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van Haeringen, H

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Off-shell scattering by Coulomb-like potentials

H. van Haeringen
Natuurkundig Laboratorium der Vrije Universiteit, Amsterdam, The Netherlands and Institute for Theoretical Physics, P. O. Box 800, University of Groningen, The Netherlands

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We derive closed expressions for and interrelationships between off-shell and on-shell scattering quantities for Coulomb plus short-range potentials. In particular we introduce off-shell Jost states and show how the transition matrices are obtained from these states. We discuss some formulas connecting the coordinate and momentum representatives of certain quantities. For the pure Coulomb case we derive analytic expressions for the Jost state and the off-shell Jost state for $l = 0$ in the momentum representation.

1. INTRODUCTION

In this paper we study off-shell scattering by a potential which is the sum of the Coulomb potential and a local central potential of short range. We derive many interesting expressions, notably for the Jost functions, the off-shell Jost functions, and the on-shell and off-shell "Jost states." These quantities are closely connected with the transition matrix which plays such an important role in scattering theory.

First, in Sec. 2, we confine ourselves to a general local short-range central potential. Here we derive many interrelationships between the above quantities. Only a few of these are well known, e.g., the defining expression for the (off-shell) Jost functions in terms of the (off-shell) Jost solutions in the coordinate representation. We give the momentum representation equivalents of these expressions which have a somewhat simpler form.

Some of the equations given in Sec. 2 are also valid for Coulomblike potentials. However, some have to be modified for such potentials with a long range. To this end we consider in Sec. 3 the pure Coulomb case. By working out a number of explicit expressions we pave the way for the treatment of the general case of Coulomb plus shortrange potentials, which will be given in Sec. 4. We also prove the validity of two conjectures made in Ref. 1.

Furthermore, in Sec. 3 we derive some interesting analytic expressions, notably for the $l = 0$ Coulomb Jost state and the off-shell Jost state in the momentum representation. In these expressions we encounter a certain hypergeometric function which appears in many other Coulomb quantities. Only its argument is different for the various different cases.

We will use mainly the notation of Refs. 1 and 2. In particular the energy is k^2 with $\mathrm{Im}k \downarrow 0$ and the energy dependence of G, G_0 and T will be suppressed. However, instead of the Jost solution $f_l(k,r)$ and the off-shell Jost solution $f_l(k,q,r)$ of the *radial* differential equations we will use the Jost solution $\langle r|kl\uparrow\rangle$ and the off-shell Jost solution $\langle r|kql\uparrow\rangle$ of the partial-wave projected equations. Here q is an off-shell momentum variable for which we assume $\mathrm{Im}q \geqslant 0$. We shall also consider the Hankel transforms of the above Jost solutions. These are denoted by $\langle p|kl\uparrow\rangle$ and $\langle p|kql\uparrow\rangle$, respectively. We call $|kl\uparrow\rangle$ the Jost state and $|kql\uparrow\rangle$ the off-shell Jost state.

2. THE SHORT-RANGE POTENTIAL CASE

In this section we confine ourselves to a local central potential $V(r)$, having a short range. Let us first recall Fuda's definition of the off-shell Jost solution[3]: $f_l(k,q,r)$ is that solution of the inhomogeneous differential equation

$$\left(k^2 + \frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - V(r)\right) f_l(k,q,r) = (k^2 - q^2) i^l qr h_l^{(+)}(qr), \tag{2.1}$$

which satisfies the asymptotic condition

$$\lim_{r\to\infty} f_l(k,q,r) e^{-iqr} = 1.$$

We introduce the "state" $|kql\uparrow\rangle$ by

$$\langle r|kql\uparrow\rangle \equiv (2/\pi)^{1/2} (qr)^{-1} f_l(k,q,r). \tag{2.2}$$

This may be compared with the "state" $|kl\uparrow\rangle$ that we introduced before,

$$\langle r|kl\uparrow\rangle \equiv (2/\pi)^{1/2} (kr)^{-1} f_l(k,r). \tag{2.3}$$

Let $H_l = H_{0l} + V_l$ be the partial-wave Hamiltonian, then we obtain from Eq. (2.1),

$$(k^2 - H_l)|kql\uparrow\rangle = (k^2 - q^2)|ql\uparrow\rangle_0, \tag{2.4a}$$

that is,

$$G_l^{-1}|kql\uparrow\rangle = G_{0l}^{-1}|ql\uparrow\rangle_0. \tag{2.4b}$$

Here $|ql\uparrow\rangle_0$ is the Jost state corresponding to $V \equiv 0$. In the coordinate representation one has

$$\langle r|ql\uparrow\rangle_0 = (2/\pi)^{1/2} i^l h_l^{(+)}(qr).$$

Furthermore, we denote the scattering state for $V \equiv 0$ and energy k^2 by $|kl\rangle$, or by $|k\rangle$ when no confusion arises, e.g., $T_l|kl\rangle = T_l|k\rangle$. It should be noted that Eqs. (2.4) are valid only in the coordinate representation. We shall call $\langle r|kql\uparrow\rangle$ the off-shell Jost solution of the "inhomogeneous Schrödinger equation" corresponding to Eqs. (2.4).

We would like to have a closed expression for $|kql\uparrow\rangle$. It is easily seen from Eq. (2.4b) that $G_l G_{0l}^{-1}|ql\uparrow\rangle_0$ is a particular solution of an *in*homogeneous differential equation. If one adds to this quantity an arbitrary solution of the corresponding homogeneous differential equation it remains a solution of (2.4). Now we have, again in the coordinate representation only,

$$(k^2 - H_l)|kl\uparrow\rangle = (k^2 - H_l)|kl\downarrow\rangle = 0. \tag{2.5}$$

Furthermore, any solution is a linear combination of $|kl\uparrow\rangle$ and $|kl\downarrow\rangle$. Therefore, using $G_l = G_{0l} + G_{0l} T_l G_{0l}$ we obtain,

$$|kql\uparrow\rangle = (1 + G_{0l}T_l)|ql\uparrow\rangle_0 + c_1|kl\uparrow\rangle + c_2|kl\downarrow\rangle. \tag{2.6}$$

In order to determine c_1 and c_2 we consider the asymptotic behavior of the right-hand side. By using

$$\langle r|G_l|r'\rangle = -\tfrac{1}{2}\pi k\langle r_>|kl\uparrow\rangle\langle kl-|r_<\rangle,$$

we obtain, for $r\to\infty$,

$$\langle r|G_{0l}T_l|ql\uparrow\rangle_0 = \langle r|G_l V_l|ql\uparrow\rangle_0$$

$$\sim -\tfrac{1}{2}\pi k\langle r|kl\uparrow\rangle\langle kl-|V_l|ql\uparrow\rangle_0,\quad r\to\infty.$$

Since $\langle r|kql\uparrow\rangle$ has by definition the same asymptotic behavior as $\langle r|ql\uparrow\rangle_0$, namely,

$$\lim_{r\to\infty}\langle r|kql\uparrow\rangle qre^{-iqr} = (2/\pi)^{1/2},$$

we find

$$c_1 = \tfrac{1}{2}\pi k\langle kl-|V_l|ql\uparrow\rangle_0,$$
$$c_2 = 0.$$

It is convenient to rewrite c_1 in terms of the *off-shell Jost function* $f_l(k,q)$. Fuda[4] has given a closed expression which in our notation reads,

$$f_l(k,q) = 1 + \tfrac{1}{2}\pi q(q/k)^l f_l(k)\langle kl-|V_l|ql\uparrow\rangle_0. \tag{2.7a}$$

Some equivalent expressions are

$$f_l(k,q) = 1 + \tfrac{1}{2}\pi q(q/k)^l f_l(k)\langle kl|T_l|ql\uparrow\rangle_0, \tag{2.7b}$$

$$= 1 + \tfrac{1}{2}\pi q(q/k)^l f_l(k)\,{}_0\langle ql\downarrow|T_l|kl\rangle, \tag{2.7c}$$

$$= 1 + \tfrac{1}{2}\pi q(q/k)^l f_l(k)\,{}_0\langle ql\downarrow|V_l|kl+\rangle. \tag{2.7d}$$

By substituting c_1 in Eq. (2.6) and using (2.7a) we obtain the convenient expression,

$$|kql\uparrow\rangle = (1+G_{0l}T_l)|ql\uparrow\rangle_0 + |kl\uparrow\rangle(k/q)^{l+1}$$

$$\times\frac{f_l(k,q)-1}{f_l(k)}. \tag{2.8}$$

From now on we shall suppress the argument k of the Jost function, so we write f_l instead of $f_l(k)$.

When the potential has a short range the off-shell Jost function and solution are continuous in $q=k$, (cf. Ref. 3)

$$\lim_{q\to k} f_l(k,q) = f_l, \tag{2.9}$$

$$\lim_{q\to k}|kql\uparrow\rangle = |kl\uparrow\rangle. \tag{2.10}$$

By taking the limit $q\to k$ in Eq. (2.8) we obtain

$$|kl\uparrow\rangle = (1+G_{0l}T_l)|kl\uparrow\rangle_0 f_l. \tag{2.11}$$

We multiply both sides of this equation by V_l and get

$$V_l|kl\uparrow\rangle = T_l|kl\uparrow\rangle_0 f_l. \tag{2.12}$$

This equality turns out to be very useful below.

Multiplying Eq. (2.8) by V_l and using Eq. (2.12), we obtain

$$V_l|kql\uparrow\rangle = T_l|ql\uparrow\rangle_0 + T_l|kl\uparrow\rangle_0(k/q)^{l+1}$$

$$\times[f_l(k,q)-1]. \tag{2.13}$$

Further we get some closed formulas for the Jost function from Eqs. (7) by taking $q=k$. We have

$$f_l = f_l(k,k) = 1 + \tfrac{1}{2}\pi k f_l\langle kl-|V_l|kl\uparrow\rangle_0,$$

and therefore

$$f_l^{-1} = 1 - \tfrac{1}{2}\pi k\langle kl-|V_l|kl\uparrow\rangle_0$$

$$= 1 - \tfrac{1}{2}\pi k\langle kl|T_l|kl\uparrow\rangle_0$$

$$= 1 - \tfrac{1}{2}\pi k\,{}_0\langle kl\downarrow|V_l|kl+\rangle$$

$$= 1 - \tfrac{1}{2}\pi k\,{}_0\langle kl\downarrow|T_l|kl\rangle. \tag{2.14}$$

By using Eq. (2.12) we obtain from Eq. (2.14),

$$f_l = 1 + \tfrac{1}{2}\pi k\langle kl|V_l|kl\uparrow\rangle = 1 + \tfrac{1}{2}\pi k\langle kl\downarrow|V_l|kl\rangle. \tag{2.15}$$

We shall need the connection between $\langle p|kl\uparrow\rangle$ and $\langle p|V|kl\uparrow\rangle$. From Eq. (2.11) we have

$$|kl\uparrow\rangle f_l^{-1} = |kl\uparrow\rangle_0 + G_{0l}T_l|kl\uparrow\rangle_0,$$

$$= |kl\uparrow\rangle_0 + G_{0l}V_l|kl\uparrow\rangle f_l^{-1}. \tag{2.16}$$

Therefore,

$$\langle p|kl\uparrow\rangle = \langle p|kl\uparrow\rangle_0 f_l + \langle p|G_{0l}V_l|kl\uparrow\rangle. \tag{2.17}$$

The free "state" $\langle p|kl\uparrow\rangle_0$ is given explicitly by

$$\langle p|kl\uparrow\rangle_0 = \frac{2}{\pi k}\frac{(p/k)^l}{p^2-k^2},\quad \mathrm{Im}k>0. \tag{2.18}$$

By inserting this in Eq. (2.17) one easily obtains

$$\langle p|V_l|kl\uparrow\rangle = (k^2-p^2)\langle p|kl\uparrow\rangle + 2(\pi k)^{-1}(p/k)^l f_l, \tag{2.19}$$

which is the relation we wanted.

The connection between the *off-shell* quantities, corresponding to the one of Eq. (2.19), can be obtained from Eqs. (2.8), (2.13), and (2.19),

$$\langle p|V_l|kql\uparrow\rangle = (k^2-p^2)\langle p|kql\uparrow\rangle$$
$$+ 2(\pi q)^{-1}(p/q)^l[f_l(k,q)$$
$$-(k^2-q^2)/(p^2-q^2)]. \tag{2.20}$$

It is interesting to consider the limit of $\langle p|kl\uparrow\rangle$ for $p\to\infty$. This limit could be used for an alternative definition of f_l (cf. Refs. 5 and 6). By using the fact that $\langle r|kl\uparrow\rangle = O(r^{-l-1})$ as $r\to 0$, we obtain

$$\langle p|V_l|kl\uparrow\rangle = (2/\pi)^{1/2}i^{-l}\int_0^\infty j_l(pr)V_l(r)\langle r|kl\uparrow\rangle r^2\,dr$$

$$\simeq cp^{l-2}\int_0^\infty j_l(x)V_l(x/p)x^{1-l}\,dx,\quad p\to\infty.$$

In this way we find that

$$\lim_{p\to\infty} p^{-l}\langle p|V_l|kl\uparrow\rangle = 0, \quad (2.21)$$

when the potential is nonsingular, i.e.,

$$V(r) = O(r^{-\alpha}), \quad \alpha<2, \quad r\to 0.$$

It is easily seen from Eqs. (2.19) and (2.21) that

$$f_l = \tfrac{1}{2}\pi k^{l+1} \lim_{p\to\infty} p^{2-l}\langle p|\, kl\uparrow\rangle. \quad (2.22)$$

This may be compared with the usual definition of f_l,

$$f_l \equiv (\pi/2)^{1/2}(2/i)^l (l!/(2l)!) \lim_{r\to 0}(kr)^{l+1}\langle r|kl\uparrow\rangle. \quad (2.23)$$

Similar equations hold for the *off-shell* Jost function and solution. The analog of Eq. (2.23) is (Ref. 3)

$$f_l(k,q) = (\pi/2)^{1/2}(2/i)^l\,[l!/(2l)!]\,\lim_{r\to 0}(qr)^{l+1}\langle r|kql\uparrow\rangle. \quad (2.24)$$

The off-shell analog of Eq. (2.22) follows by using Eq. (2.8). We have [cf. Eq. (2.21)]

$$\lim_{p\to\infty} p^{2-l}\langle p|G_{0l}T_l|ql\uparrow\rangle_0 = \lim_{p\to\infty} p^{-l}\langle p|T_l|ql\uparrow\rangle_0 = 0,$$

and so we obtain from (2.8),

$$f_l(k,q) = \tfrac{1}{2}\pi q^{l+1}\lim_{p\to\infty} p^{2-l}\langle p|kql\uparrow\rangle. \quad (2.25)$$

This expression can also be derived with the help of Eq. (2.20).

It is interesting to note that Eq. (2.25) is obtained in a different way, by using Eq. (2.24) in the expression

$$\langle p|kql\uparrow\rangle = (2/\pi)^{1/2} i^{-l}\int_0^\infty j_l(pr)\langle r|kql\uparrow\rangle r^2\,dr, \quad (2.26)$$

and applying the equality

$$\int_0^\infty j_l(x)x^{1-\lambda}\,dx = \pi^{1/2}2^{-\lambda}\frac{\Gamma(1+\frac{1}{2}l-\frac{1}{2}\lambda)}{\Gamma(\frac{1}{2}+\frac{1}{2}l+\frac{1}{2}\lambda)},$$
$$0<\mathrm{Re}\lambda<l+2. \quad (2.27)$$

On the other hand, we shall now derive Eq. (2.24) from Eq. (2.25). We have

$$\langle r|kql\uparrow\rangle = (2/\pi)^{1/2}\, i^l \lim_{\epsilon\downarrow 0}\int_0^\infty j_l(pr)\langle p|kql\uparrow\rangle e^{-\epsilon p}p^2\,dp, \quad (2.28)$$

where $e^{-\epsilon p}$ has been inserted to guarantee the convergence of the integral. It turns out that, when r goes to zero, $\langle p|kql\uparrow\rangle$ may be replaced by its asymptotic value, which is given by Eq. (2.25). Then we obtain from (2.28), using the new variable of integration $x=pr$,

$$\lim_{r\to 0}(qr)^{l+1}\langle r|kql\uparrow\rangle = f_l(k,q)(2/\pi)^{3/2}i^l$$
$$\times\lim_{\epsilon\downarrow 0}\int_0^\infty j_l(x)x^l e^{-\epsilon x/r}\,dx. \quad (2.29)$$

In order to evaluate the integral here, we note that

$$\int_0^\infty e^{-\alpha x}J_\mu(\beta x)x^\nu\,dx$$
$$= (\alpha^2+\beta^2)^{-(1/2)-(1/2)\nu}$$
$$\times\Gamma(1+\mu+\nu)P_\nu^{-\mu}(\alpha(\alpha^2+\beta^2)^{-(1/2)}),$$
$$\alpha>0,\quad \beta>0,\quad \mathrm{Re}(1+\mu+\nu)>0. \quad (2.30)$$

Here $P_\nu^{-\mu}(\zeta)$ is the Legendre function of the first kind "on the cut": $-1<\zeta<1$. Its value for $\zeta=0$ is given by

$$\Gamma(1+\mu)P_\nu^{-\mu}(0) = {}_2F_1(-\nu,\nu+1;\mu+1;\tfrac{1}{2})$$
$$= \frac{\Gamma(1+\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\mu)}{\Gamma(1+\frac{1}{2}\mu+\frac{1}{2}\nu)\Gamma(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu)}. \quad (2.31)$$

By using this expression we get

$$\lim_{\epsilon\downarrow 0}\int_0^\infty e^{-\epsilon x}J_\mu(x)x^\nu\,dx = 2^\nu\frac{\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)}{\Gamma(\frac{1}{2}+\frac{1}{2}\mu-\frac{1}{2}\nu)},$$
$$\mathrm{Re}(1+\mu+\nu)>0,$$

and so

$$\lim_{\epsilon\downarrow 0}\int_0^\infty e^{-\epsilon x}j_l(x)x^l\,dx = \pi^{1/2}2^{l-1}\Gamma(l+\tfrac{1}{2})$$
$$= \pi 2^{-l-1}(2l)!/l!. \quad (2.32)$$

By inserting this in Eq. (2.29) we just obtain Eq. (2.24).

We note that the above limiting procedures constitute in fact a generalization of the well-known Riemann–Lebesgue lemma, i.e.,

$$\lim_{y\to\pm\infty}\int f(x)e^{ixy}\,dx = 0,$$

where f is any summable function.

3. THE COULOMB CASE

Some of the expressions derived in Sec. 2 do not hold when the potential has a Coulomb range. Especially Eqs. (2.9), (2.10), (2.14), and (2.15) need modification. In this section we shall derive the analogs of these equations for the case of the pure Coulomb potential. Further we shall develop some explicit expressions, in terms of hypergeometric functions, for the particular case when $l=0$. In Sec. 4 we shall derive interesting formulas for the case when the potential is the sum of the Coulomb potential and a short-range potential, by using the results obtained in Secs. 2 and 3.

In the first place we note that the important equations (2.11) and (2.12) *do* hold for the Coulomb case, i.e.,

$$|kl\uparrow\rangle_c = (1+G_{0l}T_{cl})|kl\uparrow\rangle_0 f_{cl}, \quad (3.1)$$

and so

$$V_{cl}|kl\uparrow\rangle_c = T_{cl}|kl\uparrow\rangle_0 f_{cl}. \quad (3.2)$$

We shall prove Eq. (3.1) in an independent way. To start with, we observe that the *existence* of the quantity

$G_{0l}\,T_{cl}\,|kl\uparrow\rangle_0 = G_{cl}\,V_{cl}\,|kl\uparrow\rangle_0$ is easily confirmed by using

$$\langle r|G_{cl}|r'\rangle = -\tfrac{1}{2}\pi k\,\langle r_>|kl\uparrow\rangle_{cc}\langle kl-|r_<\rangle. \tag{3.3}$$

One can also show in this way that $G_{0l}T_{cl}|kl\downarrow\rangle_0$ is *not* defined, i.e., that it contains a divergent integral.

In order to prove Eq. (3.1), we note that its right-hand side equals some solution ψ of the equation $(k^2 - H_{cl})\psi = 0$ (in the coordinate representation). Further, by using Eq. (3.3) and by considering the asymptotic behavior $(r\rightarrow\infty)$ of the right-hand side of (3.1), we find that it must be proportional to $|kl\uparrow\rangle_c$. The next step is to consider the behavior for small r. By again using (3.3) one has

$$\langle r|G_{cl}\,V_{cl}|kl\uparrow\rangle_0 = O(\ln r),\quad l=0,\quad r\rightarrow 0,$$

$$= O(r^{-l}),\quad l>0,\quad r\rightarrow 0. \tag{3.4}$$

Therefore,

$$\lim_{r\rightarrow 0} r^{l+1}\langle r|G_{0l}\,T_{cl}\,|kl\uparrow\rangle_0 = 0,\quad l=0,1,\cdots.$$

By using Eq. (2.23) the proof of Eq. (3.1) now follows easily.

In a previous paper[2] we have derived the Coulomb analog of Eq. (2.9), by using an explicit expression for $f_{cl}(k,q)$. The following equality holds,

$$\lim_{q\rightarrow k}\omega f_{cl}(k,q) = f_{cl},\quad k>0. \tag{3.5}$$

Here

$$\omega \equiv \left(\frac{q-k}{q+k}\right)^{i\gamma}\frac{e^{\pi\gamma/2}}{\Gamma(1+i\gamma)} = \frac{f_{c0}}{f_{c0}(k,q)},\quad \mathrm{Im}\,q>0. \tag{3.6}$$

The Coulomb analog of Eq. (2.10) is now easily obtained by using Eqs. (2.8), (3.1), and (3.5),

$$\lim_{q\rightarrow k}\omega|kql\uparrow\rangle_c = |kl\uparrow\rangle_c,\quad k>0. \tag{3.7}$$

It is interesting to note that we are now able to prove the validity of two conjectures from a preceding paper.[1] The first one, Eq. (40.l), is in fact just (3.7). The second one, Eq. (40.k), is easily proved by using Eqs. (40.h)–(40.j) of Ref. 1 and Eq. (2.8).

We note that Eqs. (2.7a) and (2.7d) are valid in the Coulomb case, whereas Eqs. (2.7b) and (2.7c) are not. By using Eqs. (2.7a) and (3.5) we have obtained the interesting equality,

$$f_{cl}^{-1} = \lim_{q\rightarrow k}(\omega^{-1} - \tfrac{1}{2}\pi k\,{}_c\langle kl-|V_{cl}|ql\uparrow\rangle_0),\quad k>0. \tag{3.8}$$

Obviously, this can be considered as a Coulomb analog of the short-range potential formulas given in Eq. (2.14).

It would be interesting to have available explicit expressions for the above quantities. Only a few such formulas are known. The quantities $\langle r|kl\uparrow\rangle_c$ and $f_{cl}(k)$ for $l=0,1,2,\cdots$ have been known for a long time. We have obtained a number of interesting analytic expressions for $f_{cl}(k,q)$, $l=0,1,2,\cdots$ (see Ref. 7). Further, $\langle p|T_{cl}|p'\rangle$ is known in closed form for $l=0$[8] and for $l=1$ only.[9] Below we shall derive analytic expressions for $\langle p|kl\uparrow\rangle_c$, $\langle p|kql\uparrow\rangle_c$, and for $\langle p|T_{cl}|ql\uparrow\rangle_0$, in the case $l=0$ only.

Before we start with the derivations we would like to make some remarks on the interrelationships between the above quantities. It is important to note that $\langle p|T_{cl}|ql\uparrow\rangle_0$ can be considered as the general object from which all other quantities can be obtained in a *simple* way. This is true as well for Coulomblike potentials, and of course also for short-range potentials. Indeed, by taking $q=k$ we have $\langle p|T_{cl}|kl\uparrow\rangle_0$ from which $\langle p|kl\uparrow\rangle_c$ follows with the help of Eqs. (2.12) and (2.19). Once $\langle p|kl\uparrow\rangle_c$ is known, $\langle p|kql\uparrow\rangle_c$ is obtained by using Eq. (2.8). The ordinary off-shell Coulomb T matrix $\langle p|T_{cl}|p'\rangle$ follows from $\langle p|T_{cl}|ql\uparrow\rangle_0$ by noting that

$$2iT_{cl}|p'l\rangle = T_{cl}|p'l\uparrow\rangle_0 + (-)^l T_{cl}|(-p')l\uparrow\rangle_0. \tag{3.9}$$

Furthermore, $f_{cl}(k,q)$ can be obtained from $\langle p|kql\uparrow\rangle_c$ and f_{cl} from $\langle p|kl\uparrow\rangle_c$ by using Eqs. (2.25) and (2.22), respectively. Finally we note that application of the Coulombian asymptotic states (see Ref. 10) to $\langle p|T_{cl}|ql\uparrow\rangle_0$ and $\langle p|T_{cl}|p'\rangle$ yields $f_{cl}(k,q)$ and $\langle p|kl+\rangle_c$, respectively. Since, therefore, $\langle p|T_{cl}|ql\uparrow\rangle_0$ appears to be the object of central important, we are interested in the general structure of an analytic expression for this quantity.

For the moment we restrict ourselves to the case $l=0$ and we suppress l. Let us first recall the expression for $\langle p|T_c|p'\rangle$ given in Ref. 8,

$$\langle p|T_c|p'\rangle = ik(\pi pp')^{-1}[F_{i\gamma}(aa') + F_{i\gamma}((aa')^{-1})$$

$$- F_{i\gamma}(a/a') - F_{i\gamma}(a'/a)]. \tag{3.10}$$

Here

$$F_{i\gamma}(z) \equiv {}_2F_1(1,i\gamma;\,1+i\gamma;\,z)$$

and

$$a \equiv \frac{(p-k)}{(p+k)},\quad a' \equiv \frac{(p'-k)}{(p'+k)}.$$

By using a well-known integral representation for the hypergeometric function,

$$F_{i\gamma}(z) = i\gamma\int_0^1 t^{i\gamma-1}(1-tz)^{-1}\,dt, \tag{3.11}$$

we are able to evaluate

$$\langle p|T_c|q\uparrow\rangle_0 = \int_0^\infty \langle p|T_c|p'\rangle\langle p'|q\uparrow\rangle_0 p'^2\,dp',$$

where [cf. Eq. (2.18)]

$$\langle p|q\uparrow\rangle_0 = 2(\pi q)^{-1}(p^2-q^2)^{-1},\quad \mathrm{Im}\,q>0.$$

After a number of manipulations we arrive at

$$\langle p|T_c|q\uparrow\rangle_0 = -2k(\pi pq)^{-1}[F_{i\gamma}(ab) - F_{i\gamma}(b/a)$$

$$- F_{i\gamma}(a) + F_{i\gamma}(1/a)], \tag{3.12}$$

with

$$b \equiv \frac{(q-k)}{(q+k)}.$$

Equation (3.9) provides us with a check on this result. It can be seen by inspection that we have indeed

$$\lim_{q\to p'}(\langle p|T_c|q\uparrow\rangle_0+\langle p|T_c|(-q)\uparrow\rangle_0)$$
$$=2i\langle p|T_c|p'\rangle, \tag{3.13}$$

note that $b\to 1/b$ when q is replaced by $-q$.

Further, we clearly have from (3.12),

$$\lim_{q\to k}\langle p|T_c|q\uparrow\rangle_0=\langle p|T_c|k\uparrow\rangle_0$$
$$=2(\pi p)^{-1}[F_{i\gamma}(a)-F_{i\gamma}(1/a)]. \tag{3.14}$$

By using Eq. (3.1) one easily obtains

$$\langle p|V_c|k\uparrow\rangle_c=2(\pi p)^{-1}f_c[F_{i\gamma}(a)-F_{i\gamma}(1/a)], \tag{3.15}$$

and with the help of Eq. (2.19),

$$\langle p|k\uparrow\rangle_c=2(\pi p)^{-1}(p^2-k^2)^{-1}f_c[p/k-F_{i\gamma}(a)+F_{i\gamma}(1/a)]. \tag{3.16}$$

Finally we note that $\langle p|kq\uparrow\rangle_c$ can now easily be given. We only need to insert the known expressions for the terms on the right-hand side of Eq. (2.8). In particular, we have

$$f_c(k,q)=b^{-i\gamma}. \tag{3.17}$$

Let us, for completeness, write out this expression for the Hankel transform of the Coulomb off-shell Jost solution for $l=0$,

$$\langle p|kq\uparrow\rangle_c=\frac{2}{\pi q(p^2-q^2)}+\frac{2k}{\pi pq(p^2-k^2)}$$
$$\times[F_{i\gamma}(ab)-F_{i\gamma}(b/a)-p/k+b^{-i\gamma}$$
$$\times\{p/k-F_{i\gamma}(a)+F_{i\gamma}(1/a)\}]. \tag{3.18}$$

By using Eq. (2.13) or Eq. (2.20) we have

$$\langle p|V_c|kq\uparrow\rangle_c=2k(\pi pq)^{-1}[F_{i\gamma}(b/a)-F_{i\gamma}(ab)$$
$$+b^{-i\gamma}\{F_{i\gamma}(a)-F_{i\gamma}(1/a)\}]. \tag{3.19}$$

By taking here the limit $q\to k$ we get, with $\omega=f_c b^{i\gamma}$,

$$\lim_{q\to k}\omega\langle p|V_c|kq\uparrow\rangle_c=\langle p|V_c|k\uparrow\rangle_c.$$

Such a relation holds in fact for all l. Indeed, with the help of Eqs. (2.19), (2.20), (3.5), and (3.7) the proof of

$$\lim_{q\to k}f_c b^{i\gamma}\langle p|V_{cl}|kql\uparrow\rangle_c=\langle p|V_{cl}|kl\uparrow\rangle_c,$$
$$l=0,1,2,\cdots, \tag{3.20}$$

is easily obtained.

A final remark concerning the generalization of the $l=0$ expression for $\langle p|T_c|q\uparrow\rangle_0$ to general values of l is appropriate here. In view of Eq. (3.12) it can be expected that $\langle p|T_{cl}|ql\uparrow\rangle_0$ where $l=0,1,2,\cdots$, can be expressed in terms of simple functions and the hypergeometric function $F_{i\gamma}$ with *exactly* the same arguments as in (3.12), notably ab, b/a, a, and $1/a$.

4. THE COULOMBLIKE POTENTIAL CASE

In this section we assume that the potential is the sum of the Coulomb potential and a short-range potential, $V=V_c+V_s$. We shall discuss the necessary modifications of the equations given in Sec. 2 by using the appropriate results obtained in Sec. 3. In particular, we will derive the analogs of Eqs. (2.9), (2.10), and (2.14).

We shall use the well-known two-potential formalism. The T operator corresponding to $V=V_c+V_s$ is given by

$$T=T_c+(1+T_cG_0)t_{cs}(1+G_0T_c), \tag{4.1}$$

where t_{cs} is the solution of

$$t_{cs}=V_s+V_sG_ct_{cs}. \tag{4.2}$$

The partial-wave analogs of these equations have exactly the same form. For the partial-wave "outgoing" scattering state $|kl+\rangle$ the following equation can be obtained,

$$|kl+\rangle=|kl+\rangle_c+G_{cl}\,t_{csl}|kl+\rangle_c. \tag{4.3}$$

In order to derive relations for the "Jost states," we use Eqs. (2.11) and (2.12). These are also valid for a Coulomblike potential. We insert (4.1) in (2.11),

$$|kl\uparrow\rangle f_l^{-1}=(1+G_{0l}T_l)|kl\uparrow\rangle_0,$$

and obtain

$$|kl\uparrow\rangle f_l^{-1}=|kl\uparrow\rangle_c f_{cl}^{-1}+G_{cl}t_{csl}|kl\uparrow\rangle_c f_{cl}^{-1}$$
$$=(1+G_{cl}t_{csl})|kl\uparrow\rangle_c f_{cl}^{-1}. \tag{4.4}$$

Further, by inserting (4.1) in (2.12),
$V_l|kl\uparrow\rangle f_l^{-1}=T_l|kl\uparrow\rangle_0$,

we get

$$V_l|kl\uparrow\rangle f_l^{-1}=V_{cl}|kl\uparrow\rangle_c f_{cl}^{-1}+G_{0l}^{-1}G_{cl}t_{csl}|kl\uparrow\rangle_c f_{cl}^{-1}. \tag{4.5}$$

We are now going to derive a connection between the Jost function f_l and the Coulomb Jost function f_{cl}. To this end we write Eq. (4.3) in the coordinate representation. In the resulting equation we insert the equality [cf. Eq. (3.3)]

$$\langle r|G_{cl}|r'\rangle=-\tfrac{1}{2}\pi k\langle r_<|kl+\rangle_c\,{}_c\langle kl\downarrow|r_>\rangle.$$

We note that ${}_c\langle kl\downarrow|t_{csl}|kl+\rangle_c$ is a well-defined quantity since t_{csl} is a short-range operator. By using

$$\langle r|t_{csl}|kl+\rangle_c=O(r^{l-\alpha}),\quad \alpha<2,\quad r\to 0,$$

we obtain from Eq. (4.3)

$$\langle r|kl+\rangle=\langle r|kl+\rangle_c-\tfrac{1}{2}\pi k\langle r|kl+\rangle_c$$
$$\times{}_c\langle kl\downarrow|t_{csl}|kl+\rangle_c$$
$$+O(r^{l+2-\alpha}),\quad r\to 0.$$

The Jost functions can be obtained from the scattering states by considering their small-r behavior. We have (e.g., Refs. 5 and 6)

$$\lim_{r\to 0} r^{-l}\langle r|kl+\rangle = f_l^{-1}(2/\pi)^{1/2}(2ik)^l l!/(2l+1)!. \qquad (4.6)$$

With the help of this relation one obtains

$$f_l^{-1} = f_{cl}^{-1} - \tfrac{1}{2}\pi k f_{cl}^{-1}\ {}_c\langle kl\downarrow|t_{csl}|kl+\rangle_c,$$

as can be seen by inspection. We rewrite this equation in the more convenient form,

$$f_{cl} f_l^{-1} = 1 - \tfrac{1}{2}\pi k\ {}_c\langle kl\downarrow|t_{csl}|kl+\rangle_c. \qquad (4.7a)$$

If we take here $V_c \to 0$ we get back one of the expressions of Eq. (2.14), since in this case $t_{csl} \to T_{sl}$ and $f_{cl} \to 1$. Just as in (2.14) there are three different equivalent expressions,

namely

$$f_{cl} f_l^{-1} = 1 - \tfrac{1}{2}\pi k\ {}_c\langle kl\downarrow|V_{sl}|kl+\rangle, \qquad (4.7b)$$

$$= 1 - \tfrac{1}{2}\pi k\langle kl-|V_{sl}|kl\uparrow\rangle_c \qquad (4.7c)$$

$$= 1 - \tfrac{1}{2}\pi k\ {}_c\langle kl-|t_{csl}|kl\uparrow\rangle_c. \qquad (4.7d)$$

These are easily derived with the help of Eqs. (4.1) and (4.3).

In order to derive the analog of Eq. (2.15), we first multiply both sides of Eq. (4.4) by V_{sl}. This yields

$$V_{sl}|kl\uparrow\rangle f_l^{-1} = t_{csl}|kl\uparrow\rangle_c f_{cl}^{-1}. \qquad (4.8)$$

By inserting this equation in (4.7d) we get

$$f_{cl}^{-1} f_l = 1 + \tfrac{1}{2}\pi k\ {}_c\langle kl-|V_{sl}|kl\uparrow\rangle$$

$$= 1 + \tfrac{1}{2}\pi k\ \langle kl\downarrow|V_{sl}|kl+\rangle_c. \qquad (4.9)$$

Obviously this is the two-potential analog of Eq. (2.15).

It is interesting to consider the analog of Eq. (3.8), i.e.,

$$f_l^{-1} = \lim_{q\to k}(\omega^{-1} - \tfrac{1}{2}\pi k\langle kl-|V_l|ql\uparrow\rangle_0), \quad k>0. \qquad (4.10)$$

In order to prove this equation, we first note that one has from Eqs. (4.1)–(4.3),

$$\langle kl-|V_l = {}_c\langle kl-|V_{cl} + {}_c\langle kl-|t_{csl}(1+G_{0l}T_{cl}). \qquad (4.11)$$

We insert this expression in (4.10) and use

$$\lim_{q\to k}(1+G_{0l}T_{cl})|ql\uparrow\rangle_0 = |kl\uparrow\rangle_c f_{cl}^{-1}. \qquad (4.12)$$

By applying finally Eqs. (3.8) and (4.7d) the proof of Eq. (4.10) is completed.

Now we turn to the *off-shell* Jost function. In Eq. (2.7a) the following general formula,

$$f_l(k,q) = 1 + \tfrac{1}{2}\pi q(q/k)^l f_l\langle kl-|V_l|ql\uparrow\rangle_0, \qquad (4.13)$$

has been given.[4] This equation is also valid for a Coulomblike potential. By inserting Eq. (4.11) in (4.13), and by using (4.13) for the pure Coulomb case, we obtain

$$[f_l(k,q)-1]f_l^{-1} = [f_{cl}(k,q)-1]f_{cl}^{-1} + \tfrac{1}{2}\pi q(q/k)^l \times {}_c\langle kl-|t_{csl}(1+G_{0l}T_{cl})|ql\uparrow\rangle_0. \qquad (4.14)$$

Herewith we have obtained a useful relation between the Coulomb off-shell Jost function and the off-shell Jost function for a Coulomblike potential. Indeed, from Eq. (4.14) one obtains, by using Eqs. (4.12) and (3.5), the analog of the pure Coulomb formula (3.5),

$$\lim_{q\to k}\omega f_l(k,q) = f_l, \quad k>0. \qquad (4.15)$$

Here ω is given by Eq. (3.6).

Finally, we are going to prove

$$\lim_{q\to k}\omega|kql\uparrow\rangle = |kl\uparrow\rangle, \quad k>0. \qquad (4.16)$$

This is just the Coulomblike analog of the pure Coulomb formula (3.7). From Eqs. (2.8) and (2.11) we obtain

$$|kql\uparrow\rangle \underset{q\to k}{\to} |kl\uparrow\rangle f_l^{-1} f_l(k,q).$$

Application of Eq. (4.15) then completes the proof of Eq. (4.16).

So we see that the singular behavior of the off-shell Jost function and of the off-shell Jost state in $q=k$ is just the same as for the pure Coulomb potential. This result is as might be expected, since this singularity is generated by the asymptotic part of the potential only.

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