Stability analysis of monotone systems via max-separable Lyapunov functions

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Stability Analysis of Monotone Systems via Max-Separable Lyapunov Functions

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***Abstract*—We analyze stability properties of monotone nonlinear systems via max-separable Lyapunov functions, motivated by the following observations: first, recent results have shown that asymptotic stability of a monotone nonlinear system implies the existence of a max-separable Lyapunov function on a compact set; second, for monotone linear systems, asymptotic stability implies the stronger properties of D-stability and insensitivity to time delays. This paper establishes that for monotone nonlinear systems, equivalence holds between asymptotic stability, the existence of a max-separable Lyapunov function, D-stability, and insensitivity to bounded and unbounded time-varying delays. In particular, a new and general notion of D-stability for monotone nonlinear systems is discussed, and a set of necessary and sufficient conditions for delay-independent stability are derived. Examples show how the results extend the state of the art.**

***Index Terms*—Delay systems, D-stability, Lyapunov methods, monotone systems, positive systems.**

I. Introduction

Monotone systems are dynamical systems whose trajectories preserve a partial order relationship on their initial states. Such systems appear naturally in, for example, chemical reaction networks [2], consensus dynamics [3], systems biology [4], wireless networks [5], and as comparison systems in stability analysis of large-scale interconnected systems [6]–[8]. Due to their wide applicability, monotone systems have attracted considerable attention from the control community (see, e.g., [9]–[12]). Early references on the theory of monotone systems include the papers [13]–[15] by Hirsch and the excellent monograph [16] by Smith.

For monotone *linear* systems (also called positive linear systems), it is known that asymptotic stability of the origin implies further stability properties. First, asymptotically stable monotone linear systems always admit a Lyapunov function that can be expressed as a weighted max-norm [17]. Such Lyapunov functions can be written as a maximum of functions with 1-D arguments, so they are a particular class of max-separable Lyapunov functions [18]–[20]. Second, asymptotic stability of monotone linear systems is robust with respect to scaling of the dynamics with a diagonal matrix, leading to the so-called D-stability property. This notion appeared first in [21] and [22], with additional early results given in [23]. Third, monotone linear systems possess strong robustness properties with respect to time delays [24]–[31]. Namely, for these systems, asymptotic stability of a time-delay system can be concluded from stability of the corresponding delay-free system, simplifying the analysis. This is a significant property of monotone linear systems, since the stability of general linear systems typically depends on the magnitude and variation of the time delays [32]–[36].

For monotone *nonlinear* systems, it is, in general, unknown whether asymptotic stability of the origin implies notions of D-stability or delay-independent stability, even though results exist for certain classes of monotone systems, such as homogeneous and subhomogeneous systems [37]–[41]. Recent results in [42] and [43] show that for monotone nonlinear systems, asymptotic stability of the origin implies the existence of a max-separable Lyapunov function on every compact set in the domain of attraction. Motivated by this result and the strong robustness properties of monotone *linear* systems, we study stability properties of monotone *nonlinear* systems using max-separable Lyapunov functions. Here, monotone nonlinear systems are neither restricted to be homogeneous nor subhomogeneous.

The main contribution of this paper is to extend the stability properties of monotone linear systems discussed above to monotone nonlinear systems. Specifically, we demonstrate that asymptotic stability of the origin for a monotone nonlinear system leads to D-stability and asymptotic stability in the presence of bounded and unbounded time-varying delays. Furthermore, this paper has the following four contributions.

First, we show that for monotone nonlinear systems, the existence of a max-separable Lyapunov function on a compact set is equivalent to the existence of a path in this compact set such that, on this path, the vector field defining the system is negative in all its components. This allows for a simple evaluation of asymptotic stability for such systems.

Second, we define a novel and natural notion of D-stability for monotone nonlinear systems that extends earlier concepts in the literature [37]–[39]. Our notion of D-stability is based

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on the composition of the components of the vector field with an arbitrary monotonically increasing function that plays the role of a scaling. We then show that for monotone nonlinear systems, this notion of D-stability is equivalent to the existence of a max-separable Lyapunov function on a compact set.

Third, we demonstrate that for monotone nonlinear systems, asymptotic stability of the origin is insensitive to a general class of time delays, which includes bounded and unbounded time-varying delays. Again, this provides an extension of the existing results for monotone linear systems to the nonlinear case. In order to impose minimal restrictions on time-varying delays, our proof technique makes use of the max-separable Lyapunov function that guarantees asymptotic stability of the origin without time delays as a Lyapunov–Razumikhin function.

Fourth, we derive a set of necessary and sufficient conditions for establishing delay-independent stability of monotone nonlinear systems. These conditions can also provide an estimate of the region of attraction for the origin. As in the case of D-stability, we extend several existing results on analysis of monotone systems with time delays, which often rely on homogeneity and subhomogeneity of the vector field (see, e.g., [38]–[41]), to general monotone systems.

The remainder of this paper is organized as follows. Section II reviews some preliminaries on monotone nonlinear systems and max-separable Lyapunov functions and discusses stability properties of monotone linear systems. In Section III, our main results for stability properties of monotone nonlinear systems are presented, whereas, in Section IV, delay-independent stability conditions for monotone systems with time-varying delays are derived. Section V demonstrates through a number of examples how these results extend earlier work in the literature. Finally, conclusions and potential extensions are stated in Section VI.

Notation: The set of real numbers is denoted by $\mathbb{R}$, whereas $\mathbb{R}_+ = [0,\infty)$ represents the set of nonnegative real numbers. We let $\mathbb{R}^n_+$ denote the positive orthant in $\mathbb{R}^n$. The associated partial order is given as follows. For vectors x and y in $\mathbb{R}^n$, $x < y$ ($x \leq y$) if $x_i < y_i$ ($x_i \leq y_i$) for all $i \in \mathcal{I}_n$, where $\mathcal{I}_n = \{1, \ldots, n\}$ and $x_i \in \mathbb{R}$ represents the ith component of x. A continuous function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if $\omega(0) = 0$ and ω is strictly increasing. It belongs to class $\mathcal{K}_\infty$ if, in addition, $\omega(x) \to \infty$ as $x \to \infty$. A continuous function $\omega : \mathbb{R}^n \to \mathbb{R}_+$ is called positive definite if $\omega(0) = 0$ and $\omega(x) > 0$ for $x \neq 0$. For a real interval $[a,b]$, $\mathcal{C}\big([a,b],\mathbb{R}^n\big)$ denotes the space of real-valued continuous functions on $[a,b]$ taking real values in $\mathbb{R}^n$. Finally, $\mathbf{1}_n \in \mathbb{R}^n$ denotes the vector whose components are all one.

II. Problem Statement and Preliminaries

Consider dynamical systems on the positive orthant $\mathbb{R}^n_+$ described by the ordinary differential equation

$$\dot{x} = f(x). \tag{1}$$

Here, x is the system state, and the vector field $f : \mathbb{R}^n_+ \to \mathbb{R}^n$ is locally Lipschitz so that local existence and uniqueness of solutions is guaranteed [44]. Let $x(t, x_0)$ denote the solution to (1) starting from the initial condition $x_0 \in \mathbb{R}^n_+$ at the time $t \in \mathbb{R}_+$. We further assume that (1) has an equilibrium point at the origin, *i.e.*, $f(0) = 0$.

A. Preliminaries on Monotone Systems

In this paper, *monotone systems* will be studied according to the following definition.

Definition 2.1: The system (1) is called *monotone* with respect to the positive orthant if the implication

$$x_0' \leq x_0 \quad \Rightarrow \quad x(t, x_0') \leq x(t, x_0) \quad \forall t \in \mathbb{R}_+ \tag{2}$$

holds, for any initial conditions $x_0, x_0' \in \mathbb{R}^n_+$.

The definition states that trajectories of monotone systems starting at ordered initial conditions preserve the same ordering during the time evolution. By choosing $x_0' = 0$ in (2), since $x(t, 0) = 0$ for all $t \in \mathbb{R}_+$, it is easy to see that

$$x_0 \in \mathbb{R}^n_+ \quad \Rightarrow \quad x(t, x_0) \in \mathbb{R}^n_+ \quad \forall t \in \mathbb{R}_+. \tag{3}$$

This shows that the positive orthant $\mathbb{R}^n_+$ is an invariant set for the monotone system (1). Thus, monotone systems with an equilibrium point at the origin define positive systems.[1]

Monotonicity of dynamical systems is equivalently characterized by the so-called *Kamke–Müller condition*, stated next.

Proposition 2.2 (see [16]): The system (1) is monotone with respect to the positive orthant if and only if the following implication holds for all $x, x' \in \mathbb{R}^n_+$ and all $i \in \mathcal{I}_n$:

$$x' \leq x \text{ and } x_i' = x_i \quad \Rightarrow \quad f_i(x') \leq f_i(x). \tag{4}$$

Vector fields satisfying (4) are also referred to as *quasi-monotone nondecreasing* in $\mathbb{R}^n_+$ [45]. Note that if f is continuously differentiable on $\mathbb{R}^n_+$, then condition (4) is equivalent to the requirement that f has a Jacobian matrix with nonnegative off-diagonal elements, i.e.,

$$\frac{\partial f_i}{\partial x_j}(x) \geq 0, \quad x \in \mathbb{R}^n_+ \tag{5}$$

holds for all $i \neq j$, $i, j \in \mathcal{I}_n$ [16, Remark 3.1.1]. A vector field satisfying (5) is called *cooperative*.

Remark 2.3: In [46], the Kamke–Müller condition has been extended to a partial ordering with respect to an arbitrary cone (not necessarily the positive orthant).

In this paper, we will consider stability properties of monotone nonlinear systems. To this end, we use the following definition of (asymptotic) stability, tailored for monotone systems on the positive orthant.

Definition 2.4: The equilibrium point $x = 0$ of the monotone system (1) is said to be *stable* if, for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 \leq x_0 < \delta \mathbf{1}_n \quad \Rightarrow \quad 0 \leq x(t, x_0) < \varepsilon \mathbf{1}_n \quad \forall t \in \mathbb{R}_+. \tag{6}$$

The origin is called *asymptotically stable* if it is stable, and in addition, δ can be chosen such that

$$0 \leq x_0 < \delta \mathbf{1}_n \quad \Rightarrow \quad \lim_{t \to \infty} x(t, x_0) = 0.$$

[1] A dynamical system given by (1) is called *positive* if any trajectory of (1) starting from nonnegative initial conditions remains forever in the positive orthant, i.e., if $x_0 \in \mathbb{R}^n_+$, then $x(t) \in \mathbb{R}^n_+$ for all $t \in \mathbb{R}_+$.

We say that the origin is *globally asymptotically stable* if it is asymptotically stable for all nonnegative initial conditions:

$$\forall x_0 \in \mathbb{R}^n_+ \quad : \quad \lim_{t\to\infty} x(t, x_0) = 0.$$

Note that, due to the equivalence of norms on $\mathbb{R}^n$ and forward invariance property of the positive orthant for monotone systems, Definition 2.4 is equivalent to the usual notion of Lyapunov stability [44].

B. Preliminaries on Max-Separable Lyapunov Functions

We will characterize and study asymptotic stability of monotone nonlinear systems by means of the so-called *max-separable Lyapunov functions*

$$V(x) = \max_{i\in\mathcal{I}_n} V_i(x_i) \tag{7}$$

with scalar functions $V_i : \mathbb{R}_+ \to \mathbb{R}_+$. Since the Lyapunov function (7) is not necessarily continuously differentiable even if the functions V_i are smooth, we consider its upper-right Dini derivative along solutions of (1) (see, e.g., [47]) as

$$D^+V(x) = \limsup_{h\to 0^+} \frac{V\big(x + hf(x)\big) - V\big(x\big)}{h}. \tag{8}$$

The following result shows that if the functions V_i in (7) are continuously differentiable, then (8) admits an explicit expression.

Proposition 2.5 (see [48]): Consider $V : \mathbb{R}^n_+ \to \mathbb{R}_+$ in (7) and let $V_i : \mathbb{R}_+ \to \mathbb{R}_+$ be continuously differentiable for $i \in \mathcal{I}_n$. Then, the upper-right Dini derivative (8) is given by

$$D^+V(x) = \max_{j\in\mathcal{J}(x)} \frac{\partial V_j}{\partial x_j}(x_j) f_j(x) \tag{9}$$

where $\mathcal{J}(x)$ is the set of indices, for which the maximum in (7) is attained, *i.e.*,

$$\mathcal{J}(x) = \big\{ j \in \mathcal{I}_n \mid V_j(x_j) = V(x) \big\}. \tag{10}$$

C. Preliminaries on Monotone Linear Systems

Let $f(x) = Ax$ with $A \in \mathbb{R}^{n\times n}$. Then, the system (1) reduces to the linear system

$$\dot{x} = Ax. \tag{11}$$

It is well known that (11) is monotone in the sense of Definition 2.1 (and, hence, positive) if and only if A is Metzler, i.e., all off-diagonal elements of A are nonnegative [17]. We summarize some important stability properties of monotone linear systems in the next result.

Proposition 2.6 (see [20] and [49]): For the linear system (11), suppose that A is Metzler. Then, the following statements are equivalent.

1) The monotone linear system (11) is asymptotically stable, i.e., A is Hurwitz.
2) There is a max-separable Lyapunov function of the form
$$V(x) = \max_{i\in\mathcal{I}_n} \frac{x_i}{v_i} \tag{12}$$
on $\mathbb{R}^n_+$, with $v_i > 0$ for each $i \in \mathcal{I}_n$.
3) There exists a vector $w > 0$ such that $Aw < 0$.
4) For any diagonal matrix $\Delta \in \mathbb{R}^{n\times n}$ with positive diagonal entries, the linear system
$$\dot{x} = \Delta A x$$
is asymptotically stable, i.e., ΔA is Hurwitz.

In Proposition 2.6, the equivalence of statements 1) and 2) demonstrates that the existence of a max-separable Lyapunov function is a necessary and sufficient condition for asymptotic stability of monotone linear systems. The positive scalars v_i in the max-separable Lyapunov function (12) in the second item can be related to the positive vector w in the third item as $v_i = w_i$ for all $i \in \mathcal{I}_n$. Statement 4) shows that stability of monotone linear systems is robust with respect to scaling of the rows of matrix A. This property is known as *D-stability* [21]. Note that the notions of asymptotic stability in Proposition 2.6 hold globally due to linearity.

Another well-known property of monotone linear systems is that their asymptotic stability is insensitive to bounded and certain classes of unbounded time delays. This property reads as follows.

Proposition 2.7 (see [50]): Consider the delay-free monotone (positive) system

$$\dot{x}(t) = (A + B)x(t) \tag{13}$$

with A Metzler and B having nonnegative elements. If (13) is asymptotically stable, then the time-delay system

$$\dot{x}(t) = Ax(t) + Bx(t - \tau(t)) \tag{14}$$

is asymptotically stable for all bounded and potentially unbounded time-varying delays satisfying

$$\lim_{t\to+\infty} t - \tau(t) = +\infty.$$

Note that when A is Metzler and B is nonnegative, the time-delay system (14) is monotone [40, Proposition 3.2], i.e., solutions to the linear system (14) starting from nonnegative initial conditions $\varphi, \varphi' \in \mathcal{C}\big([-\tau_{\max}, 0], \mathbb{R}^n_+\big)$ with $\varphi'(t) \le \varphi(t)$ for all $t \in [-\tau_{\max}, 0]$ satisfy

$$x(t, \varphi') \le x(t, \varphi) \quad \forall t \in \mathbb{R}_+. \tag{15}$$

Proposition 2.7 shows that asymptotic stability of the monotone linear system (13) without time delays implies that the monotone system (14) with time-varying delays is also asymptotically stable. This is a surprising property, since the introduction of time delays may, in general, render a stable linear system unstable. The following example illustrates this point.

Example 2.8: Consider a time-delay linear system given by

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -11 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x(t - \tau_{\max}) \tag{16}$$

where $\tau_{\max}$ is a constant delay. In terms of (14),

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -11 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and $\tau(t) = \tau_{\max}$ for all $t \in \mathbb{R}_+$. The matrix B is nonnegative. However, as A is not Metzler, the time-delay linear system (16) is not monotone (positive).

Let $\tau_{\max} = 0$. Then, (16) reduces to the linear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10 & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix} x(t).$$

It is easy to verify that this delay-free system has an asymptotically stable equilibrium point at the origin. However, when $\tau_{\max}$ is larger than 1.5, the origin for the time-delay system (16) becomes unstable [32]. This simple example shows that for general linear systems, an equilibrium can lose its stability in the presence of time delays.

D. Main Goals

The main objectives of this paper are 1) to derive a counterpart of Proposition 2.6 for monotone *nonlinear* systems of the form (1); and 2) to extend the delay-independent stability property of monotone *linear* systems stated in Proposition 2.7 to monotone *nonlinear* systems with bounded and unbounded time-varying delays.

III. Stability of Monotone Nonlinear Systems

The following theorem is our first key result, which establishes a set of necessary and sufficient conditions for asymptotic stability of monotone nonlinear systems.

Theorem 3.1: Assume that the nonlinear system (1) is monotone. Then, the following statements are equivalent.

1) The origin is asymptotically stable.
2) For some compact set of the form

$$\mathcal{X} = \{x \in \mathbb{R}^n_+ \mid 0 \le x \le v\}. \tag{17}$$

With $v > 0$, there exists a max-separable Lyapunov function $V : \mathcal{X} \to \mathbb{R}_+$ as in (7) with $V_i : [0, v_i] \to \mathbb{R}_+$ differentiable for each $i \in \mathcal{I}_n$ such that

$$\nu_1(x_i) \le V_i(x_i) \le \nu_2(x_i) \tag{18}$$

holds for all $x_i \in [0, v_i]$ and for some functions ν_1, ν_2 of class $\mathcal{K}$, and that

$$D^+V(x) \le -\mu(V(x)), \tag{19}$$

holds for all $x \in \mathcal{X}$ and some positive definite function μ.

3) For some positive constant $\bar{s} > 0$, there exists a function $\rho : [0, \bar{s}] \to \mathbb{R}^n_+$ with ρ_i of class $\mathcal{K}$, ρ_i^{-1} differentiable on $[0, \rho_i(\bar{s})]$ and satisfying

$$\frac{d\rho_i^{-1}}{ds}(s) > 0 \tag{20}$$

for all $s \in (0, \rho_i(\bar{s})]$ and all $i \in \mathcal{I}_n$, such that

$$f \circ \rho(s) \le -\alpha(s) \tag{21}$$

holds for $s \in [0, \bar{s}]$ and some function $\alpha : [0, \bar{s}] \to \mathbb{R}^n_+$ with α_i positive definite for all $i \in \mathcal{I}_n$.

4) For any locally Lipschitz function $\psi : \mathbb{R}^n_+ \times \mathbb{R}^n \to \mathbb{R}^n$ given by $\psi(x, y) = \big(\psi_1(x_1, y_1), \ldots, \psi_n(x_n, y_n)\big)^{\mathrm{T}}$ where
   1) $\psi_i : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ for $i \in \mathcal{I}_n$,
   2) $\psi_i(x_i, 0) = 0$ for any $x_i \in \mathbb{R}_+$ and all $i \in \mathcal{I}_n$,
   3) $\psi_i(x_i, y_i)$ is monotonically increasing in y_i for each nonzero x_i, i.e., the implication

   $$y_i' < y_i \;\Rightarrow\; \psi_i(x_i, y_i') < \psi_i(x_i, y_i) \tag{22}$$

   holds for any $x_i > 0$ and all $i \in \mathcal{I}_n$,

   the nonlinear system

   $$\dot{x} = \psi\big(x, f(x)\big) \tag{23}$$

   has an asymptotically stable equilibrium point at the origin.

Proof: The proof is given in Appendix A. ■

Theorem 3.1 can be regarded as a nonlinear counterpart of Proposition 2.6. Namely, choosing the functions V_i in the second statement of Theorem 3.1 as $V_i(x_i) = x_i/v_i$, $v_i > 0$, and the function ρ in the third statement as $\rho(s) = ws$, $w > 0$, recovers statements 2) and 3) in Proposition 2.6, respectively. Note that we can let $v_i = w_i$ for each $i \in \mathcal{I}_n$, since, according to the proof of Theorem 3.1, the relation between V and ρ is

$$V_i(x_i) = \rho_i^{-1}(x_i) \;\text{ and }\; \rho_i(s) = V_i^{-1}(s) \quad i \in \mathcal{I}_n.$$

Statement 4) of Theorem 3.1, which can be regarded as a notion of D-stability for monotone nonlinear systems, is a counterpart of the fourth statement of Proposition 2.6. To see this, let us choose $\psi(x, y) = \Delta y$ for some diagonal matrix Δ with positive diagonal entries Δ_i, $i \in \mathcal{I}_n$. Clearly:

1) $\psi_i(x_i, 0) = \Delta_i \times 0 = 0$ for any $x_i \in \mathbb{R}_+$ and all $i \in \mathcal{I}_n$;
2) $\psi_i(x_i, y_i) = \Delta_i y_i$ is monotonically increasing in y_i.

Since $\psi(x, Ax) = \Delta Ax$, it follows from statement 4) of Theorem 3.1 that this choice of ψ recovers the D-stability result for monotone linear systems in Proposition 2.6.

Remark 3.2: According to the proof of Theorem 3.1, if there is a function ρ satisfying the third statement for $s \in [0, \bar{s}]$, then the max-separable Lyapunov function (7) with components $V_i(x_i) = \rho_i^{-1}(x_i)$ guarantees asymptotic stability of the origin for any initial condition

$$x_0 \in \mathcal{X} = \{x \in \mathbb{R}^n_+ \mid 0 \le x \le \rho(\bar{s})\}.$$

This means that the set $\mathcal{X}$ defined above is an estimate of the region of attraction for the origin.

We now present a simple example to illustrate the use of Theorem 3.1.

Example 3.3: Consider the nonlinear dynamical system

$$\dot{x} = f(x) = \begin{bmatrix} -5x_1 + x_1 x_2^2 \\ x_1 - 2x_2^2 \end{bmatrix}. \tag{24}$$

This system has an equilibrium point at the origin. As the Jacobian matrix of f is Metzler for all $(x_1, x_2) \in \mathbb{R}^2_+$, f is cooperative, i.e., it satisfies the Kamke–Müller condition. Thus, according to Proposition 2.2, (24) is monotone on $\mathbb{R}^2_+$.

First, we will show that the origin is asymptotically stable. Let $\rho(s) = (s, \sqrt{s})$, $s \in [0, 4]$. For each $i \in \{1, 2\}$, ρ_i is of class $\mathcal{K}$. It is easy to verify that

$$\rho_1^{-1}(s) = s, \quad \rho_2^{-1}(s) = s^2.$$

Thus, ρ_i^{-1}, $i \in \{1, 2\}$, is continuously differentiable and satisfies (20). In addition,

$$f \circ \rho(s) = \begin{bmatrix} -5s + s^2 \\ -s \end{bmatrix} \leq - \begin{bmatrix} s \\ s \end{bmatrix}$$

for all $s \in [0, 4]$, which implies that (21) holds. It follows from the equivalence of statements 1) and 3) in Theorem 3.1 that the origin is asymptotically stable.

Next, we will estimate the region of attraction of the origin by constructing a max-separable Lyapunov function. According to Remark 3.2, the monotone system (24) admits the max-separable Lyapunov function

$$V(x) = \max\left\{x_1, x_2^2\right\}$$

that guarantees the origin is asymptotically stable for

$$x_0 \in \left\{x \in \mathbb{R}^2_+ \mid 0 \leq x \leq (4, 2)\right\}.$$

Finally, we discuss D-stability of the monotone system (24). Consider ψ given by

$$\psi(x, y) = \left(\tfrac{x_1}{x_1^2 + 1} y_1^3 \ , \ x_2^2 y_2 \right)^{\mathrm{T}}.$$

For any $x \in \mathbb{R}^2_+$, $\psi(x, 0) = (0, 0)$. Moreover, each component $\psi_i(x_i, y_i)$, $i \in \{1, 2\}$, is monotonically increasing for any $x_i > 0$. Since the origin is an asymptotically stable equilibrium point of (24), by the equivalence of statements 1) and 4) in Theorem 3.1, the monotone nonlinear system

$$\dot{x} = \psi\big(x, f(x)\big) = \begin{bmatrix} \frac{x_1}{x_1^2+1} \left(-5x_1 + x_1 x_2^2\right)^3 \\ x_2^2 \left(x_1 - 2x_2^2\right) \end{bmatrix}$$

has an asymptotically stable equilibrium point at the origin.

Remark 3.4: A consequence of the proof of Theorem 3.1 is that all statements 1)–4) are equivalent to the existence of a vector $w > 0$ such that $f(w) < 0$ and

$$\lim_{t \to \infty} x(t, w) = 0. \tag{25}$$

In contrast to statement 3) of Proposition 2.6 for monotone linear systems, the condition $f(w) < 0$ without the additional assumption (25) does not necessarily guarantee asymptotic stability of the origin for monotone nonlinear systems. To illustrate the point, consider, for example, a scalar monotone system described by (1) with $f(x) = -x(x-1)$, $x \in \mathbb{R}_+$. This system has two equilibrium points: $x^\star = 0$ and $x^\star = 1$. Although $f(2) < 0$, it is easy to verify that any trajectory starting from the initial condition $x_0 > 0$ converges to $x^\star = 1$. Hence, the origin is not stable.

Remark 3.5: According to Lyapunov theory for *global* asymptotic stability [44], if the monotone system (1) admits a *global* max-separable Lyapunov function on $\mathbb{R}^n_+$, i.e., a max-separable Lyapunov function of the form (7) satisfying (18) and (19) for some functions ν_1, ν_2 of class $\mathcal{K}_\infty$ and for all $x \in \mathbb{R}^n_+$, then the origin is *globally* asymptotically stable. This means that by considering global max-separable Lyapunov functions, the implication 2) $\Rightarrow$ 1) in Theorem 3.1 also holds for the case of global asymptotic stability of monotone nonlinear systems. However, [42, Example IV.A] shows that the converse is, in general, not true. More precisely, for monotone nonlinear systems, global asymptotic stability of the origin does not necessarily guarantee the existence of a global max-separable Lyapunov function on $\mathbb{R}^n_+$.

Remark 3.6: The implication 1) $\Rightarrow$ 2) was shown in [42] (see also [43]) by construction of a max-separable Lyapunov function whose components are *lower-semicontinuous*. This construction may lead to *discontinuous* Lyapunov functions for monotone systems, which are undesirable for some applications [43]. However, Theorem 3.1 shows that monotone systems with asymptotically stable equilibrium point at the origin admit *Lipschitz-continuous* max-separable Lyapunov functions with *continuously differentiable* components. The existence of such a max-separable Lyapunov function plays a key role in our stability analysis of monotone systems with time-varying delays.

Remark 3.7: In [51, Th. III.2], the implication 3) $\Rightarrow$ 2) was proven for the case of global asymptotic stability of monotone nonlinear systems considering a max-separable Lyapunov function with possibly *nonsmooth* components V_i.

Remark 3.8: The implication 1) $\Rightarrow$ 4) was shown in [37]–[39] for particular classes of scaling functions ψ. If we choose $\psi_i(x_i, y_i) = d_i(x_i) y_i$ with $d_i(x_i) > 0$ for $x_i > 0$, then statement 4) recovers the results in [37] and [39]. If we choose $\psi(x, y) = \Delta y$ for some diagonal matrix Δ with positive diagonal entries, statement 4) recovers the result in [38]. Note also that in contrast to [37]–[39], neither homogeneity nor subhomogeneity assumption for monotone nonlinear systems is required in Theorem 3.1.

IV. Stability of Monotone Systems With Delays

Time delays are omnipresent in engineering systems and, in particular, distributed systems where exchange of information is involved. For this reason, the study of stability and control of dynamical systems with delayed states is essential and of practical importance [52]–[54]. The existing results on stability of time-delay systems can be classified into two major categories: 1) delay-independent stability and 2) delay-dependent stability. Delay-independent criteria guarantee stability regardless of the size of delays [55], whereas delay-dependent criteria include information on the delay margin and provide a maximal allowable delay that can be tolerated by the system [56].

In this section, delay-independent stability of nonlinear systems of the form

$$\begin{cases} \dot{x}(t) = g\big(x(t), x(t - \tau(t))\big), & t \geq 0 \\ x(t) = \varphi(t), & t \in [-\tau_{\max}, 0] \end{cases} \tag{26}$$

is considered. Here, $g : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$ is locally Lipschitz continuous with $g(0,0) = 0$, $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathbb{R}^n_+\big)$ is the vector-valued function specifying the initial state of the system, and τ is the time-varying delay, which satisfies the following assumption:

Assumption 4.1: The delay $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous with respect to time and satisfies

$$\lim_{t\to+\infty} t - \tau(t) = +\infty. \tag{27}$$

Note that τ is not necessarily continuously differentiable and that no restriction on its derivative (such as $\dot{\tau}(t) < 1$) is imposed. Roughly speaking, condition (27) implies that as t increases, the delay $\tau(t)$ grows slower than time itself. It is easy to verify that all bounded delays, irrespectively of whether they are constant or time varying, satisfy Assumption 4.1. Moreover, delays satisfying (27) may be unbounded (take, for example, $\tau(t) = \gamma t$ with $\gamma \in (0, 1)$).

Unlike the nondelayed system (1), the solution of the time-delay system (26) is not uniquely determined by a pointwise initial condition x_0, but by the continuous function φ defined over the interval $[-\tau_{\max}, 0]$. Since g is Lipschitz continuous and τ is a continuous function of time, the existence and uniqueness of solutions to (26) follow from [57, Th. 2]. We denote the solution to (26) corresponding to the initial condition φ by $x(t, \varphi)$.

Remark 4.2: Assumption 4.1 implies that there is a sufficiently large $T > 0$ such that $t - \tau(t) > 0$ for $t > T$. Define

$$\tau_{\max} = -\inf_{0\le t\le T}\{t - \tau(t)\}.$$

Since τ is continuous with respect to t, $\tau_{\max} \in \mathbb{R}_+$ is bounded ($\tau_{\max} < +\infty$). Therefore, $t - \tau(t) \in [-\tau_{\max}, t]$ for $t \in \mathbb{R}_+$ and the initial condition φ is defined on a bounded set $[-\tau_{\max}, 0]$ for any delay satisfying Assumption 4.1, even if it is unbounded.

From this point on, it is assumed that the time-delay system (26) satisfies the next assumption.

Assumption 4.3: The following properties hold.

1) $g(x, y)$ satisfies Kamke–Müller condition in x for each y, that is

$$x' \le x \text{ and } x'_i = x_i \quad\Rightarrow\quad g_i(x', y) \le g_i(x, y) \tag{28}$$

for any $y \in \mathbb{R}^n_+$ and all $i \in \mathcal{I}_n$.

2) $g(x, y)$ is order preserving in y for each x, i.e.,

$$y' \le y \quad\Rightarrow\quad g(x, y') \le g(x, y) \tag{29}$$

holds for any $x \in \mathbb{R}^n_+$.

The time-delay system (26) is said to be monotone if given two nonnegative initial conditions $\varphi, \varphi' \in \mathcal{C}\big([-\tau_{\max}, 0], \mathbb{R}^n_+\big)$ with $\varphi'(t) \le \varphi(t)$ for all $t \in [-\tau_{\max}, 0]$, then

$$x(t, \varphi') \le x(t, \varphi) \quad \forall t \in \mathbb{R}_+. \tag{30}$$

It follows from [16, Th. 5.1.1] that Assumption 4.3 ensures the monotonicity of (26). Furthermore, as the origin is an equilibrium for (26), the positive orthant $\mathbb{R}^n_+$ is forward invariant for the time-delay monotone system (26), i.e., $x(t, \varphi) \in \mathbb{R}^n_+$ for all $t \in \mathbb{R}_+$ when $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathbb{R}^n_+\big)$.

We are interested in stability of the time-delay system (26) under the assumption that the delay-free system

$$\dot{x}(t) = g\big(x(t), x(t)\big) =: f\big(x(t)\big) \tag{31}$$

has an asymptotically stable equilibrium point at the origin. Since the existence of time delays can induce instability in general dynamical systems (not necessarily monotone), the time-delay system (26) may have an unstable equilibrium point at the origin. However, the following theorem shows that asymptotic stability of the origin for *monotone* nonlinear systems is insensitive to bounded and unbounded time delays satisfying Assumption 4.1.

Theorem 4.4: Consider the time-delay system (26) under Assumption 4.3. Then, the following statements are equivalent.

1) The time-delay monotone system (26) has an asymptotically stable equilibrium point at the origin for all time-varying delays satisfying Assumption 4.1.
2) For the nondelayed monotone system (31), any of the equivalent conditions in the statement of Theorem 3.1 holds.

Proof: The proof is given in Appendix B. ■

According to Theorem 4.4, asymptotic stability of the origin for a delay-free monotone system of the form (31) implies asymptotic stability of the origin also for (26) with time-varying delays. Let us choose

$$g(x, y) = Ax + By \tag{32}$$

with A Metzler and B having nonnegative elements. Then, the nonlinear system (26) reduces to the time-delay linear system (14). Since $a_{ij} \ge 0$ for all $i \ne j$, then for $x, x' \in \mathbb{R}^n_+$ with $x' \le x$ and $x'_i = x_i$, we have

$$\begin{aligned} g_i(x', y) &= a_{ii}x'_i + \sum_{j\ne i} a_{ij}x'_j + \sum_{j=1}^{n} b_{ij}y_j \\ &\le a_{ii}x_i + \sum_{j\ne i} a_{ij}x_j + \sum_{j=1}^{n} b_{ij}y_j = g_i(x, y) \end{aligned}$$

for any $y \in \mathbb{R}^n_+$ and all $i \in \mathcal{I}_n$. Moreover, as $b_{ij} \ge 0$ for all $i, j \in \mathcal{I}_n$, then for any $y, y' \in \mathbb{R}^n_+$ with $y' \le y$, the inequality

$$g(x, y') = Ax + By' \le Ax + By = g(x, y)$$

holds for any $x \in \mathbb{R}^n_+$. The above observations show that vector-valued functions of the form (32) satisfy Assumption 4.3. Therefore, according to Theorem 4.4, asymptotic stability of the delay-free monotone linear system (13) implies that of (14) with time delays satisfying Assumption 4.1. Hence, Theorem 4.4 recovers the result of Proposition 2.7 as a special case.

Theorem 4.4 does not explicitly give any estimate of the region of attraction for the origin. However, its proof shows that the stability conditions presented in Theorem 3.1 for nondelayed monotone systems can provide such estimates, leading to the following practical tests.

Stability Test 1: Assume that for the delay-free monotone system (31), we can characterize asymptotic stability of the origin through a max-separable Lyapunov function V satisfying the second statement of Theorem 3.1. Then, for the time-delay system (26), the origin is asymptotically stable with respect to initial conditions $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathbb{R}^n_+\big)$ satisfying

$$\varphi(t) \in \mathcal{A} := \big\{x \in \mathbb{R}^n_+ \mid 0 \le x_i \le V_i^{-1}(c),\ i \in \mathcal{I}_n\big\}$$

for $t \in [-\tau_{\max}, 0]$ with

$$c = \min_{i \in \mathcal{I}_n} V_i(v_i).$$

Since V_i is continuous and strictly increasing for $x_i \in [0, v_i]$,

$$\mathcal{A} \subseteq \mathcal{X} = \big\{x \in \mathbb{R}^n_+ \mid 0 \le x_i \le v_i,\ i \in \mathcal{I}_n\big\}.$$

In other words, $\mathcal{A}$ is a level set of the Lyapunov function V, in which its time derivative is negative definite on $\mathcal{X}$. Note that when $V_i(v_i) = V_j(v_j)$ for all i and j, then $\mathcal{A} = \mathcal{X}$.

Stability Test 2: If we demonstrate the existence of a function ρ such that the nondelayed monotone system (31) satisfies the third statement of Theorem 3.1, then (26) with time delays satisfying Assumption 4.1 has an asymptotically stable equilibrium point at the origin for which the region of attraction includes initial conditions $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathbb{R}^n_+\big)$ that satisfy

$$0 \le \varphi(t) \le \rho(\bar{s}), \quad t \in [-\tau_{\max}, 0].$$

Stability Test 3: If we find a vector $w > 0$ such that $g(w, w) < 0$ and that the solution $x(t, w)$ to the delay-free monotone system (31) converges to the origin, then the solution $x(t, \varphi)$ to the time-delay system (26) converges to the origin for any initial condition $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathbb{R}^n_+\big)$ that satisfies

$$0 \le \varphi(t) \le w, \quad t \in [-\tau_{\max}, 0].$$

The following example illustrates the results of Theorem 4.4.

Example 4.5: Consider the time-delay system

$$\begin{aligned}\dot{x}(t) &= g\big(x(t), x(t - \tau(t))\big) \\ &= \begin{bmatrix} -5x_1(t) + x_1(t)x_2^2\big(t - \tau(t)\big) \\ x_1\big(t - \tau(t)\big) - 2x_2^2(t) \end{bmatrix}. \end{aligned} \tag{33}$$

One can verify that g satisfies Assumption 4.3. Thus, the nonlinear system (33) is monotone on $\mathbb{R}^2_+$. According to Example 3.3, this system without time delays has an asymptotically stable equilibrium at the origin. Therefore, Theorem 4.4 guarantees that for the time-delay system (33), the origin is still asymptotically stable for any bounded and unbounded time-varying delays satisfying Assumption 4.1.

We now provide an estimate of the region of attraction for the origin. Example 3.3 shows that for the system (33) without delays, the function $\rho(s) = (s, \sqrt{s})$, $s \in [0, 4]$, satisfies the third statement of Theorem 3.1. It follows from *Stability Test 2* that the solution $x(t, \varphi)$ to (33) converges to the origin for any initial condition $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathcal{X}\big)$ with

$$\mathcal{X} = \big\{x \in \mathbb{R}^2_+ \mid 0 \le x \le (4, 2)\big\}.$$

Remark 4.6: Our results can be extended to monotone nonlinear systems with heterogeneous delays of the form

$$\dot{x}_i(t) = g_i\big(x(t), x^{\tau_i}(t)\big), \quad i \in \mathcal{I}_n \tag{34}$$

where $g(x, y) = \big(g_1(x, y), \ldots, g_n(x, y)\big)^{\mathrm{T}}$ satisfies Assumption 4.3, and

$$x^{\tau_i}(t) := \big(x_1(t - \tau_i^1(t)), \ldots, x_n(t - \tau_i^n(t))\big)^{\mathrm{T}}.$$

If the delays τ_i^j, $i, j \in \mathcal{I}_n$, satisfy Assumption 4.1, then asymptotic stability of the origin for the delay-free monotone system (31) ensures that (34) with heterogeneous time-varying delays also has an asymptotically stable equilibrium point at the origin

Remark 4.7: In [58], several stability tests for asymptotic stability of monotone nonlinear systems without time-delays were established. Stability Test 1–Stability Test 3 extend these results to monotone nonlinear systems of the form (26) with time-varying delays.

Remark 4.8: Using max-separable Lyapunov functions in our proof techniques allows us 1) to establish asymptotic stability of monotone nonlinear systems under a general notion of D-stability, and 2) to show delay insensitivity to a general class of time delays, which includes bounded and several type of unbounded delays as a special case. Note, however, that the implications $1) \Rightarrow 4)$ in Theorem 3.1 and $2) \Rightarrow 1)$ in Theorem 4.4 demonstrate that to apply these results to monotone nonlinear systems, we can use other types of (possibly continuously differentiable) Lyapunov functions. In other words, if the asymptotic stability of a monotone nonlinear system is guaranteed using a Lyapunov function, which is not necessarily max-separable, then the system is D-stable and asymptotic stability of the origin is preserved in the presence of time delays.

V. Subhomogeneous Systems

In this section, we will illustrate how our main results recover and generalize previous results on delay-independent stability of particular classes of positive nonlinear systems.

A. Subhomogeneous Monotone Systems

Subhomogeneous monotone systems are a class of monotone nonlinear systems, which include monotone linear systems as a special case. We first define subhomogeneous vector fields.

Definition 5.1: Given an n-tuple $r = (r_1, \ldots, r_n)$ of positive real numbers and $\lambda > 0$, the *dilation map* $\delta_\lambda^r : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$\delta_\lambda^r(x) := \big(\lambda^{r_1} x_1, \ldots, \lambda^{r_n} x_n\big).$$

When $r = \mathbf{1}_n$, the dilation map is called the *standard dilation map*. A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *subhomogeneous of degree* $p \in \mathbb{R}$ with respect to the dilation map δ_λ^r if

$$f\big(\delta_\lambda^r(x)\big) \le \lambda^p \delta_\lambda^r\big(f(x)\big) \quad \forall x \in \mathbb{R}^n,\ \forall \lambda \ge 1.$$

For example, the linear mapping $f(x) = Ax$ is subhomogeneous of degree zero with respect to the standard dilation map.

Theorem 4.4 allows us to show that *global* asymptotic stability of the origin for monotone nonlinear systems whose vector fields are subhomogeneous is insensitive to bounded and unbounded time-varying delays.

Corollary 5.2: Consider the time-delay system (26) under Assumption 4.3. Suppose that $f(x) := g(x,x)$ is subhomogeneous of degree $p \in \mathbb{R}_+$ with respect to the dilation map δ_λ^r. If the origin for the delay-free monotone system (31) is globally asymptotically stable, then the subhomogeneous monotone system (26) has a globally asymptotically stable equilibrium at the origin for all time-varying delays satisfying Assumption 4.1.

Proof: Suppose the origin for the subhomogeneous monotone system (31) is globally asymptotically stable. Then, for any constant $\alpha > 0$, there exists a vector $w > 0$ such that $\alpha \mathbf{1}_n \leq w$, $g(w,w) < 0$, and the solution $x(t,w)$ to the delay-free system (31) converges to the origin [37, Th. 4.1]. It follows from Theorem 4.4 and *Stability Test 3* that the time-delay system (26) is asymptotically stable with respect to initial conditions satisfying $0 \leq \varphi(t) \leq \alpha \mathbf{1}_n$, $t \in [-\tau_{\max}, 0]$.

To complete the proof, let $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathbb{R}_+^n\big)$ be an arbitrary initial condition. As $\alpha > 0$ and φ is continuous (hence, bounded) on $[-\tau_{\max}, 0]$, we can find some sufficiently large $\alpha > 0$ such that $\varphi(t) \leq \alpha \mathbf{1}_n$ for $t \in [-\tau_{\max}, 0]$. This together with the above observations implies that the origin is asymptotically stable for all nonnegative initial conditions. ■

Remark 5.3: In [41], it was shown that global asymptotic stability of the origin for subhomogeneous monotone systems is independent of *bounded* time-varying delays. In this work, we establish insensitivity of subhomogeneous monotone systems to the general class of possibly *unbounded* delays described by Assumption 4.1, which includes bounded delays as a special case.

Another important class of monotone nonlinear systems are those with *homogeneous* vector fields.

Definition 5.4: A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *homogeneous of degree* $p \in \mathbb{R}$ with respect to the dilation map δ_λ^r if

$$f\big(\delta_\lambda^r(x)\big) = \lambda^p \delta_\lambda^r\big(f(x)\big) \quad \forall x \in \mathbb{R}^n,\ \forall \lambda > 0.$$

It is clear that every homogeneous vector field is also subhomogeneous. Therefore, according to Corollary 5.2, a homogeneous monotone system with time delays satisfying Assumption 4.1 is globally asymptotically stable if and only if the corresponding delay-free system is globally asymptotically stable.

Remark 5.5: Delay-independent stability of homogeneous monotone systems with time delays satisfying Assumption 4.1 was previously considered in [40] by using max-separable Lyapunov functions. Note, however, that the proof of Theorem 4.4 differs significantly from the analysis in [40]. The main reason for this is that the homogeneity assumption, which plays a key role in the stability proof in [40], is not satisfied for general monotone systems.

B. Subhomogeneous (Nonmonotone) Positive Systems

Finally, motivated by results in [59], we consider the time-delay system

$$\dot{x}(t) = g(x(t), x(t-\tau(t))) = h\big(x(t)\big) + d\big(x(t-\tau(t))\big). \tag{35}$$

We assume that the vector fields h and d satisfy Assumption 5.6.

Assumption 5.6: The following properties hold.

1) For each $i \in \mathcal{I}_n$, $h_i(x) \geq 0$ for $x \in \mathbb{R}_+^n$ with $x_i = 0$.
2) For all $x \in \mathbb{R}_+^n$, $d(x) \geq 0$.
3) Both h and d are subhomogeneous of degree $p \in \mathbb{R}_+$.
4) For any $x \in \mathbb{R}_+^n \setminus \{0\}$, there is $i \in \mathcal{I}_n$ such that

$$\sup\big\{d_i(z') \mid 0 \leq z' \leq x\big\} < -\sup\big\{h_i(z) \mid 0 \leq z \leq x, z_i = x_i\big\}.$$

Note that under Assumption 5.6, the time-delay system (35) is not necessarily monotone. However, Assumptions 5.6.1 and 5.6.2 ensure the positivity of (35) [16, Th. 5.2.1].

In [59], it was shown that if Assumption 5.6 holds, then the positive system (35) with *constant* delays ($\tau(t) = \tau_{\max}$ for $t \in \mathbb{R}_+$) has a globally asymptotically stable equilibrium at the origin for all $\tau_{\max} \in \mathbb{R}_+$. Theorem 4.4 helps us to extend the result in [59] to *time-varying* delays satisfying Assumption 4.1.

Corollary 5.7: For the time-delay system (35), suppose Assumption 5.6 holds. Then, the origin is globally asymptotically stable for all time delays satisfying Assumption 4.1.

Proof: For any $x, y \in \mathbb{R}_+^n$ and each $i \in \mathcal{I}_n$, define

$$\bar{g}_i(x,y) = \sup\big\{h_i(z) + d_i(z') \mid 0 \leq z \leq x, z_i = x_i, 0 \leq z' \leq y\big\}.$$

It is straightforward to show that $\bar{g}(x,y)$ satisfies Assumption 4.3. Thus, the time-delay system

$$\dot{x}(t) = \bar{g}\big(x(t), x(t-\tau(t))\big) \tag{36}$$

is monotone. Under Assumption 5.6, the subhomogeneous monotone system (36) without delays ($\tau(t) = 0$) has a globally asymptotically stable equilibrium at the origin [59, Th. III.2]. Therefore, according to Corollary 5.2, the origin for the time-delay system (36) is also globally asymptotically stable for any time delays satisfying Assumption 4.1.

As $g(x,y) \leq \bar{g}(x,y)$ for any $x, y \in \mathbb{R}_+^n$, it follows from [16, Th. 5.1.1] that for any initial condition φ,

$$x(t,\varphi,g) \leq x(t,\varphi,\bar{g}), \quad t \in \mathbb{R}_+ \tag{37}$$

where $x(t,\varphi,g)$ and $x(t,\varphi,\bar{g})$ are solutions to (35) and (36), respectively, for a common initial condition φ. Since the origin is a globally asymptotically stable equilibrium for (36), $x(t,\varphi,\bar{g}) \to 0$ as $t \to \infty$. Moreover, as (35) is a positive system, $x(t,\varphi,g) \geq 0$ for $t \in \mathbb{R}_+$. We can conclude from (37) and the above observations that for any nonnegative initial condition φ, $x(t,\varphi,g)$ converges to the origin. Hence, for the time-delay system (35), the origin is globally asymptotically stable. ■

VI. Conclusion and Future Directions

In this paper, we have presented a number of results that extend fundamental stability properties of monotone linear systems to monotone nonlinear systems. Specifically, we have shown that for such nonlinear systems, equivalence holds between asymptotic stability of the origin, the existence of a Lipschitz-continuous max-separable Lyapunov function on a compact set, and D-stability. In addition, we have demonstrated that if the origin for a delay-free monotone system is asymptotically stable, then the corresponding system with bounded and unbounded time-varying delays also has an asymptotically stable equilibrium point at the origin. We have derived a set of necessary and sufficient conditions for establishing delay-independent stability of monotone nonlinear systems, which allow us to extend several earlier works in the literature. We have illustrated the main results with several examples.

In this paper, we have used max-separable Lyapunov functions to derive delay-independent stability conditions and estimate the region of attraction for monotone nonlinear systems. There exists, however, another special kind of Lyapunov functions, called *sum-separable Lyapunov functions*, that have been widely used for stability analysis of linear and nonlinear monotone systems without time delays [60]–[62]. Sum-separable Lyapunov functions do not suffer from some limitations of max-separable Lyapunov functions in estimating the domain of attraction [19]. Therefore, it would be an interesting future work to analyze delay-independent stability of monotone nonlinear systems via sum-separable Lyapunov functions and obtain the region of attraction given by the level sets of these functions.

Appendix

Before proving the main results of the paper, namely, Theorems 3.1 and 4.4, we first state a key lemma which shows that all components of a max-separable Lyapunov function are necessarily monotonically increasing.

Lemma A.1: Consider the monotone system (1) and the max-separable function $V : \mathbb{R}_+^n \to \mathbb{R}_+$ as in (7) with components $V_i : \mathbb{R}_+ \to \mathbb{R}_+$ differentiable for all $i \in \mathcal{I}_n$. Suppose that there exist functions ν_1, ν_2 of class $\mathcal{K}$ such that

$$\nu_1(x_i) \leq V_i(x_i) \leq \nu_2(x_i) \tag{38}$$

for all $x_i \in \mathbb{R}_+$ and all $i \in \mathcal{I}_n$. Suppose also that there exists a positive-definite function μ such that

$$D^+V(x) \leq -\mu(V(x)) \tag{39}$$

for all $x \in \mathbb{R}_+^n$. Then, the functions V_i satisfy, for all $x_i > 0$,

$$\frac{\partial V_i}{\partial x_i}(x_i) > 0. \tag{40}$$

Proof: For some $j \in \mathcal{I}_n$, consider the state $x = e_j x_j$, where e_j is the jth column of the identity matrix $I \in \mathbb{R}^{n\times n}$, and $x_j \in \mathbb{R}_+$. From (38), $V_j(x_j) > 0$ for any $x_j > 0$ and $V_i(x_i) = 0$ for $i \neq j$. Thus, the set $\mathcal{J}$ in (10) satisfies $\mathcal{J}(e_j x_j) = \{j\}$ for all $x_j > 0$. Evaluating (39) through Proposition 2.5 leads to

$$D^+V(e_j x_j) = \frac{\partial V_j}{\partial x_j}(x_j) f_j(e_j x_j) \leq -\mu(V(e_j x_j)) < 0 \tag{41}$$

for $x_j > 0$. The strict inequality in (41) implies that $\partial V_j/\partial x_j$ is nonzero and that its sign is constant for all $x_j > 0$. As a negative sign would yield $V_j(x_j) < 0$ [and, hence violate (38)] and j is chosen arbitrarily, (40) holds. ■

A. Proof of Theorem 3.1

The theorem will be proven by first showing the equivalence $1) \Leftrightarrow 2) \Leftrightarrow 3)$. Next, this equivalence will be exploited to subsequently show $3) \Rightarrow 4)$ and $4) \Rightarrow 1)$. Consequently, we have

$$1) \Leftrightarrow 2) \Leftrightarrow 3) \Rightarrow 4) \Rightarrow 1)$$

which proves the desired result.

First, we note that the implication $2) \Rightarrow 1)$ follows directly from Lyapunov stability theory (e.g., [47]) and we proceed to prove that 1) implies 2).

Proof of $1) \Rightarrow 2)$ By asymptotic stability of the origin as in Definition 2.4, the region of attraction of $x = 0$ defined as

$$\mathcal{A} := \left\{x_0 \in \mathbb{R}_+^n \mid \lim_{t\to\infty} x(t, x_0) = 0\right\}$$

is nonempty. In fact, we can find some $\delta > 0$ such that all states satisfying $0 \leq x < \delta\mathbf{1}$ are in $\mathcal{A}$. Then, as the system (1) is monotone, the reasoning from [8, Th. 3.12] (see also [51, Th. 2.5] for a more explicit statement) can be followed to show the existence of a vector v such that $0 < v < \delta\mathbf{1}$ and $f(v) < 0$.

Let $\omega(t) = x(t, v)$, $t \in \mathbb{R}_+$, be the solution to the system (1) starting from such a v. By the local Lipschitz continuity of f, ω is continuously differentiable. As $v \in \mathcal{A}$, $w_i(t) \to 0$ when $t \to \infty$ for each $i \in \mathcal{I}_n$. Moreover, note that v is an element of the set

$$\Omega = \{x \in \mathbb{R}_+^n \mid f(x) < 0\}.$$

According to [16, Proposition 3.2.1], Ω is forward invariant so $\omega(t) \in \Omega$ for all $t \in \mathbb{R}_+$. Thus, the components $\omega_i(t)$ are strictly decreasing in t, i.e., $\dot{\omega}_i(t) < 0$ for $t \in \mathbb{R}_+$. This further implies that, for a given state component $x_i \in (0, v_i]$, there exists a unique $t \in \mathbb{R}_+$ such that $x_i = \omega_i(t)$. Let

$$T_i(x_i) = \left\{t \in \mathbb{R}_+ \mid x_i = \omega_i(t)\right\}.$$

From the definition, it is clear that $T_i(x_i) = \omega_i^{-1}(x_i)$. Since $\dot{\omega}_i(t) < 0$ for $t \in \mathbb{R}_+$, the inverse of ω_i, i.e., the function T_i, is continuously differentiable and strictly decreasing for all $x_i \in (0, v_i]$. We define, as in [42] and [43], the component functions

$$V_i(x_i) = e^{-T_i(x_i)}, \quad i \in \mathcal{I}_n. \tag{42}$$

Note that $V_i(0) = 0$. Moreover, V_i are continuously differentiable and strictly increasing for all $x_i \in (0, v_i]$ as a result of the properties of the function T_i. Therefore, the component functions V_i in (42) satisfy (18) for some functions ν_1, ν_2 of class $\mathcal{K}$. Moreover, from [43, Th. 3.2], the upper-right Dini derivative of the max-separable Lyapunov function (7) with components (42) is given by

$$D^+V(x) \leq -V(x)$$

for all $x \in \mathcal{X}$ with $\mathcal{X} = \{x \in \mathbb{R}_+^n \mid 0 \leq x \leq v\}$. This shows that (19) holds, and hence, the proof is complete. ■

Similar to the proof for implication $1) \Rightarrow 2)$, since $f(\omega(t)) < 0$ for all $t \in \mathbb{R}_+$ and $\omega(t)$ is strictly decreasing, the trajectory ω can also help us to prove the implication $1) \Rightarrow 3)$ by constructing an increasing function ρ that satisfies statement 3). However, in order to find a relation between the max-separable Lyapunov function V in the second statement and the function ρ in the third statement, we now proceed to show equivalence between 2) and 3). We begin with the implication $2) \Rightarrow 3)$.

Proof of $2) \Rightarrow 3)$ According to Lemma A.1, the functions V_i are monotonically increasing. Thus, their inverses

$$\rho_i(s) = V_i^{-1}(s) \tag{43}$$

can be defined for all $s \in [0, V_i(v_i)]$. Define

$$\bar{s} := \min_{i \in \mathcal{I}_n} V_i(v_i).$$

Since $v > 0$, it follows from (18) that $\bar{s} > 0$. Moreover, due to continuous differentiability of V_i and Lemma A.1, ρ_i is of class $\mathcal{K}$ and satisfies (20).

In the remainder of the proof, it will be shown that the function ρ with components ρ_i defined in (43) satisfies (21) for all $s \in [0, \bar{s}]$. Thereto, consider a state x^s, parameterized by s according to the definition $x^s := \rho(s)$. Since $\bar{s} \le V_i(v_i)$ for all $i \in \mathcal{I}_n$, it holds that $x^s \in \mathcal{X}$ for all $s \in [0, \bar{s}]$. Evaluating (19) for such an x^s yields

$$D^+ V\big(x^s\big) = \max_{j \in \mathcal{J}(x^s)} \frac{\partial V_j}{\partial x_j}\big(x_j^s\big) f_j\big(x^s\big) \le -\mu\big(V(x^s)\big). \tag{44}$$

The definition $x_i^s = \rho_i(s)$, $i \in \mathcal{I}_n$, implies, through (43), that $V_i(x_i^s) = s$ for all $s \in [0, \bar{s}]$. Consequently, $V(x^s) = s$ and the set $\mathcal{J}(x^s)$ in (44) satisfies $\mathcal{J}(x^s) = \mathcal{I}_n$, such that (44) implies

$$\frac{\partial V_i}{\partial x_i}\big(\rho_i(s)\big) f_i\big(\rho(s)\big) \le -\mu\big(s\big) \tag{45}$$

for all $s \in [0, \bar{s}]$ and every $i \in \mathcal{I}_n$. Since V_i is strictly increasing and μ is positive definite, $f_i(\rho_i(s)) \le 0$. Define functions $r_i : [0, \bar{s}] \to \mathbb{R}$ as

$$r_i(s) := \sup\left\{ \frac{\partial V_i}{\partial x_i}(z_i) \;\middle|\; \rho_i(s) \le z_i \le \rho_i(\bar{s}) \right\}.$$

By continuous differentiability of V_i and the result of Lemma A.1, it follows that r_i exists and satisfies $r_i(s) > 0$ for all $s \in [0, \bar{s}]$. Moreover, it is easily seen that

$$r_i(s) \ge \frac{\partial V_i}{\partial x_i}(\rho_i(s)).$$

This together with (45) implies that

$$r_i(s) f_i(\rho(s)) \le \frac{\partial V_i}{\partial x_i}(\rho_i(s)) f_i(\rho(s)) \le -\mu(s)$$

for all $s \in [0, \bar{s}]$. Here, the inequality follows from the observation that $f_i(\rho_i(s)) \le 0$. Then, strict positivity of r_i implies that

$$f_i(\rho(s)) \le -\frac{\mu(s)}{r_i(s)}$$

for all $s \in [0, \bar{s}]$ and any $i \in \mathcal{I}_n$. Since r_i is strictly positive and μ is positive definite, the function $\alpha_i(s) = \mu(s)/r_i(s)$ is positive definite and (21) holds. ■

We continue with the reverse implication $3) \Rightarrow 2)$.

Proof of $3) \Rightarrow 2)$ Define $v := \rho(\bar{s})$. Since the functions ρ_i, $i \in \mathcal{I}_n$, are of class $\mathcal{K}$ and $\bar{s} > 0$, we have $v > 0$. Let V_i be such that

$$V_i(x_i) = \rho_i^{-1}(x_i) \tag{46}$$

for $x_i \in [0, v_i]$. Note that the inverse of ρ_i exists on this compact set as ρ_i is of class $\mathcal{K}$ and, hence, is strictly increasing. Because of the same reason, it is clear that V_i as in (46) satisfies (18) for some functions ν_1, ν_2 of class $\mathcal{K}$. It follows from [51, Th. III.2] that the max-separable Lyapunov function (7) with components (46) satisfies (19). ■

We now show that $3)$ implies $4)$ by exploiting the equivalence $1) \Leftrightarrow 2) \Leftrightarrow 3)$.

Proof of $3) \Rightarrow 4)$ First, we show that the system (23) is monotone. Thereto, recall that monotonicity of (1) implies

$$\begin{aligned} x' \le x,\ x_i' = x_i \quad &\Rightarrow \quad f_i(x') \le f_i(x) \\ &\Rightarrow \quad \psi_i\big(x_i', f_i(x')\big) \le \psi_i\big(x_i, f_i(x)\big) \end{aligned} \tag{47}$$

where the latter implication follows from (22). Then, (47) represents the Kamke–Müller condition for the vector field $\psi(x, f)$, such that monotonicity of (23) follows from Proposition 2.2.

Next, note that $\varphi\big(0, f(0)\big) = 0$ implying that the origin is an equilibrium point of (23). In order to prove asymptotic stability of the origin, we recall that f satisfies, by assumption, (21) for some function ρ. From this, we have

$$\psi\big(\rho(s), f(\rho(s)\big) \le \psi\big(\rho(s), -\alpha(s)\big) \tag{48}$$

where the inequality is maintained due to the fact that ψ_i is monotonically increasing in the second argument for all $i \in \mathcal{I}_n$. The functions α_i and ρ are positive definite; hence, $-\alpha(s) < 0$ and $\rho(s) > 0$ for all $s \in (0, \bar{s}]$. Then, from (22), we have

$$\psi\big(\rho(s), -\alpha(s)\big) < \psi\big(\rho(s), 0\big) = 0$$

for all $s \in (0, \bar{s}]$, where the equality follows from $\varphi(x, 0) = 0$ for any $x \in \mathbb{R}_+^n$. Hence, $\psi\big(\rho(s), -\alpha(s)\big)$ is negative definite and (48) is again of the form (21). From the implication $3) \Rightarrow 2) \Rightarrow 1)$, we conclude the origin is asymptotically stable. ■

Finally, we prove that $4)$ implies $1)$.

Proof of $4) \Rightarrow 1)$*:* Assume that the system (23) is asymptotically stable for any Lipschitz continuous function ψ satisfying statement $4)$. Particularly, let $\psi(x, y) = y$. Then, the monotone system (1) is asymptotically stable. ■

B. Proof of Theorem 4.4

Proof of $1) \Rightarrow 2)$*:* Assume that the time-delay monotone system (26) is asymptotically stable for all delays satisfying Assumption 4.1. Particularly, let $\tau(t) = 0$, $t \in \mathbb{R}_+$. Then, the nondelayed monotone system (31) is asymptotically stable. ■

Proof of $2) \Rightarrow 1)$*:* For the delay-free monotone system (31), assume that the third statement of Theorem 3.1 holds. Let ρ be a function such that (31) satisfies (21), i.e.,

$$g\big(\rho(s), \rho(s)\big) \le -\alpha\big(s\big) \tag{49}$$

holds for all $s \in [0, \bar{s}]$. Define

$$v := \rho(\bar{s}) \tag{50}$$

and let $\mathcal{X} = \big\{x \in \mathbb{R}_+^n \mid 0 \le x \le v\big\}$. First, we show that for any $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathcal{X}\big)$, the solution $x(t, \varphi)$ to the time-delay system (26) satisfies

$$x(t, \varphi) \in \mathcal{X}, \quad t \in \mathbb{R}_+. \tag{51}$$

Clearly, $x(0, \varphi) = \varphi(0) \in \mathcal{X}$. In order to establish a contradiction, suppose that the statement $x(t, \varphi) \in \mathcal{X}, t \in \mathbb{R}_+$, is not true.

Then, there is $i \in \mathcal{I}_n$ and a time $\hat{t} \in \mathbb{R}_+$ such that $x(t, \varphi) \in \mathcal{X}$ for all $t \in [0, \hat{t}]$, $x_i(\hat{t}, \varphi) = v_i$, and

$$D^+ x_i(\hat{t}, \varphi) \geq 0. \tag{52}$$

As $x(\hat{t}, \varphi) \leq v$, it follows from Assumption 4.3.1 that

$$g_i\big(x(\hat{t}, \varphi), y\big) \leq g_i\big(v, y\big) \tag{53}$$

for any $y \in \mathbb{R}^n_+$. Since $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathcal{X}\big)$ and, by Remark 4.2, $\hat{t} - \tau(\hat{t}) \in [-\tau_{\max}, \hat{t}]$, we have $x(\hat{t} - \tau(\hat{t}), \varphi) \in \mathcal{X}$ irrespectively of whether $\hat{t} - \tau(\hat{t})$ is nonnegative or not. Thus, Assumption 4.3.2 implies that

$$g_i\big(y, x(\hat{t} - \tau(\hat{t}), \varphi)\big) \leq g_i\big(y, v\big) \tag{54}$$

for any $y \in \mathbb{R}^n_+$. Using (53) and (54), the Dini derivative of $x_i(t, \varphi)$ along the trajectories of (26) at $t = \hat{t}$ is given by

$$\begin{aligned} D^+ x_i(\hat{t}, \varphi) &= g_i\big(x(\hat{t}, \varphi), x(\hat{t} - \tau(\hat{t}), \varphi)\big) \\ &\leq g_i(v, v) \overset{(50)}{=} g_i\big(\rho(\bar{s}), \rho(\bar{s})\big) \\ &\overset{(49)}{\leq} -\alpha_i(\bar{s}) < 0 \end{aligned}$$

which contradicts (52). Therefore, $x(t, \varphi) \in \mathcal{X}$ for all $t \in \mathbb{R}_+$. ■

Next, we will prove the asymptotic stability of the origin for the time-delay system (26). In other words, we will show that for any $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathcal{X}\big)$, $x(t, \varphi) \to 0$ as $t \to \infty$. According to the proof of Theorem 3.1, if the delay-free monotone system (31) satisfies (49), then it admits a max-separable Lyapunov function (7) with components

$$V_i(x_i) = \rho_i^{-1}(x_i) \tag{55}$$

defined on $\mathcal{X}$, see (46), such that

$$D^+ V(x) = \max_{j \in \mathcal{J}(x)} \frac{\partial V_j}{\partial x_j}(x_j) g_j\big(x, x\big) \leq -\mu(V(x)) \tag{56}$$

holds for $x \in \mathcal{X}$. Recall from (51) that $x(t, \varphi) \leq v$ for all $t \in \mathbb{R}_+$ when $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathcal{X}\big)$. Thus

$$\begin{aligned} V(x(t, \varphi)) = \max_{i \in \mathcal{I}_n} V_i(x_i(t, \varphi)) &\overset{(55)}{=} \max_{i \in \mathcal{I}_n} \rho_i^{-1}(x_i(t, \varphi)) \\ &\leq \max_{i \in \mathcal{I}_n} \rho_i^{-1}(v_i) \overset{(50)}{=} \bar{s} \end{aligned} \tag{57}$$

where the inequality follows from the fact that ρ_i and, hence, ρ_i^{-1} are functions of class $\mathcal{K}$. This, in turn, implies that

$$\begin{aligned} \bar{x}(t) &:= \rho(V(x(t, \varphi))) \\ &\leq \rho(\bar{s}) \overset{(50)}{=} v \end{aligned} \tag{58}$$

or, equivalently, $\bar{x}(t) \in \mathcal{X}$ for $t \in \mathbb{R}_+$. Hence, from (56),

$$\max_{j \in \mathcal{J}(\bar{x}(t))} \frac{\partial V_j}{\partial x_j}\big(\bar{x}_j(t)\big) g_j\big(\bar{x}(t), \bar{x}(t)\big) \leq -\mu(V(\bar{x}(t))) \tag{59}$$

for all $t \in \mathbb{R}_+$. Note also that for each $i \in \mathcal{I}_n$,

$$V_i\big(\bar{x}_i(t)\big) = V_i\big(\rho_i(V(x(t, \varphi)))\big) \overset{(55)}{=} V\big(x(t, \varphi)\big)$$

which means that

$$V\big(\bar{x}(t)\big) = V\big(x(t, \varphi)\big), \quad \text{and} \quad \mathcal{J}(\bar{x}(t)) = \mathcal{I}_n. \tag{60}$$

Since $V_i(x_i) \leq V(x)$ for $i \in \mathcal{I}_n$ and $V_j(x_j) = V(x)$ for $j \in \mathcal{J}(x)$, we obtain from (55) that

$$x_i\big(t, \varphi\big) \leq \rho_i\big(V(x(t, \varphi))\big) = \bar{x}_i(t)$$

for any $i \in \mathcal{I}_n$, and

$$x_j\big(t, \varphi\big) = \rho_j\big(V(x(t, \varphi))\big) = \bar{x}_j(t) \tag{61}$$

for $j \in \mathcal{J}(x(t, \varphi))$. Thus, according to Assumption 4.3.1,

$$g_j\big(x(t, \varphi), y\big) \leq g_j\big(\bar{x}(t), y\big), \quad j \in \mathcal{J}(x(t, \varphi)) \tag{62}$$

for any $\varphi \in \mathcal{C}\big([-\tau_{\max}, 0], \mathcal{X}\big)$ and any $y \in \mathbb{R}^n_+$. This condition will be exploited later in the proof.

In the remainder of the proof, the max-separable Lyapunov function V of the delay-free system (31) will be used as a candidate Lyapunov–Razumikhin function for the time-delay system (26). To establish a Razumikhin-type argument [57], it is assumed that

$$V\big(x(\sigma, \varphi)\big) < q\big(V(x(t, \varphi))\big) \tag{63}$$

for all $\sigma \in [t - \tau(t), t]$, where $q : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function satisfying $q(s) > s$ for $s > 0$. We are going to construct a function q that satisfies the above assumptions. In order to do so, we will introduce a compact set $\tilde{\mathcal{X}}$, constants $L_{\tilde{\mathcal{X}}}$ and D, and a class $\mathcal{K}$ function $\tilde{\mu}$. Specifically, the compact set $\tilde{\mathcal{X}}$ is defined as

$$\tilde{\mathcal{X}} := \{x \in \mathbb{R}^n_+ \mid 0 \leq x \leq 2v\}.$$

Clearly, $\mathcal{X} \subset \tilde{\mathcal{X}}$. Since g in (26) is locally Lipschitz, there is a constant $L_{\tilde{\mathcal{X}}} > 0$ such that

$$\|g(x, y') - g(x, y)\|_\infty \leq L_{\tilde{\mathcal{X}}} \|y' - y\|_\infty \tag{64}$$

for $x, y, y' \in \tilde{\mathcal{X}}$ and with $\|x\|_\infty = \max_i |x_i|$. By continuous differentiability of V_i, $i \in \mathcal{I}_n$, and the fact that $\mathcal{X}$ is a compact set, there exists $D > 0$ such that

$$\frac{\partial V_i}{\partial x_i}(x_i) \leq D, \quad x \in \mathcal{X}. \tag{65}$$

Because the function μ satisfying (56) is positive definite on the compact set $[0, \bar{s}]$, there is a function $\tilde{\mu} : [0, \bar{s}] \to \mathbb{R}_+$ of class $\mathcal{K}$ that lower bounds μ [63], i.e.,

$$\tilde{\mu}(s) \leq \mu(s), \quad s \in [0, \bar{s}]. \tag{66}$$

We now define the function $q : [0, \bar{s}] \to \mathbb{R}_+$ as

$$q(s) := \min_{i \in \mathcal{I}_n} \rho_i^{-1}\left(\rho_i(s) + \frac{\tilde{\mu}(s)}{2kDL_{\tilde{\mathcal{X}}}}\right) \tag{67}$$

where

$$k = \max\left\{1, \max_{i \in \mathcal{I}_n}\left\{\frac{\tilde{\mu}(\bar{s})}{2DL_{\tilde{\mathcal{X}}} v_i}\right\}\right\}. \tag{68}$$

Since the functions ρ_i and $\tilde{\mu}$ are of class $\mathcal{K}$, q is nondecreasing. Also, one can verify that $q(s) > s$ for all $s > 0$. Therefore, all conditions on q are satisfied. Using this function, we are ready to employ the Lyapunov–Razumikhin method.

As V is max-separable, (63) implies that

$$V_i(x_i(t-\tau(t),\varphi)) \leq V\big(x(t-\tau(t),\varphi)\big) \leq q(V(x(t,\varphi)))$$

for all $i \in \mathcal{I}_n$. From (55), we then have

$$x\big(t-\tau(t),\varphi\big) \leq \rho\big(q(V(x(t,\varphi)))\big) =: \tilde{x}(t). \tag{69}$$

The application of Assumption 4.3.2 yields

$$g_i\big(y, x(t-\tau,\varphi)\big) \leq g_i\big(y,\tilde{x}(t)\big) \tag{70}$$

for all $i \in \mathcal{I}_n$ and any $y \in \mathbb{R}^n_+$. Returning to the candidate Lyapunov–Razumikin function V, its upper-right Dini derivative along trajectories $x(\cdot,\varphi)$ reads

$$\begin{aligned}
&D^+V\big(x(t,\varphi)\big)\\
&= \max_{j\in\mathcal{J}(x(t,\varphi))} \frac{\partial V_j}{\partial x_j}\big(x_j(t,\varphi)\big) g_j\big(x(t,\varphi), x(t-\tau(t),\varphi)\big)\\
&\overset{(62),(70)}{\leq} \max_{j\in\mathcal{J}(x(t,\varphi))} \frac{\partial V_j}{\partial x_j}\big(x_j(t,\varphi)\big) g_j\big(\bar{x}(t),\tilde{x}(t)\big)\\
&\overset{(61)}{=} \max_{j\in\mathcal{J}(x(t,\varphi))} \frac{\partial V_j}{\partial x_j}\big(\bar{x}_j(t)\big) g_j\big(\bar{x}(t),\tilde{x}(t)\big).
\end{aligned} \tag{71}$$

Since $\mathcal{J}(x(t,\varphi)) \subseteq \mathcal{I}_n$, we obtain from (60) that $\mathcal{J}(x(t,\varphi)) \subseteq \mathcal{J}(\bar{x}(t))$. This together with (71) implies that

$$D^+V\big(x(t,\varphi)\big) \leq \max_{j\in\mathcal{J}(\bar{x}(t))} \frac{\partial V_j}{\partial x_j}\big(\bar{x}_j(t)\big) g_j\big(\bar{x}(t),\tilde{x}(t)\big). \tag{72}$$

We next find an upper bound on $g_j\big(\bar{x}(t),\tilde{x}(t)\big)$, $j \in \mathcal{J}(\bar{x}(t))$.

Since $q(s) > s$ for $s > 0$ and ρ_i are of class $\mathcal{K}$,

$$\bar{x}_i(t) = \rho_i(V(x(t,\varphi)) \leq \rho_i(q(V(x(t,\varphi))) = \tilde{x}_i(t)$$

holds for each $i \in \mathcal{I}_n$. Thus, from Assumption 4.3.2,

$$g_i\big(\bar{x}(t),\bar{x}(t)\big) \leq g_i\big(\bar{x}(t),\tilde{x}(t)\big), \quad i \in \mathcal{I}_n. \tag{73}$$

In order to use the Lipschitz condition (64), $\bar{x}(t)$ and $\tilde{x}(t)$ are required to belong to $\tilde{\mathcal{X}}$. As $\bar{x}(t) \in \mathcal{X}$ and $\mathcal{X} \subset \tilde{\mathcal{X}}$, $\bar{x}(t) \in \tilde{\mathcal{X}}$. Moreover, by recalling the definition of $\tilde{x}$ in (69) and using (67), we obtain

$$\begin{aligned}
\tilde{x}_i(t) &\leq \rho_i(V(x(t,\varphi))) + \frac{\tilde{\mu}(V(x(t,\varphi)))}{2kDL_{\tilde{\mathcal{X}}}}\\
&\overset{(57)}{\leq} \rho_i(\bar{s}) + \frac{\tilde{\mu}(\bar{s})}{2kDL_{\tilde{\mathcal{X}}}}\\
&\overset{(50)}{=} v_i + \frac{\tilde{\mu}(\bar{s})}{2kDL_{\tilde{\mathcal{X}}}}\\
&\overset{(68)}{\leq} v_i + v_i = 2v_i
\end{aligned} \tag{74}$$

implying that $\tilde{x}(t) \in \tilde{\mathcal{X}}$. Making use of (64) and (73), we have

$$\begin{aligned}
&g_j\big(\bar{x}(t),\tilde{x}(t)\big) - g_j\big(\bar{x}(t),\bar{x}(t)\big)\\
&\qquad = |g_j\big(\bar{x}(t),\tilde{x}(t)\big) - g_j\big(\bar{x}(t),\bar{x}(t)\big)|\\
&\qquad \leq \|g\big(\bar{x}(t),\tilde{x}(t)\big) - g\big(\bar{x}(t),\bar{x}(t)\big)\|_\infty\\
&\qquad \leq L_{\tilde{\mathcal{X}}} \max_{i\in\mathcal{I}_n} \{|\tilde{x}_i(t) - \bar{x}_i(t)|\}.
\end{aligned}$$

Since, from (58) and (74),

$$\begin{aligned}
\tilde{x}_i(t) \leq \bar{x}_i + \frac{\tilde{\mu}(V(x(t,\varphi)))}{2kDL_{\tilde{\mathcal{X}}}} &\overset{(60)}{=} \bar{x}_i + \frac{\tilde{\mu}(V(\bar{x}(t))}{2kDL_{\tilde{\mathcal{X}}}}\\
&\overset{(66)}{\leq} \bar{x}_i + \frac{\mu(V(\bar{x}(t))}{2kDL_{\tilde{\mathcal{X}}}}
\end{aligned}$$

it follows that

$$g_j\big(\bar{x}(t),\tilde{x}(t)\big) - g_j\big(\bar{x}(t),\bar{x}(t)\big) \leq \frac{\mu(V(\bar{x}(t)))}{2kD}. \tag{75}$$

Substituting (75) into the right-hand side of (72) yields

$$\begin{aligned}
D^+V\big(x(t,\varphi)\big) \leq &\max_{j\in\mathcal{J}(\bar{x}(t))} \Big\{\frac{\partial V_j}{\partial x_j}\big(\bar{x}_j(t)\big) g_j\big(\bar{x}(t),\bar{x}(t)\big)\\
&+ \frac{\partial V_j}{\partial x_j}\big(\bar{x}_j(t)\big) \frac{\mu\big(V(\bar{x}(t))\big)}{2kD}\Big\}\\
&\leq -\Big(1 - \frac{1}{2k}\Big)\mu\big(V(\bar{x}(t))\big)
\end{aligned} \tag{76}$$

where (59) as well as the bound (65) were used to get the second inequality. From (68), $k \geq 1$, implying that the right-hand side of (76) is negative definite for all $t \in \mathbb{R}_+$. Therefore, according to Razumikhin stability theorem [57, Th. 7], the origin for the time-delay system (26) is asymptotically stable. This finalizes the proof of the implication 2) $\Rightarrow$ 1).

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