasing attention over the years [1]. At the same time, memristive devices have been proposed, and effectively used, as the basic computational units in neuromorphic applications. They offer multiple resistance values making them suitable candidates for building synapses and neurons, either when they show non-volatile or volatile behaviour, respectively.

Wiring together memristive devices enables the realization of networks and the consequent exploitation of higher computational and learning capabilities similarly to those observed in the brain [2, 3]. Memristive networks have been proposed in a top-down design to build cross-bar arrays to implement a fast and energy-efficient hardware replacement for computing steps in modern neural-network models in artificial intelligence [4]. Nevertheless, the biological world has no topdown design. It relies on self-organization emerging from a combination of topology and functions in a way that is not fully understood. Another key
role is indeed played by the number of neurons and synapses, enabling fault-tolerance and noise resistant computation in the brain. Self-assembled networks of memristive devices are a new class of substrates that show similar features to the brain and that are receiving growing attention in the neuromorphic community. Among the many candidate hardware substrates showing collective behaviour and non-linear response, we mention highly dense self-assembled nanowire networks [5–7], metallic nanojunctions [8, 9] and domain wall networks in ferroelectric-ferroelastic thin film crystals that have shown memristive capabilities and the potential for information processing task [10].

It has been demonstrated that networks of memristors effectively behave as a single memristor [11] with the advantage that they can realize a larger number of resistance values than a single memristor, but they also provide new advantages as the tunability of the voltage threshold as consequence of the choice of the input/output electrode location on the network, and the tunability of the strength of interaction by modulating the $G_{\text{max}}/G_{\text{min}}$ ratio, where $G_{\text{max}}$ and $G_{\text{min}}$ are the maximum and minimum states of conductance of the single memristor.

To exploit the behaviour of these systems, a completely different approach to computation needs to be taken into consideration, leading to the concept of analogue parallel memristive-hardware [12]. In such an analogue parallel processor, a key feature is represented by the collective dynamics of the elements in the system and the possibility of tuning it to the requirement of the task to be performed. In this sense, the possibility of tuning a sufficiently high interaction strength proves to be essential in boosting the cooperation of the elements in the parallel processor [13], thus making the state of one of them dependent on the state of many other elements.

The presence of disorder also plays a role in modulating the collective dynamics of the system. Disorder can be included through the topological constraints the evolving system copes with. In the context of finding the shortest path between the two nodes of a graph, it has been shown that networks with random geometries are more prone to provide fewer active traces, making such systems more suited to use in solving mazes in a parallel efficient way when compared to more regular topological structures such as lattices [13, 14].

In this work, we focus on quenched disorder by diluting a network of memristors with Ohmic conductor (OC) impurities. The presence of impurities is a more realistic approach to modelling self-assembled conductive networks in materials where, as experimental evidence suggests [10, 15], a mixture of different electrical properties coexist. Simulations are still highly needed to shed light on potential applications and in general to understand the behaviour of these highly complex non-linear systems. With this aim, we develop a model where we make use of graph theory to simulate mixed networks where dynamic, i.e. memristive and static, i.e. OCs, edges coexist.

In our work, we focus on the role of the strength of interaction, i.e. $G_{\text{max}}/G_{\text{min}}$ ratio, combined with the role of disorder introduced in the network by the density of memristive/resistive edges. To comprehend the collective behaviour of these systems and how the functional connectivity transforms under the application of external stimuli, we measure the global conductance and the entropy of the internal distribution of conductance states of the network. Our aim is to quantify the transformation of the whole network at the steady state induced by the DC voltage bias, such that when the source is active the network evolves towards a new equilibrium state. This corresponds to the limit of slow frequency input signal where each memristor in the network has time to adjust its value to both the value of the input signal and the other memristors in the network.

1.1. Related works in the literature

Previous work in the literature by Oskooee and Sahimi [16] addressed networks diluted with either ‘poor’ or ‘good’ OCs. Poor_OCs refer to the edges where the conductance is set to the minimum, while Good_OCs have a good, even maximum conductance of the individual memristors in the network, respectively. The response of the network was analysed under the application of an AC input voltage in relation to the density, $p$, of memristive elements. The authors found that while memristive networks diluted with Poor_OCs showed a global memristive behaviour with the characteristic pinched $I$–$V$ curve only above the percolation threshold, networks that are diluted with Good_OCs also showed a strong memristive phase above the percolation threshold together with a weak memristive phase below the percolation threshold.

In a similar work by Sheldon and Di Ventra [17], quenched disorder has been introduced in a network of memristors through various distributions of current thresholds. Such networks were investigated when subjected to a slowly increasing voltage bias in the adiabatic limit and in relation to the interaction strength $G_{\text{max}}/G_{\text{min}}$. A switching of the whole network to the high-conductive state was observed as a consequence of the formation of a backbone trace in the network. For higher values of the interaction strength, the switching of the network abruptly resembles a first order phase transition and the produced backbone pathways inhibit the switching of other elements thus reducing the number of available states to the network.
We mention the work of Manning et al [15] where winner-take-all path (WTAP) behaviour was deeply investigated through both measurements on nanowire networks in an experimental setup and in simulations. In the latter, both memristive junctions and Ohmic wires were taken into consideration. Their work showed that the conductance state of the entire network due to the application of a current signal in the input is encoded in a few active traces composed of memristors in the high-conductance state. Furthermore, even though the authors showed that in the case where the wires were more conductive than the memristive elements in the system, the WTAP would not take place. However, a systematic study on the role of dilution on the functional connectivity that emerges from the formation of such backbone pathways has not been addressed as yet.

2. Methods

2.1. Simulation scheme

Numerical simulations of neuromorphic hardware are essential both to understand the behaviour of materials showing memristive properties and to study their potential as neuromorphic hardware candidates in applications. Nevertheless, numerical simulation of large networks of memristive devices still represents a challenge and a variety of approaches have been investigated. Algebraic graph theory and, in particular, the use of Modified Voltage Nodal Analysis (MVNA) [18] provides a powerful and versatile tool to simulate the electrical behaviour of self-assembled networks with memristive properties (see e.g. [5, 6, 19–21]).

In the simulation scheme used in this work, a network topology is instantiated. Edges are either memristors, with a probability \( p \), or resistors with a probability \( 1 - p \). Conductance of the Ohmic edges is set to a fixed value that does not change during the simulation. On the other hand, at time \( t = 0 \), the conductance of the memristor edges is set to the minimum conductance allowed, i.e. \( G_{\text{min}} \). A voltage signal is then applied to drive the network out of equilibrium. The network is frozen at each time step and MVNA solves for the currents flowing into the sources and the voltages over each node of the network. The conductance value of each memristive element is updated according to the voltage distribution computed by MVNA and to the dynamical model chosen to describe the memristor edge (see section 2.2). We mention that MVNA is the formalism used in SPICE-type simulators for electronic circuits [22]. With our implementation, we provide a simulator that is exponentially faster in the number of memristors in the network compared to previous simulators in the literature [6] (see figure S1 in supplementary material).

Compared to studies that focus on the topological structure of the system [23, 24] and show results on how the topology influences the evolution of the functional connectivity between two electrodes on the network, we rather focus on how dilution influences such connectivity on the substrate. In fact, we show in section 3 that diluting a network with non-memristive impurities can boost the selectivity of the trace formation, when combined with sufficient strength of interaction \( (G_{\text{max}}/G_{\text{min}}) \). In our grid-graph approach, we interpret each node not as a single unit, e.g. a junction between two nanowires as in [23, 24], but as a coarse-grained description of an area of a physical sample with an abundance of connectivity between neighbouring regions, and where the connectivity may show predominantly memristive or non-memristive properties. It is evident that the topology at the microscale in self-assembled materials is more complicated than in a regular grid. However, the conductive properties of regions in the 2D material can be aggregated, for the purpose of obtaining a tractable computation of the effects of purely resistive dilution. A similar approach has been taken by the authors of [6, 7, 10], where first a grid-graph is instantiated and then random diagonal edges are added between the nodes. In our case, the randomness in the instantiated topology is introduced by the random deposition of Ohmic resistors in the system. Furthermore, the choice of the grid graph enables us to show that even in the simple case of a scale-invariant network, the functional connectivity can be tuned by a combination of hyperparameters, i.e. strength of interaction and fraction of memristive components, without requiring a more complex topology. The results are extended to a grid-graph network with random diagonal edges in supplementary material S5.

2.2. Memristor model

Each memristor in the network is modelled with the potentiation-depression rate balance equation originally proposed for a single memristive nanowire [25] and then implemented for a homogeneous network of nanowires [6, 7]:

\[
\frac{dg}{dt} = (1 - g)k_p(V) - gk_d(V)
\]

(1)
where $0 \leq g \leq 1$ is the normalized conductance and $k_p, k_d$ are the potentiation and depression rate coefficients respectively:

$$k_{p,d}(V) = k_{p,d,0}e^{\pm \eta_{p,d} V}$$

(2)

where + and − are associated with $k_p$ and $k_d$ respectively. $k_{p,d,0} > 0$ are fitting constants and $\eta_{p,d} > 0$ are transition rates. For simplicity, the rate coefficients $k_p$, $k_d$ are assumed to be dependent only on the voltage drop between the two terminals of the memristor, as expected for ionic diffusion processes [25–28].

The current flowing through the device follows Ohm’s law for electrical transport:

$$I(t) = [g(t)G_{\text{max}} + (1 - g(t))G_{\text{min}}] V(t)$$

(3)

where $G_{\text{min}}$ and $G_{\text{max}}$ are the maximum and minimum values of the conductance, respectively. These conductance values can be set to empirically measured ones. The advantage of this model is the low computational cost due to the analytical solution available for discrete time steps $\Delta t > 0$ for the differential equation (1):

$$g(t + \Delta t) = \frac{k_p}{k_p + k_d} \left(1 - e^{-(k_p + k_d)\Delta t}\right) + g(t)e^{-(k_p + k_d)\Delta t}.$$  

(4)

The model comes with seven parameters \{\$g_0, G_{\text{max}}, G_{\text{min}}, \eta_p, \eta_d, k_p, k_d\}, where $g_0$ is the value at time $t = 0$ of the normalized conductance $g(t = 0) = g_0$.

This model enables the integration of ionic diffusion through the potentiation-depression rate balance and electrical transport by the linear Ohmic conduction. It is able to emulate short-term plasticity effects including potentiation, depression and relaxation, paired-pulse-facilitation and heterosynaptic plasticity, when a network of these elements is considered [7, 25]. Additionally, temporal processing of multiple spatial inputs in the context of reservoir computing has been successfully demonstrated [6] in a multi-terminal configuration thanks to the nonlinear dynamics and fading memory properties of the network.

As a final remark, for edges that do not show memristive properties but only linear electrical conduction, equations (1) and (3) reduce to $g \simeq 0$ and $I(t) \simeq G_{\text{min}} V(t)$, respectively. See table 1 in the supplementary material for the parameters used in all simulations. In all plots of this work, quantities are computed after the transient due to the application of the input voltage signal, i.e. at the new equilibrium state induced after the application of the steady input voltage signal.

2.3. Edge network entropy (Pershin and Di Ventra [29])

To investigate how the functional connectivity of the network transforms in relation to both the fraction of memristors and resistors and the $G_{\text{max}}/G_{\text{min}}$ ratio, we take an ideal observer approach. We define the edge network entropy inspired by the Shannon based entropy proposed by Pershin et al [29]:

$$\sigma = -\frac{1}{\log m} \sum_{i,j \in E} \hat{G}_{i,j} \log \hat{G}_{i,j}$$

(5)

where $m$ is the number of edges and the sum is over the set of edges, $E$, in the graph. $\hat{G}_{i,j} = G_{i,j}/\sum_{i,j \in E} G_{i,j}$ is the edge conductance normalized over the sum of all the conductances in the network. It is straightforward to show that 1 corresponds to the maximum network entropy of a given graph, i.e. when all the edges of the graph have equal conductance, the random walk becomes the most unpredictable. In statistical physics, the entropy is a measure of the number of possible configurations of a system. As suggested by the authors of [29], we can interpret the decrease of entropy defined in equation (5) as the consequence of the decreased number of equal resistance paths available in the network and the emergence of fewer low resistive paths for the current flow. In the following, we will refer to the low-resistive paths as active traces.

3. Results and discussion

We study the 2D regular grid-graph as the simplest trivial case of a scale-invariant network of linear size 21. The linear size was chosen as the best trade-off between computational time and regular results over the distribution of disorder. In this work we do not focus on the behaviour of the model at the percolation threshold, finite size effects are not addressed and are left to be dealt with in future work. In the defined settings, the percolation threshold for a spanning cluster of memristors connecting the source and the ground node centred on two opposite sides of the grid is equal to $p_c \simeq 0.57$ (see section S2 in supplementary material). We remark that the results that will be presented in this section are not qualitatively dependent on
Figure 1. Figure (a) shows the network effective conductance, $G_{nw}$, and its second order derivative with respect to the input voltage, $d^2 G_{nw}/dV^2$, in log-linear plot against the applied voltage at the steady state. Several values of the $G_{max}/G_{min}$ ratio are addressed (see colour in legend). We observe a continuous transition for all curves from insulating to conductive state of the network. Almost all curves show two peaks in the second derivative, one at low voltages (black dashed line) and one at higher voltages (red dashed line). Network entropy against the input voltage at the steady state in (b). Insets show exemplary steady state schematic of the memristor grid for $G_{max}/G_{min} = 1 \times 10^6$. Conductance of memristors is proportional to red intensity. Blue dots are proportional to the voltage distribution over the nodes with respect to the ground. Ground and source electrode locations are on the two opposite sides of the grid along the symmetry of the graph.

3.1. Insulating-conducting transition in pure memristor network

Figure 1(a) shows the insulating-conducting transition for a pure memristor grid-graph network. To obtain the curves we applied an adiabatic protocol inspired by [17]. The protocol starts applying a sufficiently low input voltage in order to not activate the memristors in the network, ($V_{start} = 0.1$ [a.u.]). We raise the input voltage by a finite step ($\Delta V = 0.2$ [a.u.]) and hold the new input voltage unchanged until the network has reached a new steady state. The total network conductance is recorded at the steady state. The procedure is then iterated by raising the input of a voltage step $\Delta V$.

We observe continuous slope switching for all curves. This is in contrast with previous work by Sheldon et al [17], where abrupt switching was reported for sufficiently high values of the $G_{max}/G_{min}$. We speculate that such a distinct behaviour is mainly due to the differences between the memristor model implemented in
our work and their memristor model, particularly in their work the memristor elements were modelled through discrete switching dynamics subject to a current threshold. Furthermore, for all curves, the switching is followed by a saturated regime of conductance as a function of the voltage applied.

In figure 1(a), below each curve for the conductance, we plot the second order derivative of the conductance with respect to the voltage applied. Mizrahi et al [14] proved the second order derivative with respect to time to be a solid method to identify the single trace formation in optimization problems. Two peaks appear for almost every $d^2 G_{nw}/dV^2$ curve in figure 1(a). For low $G_{max}/G_{min}$ the low voltage peak, black dashed vertical line, is higher compared to the one at higher voltages, red dashed line, while for increasing $G_{max}/G_{min}$ the height of the peaks switches order.

Figure 1(b) shows the entropy of the conductance distribution as defined in section 2.3. Deep minima are observable in the entropy curves only for sufficiently high ratio $G_{max}/G_{min} > 1 \times 10^2$. In accordance with previous work in the literature [12], high $G_{max}/G_{min}$ ratio, hence strong interaction, is crucial to have a strong mechanism of the collective dynamics in the network and to enable the reinforcement behaviour and ultimately the WTAP. For high $G_{max}/G_{min}$, entropy minima appear in correspondence to the voltage required to switch the network from the insulating to the conductive state, identified by the highest peak in the second order derivative of the conductance with respect to the input voltage, as shown in figure 1 and in figure S6. An optimal voltage is in fact required to produce the minimum number of active traces in the network. In the context of nondeterministic polynomial time (NP) hard problem, the low number of active traces corresponds to a successful computation of the shortest path between the two electrode locations [13, 14, 29] by the internal network dynamics. Insets show that the entropy effectively captures the number of active traces between the source and ground electrodes at the steady state along the curve $G_{max}/G_{min} = 1 \times 10^6$. We note that network entropy in equation (5) might not be sufficiently descriptive when multiple electrode configurations are in use as it does not measure the direction of the active traces. Additionally, the conduction-based network entropy does not address the dynamic properties of the memristive material. Spectral entropy [30, 31] may be a potentially more sophisticated theoretical descriptor for a given memristive network.

3.2. Insulating-conducting transition in mixed memristor-resistor network

We now discuss simulations of memristor networks diluted with either poor Ohmic conductors (Poor OCs), i.e. a fraction of the edges in the network has fixed conductance equal to $G_{min}$, or good Ohmic conductors (Good OCs), i.e. a fraction of the edges have conductance equal to $G_{max}$. All results in this section are obtained by applying a slowly raising input voltage up to the desired value which is then held constant in order to let the network relax to the steady state induced by the applied input.

Pershin and Di Ventra [13] observed that random networks are less prone to activate multiple traces between electrodes compared to grid-graph networks as a consequence of the symmetry breaking in the topology. In their work, they provide an example of a grid-graph network of memristors with strong interaction. They showed that if electrode locations are not selected along a symmetry direction multiple traces are activated. On the other hand, if the memristor network has a random topology, the solution to the shortest path problem provided at the steady state of the dynamics will exhibit an (almost) unique solution. Hence, the topological disorder over which the dynamics evolve is responsible for modulating the collective dynamics of the memristive system. Topological disorder is most usually introduced in the system by pruning components of the network, e.g. pruning nodes, bonds, leading to stronger finite size effects that can be alleviated only by larger network sizes and by increasing the number of simulations in order to obtain solid results. On the other hand, disorder can be introduced in the system by replacing some elements with elements of dissimilar properties. In our case, disorder is introduced by diluting the network of memristors with OC impurities.

Along this line of research we investigate the role of Ohmic impurities in the activation of traces in relation to both the strength of interaction modulated through the ratio $G_{max}/G_{min}$ and the input voltage applied. In our simulations electrodes will be placed along the symmetry of the grid-graph in order to compare to the case where the pure memristor grid-graph could provide a single trace formation connecting the two electrodes at an optimal input voltage as seen in the previous subsection.

3.2.1. Poor Ohmic conductance (Poor OC)

Network conductance as a function of the input voltage is shown in figure 2. Each curve is evaluated for a fixed density of memristive edges, $p$. The interaction strength is tuned at $G_{max}/G_{min} = 1 \times 10^4$ in order to enable a strong mechanism of collective dynamics. Above the percolation threshold, $p \gg 0.57$, the global conductance is a non-linear function of the voltage applied. In fact, above the percolation threshold there exists a path connecting source and ground electrodes composed mainly of memristors which enables the collective dynamics to take full advantage of both the non-linearity of the single memristors and the
Figure 2. Network conductance as a function of the input voltage in log-linear plot. Fraction $p$ of the elements in the network (colour code: see legend) are memristors with $G_{\text{max}}/G_{\text{min}} = 1 \times 10^4$. Density $1 - p$ of the elements is Poor_OC whose conductance is fixed to $G_{\text{min}}$. Below the percolation threshold, $p \lesssim 0.4$, a slow sub-linear relation is observed in the conductance against the voltage applied. Above the percolation threshold, $p \gtrsim 0.6$, a steep transition between the global-insulating towards the global-conducting state is observed, with a decreasing network conductance for lower densities at high voltage. The mean and the margin of error in the 95% confidence interval (shaded area) are computed over 100 network realizations.

Figure 3. (a) shows the conductance for fixed voltage $V = 8$ [a.u.] as a function of the density of memristors in the network, $p$, for multiple values of the $G_{\text{max}}/G_{\text{min}}$ ratio (in colour). All plots are in log-linear scale. For the curves corresponding to $G_{\text{max}}/G_{\text{min}} \gtrsim 1 \times 10^3$, a steep transition in the conductance is observed. In (b), for low DC voltages, $V = 2.5$ [a.u.], the conductance regresses to a linear dependence on the density of memristors, $p$. The mean and the margin of error in the 95% confidence interval (shaded area) are computed over 100 network realizations.
values of the ratio $G_{\text{max}}/G_{\text{min}} \lesssim 10$, the global conductance increases (almost) linearly as a function of $p$. This holds for either high (see figure 3(a)) or low voltages in input (see figure 3(b)). For values of $G_{\text{max}}/G_{\text{min}} \gtrsim 1 \times 10^3$ we observe a transition in the global conductance for two domains of memristor density in the network, $p$. The network transitions from the low density regime with low global conductance to the high density regime with the conductance steeply increasing. We highlight that such a transition appears with a non-zero slope, similarly to what would be expected for a second-order phase transition common in percolation problems [32]. Nevertheless, we leave further investigation of the type of transition involved and the role of finite size effects close to the threshold region to future work. For smaller values of the DC voltage $V \lesssim 2.5$ [a.u.] the conductance shows a trend to grow (almost) linearly in the fraction of memristors, $p$, (we only report the case $V \sim 2.5$ [a.u.] in figure 3(b)). Because of the lack of power supplied, few memristors are activated even in the case of strong interactions. The regression to the linear relation between conductance and density of memristive edges for low voltage applied is more pronounced for lower values of $G_{\text{max}}/G_{\text{min}}$. Thus, a minimum DC input voltage combined with a sufficiently strong interaction is required to observe the transition in figure 3(a) and such a minimum required voltage is observed to be higher for decreasing values of the $G_{\text{max}}/G_{\text{min}}$ ratio.

### 3.2.1.1. Edge network entropy

We now measure the edge network entropy (see section 2.3) at the steady state induced by the application of the DC input signal. We fix $G_{\text{max}}/G_{\text{min}} = 1 \times 10^6$ tuning strong interaction in the network, thus virtually enabling the path reinforcing behaviour discussed in the previous sections.

Figure 4(a) shows a region (bright yellow) of low entropy that is the result of the combined effect of the optimal DC voltage and the quenched disorder in the network due to the presence of the OC elements. In fact, above the percolation threshold, there exists a range $p \sim [0.6, 0.8]$ where diluting the network with Poor_OC impurities provides a strong or stronger trace reinforcing mechanism for a wider range of voltage bias, thus boosting the robustness of this behaviour to the applied voltage. On the other hand, higher sensitivity to the optimal voltage is observed for small fraction of disorder and no disorder at all, i.e. $p \sim 1$, corresponding to higher entropy. We anticipate that this is peculiar of Poor_OC, while a completely different behaviour is observed when the network is diluted with Good_OCs (see next section 3.2.2). Furthermore, in the context of self-assembled memristive networks, for high $G_{\text{max}}/G_{\text{min}} = 1 \times 10^6$ and Poor_OCs, only a sufficient memristive density is required in the network to observe WTAP behaviour. As expected, the role of quenched (Poor_OC) disorder is similar to that of the purely topological disorder observed by Pershin and Di Ventra [13] especially in the limit of high $G_{\text{max}}/G_{\text{min}}$ ratios. Nevertheless, dilution is to be preferred over removal of bonds as it allows large connected components to be present in the system, thus providing stronger resilience of the self-assembled neuromorphic hardware.
Figure 5. The memristor network is diluted with Good_OCs. Figure (a) shows network conductance as a function of the input voltage in log-linear plot. In (b), entropy is an increasing function of the input voltage as more and more memristors are switched to the high conductance state for stronger input applied. The apparent discontinuity between $p \to 1$ and $p = 1$ is due to the fact that entropy is measured over all edges, thus at $V = 1$ [a.u.] entropy is dominated by the Good_OCs. Insets in (c) depict exemplary network state and the conductance of the edges in grades of red along the curves $p = 0.3$ and $p = 0.7$, well below and well above the percolation threshold, respectively. In all figures $G_{\text{max}} / G_{\text{min}} = 1 \times 10^6$. Curves are averaged over 100 network realizations. Shaded area represents the margin of error in the 95% confidence interval and the standard error of the mean in figure (a) and in figures (b) and (c) respectively.

Figure 4(b), shows the combined effect of quenched disorder and optimal input voltage in the two cases of well-above percolation threshold, $p = 0.7$, and well-below percolation threshold, $p = 0.3$. Only the curve $p = 0.7$ (red) shows a minimum. In correspondence to that minimum, we observe an (almost) single path of active memristors whose conductance is depicted in a scale of red, that is connecting source and ground electrodes. For densities of memristors in the network well below the percolation threshold (see red curve for $p = 0.3$) no entropy minimum is observed and the insets show no winning trace between the two electrodes. This corresponds to the dark area in figure 4(a) below the percolation threshold, $p \ll 0.57$.

### 3.2.2. Good Ohmic conductance (Good_OC)

We now discuss the complementary case, i.e. Good_OC dilution. Here, the conductance of OCs is fixed to $G_{\text{max}}$. The ratio between the two limiting states is set to $G_{\text{max}} / G_{\text{min}} = 1 \times 10^6$ for all memristors in the network. Figure 5(a) shows the network conductance as a function of the DC input voltage. We observe the same qualitative behaviour for all fractions of memristors in the network (see legend) in contrast to what is shown for Poor_OC dilution in figure 2. As expected, under the condition of Good_OC conductance dilution, lower voltages are required to bring the memristors into the conductive state. For a sparse density of memristors, this behaviour is more evident.

For all dilutions in figure 5(b), the entropy is a monotonic increasing function of the density of memristors. A pure-memristor network (dark-red curve for $p = 1$) is plotted for comparison. The behaviour of the network can be qualitatively understood by observing the insets of figure 5(c). The presence of Good_OCs randomly distributed in the network does not enable the collective reinforcement mechanism observed for the Poor_OC case. In fact, as the voltage increases, memristors become active and fill the gaps
between the multiple good conductive elements already present in the system. As a consequence, the edge-conductance distribution becomes less irregular as more and more memristors are in the higher conductance state.

4. Conclusion

A model for mixed memristor-resistor networks for neuromorphic computing was developed, starting from the pure-memristor network model described in [6, 25]. We measure the entropy of the conductance distribution to quantify the trace formation in the network, assuming an ideal observer approach. High entropy is associated with multiple, equally conductive traces, while low-entropy is associated with the presence of a small number of highly conductive traces in the network.

We measured the conductance and the entropy in the insulating-conducting transition in a network of pure memristive elements and identified an optimal voltage range that enables the collective reinforcement mechanism leading to WTAP and single-trace formation. In accordance with previous findings [13, 14], the WTAP mechanism emerges for high values of the ratio between the two limiting states of the memristors, enabling sufficiently strong interaction and cooperative behaviour in the system.

We continued by diluting the network of memristors with OCs. The conductance of the OCs was fixed to either the minimum or the maximum of the conductance state of the memristors. We showed that the density of impurities in the network determines the conductance dependence on the input voltage. In the case where the network was diluted with Poor_OCs and below the percolation threshold, we measured a sub-linear dependence of the conductance as a function of the voltage applied. On the other hand, above the percolation threshold, we measured a steep non-linear dependence of the global conductance versus the applied DC voltage. Furthermore, non-linearity is shown to be modulated by the fraction of impurities in the above percolation threshold domain. In the case of Good_OCs diluting the network, we did not observe any significant difference in the qualitative behaviour of the curves of the global conductance versus the input voltage. Similarly to the case of a pure memristor network in section 3.1, the entropy of the conductance distribution was measured at the equilibrium state induced by the input voltage. In the case of Good_OCs dilution, no self-reinforcing mechanism leading to single trace formation was observed, but a rather non-selective distributed functional connectivity was evolved by the network connecting the two electrodes on the substrate. On the contrary, in the case where the dilution was done by introducing Poor_OCs, a self-reinforcing mechanism leading to single trace formation, selective functional connectivity between the ground and the source electrode was observed. Furthermore, resilience to the applied voltage was boosted in the single trace formation by augmenting the optimal voltage range for densities of Poor_OCs present in the system above the percolation threshold but below the limit of a pure memristor network. Thus, Poor_OCs diluted networks prove to be suited for applications where the trace formation is required to be limited but selective between the electrodes in order to establish independent memories or associations.

In fact, in a complex conduction network, an association is present if the activity of a (group of) electrode(s) provokes the activity of another without significantly activating another electrode (or groups of them) [33]. In other words, a precondition for building association relations between electrodes placed on the substrate is that only a fraction of the memristive network, i.e. the part that encodes the desired memory association is activated once the corresponding electrodes are active. As demonstrated in our study, non-memristive impurities may be beneficial in creating associations, rather than constituting a nuisance. Furthermore, we highlight that dilution is in general to be preferred over removal of elements as it prevents the system from becoming unconnected, leading to interruption of electrical transport.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Code availability

The code used to generate the dataset of simulations can be accessed on GitHub: https://github.com/CipolliniDavide/Percolation-and-Conduction-in-MemNet.git.

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