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From a standard factorization to a J-spectral factorization for a class of infinite-dimensional systems☆

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ABSTRACT

Matrix-valued functions in the Wiener class on the imaginary line are considered in this note. This class of functions is large enough to be suitable for many applications in systems and control of infinite-dimensional systems. For this class of functions three kinds of factorization are discussed: (right-)standard factorization, canonical factorization, and J-spectral factorization. In particular, we focus on an algorithmic procedure to find a (right-)standard factorization and a J-spectral factorization for matrix-valued functions in the Wiener class under the assumption that such factorizations exists. In practice, the J-spectral factors for irrational functions are usually calculated using rational approximations. We show that approximation using rational functions may be achieved in the Wiener norm.

1. Introduction

Roughly speaking, the J-spectral factorization problem on the imaginary axis can be formulated as follows: given a matrix-valued function Z defined on the imaginary axis, find a bistable (stable with the inverse stable) matrix-valued function V such that $Z = V^*JV$, where J is a signature matrix and V^* denotes the conjugate transpose of V. The bistable matrix-valued function V is known as a J-spectral factor for Z. The J-spectral factorization has various applications in several areas of control theory and practice. For example, solving an H_∞-control problem for systems with the transfer function in the Wiener class is known to be equivalent to solving two nested J-spectral factorizations (see Iftime, 2002).

Two important problems arise when a J-spectral factorization approach is considered for systems with an infinite-dimensional transfer function. Problem 1: construct algorithms to solve a J-spectral factorization for a given infinite-dimensional matrix-valued function. Problem 2: if one uses rational matrix-valued functions to approximate a J-spectral factorization associated to infinite-dimensional systems, how good is this approximation? In this note we address the above two problems for matrix-valued functions in the Wiener class on the imaginary line and consider a frequency-domain approach.

A solution to Problem 1 is provided by Theorem 4.1. It provides a detailed, self-contained presentation of the constructive procedures of a J-spectral factorization and of the, more general, (right-)standard factorization. We build the algorithms on the operator theoretic methods from Clancey and Gohberg (1981, Ch. II) under the assumption that the matrix-valued functions to be factorized admit a (right-)standard factorization and a J-spectral factorization, respectively. Necessary and sufficient conditions for the existence of a J-spectral factorization for matrix-valued functions in the Wiener class have been studied in Iftime and Zwart (2001).

An answer to Problem 2 is given by Theorem 4.4: the J-spectral factor depends continuously in the Wiener norm on the matrix-valued function to be J-factorized. In practice, it is very common to approximate J-spectral factors for irrational function by using rational approximations. Explicit procedures to construct a J-spectral factorization make usually use of a state-space representation of the transfer function (Meinsma, 1995; Ran, 2003; Zong, 2005). In the scalar case, the J-spectral factorization is a spectral factorization. It is known that the mapping which maps the spectral densities to the spectral factor is not continuous in general (Anderson, 1985). Positive results have also been obtained in the scalar case (see Jacob & Partington, 2001 and Jacob, Winkin, & Zwart, 1999).

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $i\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_-$ be the set of natural numbers, the set of integer numbers, the set of real numbers, the imaginary axis, the set of complex numbers, the right half-plane and the left

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half-plane, respectively. Denote by $\overline{\mathbb{C}_+}$ the right half complex plane including the imaginary axis. Equip $i\mathbb{R} \cup \{\infty\}$, the imaginary axis including infinity, with its one point compactification topology which is induced by the topology of the unit circle through the Cayley transform $\phi : \overline{\mathbb{C}_+} \to \overline{\mathbb{D}}$, where $\overline{\mathbb{D}}$ is the unit disc and $\phi(s) = \frac{1-s}{1+s}$. Consequently $+i\infty$ and $-i\infty$ (the ends of $i\mathbb{R}$) are identified. Similarly $\overline{\mathbb{C}_+}$ is equipped with the topology induced via Cayley transform by the topology on the closed unit disc.

Consider the set $\mathcal{A}$ of functions f with the representation $f(t) = \begin{cases} f_a(t) + f_0\delta(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$, where $f_a(t)$ and f_0 are in $\mathbb{C}$, $\int_0^\infty |f_a(t)|dt < \infty$, and δ is the delta distribution at zero. The Laplace transform of $f \in \mathcal{A}$ is well defined and it is given by $\hat{f}(s) = \int_0^\infty e^{-st} f_a(t)dt + f_0$, for $s \in \overline{\mathbb{C}_+}$. The set $\hat{\mathcal{A}} = \{\hat{f} \mid f \in \mathcal{A}\}$ of Laplace transforms of the functions in $\mathcal{A}$ is contained in the Hardy space H_∞ and it is known in the literature as the *causal Wiener class*. From the definition of $\mathcal{A}$ and the properties of the Laplace transforms it follows that, under pointwise addition and multiplication, $(\hat{\mathcal{A}}, \|\cdot\|_\infty)$ is a commutative Banach algebra with identity. Every function $\hat{f} \in \hat{\mathcal{A}}$ has the limit f_0 at infinity, i.e. $\left\{|\hat{f}(s) - f_0|; s \in \overline{\mathbb{C}_+}, |s| \geq \rho\right\} \to 0$, as $\rho \to \infty$.

Let $f^\sim$ be defined as $f^\sim(s) = \overline{f(-\bar{s})}$, where by $\bar{z}$ we mean the complex conjugate of the complex number z. The *Wiener class* of infinite-dimensional transfer functions is defined as $\hat{\mathcal{W}} = \left\{g \in L_\infty(i\mathbb{R}, \mathbb{C}) \mid g(i\cdot) = g_1(i\cdot) + g_2(i\cdot),\ g_1,\ g_2^\sim \in \hat{\mathcal{A}}\right\}$, where L_∞ is the space of functions bounded almost everywhere on the imaginary axis ($i\mathbb{R}$). Let $C(i\mathbb{R})$ denote the space of continuous functions on $i\mathbb{R}$ with the sup-norm. The Wiener class is a decomposing Banach algebra continuously embedded in $C(i\mathbb{R})$ (i.e., $\|\cdot\|_{C(i\mathbb{R})} \leq \|\cdot\|_{\hat{\mathcal{W}}}$). Moreover, $\hat{\mathcal{A}}$ is a closed subalgebra of $\hat{\mathcal{W}}$, $\hat{\mathcal{A}} \subset H_\infty$ (the set of stable functions in L_∞). For more background on the Wiener class and the causal Wiener class we refer to Jacob et al. (1999) and the references therein (note that in this paper we consider only functions with no delayed impulses). Denote by RL_∞ the set of proper rational functions which have no poles on the imaginary axis. Let $L_\infty^{n\times n}$, $RL_\infty^{n\times n}$, $H_\infty^{n\times n}$, $\hat{\mathcal{W}}^{n\times n}$ and $\hat{\mathcal{A}}^{n\times n}$, the classes of $n \times n$ matrix-valued functions with entries in L_∞, RL_∞, H_∞, $\hat{\mathcal{W}}$ and $\hat{\mathcal{A}}$, respectively, with an appropriate norm. If one defines $\hat{\mathcal{A}}_0^\sim = \{\hat{f} \in L_\infty \mid \hat{f}^\sim \in \hat{\mathcal{A}}$ and $\lim_{|s|\to\infty, s\in i\mathbb{R}} \hat{f}(s) = 0\}$, then $\hat{\mathcal{W}} = \hat{\mathcal{A}} \oplus \hat{\mathcal{A}}_0^\sim$. Denote by P the projection of $\hat{\mathcal{W}}$ onto $\hat{\mathcal{A}}$ and consider $Q = I - P$. Denote by $G\hat{\mathcal{A}}^{n\times n}$ and $G\hat{\mathcal{W}}^{n\times n}$ the set of invertible elements over $\hat{\mathcal{A}}^{n\times n}$ and $\hat{\mathcal{W}}^{n\times n}$, respectively (i.e. with the inverse also in $\hat{\mathcal{A}}^{n\times n}$ and $\hat{\mathcal{W}}^{n\times n}$, respectively). It is well known (see Callier & Desoer, 1978) that $G\hat{\mathcal{A}}^{n\times n} = \{\hat{F} \in \hat{\mathcal{A}}^{n\times n} \mid \inf_{s\in\overline{\mathbb{C}_+}}\{|\det \hat{F}(s)| > 0\}\}$ and $G\hat{\mathcal{W}}^{n\times n} = \{\hat{F} \in \hat{\mathcal{W}}^{n\times n} \mid \inf_{s\in i\mathbb{R}}\{|\det \hat{F}(s)| > 0\}\}$. We omit the size of a matrix when it can be deduced from the context.

Recall now the definitions of a standard factorization, canonical factorization and J-spectral factorization relative to the imaginary axis (see Iftime & Zwart, 2001 and the references therein).

Definition 2.1. The matrix valued function $Z \in G\hat{\mathcal{W}}^{n\times n}$ is said to admit a *(right-) standard factorization relative to the imaginary axis* if Z can be decomposed as

$$Z = Z_- D Z_+, \tag{1}$$

with $Z_+, Z_-^\sim \in G\hat{\mathcal{A}}^{n\times n}$, and D a diagonal matrix function

$$D(s) = diag\left[\left(\frac{s - s_{+,1}}{s - s_{-,1}}\right)^{k_1}, \ldots, \left(\frac{s - s_{+,n}}{s - s_{-,n}}\right)^{k_n}\right], \quad s \in i\mathbb{R}, \tag{2}$$

with $s_{+,i} \in \mathbb{C}_-, s_{-,i} \in \mathbb{C}_+, k_i \in \mathbb{Z}$ and $k_1 \geq \cdots \geq k_n$. The integers k_i are called *(the right-) partial indices* of the factorization. In the case $k_1 = \cdots = k_n = 0$, so that, $Z = Z_- Z_+$, then Z is said to admit a *(right-) canonical factorization* relative to the imaginary axis.

The following theorem provides equivalent conditions for the existence of a canonical factorization for $Z \in \hat{\mathcal{W}}^{n\times n}$.

Theorem 2.2 (*Clancey & Gohberg, 1981, Theorem 1.1, page 35*). *Let $Z \in \hat{\mathcal{W}}^{n\times n}$. The following statements are equivalent:*

(1) *The matrix-valued function Z admits a canonical factorization $Z = Z_- Z_+$.*
(2) *Each of the equations*

$$X - P((I - Z)X) = I \tag{3}$$
$$Y - Q(Y(I - Z)) = I \tag{4}$$

is solvable in $\hat{\mathcal{W}}^{n\times n}$.
(3) *For any $F, G \in \hat{\mathcal{W}}^{n\times n}$, the following equations*

$$X - P((I - Z)X) = F \tag{5}$$
$$Y - Q(Y(I - Z)) = G \tag{6}$$

are uniquely solvable in $\hat{\mathcal{W}}^{n\times n}$.

Note that in Clancey and Gohberg (1981) the result is stated for general decomposing matrix algebras of functions on a contour. The Wiener class of infinite-dimensional transfer functions on the imaginary axis is a decomposing Banach algebra. For other examples of decomposing Banach algebras see Clancey and Gohberg (1981, pages 59–63).

For the definition of the J-spectral factorization we shall introduce the signature matrix $J_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, where $p, q \in \mathbb{N}$. We use J when there is no danger of confusion.

Definition 2.3. Let $Z = Z^\sim \in \hat{\mathcal{W}}^{n\times n}$ be a matrix-valued function. Z has a *J-spectral factorization* if there exists a matrix-function $V \in G\hat{\mathcal{A}}^{n\times n}$ such that $Z(s) = V^\sim(s) J V(s)$ for all $s \in i\mathbb{R}$. Such a matrix V is called a *J-spectral factor.*

For necessary and sufficient conditions for the existence of a J-spectral factorization we refer to Iftime and Zwart (2001).

3. Technical results

To present the constructive procedures for a standard factorization and a J-spectral factorization several technical results are needed (Proposition 3.1, Lemmas 3.2 and 3.4). The first result is a consequence of Theorem 2.2.

Proposition 3.1. *Consider a matrix-valued function $Z \in \hat{\mathcal{W}}^{n\times n}$ which admits a canonical factorization. Then each of the equations*

$$X - P((I - Z)X) = I, \quad and \tag{7}$$
$$Y - Q(Y(I - Z)) = I \tag{8}$$

is uniquely solvable in $\hat{\mathcal{W}}^{n\times n}$. Moreover, a canonical factorization for Z is given by

$$Z = Y_s^{-1} X_s^{-1},$$

where X_s and Y_s are the unique solutions of Eqs. (7) and (8), respectively.

Proof. The solvability of Eqs. (7) and (8) and the uniqueness of the solutions are obtained from the implication "1. → 2." and from point "3." of Theorem 2.2, respectively.

Let X_s and Y_s the unique solutions of Eqs. (7) and (8), respectively. We need to prove that indeed $Z = Y_s^{-1} X_s^{-1}$ is a canonical factorization for Z. The proof follows the lines of the proof of the implication "2. → 1." of Theorem 2.2. Define $U := X_s - I$ and $V = Y_s - I$. Since X_s and Y_s are solutions to Eqs. (7) and (8), respectively, they satisfy

$$U + I - P((I - Z)(U + I)) = I,$$
$$V + I - Q((V + I)(I - Z)) = I.$$

It follows that

$$U = P((I-Z)(U+I)) \in \hat{\mathcal{A}}^{n\times n} \quad \text{and}$$
$$V = Q((V+I)(I-Z)) \in (\hat{\mathcal{A}}_0^{\sim})^{n\times n},$$

since $\hat{\mathcal{W}} = \hat{\mathcal{A}} \oplus \hat{\mathcal{A}}_0^{\sim}$, P is the projection of $\hat{\mathcal{W}}^{n\times n}$ onto $\hat{\mathcal{A}}^{n\times n}$ and $Q = I - P$. Now consider

$$U_- = -Q((I-Z)X_s) \in (\hat{\mathcal{A}}_0^{\sim})^{n\times n} \quad \text{and}$$
$$V_+ = -P(Y_s(I-Z)) \in (\hat{\mathcal{A}}^{n\times n})^{n\times n}.$$

It can be seen that

$$\begin{aligned} ZX_s - I &= (I-(I-Z))X_s - I = X_s - (I-Z)X_s - I \\ &= X_s - P((I-Z)X_s) - I - Q((I-Z)X_s) \\ &= -Q((I-Z)X_s), \end{aligned}$$

where Eq. (7) is used in the last equality. Therefore one has $ZX_s = I + U_-$. Similarly, using Eq. (8), one gets $Y_s Z = I + V_+$. One can write now the following double equality $Y_s Z X_s = Y_s(I+U_-) = (I+V_+)X_s$. Replace Y_s by $V+I$ and X_s by $U+I$ in the second and the third term of the above equality and write only the second equality to obtain $(V+I)(I+U_-) = (I+V_+)(U+I)$, which is the same as

$$V + U_- + VU_- = V_+ + U + V_+U.$$

The left-hand side of the above equality is in $(\hat{\mathcal{A}}_0^{\sim})^{n\times n}$ and the right-hand side is in $\hat{\mathcal{A}}^{n\times n}$. Using the decomposition $\hat{\mathcal{W}} = \hat{\mathcal{A}} \oplus \hat{\mathcal{A}}_0^{\sim}$, it follows that

$$V + U_- + VU_- = V_+ + U + V_+U = 0,$$

which is equivalent to

$$I = (V+I)(I+U_-) = (I+V_+)(U+I).$$

Then, $V+I$ and $U+I$ are invertible and

$$(V+I)^{-1} = I + U_- \quad \text{and} \quad (U+I)^{-1} = I + V_+.$$

One has now that $(V+I)Z(U+I) = I$ which leads to

$$Z = (V+I)^{-1}(U+I)^{-1} = (I+U_-)(I+V_+) = Y_s^{-1}X_s^{-1}$$

with $V_+, U \in \hat{\mathcal{A}}^{n\times n}$ and $U_-, V \in (\hat{\mathcal{A}}_0^{\sim})^{n\times n}$.

Lemma 3.2. *Let $Z \in \hat{\mathcal{W}}^{n\times n}$ be a matrix-valued function invertible over $\hat{\mathcal{W}}^{n\times n}$. Then there exist $A \in \hat{\mathcal{W}}^{n\times n}$ and a rational matrix-valued function R invertible over $RL_\infty^{n\times n}$ such that $Z = AR$ and A has a canonical factorization.*

Proof. Let R be a rational matrix-valued function invertible over $RL_\infty^{n\times n}$. One can write

$$Z = \left[I - (R-Z)R^{-1}\right]R.$$

Then $Z = AR$ for $A = I - (R-Z)R^{-1}$. It should further be proved that R can be chosen such that A has a canonical factorization. Since $\hat{\mathcal{W}}$ is an R-algebra, there exits an R invertible over $RL_\infty^{n\times n}$ which satisfies $\|R - Z\| < \epsilon_1$ and $\|R^{-1}\| < \|Z^{-1}\| + \epsilon_2$ for small positive real numbers ϵ_1 and ϵ_2. Therefore we can take R that satisfies

$$\|(R-Z)R^{-1}\| < \min\left\{\|P\|^{-1}, \|Q\|^{-1}\right\}, \tag{9}$$

where P is the projection of $\hat{\mathcal{W}}^{n\times n}$ onto $\hat{\mathcal{A}}^{n\times n}$ and $Q = I - P$. Define the operators $T_A(X)$ and $S_A(Y)$ on $\hat{\mathcal{W}}^{n\times n}$ by

$$T_A(X) := X - P((I-A)X) \quad \text{and} \quad S_A(Y) := Y - Q(Y(I-A)).$$

From Eq. (9) and the definition of A one has that

$$\|I - A\| < \min\left\{\|P\|^{-1}, \|Q\|^{-1}\right\}.$$

Using this inequality and the definition of $T_A(X)$ one has

$$\|I - T_A\| = \|P\|\,\|I - A\| < 1.$$

Thus T_A is an invertible operator on $\hat{\mathcal{W}}^{n\times n}$. Similarly one can obtain that S_A is an invertible operator on $\hat{\mathcal{W}}^{n\times n}$. Then Eqs. (3) and (4) have a unique solution (with A in place of Z) and consequently A has a canonical factorization.

Remark 3.3. Note that using Lemma 3.2 and Proposition 3.1 one can write $Z = A_-A_+R$ with $A_- = Y_s^{-1}$ and $A_+ = X_s^{-1}$, where X_s and Y_s are the unique solutions of

$$X - P((I-A)X) = I \quad \text{and} \quad Y - Q(Y(I-A)) = I.$$

Another technical result is stated in the next lemma and it can be obtained in a constructive manner as presented in the proof. An alternative (constructive) procedure is by using the language of zero and pole chains (see Ball, Gohberg, & Rodman, 1990).

Lemma 3.4. *Let $A_+ \in G\hat{\mathcal{A}}^{n\times n}$ and R a rational matrix-valued function invertible over $RL_\infty^{n\times n}$. Then there exist a matrix-valued function $\Theta_+ \in G\hat{\mathcal{A}}^{n\times n}$ and a rational matrix-valued function Λ invertible over $RL_\infty^{n\times n}$ such that*

$$A_+R = \Lambda\Theta_+.$$

Proof. The rational matrix-valued function R can be written as $R = \frac{p}{q}R_+$ where $R_+ \in \hat{\mathcal{A}}^{n\times n}$. The polynomial q has only zeros with positive real part which are the zeros (with positive real part) of all denominators of the entries in R, counted with appropriate multiplicities; the polynomial p has its degree equal to the degree of q and is chosen such that it has only zeros with negative real part. Let us denote by $s_i,\ i = 1, .., n$, the zeros (taking into account the multiplicity) of $\det(A_+(s)R_+(s))$ in the right-half plane, which are in finite number because $A_+ \in G\hat{\mathcal{A}}^{n\times n}$ and R_+ is a rational matrix-valued function. Actually they are the right-half plane zeros from $\det(R_+(s))$. One can write now

$$A_+R_+ = \left[l_1^T, \ldots, l_n^T\right]^T$$

where $l_i,\ i = 1, .., n$, are the row vectors of A_+R_+. For $s = s_1$, the vectors $l_i(s_1),\ i = 1, .., n$, are linearly dependent, i.e.,

$$a_1 l_1(s_1) + \cdots + a_k l_k(s_1) + \cdots a_n l_n(s_1) = 0, \tag{10}$$

for some $a_i,\ i = 1, \ldots, n$, not all zero. Let for example $a_k \neq 0$. Denote by

$$\bar{l}_k(s) := \left(\frac{s+s_1}{s-s_1}\right)^m [a_1 l_1(s) + \cdots + a_k l_k(s) + \cdots a_n l_n(s)]$$

where m is the multiplicity of s_1 as the root of the expression (10). Then $A_+R_+ = \Pi_1 \cdot \left[l_1^T, \ldots, \bar{l}_k^T, \ldots, l_n^T\right]^T$, where Π_1 is the identity matrix in which the kth row is replaced by $\left[-\frac{a_1}{a_k}, \ldots, (\frac{s+s_1}{s-s_1})^m, \ldots, -\frac{a_n}{a_k}\right]$. In this way we have reduced the order of the zero of the corresponding determinant at $s = s_1$ but maybe still not have brought it down to zero. Nevertheless, iteration of this procedure will achieve the desired factorization after finitely many steps. It is easy to see that $\Pi_1 \in GRL_\infty^{n\times n}$, $\left[l_1^T, \ldots, \bar{l}_k^T, \ldots, l_n^T\right]^T \in \hat{\mathcal{A}}^{n\times n}$ and its determinant is equal to zero for $s \in \{s_2, \ldots, s_n\}$. Repeating the procedure for finitely many steps, one will obtain $C_+R_+ = \Pi_1\Pi_2 \cdot \ldots \cdot \Pi_n\Theta_+$. By construction, one can see that $\Theta_+ \in G\hat{\mathcal{A}}^{n\times n}$ and that $\Lambda = \frac{p}{q}\Pi_1 \cdot \ldots \cdot \Pi_n \in GRL_\infty^{n\times n}$.

The J-spectral factor V is unique up to multiplication by a constant J-unitary matrix Q, i.e. Q satisfies $Q^{\sim}JQ = J$. The proof follows standard steps and it is presented for the completeness of the exposition.

Proposition 3.5. *Let $Z = Z^{\sim} \in \hat{\mathcal{W}}^{n\times n}$ and suppose that $V \in G\hat{\mathcal{A}}^{n\times n}$ is a J-spectral factor for Z. Then $W \in G\hat{\mathcal{A}}^{n\times n}$ satisfies $W^{\sim}JW = Z = V^{\sim}JV$ on the imaginary axis if and only if $W = QV$, where Q is a constant matrix satisfying $Q^{\sim}JQ = J$.*

Proof. Let $V \in G\hat{\mathcal{A}}^{n\times n}$ be a J-spectral factor for Z and Q a constant matrix such that $Q^{\sim}JQ = J$. Then $QV \in G\hat{\mathcal{A}}^{n\times n}$ and $W = QV$ satisfies $W^{\sim}JW = X^{\sim}Q^{\sim}JQX = X^{\sim}JX = Z$.

Conversely, let $V \in G\hat{\mathcal{A}}^{n\times n}$ and $W \in G\hat{\mathcal{A}}^{n\times n}$ such that $W^{\sim}JW = Z = V^{\sim}JV$. Multiply this equality to the right by V^{-1} and to the left by $(W^{\sim})^{-1}$ to obtain $JWV^{-1} = (W^{\sim})^{-1}V^{\sim}J$. Now, $JWV^{-1} \in \hat{\mathcal{A}}^{n\times n}$, so JWV^{-1} is bounded and continuous in the closed right half-plane and holomorphic in the open right half-plane. We also have that

$$(JWV^{-1})^{\sim} = ((W^{\sim})^{-1}V^{\sim}J)^{\sim} \in \hat{\mathcal{A}}^{n\times n},$$

therefore JWV^{-1} is bounded and continuous in the closed left half-plane and holomorphic in the open left half-plane. Using Liouville's theorem (Berenstein & Gay, 1991, page 108) one can conclude that $Q := JWV^{-1}$ is a constant matrix. It is straightforward to verify that Q satisfies $Q^{\sim}JQ = J$.

4. Main results

Now we have all the ingredients to present an algorithmic construction of a (right-) standard factorization and a J-spectral factorization.

Theorem 4.1. *Consider $Z \in G\hat{\mathcal{W}}^{n\times n}$. Then Z has a standard factorization relative to the imaginary axis. The first four steps provide a constructive procedure of a such standard factorization. Furthermore, if Z admits J-spectral factorization, one can use steps (5)–(7) to derive a particular J-spectral factor of Z.*

(1) *Find a matrix-valued function $A \in \hat{\mathcal{W}}^{n\times n}$ which admits a canonical factorization and a rational matrix-valued function R invertible over $RL_{\infty}^{n\times n}$ such that $Z = AR$.*
(2) *Find X_s and Y_s, the unique solutions of*

$$X - P((I - A)X) = I \quad \text{and} \quad Y - Q(Y(I - A)) = I.$$

Write $Z = A_-A_+R$, were $A_- = Y_s^{-1}$ and $A_+ = X_s^{-1}$.
(3) *Find a matrix-valued function $\Theta_+ \in G\hat{\mathcal{A}}^{n\times n}$ and a rational matrix-valued function Λ invertible over $RL_{\infty}^{n\times n}$ such that $A_+R = \Lambda\Theta_+$. Write $Z = A_-\Lambda\Theta_+$.*
(4) *Find a standard factorization $\Lambda = \Lambda_-D\Lambda_+$ for the rational matrix-valued function Λ, where $\Lambda_+, \Lambda_-^{\sim} \in G\hat{\mathcal{A}}^{n\times n}$ and D as in (2).*

Then $Z = (A_-\Lambda_-)D(\Lambda_+\Theta_+)$ is a standard factorization.

If Z admits a J-spectral factorization then $Z = Z^{\sim}$ and $D = I$ and $\Lambda_{\pm}$ can also be taken to be identity. One can derive a particular J-spectral factor as follows:

(5) *Write Z as $Z = Z_-Z_+$, where $Z_- = Y_s^{-1}$ and $Z_+ = X_s^{-1}$.*
(6) *Factorize the matrix $Z_-^{\sim}Z_+^{-1}$ as $Z_-^{\sim}Z_+^{-1} = U^{\sim}JU$.*
(7) *Finally, one has the J-spectral factorization $Z = V^{\sim}JV$, where the J-spectral factor is $V := UZ_+ = UX_s^{-1}$.*

Proof. Since $Z \in G\hat{\mathcal{W}}^{n\times n}$, Z has a J-spectral factorization (similar to Clancey & Gohberg, 1981, Theorem 6.3, page 63). The proof of the algorithmic construction of a J-spectral factor is a consequence of Lemmas 3.2, 3.4 and Proposition 3.5. It will be shown that all the above steps can be performed.

(1) The first step is a consequence of Lemma 3.2.
(2) It follows from Proposition 3.1 (see also Remark 3.3).
(3) Using the constructive procedure from the proof of Lemma 3.4 one can find $\Theta_+ \in G\hat{\mathcal{A}}$ and a rational matrix-valued function Λ.
(4) Since $\det \Lambda(s) \neq 0$ on the imaginary axis then there exist a standard factorization $\Lambda = \Lambda_-D\Lambda_+$ where $R_+, R_-^{\sim} \in G\hat{\mathcal{A}}$ and D as in (2). Then

$$Z = A_-\Lambda_-D\Lambda_+\Theta_+ = Z_-Z_+.$$

Since we have assumed that Z has a J-spectral factorization, it follows that Z has no equalizing vectors and consequently D is the identity (see Iftime & Zwart, 2001). One can write

$$Z = A_-\Lambda_-\Lambda_+\Theta_+ = Z_-Z_+,$$

where $Z_- = A_-\Lambda_-$ and $Z_+ = \Lambda_+\Theta_+$.
(5) Notice that $Z_+^{\sim}Z_-^{\sim} = Z^{\sim} = Z = Z_-Z_+$. Then $Z_-^{\sim}Z_+^{-1} = (Z_+^{\sim})^{-1}Z_-$, in which the right-hand side and the $\sim$ of the left-hand side are in $\hat{\mathcal{A}}$. Consequently $Z_-^{\sim}Z_+^{-1}$ is a invertible constant matrix such that $Z_-^{\sim}Z_+^{-1} = (Z_-^{\sim}Z_+^{-1})^*$. Therefore, $Z_-^{\sim}Z_+^{-1} = U^{\sim}JU$ for some unitary matrix U.
(6) Write $Z_- = Z_+^{\sim}U^{\sim}JU$, which gives $Z = Z_-Z_+ = Z_+^{\sim}U^{\sim}JUZ_+$. The choice $V := UZ_+ = U\Lambda_+\Theta_+$ gives a J-spectral factor for Z.
(7) It follows from steps (5) and (6).

Remark 4.2. Other classes of matrix-valued functions for which the standard and J-spectral factorizations are possible are: Wiener algebra on the unit circle (Clancey & Gohberg, 1981, Theorem 6.1, page 59), algebras of Holder continuous functions (Clancey & Gohberg, 1981, Theorem 6.2, page 61), Functions analytic on a contour (Clancey & Gohberg, 1981, Corollary 6.1, page 61) in Clancey and Gohberg (1981) and The Wiener algebra on the real line (Clancey & Gohberg, 1981, Theorem 6.3, page 63). These examples in Clancey and Gohberg (1981) address only the existence a standard factorization (as a consequence of Clancey & Gohberg, 1981, Theorem 3.1, page 44) and it seems to have missed to mention the algorithmic constructive procedures of the standard factorization and J-spectral factorization. Theorem 4.1 is stated for matrix-valued functions in the Wiener class on the imaginary axis which should be useful for applications. The steps (5)–(7) define a particular J-spectral factor as $V = UZ_+$, unique up to multiplication by a constant J-unitary matrix Q (see Proposition 3.5).

Remark 4.3. Assume now that it is known that Z has a J-spectral factorization and one would apply the whole Theorem 4.1 to Z. Take $R = I$ in Step (1) of the algorithm, then $A = Z$. In this case, the set of equations in Step (2) amounts to finding a canonical factorization (as in Clancey & Gohberg, 1981). In Step (3), take $\Theta_+ = A_+ = X_s^{-1}$ and $\Lambda = I$. Take now $\Lambda = \Lambda_{\pm} = I$ in Step (4). The rest is the standard procedure to convert a canonical factorization of Z to a J-spectral factorization of Z in case $Z = Z^{\sim}$. If one takes $R \neq I$, operator equations for $A := ZR^{-1}$ need to be solved in Step (2), which may be in some cases more convenient. However, the price to be paid is that Step (3) and Step (4) should also be also performed.

In practice, J-spectral factors for irrational function are approximated using rational functions. For the scalar case, it is known (Jacob et al., 1999) that the spectral factor depends continuously on the spectral density in the Wiener norm. We conclude this note by showing that the J-spectral factor also depends continuously, in the Wiener norm, on the matrix-valued function to be J-factorized.

Theorem 4.4. *Assume that $Z, Z_k \in \hat{\mathcal{W}}^{n\times n}$, $k \in \mathbb{N}$ admit J-spectral factorizations and satisfy $Z_k \to Z$ in the $\hat{\mathcal{W}}$-norm as $k \to \infty$. Consider V, V_k the particular J-spectral factors associated to Z, Z_k, respectively,* *obtained as in Theorem 4.1. Then there exist constants $c_1, c_2 > 0$ such that*

$$\|V_k - V\|_{\hat{\mathcal{W}}} \leq c_1 \|Z_k - Z\|_{\hat{\mathcal{W}}}, \quad \text{and}$$
$$\|V_k^{-1} - V^{-1}\|_{\hat{\mathcal{W}}} \leq c_2 \|Z_k - Z\|_{\hat{\mathcal{W}}}.$$

Proof. For a matrix-valued function $Z \in \hat{\mathcal{W}}^{n\times n}$ one can define the operator $T_Z : \hat{\mathcal{W}}^{n\times n} \to \hat{\mathcal{W}}^{n\times n}$ by

$$T_Z(X) = P(ZX) + Q(X),$$

where P is the continuous projection from $\hat{\mathcal{W}}^{n\times n}$ onto $\hat{\mathcal{A}}^{n\times n}$. Since Z has a J-spectral factorization it follows from Theorem 2.2 that $T_Z(\cdot)$ is invertible. As a consequence of Theorem 4.1, one can consider $V = T_Z^{-1}(I)$ without loss of generality. Following the same procedure, one can obtain similarly that the operators $T_{Z_k}(\cdot)$ are also invertible and $V_k = T_{Z_k}^{-1}(I)$. Note that

$$T_Z(X) = P(ZX) + X - P(X) = X - P((I - Z)X),$$

which gives $\|T_Z - T_{Z_k}\| \leq \|P\| \cdot \|Z - Z_k\|_{\hat{\mathcal{W}}}$.

In particular one has

$$\|T_Z(I) - T_{Z_k}(I)\|_{\hat{\mathcal{W}}} \leq \|P\| \cdot \|Z - Z_k\|_{\hat{\mathcal{W}}},$$

which implies that there exists a c_2 such that

$$\|V_k^{-1} - V^{-1}\|_{\hat{\mathcal{W}}} \leq c_2 \|Z_k - Z\|_{\hat{\mathcal{W}}}.$$

To prove the other inequality it is enough to study the continuity of the map $Z \to T_Z^{-1}(I)$. Using the inversion theorem in Banach algebras from Gohberg, Goldberg, and Kaashoek (1993, Chapter XXIX.4), there exists a positive constant c_2 such that $\|T_Z^{-1} - T_{Z_k}^{-1}\| \leq c_2 \|Z - Z_k\|_{\hat{\mathcal{W}}}$, and in particular

$$\|V_k^{-1} - V^{-1}\|_{\hat{\mathcal{W}}} = \|T_Z^{-1}(I) - T_{Z_k}^{-1}(I)\|_{\hat{\mathcal{W}}} \leq c_2 \|Z - Z_k\|_{\hat{\mathcal{W}}}.$$

Remark 4.5. Typically, J-spectral factors for irrational function are calculated using approximations by rational functions. Since $\hat{\mathcal{W}}^{n\times n}$ is an R-algebra, the above result allows the use of the J-spectral factorization of matrix-valued rational functions to compute an approximation in the Wiener norm of an irrational J-spectral factor. Note that the error inequalities in Theorem 4.4 are established for a particular choice of the J-spectral factor.

5. Conclusions

Starting from a canonical factorization for a certain matrix-valued function, we presented constructive algorithms for a (right-)standard factorization and a J-spectral factorization of a given matrix-valued function in the Wiener class of infinite-dimensional systems on the imaginary line. The algorithms are based on the operator theoretic methods from Clancey and Gohberg (1981). The constructive procedure is used to show that the J-spectral factor depends continuously on the J-spectral factor in the Wiener norm. This result should be useful in the context of approximation of irrational transfer functions in the Wiener class.

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