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The Coulomb and Coulomb-like off-shell Jost functions

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The off-shell Jost functions are studied for a potential which is the sum of the Coulomb potential and an arbitrary local short-range central potential. We derive their singular on-shell behavior and their connection with the pure Coulomb off-shell Jost functions. For the latter we derive a large variety of interesting explicit analytic expressions which are useful for various purposes.

1. INTRODUCTION

In this paper we investigate the off-shell Jost functions $f_{\omega}(k,q)$ for the Coulomb potential and the off-shell Jost functions $f_+(k,q)$ for a Coulomb plus short-range potential, $V = V_c + V_s$, where $V_c$ is assumed to be local and central. As is now well known, these off-shell Jost functions are particularly interesting in connection with the transition matrices.

In Sec. 2 we show that $f_{\omega}(k,q)$ is a basic constituent of $f_+(k,q)$. In particular, we prove that $f_+(k,q)$ has exactly the same singularity in $q = k$ as $f_{\omega}(k,q)$. In order to obtain the most convenient formula for $f_{\omega}(k,q)$, a regrouping of certain hypergeometric function expressions has to be performed. By doing this we supply the supplementary proof of the simple formula for $f_{\omega}(k,q)$ that we have given before. This formula contains Jacobi polynomials and certain polynomials of two variables, $A_{\omega}$.

In Sec. 3 we derive a large number of interesting expressions for these polynomials $A_{\omega}$. Each of these is useful for different purposes, as is clearly illustrated at the end of Sec. 3. We shall use the notation of Ref. 1.

2. THE OFF-SHELL JOST FUNCTIONS

In this section we will express the off-shell Jost function $f_{\omega}(k,q)$ for a Coulomb-like potential in terms of the Coulomb off-shell Jost function $f_+(k,q)$. By using this expression the on-shell behavior at $q = k$ is easily obtained. Further, we shall sketch the derivation of a simple closed expression for $f_{\omega}(k,q)$.

We start by noting that

\[ f_{\omega}(k,q) = 1 + \frac{i\pi q}{2} f_+(k, q) \langle q|l| V_c |k + \rangle. \]  

(2.1)

Here $f_+(k)$ is the Jost function and $|k + \rangle$ the "outgoing" scattering state, with energy $k^2$, for the potential $V_c = V_{cl} + V_{si}$. We use the Coulomb analog of Eq. (2.1) and apply the two-potential formalism. In this way we get the convenient expression,

\[ f_{\omega}^{-1}(k)f_{\omega}(k,q) = f_+^{-1}(k)f_+(k,q) + \langle kl | - V_c G_c |X_\omega \rangle. \]  

(2.2)

Here $G_c$ is the partial-wave Green operator for $V_c$, and $|X_\omega \rangle$ is defined by

\[ |X_\omega \rangle = \frac{1}{2\pi} k G_c^{-1}[|q|k]^{l+1}|q|l\rangle_0 - |k\rangle_0. \]

By inserting

\[ \langle p|q\rangle \rangle_0 = 2(\pi q)^{l}|p/q\rangle^{l}(p^2 - q^2)^{-1}, \]

we obtain a simple expression for $|X_\omega \rangle$ in the momentum representation,

\[ \langle p|X_\omega \rangle = (p/k)(q^2 - q^2)/(p^2 - q^2). \]  

(2.3)

Equation (2.2) is very interesting, since it clearly shows that $f_{\omega}(k,q)$ has exactly the same singularity in $q = k$ as $f_{\omega}(k,q)$. As a matter of fact, by using Eq. (2.3) we have

\[ \lim_{q \to k} \omega f_{\omega}(k,q) = f_{\omega}(k), \quad k > 0. \]

(2.4)

Here

\[ \omega(\frac{q-k}{q+k})^\nu e^{q/2} = \frac{f_{\omega}(k)}{f_{\omega}(k,q)}. \]

Now we are going to summarily derive explicit expressions for $f_{\omega}(k,q)$ [cf. Eqs. (4) and (7) of Ref. 1]. In order to evaluate $\omega(q|l|V_c |k + \rangle_\omega$, which occurs in

\[ f_{\omega}(k,q) = 1 + \frac{i\pi q}{2} f_+(k) \langle q|l|V_c |k + \rangle_\omega, \]

we use the well-known expressions,

\[ \langle q|l + \rangle_\omega = (2/\pi)^{l+1} h_{l+1}^{1}(q), \]

\[ \langle r|k + \rangle_\omega = (2/\pi)^{l+1} f_+(k) (2l + 1)!^{-1} (2iqr)^z e^{-ikr} \times \Gamma(l + 1 - i\nu; 2l + 2z), \]

(2.5)

By using Ref. 3, p. 278, one obtains

\[ \langle q|l|V_c |k + \rangle_\omega = 2i\pi q f_+(k) (2l + 1)!^{-1} \times \sum_{m=0}^{l} (m + 1) s_m(k/q)^{-z} \]

\[ \times \sum_{m=0}^{l} (m + 1) s_m(k/q)^{-z} (2l + 2z), \]

where $z = 2k/(q + k)$. The important step now is to separate off that part which contains the branch-point singularity in $q = k$. To this end we apply two transformations to the hypergeometric function $s_F$ on the right-hand side of Eq. (2.5) and find (Ref. 3, p. 47),

\[ s_F(l + 1 + i\nu; l - m + 1; 2l + 2z) = (1 - z)^{-m - i\nu} \left( \frac{\Gamma(2l + 2)\Gamma(i\nu - m)}{\Gamma(l + 1 - m)\Gamma(l + 1 + i\nu)} \right) z^{-2l - 1}. \]
The hypergeometric series for the $F_i$'s on the right-hand side break off. Therefore, these $F_i$'s can be rewritten in terms of Jacobi polynomials. One has, with $z = 2/(1 + x)$,

$$\begin{align*}
P_{i,+m}^{(i,y - m, -i, y)}(x) &= \left( 1 + i, y \right) z^{-m - l} F_i(1 - m, i, y - l; 1 + i, y - m; 1 + z), \\
\text{and so} \\
P_{i,-m}^{(i,y + m, i, y)}(x) &= \left( 1 - i, y \right) z^{m - l} F_i(1 - m, -i, y - l; 1 - i, y + m; 1 - z).
\end{align*}$$

When we insert all this in Eq. (2.5) we get a complicated expression. In order to simplify this expression we introduce the polynomials $A_i$, such as $A_i(q^2/k^2; x, y^2)$,

$$\begin{align*}
A_i(q^2/k^2; x, y^2) &= \sum_{m = 0}^{l} \binom{l + m}{l} F_i(1 - m, i, y - l; 1 + i, y - m; x), \\
\text{and so} \\
P_{i,-m}^{(i,y + m, i, y)}(x) &= \left( 1 - i, y \right) z^{m - l} F_i(1 - m, -i, y - l; 1 - i, y + m; 1 - z).
\end{align*}$$

Furthermore, we shall now prove that

$$\begin{align*}
\sum_{m = 0}^{l} \binom{l + m}{l} \frac{1}{4k} p_{i,m}^{(i,y + m, i, y)}(q/k) &= P_{i,-m}^{(i,y + m, i, y)}(q^2/k^2), \\
\text{for this proof we use} \\
P_n^{(a,b)}(\xi) &= \binom{n + a}{n} F_i(1 - n, n + 1 + a + b; 1 + a - 1/2, -1/2), \\
\text{and the well-known integral representation,} \\
2 F_i(a,b;c;\xi) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 \left[ t^{b-1}(1-t)^{c-b-1}(1-\xi) - a dt.
\end{align*}$$

The left-hand side of Eq. (2.7) then becomes

$$\begin{align*}
\Gamma(l + 1 - i, y) \Gamma(l - (i, y - l)) \Gamma(l + 1) &- 1 \\
\times \int_0^1 t^{l - l - 1 - l} [1 - \frac{1}{l} (1 - q/k)] t \\
\times \sum_{m = 0}^{l} \binom{l}{m} \left[ t^{m - \frac{1}{2}}(1 - q/k) \right] - m \left( \frac{k^2 - q^2}{4k^2} \right) dt.
\end{align*}$$

By performing the summation and using again Eqs. (2.8) and (2.9) we obtain the desired expression, i.e.,

$$\begin{align*}
\frac{(l - i, y)}{l} \frac{\Gamma(l + 1 - i, y)}{\Gamma(l - (i, y - l))} \\
\times \sum_{m = 0}^{l} \binom{l}{m} \left[ t^{m - \frac{1}{2}}(1 - q/k) \right] - m \left( \frac{k^2 - q^2}{4k^2} \right) dt.
\end{align*}$$

This completes the proof of Eq. (2.7).

By inserting the above expressions in Eq. (2.5) and using Eqs. (2.6) and (2.7) we obtain

$$\begin{align*}
\langle q|V_{c_i}|k l + \gamma \rangle &= 2 c_{\gamma}(q J_i(k, q))^{-1} \left[ - x - l A_i(x, y) \right] \\
&+ f_{oc}(k, q) P_{i,-m}^{(i,y,m, y)}(u), \\
\text{cf. Eq. (7) of Ref. 1. Here} x = q/k, u = (q^2 + k^2)/(2kq), \\
f_{oc}(k, q) &= \left( \frac{q + k}{q - k} \right)^\gamma, \\
\text{and}
\end{align*}$$

$$\begin{align*}
c_{\gamma} &= \frac{\Gamma(l + 1) \Gamma(l - y, y) \Gamma(l + 1 - i, y)}{\Gamma(l + 1 + i, y) \Gamma(l + 1 + i, y)}.
\end{align*}$$

In Sec. 3 we will derive a large number of useful expressions for the polynomials $A_i$.

### 3. THE TWO-VARIABLE POLYNOMIALS $A_i$

In this section we shall derive a number of interesting explicit expressions for the polynomials $A_i$ that occur in the formula (2.10) for the Coulomb off-shell Jost functions $f_{oc}(k, q)$.

To start with, we have

$$A_i \equiv A_i(x, y^2) = \sum_{n = 0}^{l} \binom{l + n}{l} (-x)^{l - n} P_{i}^{(r, n, -i, y)}(x).$$

[Eq. (2.6)], where $x = q/k$. Substitution of

$$P_{i}^{(r, n, -i, y)}(x) = \frac{\Gamma(l + 1 + i, y) \Gamma(l + n + 1) \Gamma(l + n + 1) \Gamma(l + 1 - l)}{\Gamma(l + 1 + i, y) \Gamma(l + 1 + i, y)}$$

yields

$$A_i = \frac{\Gamma(l + 1 + i, y) \Gamma(l + n + 1) \Gamma(l + 1 - l)}{\Gamma(l + 1 + i, y) \Gamma(l + 1 + i, y)}$$

$$\times \sum_{n = 0}^{l} \binom{l}{n} \int_0^1 (1 - t)^{y - l - 1} \left[ 1 - \frac{1}{l} (1 - x)^{n} (tx)^l \right]$$

$$\times \int_0^1 \left[ 1 - \frac{1}{l} (1 - x)^{l} (1 - t)^{y - l - 1} \right] dt.$$  

The summation is easily carried out. We then get

$$A_i = \frac{t^y c_{\gamma}}{4} \int_0^1 (1 - t)^{y - l - 1} [(2 - t)^{1 - y} - x^2 t^{1 - y}] dt,$$

where $c_{\gamma}$ is given by Eq. (2.11). The polynomials $A_i$ can also be expressed in terms of Gegenbauer polynomials $C^{I}_{\ell}$. Indeed, by introducing $\tau = 1 - t$ we get from Eq. (2.3a),

$$A_i = \frac{1 - x^2}{4} \int_0^1 \tau^{y - l - 1}$$

$$\times \int_0^1 \left[ 1 + \tau^2 - 2 \tau x^2 \right] d\tau.$$

It is well known that

$$(1 - 2 \xi^2 + \tau^2) = \sum_{n = 0}^{\infty} C^{-1, \ell}_{n}(\xi)^n, \quad |\tau| < 1, \quad \lambda \neq 0.$$
Because of
\[ C_n = 0, \quad n = 2l + 1, 2l + 2, \ldots, \]
we can apply the above expansion to Eq. (3.2b), the result being,
\[ A_i = \left(1 - \frac{x^2}{4}\right) \gamma c_{iy}^{-1} \sum_{n=0}^{2l} \frac{1}{n - l + iy} C_n \frac{x^2 + 1}{x^2 - 1}. \] (3.3a)
By using
\[ C_{i-n}(\xi) \equiv C_{i-n}(\xi), \quad -l < n < l, \]
we recast the above sum in the more convenient form,
\[ \sum_{n=0}^{2l} \frac{1}{n - l + iy} C_n \frac{x^2 + 1}{x^2 - 1} = -iy \sum_{n=0}^{2l} \frac{\epsilon_n}{n^2 + \gamma^2} C_{i-n}(\xi). \]
Here \( \epsilon_n \) is the Neumann symbol,
\[ \epsilon_n = \begin{cases} \{1, & n = 0, \\ 2, & n = 1, 2, 3, \ldots. \end{cases} \]
In this way we obtain from Eq. (3.3a),
\[ A_i = \left(1 - \frac{x^2}{4}\right) \gamma c_{iy}^{-1} \sum_{n=0}^{2l} \frac{\epsilon_n}{n^2 + \gamma^2} C_{i-n}(\xi). \] (3.3b)
This expression can be rewritten in terms of the Jacobi polynomials \( P_{\mu - n}^{\nu} \). By using
\[ \frac{4x}{1 - x^2} C_{i-n}(\xi) \frac{x^2 + 1}{x^2 - 1} = \begin{cases} (\lambda + \mu)(n!) \frac{x + 1}{x - 1} \mathcal{F}(\lambda + n; 1 - \lambda - n; \frac{x + 1}{x - 1}), \\ (\lambda + \mu)(n!) \frac{x + 1}{x - 1} \mathcal{F}(\lambda + n; 1 - \lambda - n; \frac{x + 1}{x - 1}), \end{cases} \]
we derive the interesting relation,
\[ \begin{aligned}
\frac{4x}{1 - x^2} C_{i-n}(\xi) \frac{x^2 + 1}{x^2 - 1} &= \frac{\Gamma(l + 1) \Gamma(l + n + 1)}{\Gamma(l - n + 1) \Gamma(l + n + 1)} \left(1 - \frac{x}{1 + x}\right)^n \times P_{\mu - n}^{\nu}(\frac{1}{2}x + \frac{1}{2}x^i), \quad |n| < l.
\end{aligned} \] (3.3c)
By inserting this into (3.3b) we get
\[ A_i = x^i c_{iy}^{-1} \gamma \sum_{n=0}^{2l} \frac{\epsilon_n}{n^2 + \gamma^2} \frac{\Gamma(l + 1)}{\Gamma(l + n + 1)} \left(1 - \frac{x}{1 + x}\right)^n \times P_{\mu - n}^{\nu}(\frac{1}{2}x + \frac{1}{2}x^i). \] (3.4a)
Further, we have
\[ P_{\mu - n}^{\nu}(z) = (l + 1)_z (z + 1)^{\nu/2} (z - 1)^{- \nu/2} \mathcal{F}_{\mu - n}(z), \]
where \( \mathcal{F}_{\mu - n} \) is Legendre's function of the first kind. Substitution of this expression yields
\[ A_i = x^i c_{iy}^{-1} \gamma \sum_{n=0}^{2l} \frac{\epsilon_n}{n^2 + \gamma^2} \frac{\Gamma(l + 1)}{\Gamma(l - n + 1)} \mathcal{F}_{\mu - n}(\frac{1}{2}x + \frac{1}{2}x^i), \]
\[ 0 < x < 1. \] (3.4b)
When \( x > 1 \) the Legendre function here has to be multiplied by \((-1)^n\).

From Eq. (3.2a) one can find an expression containing either \( \mathcal{F}(\cdot; 1) \) or \( \mathcal{F}(\cdot; -1) \) or \( \mathcal{F}(\cdot; 2) \). It turns out that the formula with \( \mathcal{F}(\cdot; 2) \) is the more convenient one. We obtain this formula by using the binomial expansion, which yields
\[ A_i = 4^{-i} \gamma c_{iy}^{-1} \sum_{n=0}^{2l} \frac{\Gamma(l + 1 - n)}{\Gamma(l + n + 1)} (1 - t)^{l - n - 1} \times \sum_{m=0}^{l} \frac{(l)!}{m!} (2 - t)^{2m}(1 - x^2)^m. \]
By again using the binomial expansion,
\[ (2 - t)^{2m} = \sum_{n=0}^{2m} \frac{2m}{n!} (2 - t)^m. \]
the integration can be performed, with the result,
\[ \int_0^1 (1 - t)^{l - n - 1} t^{2l - n} dt = \frac{\Gamma(2l - n + 1) \Gamma(\gamma - l)}{\Gamma(\gamma + l + n + 1)}. \]
In this way we get
\[ A_i = 4^{-i} \gamma c_{iy}^{-1} (2l) \Gamma(\gamma - l) \Gamma(l + 1 + iy) \times \sum_{m=0}^{l} \frac{(l)!}{m!} (-y)^m x^{2l - 2m} \times \mathcal{F}_{\mu - n}(2 - 2l, -l + iy; -2l). \] (3.5a)
The sum \( \Sigma_n \) is a terminating hypergeometric series for which we write \( \mathcal{F}(\cdot; -2l, -l + iy; -2l) \). One should be careful here, since the third parameter, \(-2l\), is a nonpositive integer. By using expression (2.11) for \( c_{iy} \) we obtain
\[ A_i = 4^{-i} \gamma c_{iy}^{-1} (2l) \Gamma(\gamma - l) \Gamma(l + 1 + iy) \times \mathcal{F}_{\mu - n}(2 - 2l, -l + iy; -2l). \] (3.5b)
We note that \( A_i \) is a function of \( y^2 \) rather than of \( y \), as can be seen from Eq. (3.3b). So we have, by replacing \( m \) by \( l - n \),
\[ A_i = 4^{-i} \gamma c_{iy}^{-1} (2l) \sum_{m=0}^{l} \frac{(l)!}{m!} (-y)^m x^{2l - 2m} \times \mathcal{F}_{\mu - n}(2 - 2l, -l + iy; -2l). \] (3.5a)
The hypergeometric function \( \mathcal{F}(\cdot; -2l) \) can be expressed in terms of a Jacobi polynomial with argument 0. By using Ref. 3, p. 212, we have
\[ A_i = \frac{(-y)^l}{l!} \sum_{n=0}^{l} \frac{(2n)!}{(l - n)!} (-\frac{1}{2}y)^n \times \mathcal{F}_{\mu - n}(2n + 1 + iy; 2n - l - iy; 0), \] (3.5b)
or

\[ A_I = \frac{1}{l!} \left( \frac{4}{x^2} \right)^l \sum_{n=0}^{l} \frac{(2n)!}{n!} \left( \frac{2l - 2n}{l - n} \right)! \left( - \frac{4}{x^2} \right)^{-n} \times P_{\frac{l}{2n}}^{(2n + iy)} (0). \] (3.5c)

Now we come to the derivation of the most elegant formula for \( A_I \), i.e., a generalized hypergeometric function \( F \) with argument 1 - \( x^2 \). From Eq. (3.2a) we have

\[ A_I = \int_0^1 (1 - t)^{iy - 1} \left[ 1 - t + \frac{1}{2}(1 - x^2)t^2 \right] dt. \]

After substitution of

\[ \left( 1 - t + \frac{1}{2}(1 - x^2)t^2 \right)^l = \sum_{n=0}^{l} \binom{l}{n} (1 - t)^{iy - 1 - n} t^{2n - 2n} \times (1 - x^2)^n, \]

we can perform the integration, the result being

\[ \int_0^1 (1 - t)^{iy - 1 - n} t^{2n} dt = \Gamma (iy - n) \Gamma (2n + 1)/\Gamma (iy + n + 1). \]

In this way we obtain

\[ A_I = c_{iy}^{-1} \sum_{n=0}^{l} \frac{(2n)!}{n!} \left( \frac{-1}{1 + iy} \right)_n \frac{2^{-2n}}{2^{-2n} (1 - x^2)^n}. \] (3.6a)

By using the doubling formula for the gamma function we have

\[ (2n)! = \left( \frac{1}{2} \right)_n 2^{2n} n!, \]

and so

\[ A_I = c_{iy}^{-1} F_i (-l, 1; 1 + iy, 1 - iy; 1 - x^2). \] (3.6b)

An alternative expression is

\[ A_I = \sum_{n=0}^{l} \frac{\Gamma (l + 1 + iy)}{\Gamma (n + 1 + iy)} \frac{\Gamma (l + 1 - iy)}{\Gamma (n + 1 - iy)} \times \frac{\left( \frac{1}{2} \right)_n (x^2 - 1)^n}{\Gamma (l + 1) \Gamma (l + 1 - n)}. \] (3.6c)

where we have inserted Eq. (2.11). Furthermore, we have the terminating hypergeometric series,

\[ A_I = \frac{\left( \frac{1}{2} \right)_l}{l!} (x^2 - 1)^l \sum_{n=0}^{l} \frac{(iy - l)_n (-iy - l)_n}{(\frac{1}{2} - l)_n} \frac{(1 - x^2)^{-n}}{n!}. \] (3.6d)

From Eq. (3.6c) one can derive an expression involving a \( F_i \) with argument 1. By inserting

\[ (x^2 - 1)^n = \sum_{m=0}^{n} \binom{n}{m} x^{2m} (-1)^n - m \]

in (3.6c) and introducing the new summation variable \( v = n - m \), we have

\[ \sum_{m=0}^{n} \sum_{v=0}^{m} \cdots = \sum_{m=0}^{l} \sum_{v=0}^{m} \cdots. \]

It turns out that the sum \( \sum \) is a \( F_i (\cdots; 1) \), and thus we obtain

\[ A_I = \sum_{n=0}^{l} \frac{x^{2n} \left( \frac{1}{2} \right)_n}{\Gamma (l + 1) \Gamma (l + 1 - n)} \times \frac{\Gamma (l + 1 + iy) \Gamma (l + 1 - iy)}{\Gamma (n + 1 + iy) \Gamma (n + 1 - iy)} \times F_i (n - l, n + 1, n + \frac{1}{2}; n + 1 + iy; n + 1 - iy; 1). \] (3.7)

We transform this \( F_i \) into a \( F_i \) with different parameters by applying a generalization of Dixon's theorem, see Slater (Ref. 4, p. 52),

\[ F_i (n - l, n + 1, n + \frac{1}{2}; n + 1 + iy; n + 1 - iy; 1) = \Gamma \left[ l - n + \frac{1}{2}, n + 1 + iy, n + 1 - iy \right] \frac{1}{\frac{1}{2}, l + 1, n + 1} \times F_i (iy, -iy, l - n + \frac{1}{2}; l + 1; 1). \]

Then we have from (3.7),

\[ A_I = \frac{\Gamma (l + 1 + iy) \Gamma (l + 1 - iy)}{\Gamma (l + 1)} \times \sum_{n=0}^{l} \frac{x^{2n} \left( \frac{1}{2} \right)_n}{\Gamma (l + 1) \Gamma (l + 1 - n)} \times F_i (iy, -iy, l - n + \frac{1}{2}; l + 1; 1). \] (3.8a)

Note that the hypergeometric series for this \( F_i \) breaks off when \( iy = 0, -1, -2, \ldots \). The case \( iy = 0 \) corresponds to no Coulomb interaction at all. On the other hand, \( iy = -1, -2, -3, \ldots \) occurs for the Coulomb bound states.

It is not difficult to derive from Eq. (3.8a) the corresponding series with decreasing powers of \( x \). This expression has almost exactly the same form as (3.8a), namely,

\[ A_I = \frac{\Gamma (l + 1 + iy) \Gamma (l + 1 - iy)}{\Gamma (l + 1)} \times \sum_{n=0}^{l} \frac{x^{2n} \left( \frac{1}{2} \right)_n}{\Gamma (l + 1) \Gamma (l + 1 - n)} \times F_i (iy, -iy, l - n + \frac{1}{2}; l + 1; 1). \] (3.8b)

By comparing this expression with Eq. (3.5c) we get the interesting equality

\[ F_i (iy, -iy, n + \frac{1}{2}; l + 1; 1) \]

\[ = \frac{(iy - l)_l (-iy - l)_l}{\Gamma (l + 1 + iy) \Gamma (l + 1 - iy)} \times \frac{\Gamma (l + 1 + iy) \Gamma (l + 1 - iy)}{\Gamma (l + 1) \Gamma (l + 1 - n)} \times \frac{\Gamma (l + 1 + iy) \Gamma (l + 1 - iy)}{\Gamma (l + 1) \Gamma (l + 1 - n)} \times F_i (iy, -iy, l - n + \frac{1}{2}; l + 1; 1). \]

In the particular case when \( l = 2n \) this expression can be simplified. By using (e.g., Ref. 3, p. 167).
\[ \Gamma(1 + \mu)P_{-\mu}(0) = J_1(-v\nu + v + 1;\mu + 1; \frac{1}{2}) \]
\[ = \Gamma(1 + \frac{1}{2} + \mu) \times [\Gamma(1 + \frac{1}{2} + \frac{1}{2} + \mu)]^{-1} \times (\frac{1}{2} - \frac{1}{2} + \mu - \frac{1}{2} + \mu) \]
we get
\[ P_{\frac{1}{2} - \mu}(0) = 2^{2\mu} \Gamma(\frac{1}{2} + \frac{1}{2} + \mu + \nu) \Gamma(\frac{1}{2} - \frac{1}{2} + \mu - \mu) \Gamma(2n + 1) \]
\[ = \left( -\gamma \right) n! \left( \frac{n - 1}{2} + \frac{1}{2} \right) \left( \frac{n - 1}{2} - \frac{1}{2} \right) \]
(3.8c)
and so
\[ J_1(\mu(\nu - \nu)n + \frac{1}{2};2n + 1; 1) \]
\[ = \frac{\pi \Gamma(n + 1)}{\Gamma(\frac{1}{2} + \frac{1}{2} + \mu) \Gamma(\mu + 1 + \frac{1}{2} + \mu) \Gamma(n + 1 - \frac{1}{2} + \mu)} \]
cf. Eqs. (2.3) and (3.13) of Ref. 4.

One can see from Eq. (3.6c) in particular that the degree of the polynomial \( A_1 \equiv A_1(x;\nu) \) is \( l \), both in \( x^2 \) and in \( \gamma^2 \),
\[ A_1 = \sum_{n = 0}^{l} x^{2l - 2n} D_n^{(1)}(\gamma^2) \]
(3.9a)
\[ A_1 = \sum_{n = 0}^{l} \gamma^{2l - 2n} F_n^{(1)}(\gamma^2) \]
(3.9b)
Here \( D_n^{(1)} \) and \( F_n^{(1)} \) are certain polynomials of degree \( n \). It turns out that Eq. (3.9b) is less suitable for practical applications, so we shall mainly restrict ourselves to the expansion in the \( D_n^{(1)} \)'s. One can also write \( A_1 \) as
\[ A_1 = \sum_{n = 0}^{l} \sum_{m = 0}^{n} x^{2l - 2n} \gamma^{2m} a_{n,m}^{(1)} \]
(3.10)
Here the coefficients \( a_{n,m}^{(1)} \) are real positive numbers, as can be proven with the help of Eq. (3.8).

It is of interest to discuss a number of special cases. In the first place we consider the zero-energy case, \( k = 0 \). Recalling \( x \equiv q/k \) and \( \gamma \equiv -s/k \), we have from Eq. (3.6c),
\[ A_1 \approx \gamma^{2l}(l! - 2)^{-2} F_1(-l, 1, 2; -x^2/\gamma^2) \]
\[ = 4^{-l} \left( \frac{1}{2} \right) \gamma^l \]
(3.11)
On the other hand, for \( k \to \infty \) we have \( x \to 0 \) and \( \gamma \to 0 \). In this case we get from Eq. (3.8),
\[ A_1(0;0) = d_{n,0}^{(1)} \]
(3.12)
For \( x = 1 \) one easily derives from Eq. (3.6b),
\[ A_1(1;\gamma^2) = c_{\gamma^2} \]
(3.13)
The numbers \( a_{n,m}^{(1)}(n,m = 0,1,\ldots,l) \) can be considered as a matrix, which is triangular because of
\[ a_{n,m}^{(1)} = 0, \quad n < m \]
The matrix elements on the principal axis are given by
\[ a_{n,m}^{(1)} = \frac{4^m}{l!} \left( \frac{2l - 2n}{l - n} \right) \]
(3.14)
In particular for \( n = l \) one has
\[ a_{n,n}^{(1)} = F_n^{(1)}(l! - 2) \]
(3.15)
Equation (3.14) is obtained by considering
\[ D_n^{(1)}(\gamma^2) = \sum_{m = 0}^{l} \gamma^{2m} a_{n,m}^{(1)} \]
and
\[ D_n^{(1)}(\gamma^2) = (-1)^{n} 4^n l! \frac{2^n - 2n + 1 + i\gamma}{l!} \]
(3.16)

It is interesting to note the connection of \( D_n^{(1)} \) with certain known polynomials, namely Krawtchouk’s polynomials \( k_n(z) \), which depend in addition on a positive variable \( p < 1 \) and a positive integer \( N \). These polynomials are associated with the binomial distribution in probability theory. According to Refs. 5 and 6 one has, with \( p = \frac{1}{2} \) and \( N = 2l \),
\[ k_{2n}(\gamma^2 + l) \]
\[ = 4^{-n} \left( \frac{1}{2} \right) \gamma^l \;
(3.17)
Since \( k_n(z) \) is defined for an integer variable \( z \) only, \( D_n^{(1)} \) may be considered as a generalization of \( k_{2n} \).

For \( \gamma = 0 \) we get from Eqs. (3.4a) and (3.6b),
\[ A_1(x^2;0) = x^l P_l(\frac{1}{2} x + \frac{1}{2} x^2) \]
(3.18)
By using these expressions we obtain
\[ a_{n,0}^{(1)} = D_n^{(1)}(0) = 4^{-l} \left( \frac{2l}{l - n} \right) \]
(3.19)
Further we derive from Eqs. (3.8c) and (3.15),
\[ D_n^{(2n)}(\gamma^2) = \left( \frac{2n}{n} \right) \left( \gamma^l \right) \]
(3.20)
which again shows the dependence on \( \gamma^l \) rather than on \( \gamma \). For \( x = 0 \) we have from Eq. (3.15),
\[ A_1(0;\gamma^2) = D_n^{(1)}(\gamma^2) \]
(3.21)
In order to obtain explicit expressions for \( A_1, A_2, \ldots \), Eq. (3.6) is very useful. We first recast Eq. (3.6c) in a more explicit form,
\[ A_1 = (l + i\gamma)^{-l} \sum_{n = 0}^{l} \frac{(l + i\gamma)_{1 + \gamma}}{l + \gamma} \left( \frac{-1}{x^2} \right)^n \]
(3.22)
Therefore, we have
\[ D^{(i)}_m (\gamma^2) = (l + 1)^{-1} \sum_{n=0}^{\infty} \left( \frac{n}{1} \right)^{l + 1 - n} \prod_{m=n+1}^{l} m^2 + \gamma^2. \]  

In particular, 

\[ D^{(i)}_0 = \frac{\Gamma(l + 1)}{l! \Gamma(\frac{l+1}{2})}, \]
\[ D^{(i)}_1 = \frac{\Gamma(l + 1)}{l! \Gamma(\frac{l+1}{2})} (\gamma^2 + \frac{1}{2}), \]
\[ D^{(i)}_2 = \frac{\Gamma(l + 1)}{l! \Gamma(\frac{l+1}{2})} \left[ \frac{1}{6} \gamma^4 + \gamma^2 (3l - 2) + \frac{3}{2} l (l - 1) \right], \]
\[ D^{(i)}_3 = \frac{\Gamma(l + 1)}{l! \Gamma(\frac{l+1}{2})} \left[ \frac{1}{4} \gamma^4 + \gamma^2 \left( \frac{15}{2} l - 10 \right) + \frac{1}{2} (45 l^2 - 105 l + 46) + \frac{15}{8} l (l - 1) (l - 2) \right]. \]

Finally, we give the first four polynomials \( A_i \) in explicit form, 

\[ A_0 = 1, \]
\[ A_1 = \frac{1}{3} (\gamma^2 + 1 + 2 \gamma^2), \]
\[ A_2 = \frac{1}{6} [3 \gamma^2 + 2 \gamma^2 (1 + \gamma^2) + 3 + 8 \gamma^2 + 2 \gamma^4], \]
\[ A_3 = \frac{1}{4} \left[ 15 \gamma^2 + 3 \gamma^2 (3 + 2 \gamma^2) + x^2 (9 + 14 \gamma^2 + 2 \gamma^4) \right] + \frac{1}{2} (45 + 136 \gamma^2 + 50 \gamma^4 + 4 \gamma^6). \]

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