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Nonperturbative confinement in quantum chromodynamics. I. Study of an approximate equation of Mandelstam

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An approximated form of the Dyson–Schwinger equation for the gluon propagator in quarkless QCD is subjected to nonlinear functional and numerical analysis. It is found that solutions exist, and that these have a double pole at the origin of the square of the propagator momentum, together with an accumulation of soft branch points. This analytic structure is strongly suggestive of confinement by infrared slavery.

PACS numbers: 11.10.Np

I. INTRODUCTION

Of the various ways that quarks and gluons might automatically be confined in quantum chromodynamics, the hypothesis that the gluon propagator has a strong singularity at the origin of the k^2 plane, where k is the gluon four-momentum, is especially attractive. Recently Mandelstam,¹ working in the Landau gauge, has approximated the Dyson equation for the gluon propagator and claims that the latter probably behaves like k^{-4} as $k \rightarrow 0$. Similarly, Baker *et al.*,² using an axial gauge, make a different sequence of approximations, obtaining a much more complicated equation. However, again they claim the same infrared behavior for the gluon propagator.

We propose to study these equations by functional and numerical methods in order to elucidate the infrared properties of QCD. In this initial paper, we consider Mandelstam's equation in its simplest form and prove that it indeed possesses a solution with the k^{-4} infrared behavior. Further, we study the analytical structure of the propagator, away from the origin, by numerical means.

Let us first clarify the sense in which a gluon propagator that behaves like k^{-4} , as $k \rightarrow 0$, corresponds to a linearly rising potential,

$$V(|\mathbf{x}|) \sim |\mathbf{x}| \text{ as } |\mathbf{x}| \rightarrow \infty, \quad (1.1)$$

acting between a quark and an antiquark in their center-of-mass system, effectively confining them. In nonrelativistic Born approximation, the scattering amplitude is given in terms of the potential by

$$A(\mathbf{k}) = -\frac{1}{4\pi} \int d^3x V(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (1.2)$$

where \mathbf{k} is the three-momentum transfer between initial and final quark. For a linear potential, the amplitude is thus proportional to

$$-\frac{1}{4\pi} \int d^3x |\mathbf{x}| e^{i\mathbf{k}\cdot\mathbf{x}}, \quad (1.3)$$

which is well defined as a tempered distribution, as we now show. For let $\Omega(\mathbf{k})$ be any infinitely differentiable test func-

tion that vanishes faster than any inverse power of $|\mathbf{k}|$, as $|\mathbf{k}| \rightarrow \infty$, and which satisfies in addition

$$\Omega(0) = \Omega'(0) = \Omega''(0) = \Omega'''(0) = 0. \quad (1.4)$$

We show in Appendix A that

$$-\frac{1}{4\pi} \int d^3x |\mathbf{x}| \int d^3k \Omega(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} = 2 \int d^3k \Omega(\mathbf{k}) |\mathbf{k}|^{-4}, \quad (1.5)$$

i.e., the amplitude is proportional to $|\mathbf{k}|^{-4}$, as a distribution on our space of test functions. In the center-of-mass system, the square of the four momentum transfer, k^2 , is equal to $-|\mathbf{k}|^2$, so we see that a linear potential indeed corresponds to a propagator that behaves like k^{-4} as $k \rightarrow 0$.

In Sec. 2, we discuss the rather drastic approximations that Mandelstam makes; and this culminates in the derivation of a deceptively simple nonlinear Volterra equation. The rest of the present work is devoted to a study of this equation; but it is appropriate to ask at this point how much of QCD really has survived. The infrared behavior may be such a survivor, since the approximations are less severe for small momenta. Moreover, there are indications that the confining singularity may be specific to the vector theory, for in massless QED, if one approximates the Dyson equation for the electron propagator by replacing the full vertex and the full photon propagator by their bare values, one finds in general that there is no singularity of the solution at $k^2 = 0$: the chiral symmetry has been broken.³ Unfortunately, the branch point nearest to the origin, which should correspond to the point $k^2 = m_e^2$, is complex,⁴ and moreover its position is gauge-dependent.⁵ Because of certain algebraic differences, the massless vector case (QCD) is quite different; and we find that a solution exists in which the propagator has a double pole at the origin, in the variable $x = -k^2$ (implying confinement), and also a branch point at $x = 0$, corresponding to soft gluon effects. Unfortunately, these desirable properties of the solution are marred by the fact that, as in the spinor case, we also find unwanted complex branch points.

In Sec. 2, we summarize Mandelstam's method of obtaining his gluon equation, and we convert it into a nonlinear Volterra equation. This equation is not suited to an application of the Banach theorem, because of cancellations for small k that are hard to handle, so in Sec. 3 we deduce a new integral equation, by way of a nonlinear differential equa-

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tion, that is so suited; and in Sec. 4 we elaborate a proof of the local uniqueness of an analytic solution of the propagator equation in a cardioid region around the origin of the k^2 plane. In Sec. 5, we outline a program for the numerical integration, by a fourth-order Runge-Kutta routine, of the nonlinear differential equation of Sec. 3, which has been used to continue out of the domain of the complex plane for which the existence proof is valid. Six complex branch points have been found close to the origin, and there may be more. In Appendices A and B we collect some technical details, while Appendix C is devoted to a further approximation of the Mandelstam equation, in which the kernel is replaced by its average value. This is a gross simplification; but the advantage is that the averaged equation is explicitly soluble, and this is of great help in testing the computer programs.

II. MANDELSTAM'S METHOD

In Ref. 1, Mandelstam considers a Dyson equation for the gluon propagator, in QCD without quarks, in which the four-gluon vertices are thrown away. Further, the three-gluon vertex is replaced by its bare value; and simultaneously one, but not both, of the internal gluon propagators is replaced by its bare value. This rather arbitrary procedure is justified on the grounds that if the propagator behaves like k^{-4} , as hoped, then the full three-gluon vertex behaves like k^2 , and therefore the replacement of the full by the bare vertex should be matched by the softening of the k^{-4} behavior of one full propagator to the k^{-2} behavior of a bare propagator. Clearly this simplification is drastic, and it is uncertain whether the physical import of the Dyson equation has been fundamentally altered. The contribution of the ghost field is expected to be fairly small, and is in the first instance neglected, although it can be included without too much trouble.

The mutilated Dyson equation is depicted in Fig. 1. It is assumed that the full gluon propagator can be written in the form

$$D'_{F\mu\nu}(k) = F(-k^2)D_{F\mu\nu}^{ab}(k), \quad (2.1)$$

where F is an unknown function of the scalar $-k^2$, and where

$$D_{F\mu\nu}^{ab}(k) = i[-g_{\mu\nu} + k_\mu k_\nu / k^2] \frac{\delta^{ab}}{k^2 + i\epsilon} \quad (2.2)$$

is the bare propagator in the Landau gauge. Here μ, ν are Lorentz indices and a, b are SU(3) color superscripts. The Dyson equation is then

$$\begin{aligned} D_F'^{-1}(k) &= D_F^{-1}(k) - \frac{g^2}{(2\pi)^4} \\ &\times \int d^4k' \Gamma(k, k' - k, -k') D_F'(k') \\ &\times \Gamma(k', k - k', -k) D_F(k' - k), \end{aligned} \quad (2.3)$$

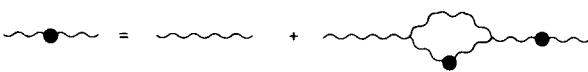


FIG. 1. Approximate Dyson equation for the gluon propagator. The wavy lines with a black dot are full, the others bare gluon propagators.

where Lorentz and color indices have been omitted, where Γ is the bare three-gluon vertex, and where g is the SU(3) coupling constant. After a Wick rotation has been performed, the angular integrations can be done, and Mandelstam finds

$$[F(x)]^{-1} = 1 - \frac{G}{x} \int_0^\infty dy K_1(x, y) F(y), \quad (2.4)$$

where $x = -k^2, y = -k'^2$, and where G is proportional to g^2 , the constant of proportionality being unimportant. In (2.4) the kernel is

$$\begin{aligned} K_1(x, y) &= 18 + \left(25 \frac{y}{x} - \frac{7}{2} \frac{y^2}{x^2}\right) \theta(x - y) \\ &+ \left(25 \frac{x}{y} - \frac{7}{2} \frac{x^2}{y^2}\right) \theta(y - x). \end{aligned} \quad (2.5)$$

One can rewrite (2.4) in the form

$$\begin{aligned} [F(x)]^{-1} &= 1 + \frac{B}{x} - \frac{G}{x} \int_0^x dy \left(25 \frac{y}{x} - \frac{7}{2} \frac{y^2}{x^2}\right) F(y) \\ &- \frac{G}{x} \int_x^\infty dy \left(25 \frac{x}{y} - \frac{7}{2} \frac{x^2}{y^2}\right) F(y), \end{aligned} \quad (2.6)$$

where

$$B = -18G \int_0^\infty dy F(y). \quad (2.7)$$

This integral needs both infrared and ultraviolet cutoffs, which we do not write explicitly. Now is it possible that the propagator (2.1) behaves like k^{-4} as $k \rightarrow 0$, i.e., that $F(x)$ behaves like x^{-1} as $x \rightarrow 0$? Let us write, following Mandelstam,

$$F(x) = A/x + F_1(x), \quad (2.8)$$

where $F_1(x)$ is to be less singular than a pole as $x \rightarrow 0$ (in fact, it will turn out to be bounded). We substitute (2.8) into (2.6) and perform the integrals over the pole term:

$$\begin{aligned} \frac{x}{A + xF_1(x)} &= 1 + \frac{1}{x} \left[B - \frac{93}{2} AG \right] \\ &- \frac{G}{x} \int_0^x dy \left(25 \frac{y}{x} - \frac{7}{2} \frac{y^2}{x^2}\right) F_1(y) \\ &- \frac{G}{x} \int_x^\infty dy \left(25 \frac{x}{y} - \frac{7}{2} \frac{x^2}{y^2}\right) F_1(y). \end{aligned} \quad (2.9)$$

Now if $F_1(x)$ is well behaved at the origin, the pole term on the right hand side of (2.9) must vanish, i.e., A is given by

$$A = 2B/93G.$$

A constant behavior of $F_1(x)$ as $x \rightarrow 0$ would be inconsistent with (2.9), since the first term under the second integral would yield $\ln x$, which could not be cancelled; and likewise a linear behavior, $F_1(x) \sim x$, is inconsistent, for now the second term under the second integral gives an insupportable $\ln x$. Mandelstam shows that these inconsistencies are removed if $F_1(x) \sim x^\alpha$, where $\alpha = -1 + (31/6)^{1/2} \approx 1.273$. He further suggests, as a reasonable first approximation, the dropping of the " $\frac{7}{2}$ terms" in (2.9). The equation then becomes

$$\begin{aligned} \frac{x}{A + xF_1(x)} &= 1 - 25 \frac{G}{x} \int_0^x dy \frac{y}{x} F_1(y) \\ &- 25 \frac{G}{x} \int_x^\infty dy \frac{x}{y} F_1(y), \end{aligned} \quad (2.10)$$

and now a behavior $F_1(x) \sim x$ as $x \rightarrow 0$ is no longer impossible, since the offending second term under the second integral in (2.9) has been thrown away. In this paper, we shall study the approximate equation (2.10) exclusively: it is hoped that the behavior x instead of $x^{1.273}$ is not too serious an error. In a future work, we propose to return to the full equation (2.9): the analysis can be completed, but it is appreciably more complicated.

The approximate equation (2.10) can be rewritten

$$\frac{x}{A + xF_1(x)} = 1 - C + 25 \frac{G}{x} \int_0^x dy \left(\frac{x}{y} - \frac{y}{x} \right) F_1(y), \quad (2.11)$$

where

$$C = 25G \int_0^\infty \frac{dy}{y} F_1(y). \quad (2.12)$$

For consistency as $x \rightarrow 0$, we must impose $C = 1$; but since the integral in (2.12) needs an ultraviolet cutoff, this imposition can be regarded as a renormalization condition. Further, if we make the scaling transformations

$$x \rightarrow 5AG^{1/2}x, \quad y \rightarrow 5AG^{1/2}y, \quad F_1(x) \rightarrow \frac{1}{5} G^{-1/2} F_1(x), \quad (2.13)$$

then Eq. (2.11) takes on the pleasing form

$$\frac{x^2}{1 + xF_1(x)} = \int_0^x dy \left(\frac{x}{y} - \frac{y}{x} \right) F_1(y), \quad (2.14)$$

in which there are no divergences left, in which the unknown constant A , and even the coupling G , have disappeared.

III. APPROXIMATE GLUON PROPAGATOR

We first rewrite the approximate equation (2.14) in terms of the new unknown function,

$$G(x) = F_1(x) / [x + x^2 F_1(x)], \quad (3.1)$$

so that

$$G(x) = \frac{1}{x^2} - \frac{1}{x^3} \int_0^x dy \left(1 - \frac{y^2}{x^2} \right) \frac{G(y)}{1 - y^2 G(y)}. \quad (3.2)$$

Unfortunately, this equation is poorly suited either to numerical iteration or to an existence proof via the Banach theorem. For example, $G(0)$ is finite, but this is a result of delicate cancellation:

$$\begin{aligned} G(x) &\sim \frac{1}{x^2} - \frac{1}{x^3} \int_0^x dy \left(1 - \frac{y^2}{x^2} \right) [G(0) + O(y)] \\ &= \frac{1}{x^2} \left(1 - \frac{2}{3} G(0) \right) + O\left(\frac{1}{x}\right), \end{aligned} \quad (3.3)$$

so that we must require

$$G(0) = \frac{3}{2}. \quad (3.4)$$

By pushing this analysis further, one can show that $G'(0)$ must vanish, that $G''(0) = -27$, and so on. In fact, in Sec. 5, we shall obtain an asymptotic (but divergent) series for $G(x)$ in the variable x^2 .

To construct an integral equation that does not involve cancellations, we differentiate $x^3 G(x)$, and then $x^3 [x^3 G(x)]'$, in order to obtain, from (3.2), the nonlinear differential equation

$$x^4 G'' + 9x^3 G' + (15x^2 + 2)G = 3 - 2x^2 G^2 / (1 - x^2 G). \quad (3.5)$$

It is possible to resolve this equation in terms of the solution of the homogeneous linear differential equation [i.e., the left-hand side of (3.5) equal to zero]; but this route involves Bessel functions and their estimation in terms of simpler functions. It is more straightforward to observe that the functions

$$g_\pm(x) = x^{-7/2} \exp(\pm i2^{1/2}/x) \quad (3.6)$$

solve the homogeneous equation

$$x^4 g_\pm'' + 9x^3 g_\pm' + (\frac{63}{4}x^2 + 2)g_\pm = 0. \quad (3.7)$$

This is almost like the left-hand side of (3.5), excepting only that the coefficient of $x^2 g_\pm$ is not quite correct! However, by adding $\frac{3}{4}x^2 G$ to both sides of (3.5), we obtain

$$\begin{aligned} x^4 G'' + 9x^3 G' + (\frac{63}{4}x^2 + 2)G \\ = 3 + \frac{3}{4}x^2 G - 2x^2 G^2 / (1 - x^2 G). \end{aligned} \quad (3.8)$$

This equation can be resolved in terms of g_\pm by the method of variation of parameters. The linear term $\frac{3}{4}x^2 G$ on the right-hand side of (3.8) will not give trouble for small x , thanks to the factor x^2 . We find

$$\begin{aligned} G(x) &= 2^{-1/2} x^{-7/2} \int_0^x dy \sin \left[2^{1/2} \left(\frac{1}{x} - \frac{1}{y} \right) \right] \\ &\times \left\{ -3y^{3/2} - \frac{3}{4}y^{7/2} G(y) + \frac{2y^{7/2} G^2(y)}{1 - y^2 G(y)} \right\}. \end{aligned} \quad (3.9)$$

It can easily be checked that this is the correct solution of the differential equation: additional multiples of g_+ or g_- would be inconsistent with (3.4), and thus with (3.2).

The form (3.9) is suitable for an existence proof; but the argument of the sine factor makes matters somewhat awkward. To simplify matters, define

$$\xi = 2^{1/2}/x, \quad \tilde{G}(\xi) = G(x) \quad (3.10)$$

and

$$\xi = 2^{1/2}/y - 2^{1/2}/x, \quad (3.11)$$

so that

$$\tilde{G}(\xi) = f(\xi) + \xi^{7/2} \int_0^\infty d\xi' \frac{\Sigma(\xi + \xi')}{(\xi + \xi')^{11/2}} \sin \xi', \quad (3.12)$$

where

$$f(\xi) = \frac{3}{2} \xi^{7/2} \int_0^\infty \frac{d\xi'}{(\xi + \xi')^{7/2}} \sin \xi', \quad (3.13)$$

and

$$\Sigma(\xi) = \frac{3}{4} \tilde{G}(\xi) - \frac{2\tilde{G}^2(\xi)}{1 - 2\tilde{G}(\xi)/\xi^2}. \quad (3.14)$$

This equation will now be used for an existence proof for $|\xi|$ large enough, i.e., $|x|$ small enough, for all $|\arg x| \leq \pi - \epsilon$, $\epsilon > 0$.

IV. PROOF OF EXISTENCE

Let $D(\rho, \epsilon)$ be the following domain in the complex ξ plane:

$$\begin{aligned} D(\rho, \epsilon) &= \{ \xi \mid \operatorname{Re} \xi \geq 0, |\xi| > \rho^{-1}; \\ &(\operatorname{Re} \xi < 0, (|\operatorname{Im} \xi| - \rho^{-1}) / |\operatorname{Re} \xi| > \tan \epsilon) \}. \end{aligned} \quad (4.1)$$

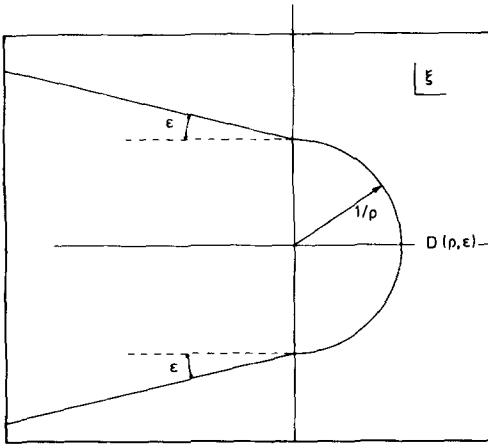


FIG. 2. The domain $D(\rho, \epsilon)$ lies outside the comet-shaped region depicted.

In words, it is the region outside a semicircle in the right half-plane and above or below lines inclined at an angle ϵ to the real axis (see Fig. 2). Let B be the Banach space of functions, $f(\xi)$, analytic in D , with norm

$$\|f\| = \sup_{\xi \in D(\rho, \epsilon)} |f(\xi)|. \quad (4.2)$$

Let P be the mapping

$$P(\tilde{G}; \xi) = f(\xi) + \xi^{7/2} \int_0^\infty d\zeta \frac{\Sigma(\zeta + \xi)}{(\zeta + \xi)^{11/2}} \sin \zeta, \quad (4.3)$$

where f and Σ were defined in (3.13) and (3.14). It is clear that, if $\xi \in D(\rho, \epsilon)$, for a given $\rho > 0$ and $0 < \epsilon < \pi/2$, then $\zeta + \xi \in D(\rho, \epsilon)$ for all $0 < \zeta < \infty$. Let \tilde{G} lie in a ball defined by

$$\{\tilde{G} | \tilde{G} \in B, \|\tilde{G}\| < b\}. \quad (4.4)$$

Then $\Sigma(\zeta + \xi)$ is analytic in ξ , for any fixed ζ in $(0, \infty)$, on condition that the denominator in (3.14) does not vanish. This can be prevented for any \tilde{G} satisfying (4.4) by restricting ρ as follows:

$$\rho^2 < 1/2b. \quad (4.5)$$

Under this condition, the integral term in (4.3) is analytic for $\xi \in D(\rho, \epsilon)$, since the integrand is so analytic, and the integral converges uniformly. Clearly $f(\xi)$, the known function (3.13), is also analytic in this domain: in fact it is analytic in the plane, cut $-\infty < \zeta < 0$, as we show in Appendix B. Thus $P(\tilde{G}; \xi)$ is analytic for $\xi \in D(\rho, \epsilon)$.

We shall now show that, if one imposes further constraints on ρ and b , P maps the ball (4.4) into itself contractively. The Banach theorem then applies, and so we assert the existence and uniqueness of a solution of the equation

$$\tilde{G}(\xi) = P(\tilde{G}; \xi) \quad (4.6)$$

in the ball (4.4).

When \tilde{G} satisfies (4.6) and ρ is restricted by (4.5), we see from (3.14) that

$$|\Sigma(\xi)| \leq \frac{3b}{4} + \frac{2b^2}{1 - 2b\rho^2} \quad (4.7)$$

for all $\xi \in D(\rho, \epsilon)$. Hence

$$|P(\tilde{G}; \xi)| \leq C_\epsilon + \left(\frac{3b}{4} + \frac{2b^2}{1 - 2b\rho^2} \right) |\xi|^{7/2} \times \int_0^\infty \frac{d\zeta}{|\zeta + \xi|^{11/2}}, \quad (4.8)$$

where C_ϵ is the bound (B4) on $|f(\xi)|$. We change the integration variable to $\omega = \zeta/|\xi|$ and define

$$D_\epsilon = \int_0^\infty \frac{d\omega}{|\omega - e^{i\epsilon}|^{11/2}}. \quad (4.9)$$

Then

$$|P(\tilde{G}; \xi)| \leq C_\epsilon + \rho \left(\frac{3b}{4} + \frac{2b^2}{1 - 2b\rho^2} \right) D_\epsilon. \quad (4.10)$$

If

$$b > C_\epsilon \quad (4.11)$$

and

$$\rho < (b - C_\epsilon) \left(\frac{3b}{4} + \frac{2b^2}{1 - 2b\rho^2} \right) D_\epsilon, \quad (4.12)$$

then the right-hand side of (4.10) is not greater than b : the ball (4.4) has been mapped into itself.

To demonstrate the contractivity, we differentiate (3.14) with respect to $\tilde{G}(\xi)$:

$$\frac{d\Sigma(\xi)}{d\tilde{G}(\xi)} = \frac{3}{4} - \frac{4\tilde{G}(\xi)[1 - \xi^{-2}\tilde{G}(\xi)]}{[1 - 2\xi^{-2}\tilde{G}(\xi)]^2}. \quad (4.13)$$

Since (4.5) implies that

$$|\xi^{-2}\tilde{G}(\xi)| < b\rho^2 < \frac{1}{2}, \quad (4.14)$$

it follows that

$$|1 - \xi^{-2}\tilde{G}(\xi)| < \frac{3}{2}. \quad (4.15)$$

Hence

$$\left| \frac{d\Sigma(\xi)}{d\tilde{G}(\xi)} \right| < \frac{3}{4} + \frac{6b}{[1 - 2b\rho^2]^2}. \quad (4.16)$$

Now let $\tilde{G}_1(\xi)$ and $\tilde{G}_2(\xi)$ be any two functions in the ball (4.4). The mean value theorem implies

$$|\Sigma_1(\xi) - \Sigma_2(\xi)| \leq \sup_{0 < \mu < 1} \left| \frac{d\Sigma(\xi)}{d\tilde{G}(\xi)} \right|_{\tilde{G} = \tilde{G}_\mu} |\tilde{G}_1(\xi) - \tilde{G}_2(\xi)|, \quad (4.17)$$

where $\tilde{G}_\mu = \mu\tilde{G}_1 + (1 - \mu)\tilde{G}_2$. Now since the ball (4.4) is quintessentially convex, \tilde{G}_μ belongs to it, and we may use the bound (4.16) for the supremum in (4.17). By the same analysis as in Eq. (4.8) *et seq.*, it follows that

$$|P(\tilde{G}_1) - P(\tilde{G}_2)| \leq \rho \left[\frac{3}{4} + 6b/(1 - 2b\rho^2)^2 \right] D_\epsilon \|\tilde{G}_1 - \tilde{G}_2\|. \quad (4.18)$$

The mapping is contractive if this Lipschitz coefficient is strictly less than unity, i.e.,

$$\rho < \left[\left(\frac{3}{4} + 6b/(1 - 2b\rho^2)^2 \right) D_\epsilon \right]^{-1}. \quad (4.19)$$

The conditions for a contraction mapping are (4.5), (4.11), (4.12), and (4.19). They are clearly consistent, since if b is not too small, then ρ can certainly be made so small that (4.5), (4.11), and (4.12) are all satisfied. Since C_ϵ and D_ϵ [Eqs. (B4) and (4.9)] tend to infinity as $\epsilon \rightarrow 0$, it follows that the permissible values of ρ tend to zero in the same limit. Accordingly, one can imagine repeating the proof for various

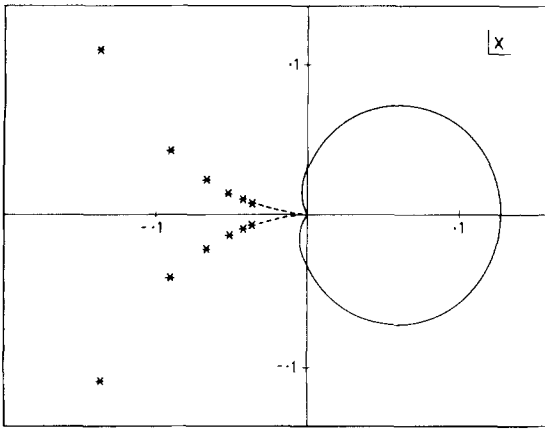


FIG. 3. Cardioid region in which the Banach theorem applies. The stars outside this region are the locations of branch points, as determined by the numerical method of Sec. 5.

values of ϵ , starting with $\epsilon = \pi/2$ and gradually allowing ϵ to decrease to zero. For ever-decreasing values of ϵ , one proves existence and uniqueness of an analytic solution in the corresponding domain $D(\rho, \epsilon)$ of Fig. 2, but with ever-increasing values of the radius $1/\rho$. This means that, in the original variable, $x = 2^{1/2}/\xi$, of Eq. (3.9), one has a proof of existence in the union of the corresponding domains, which has been plotted in Fig. 3. This curve has been obtained by computing C_ϵ and D_ϵ by a numerical integration routine, and then by finding the largest value of ρ consistent with (4.12) and (4.19), as b varies between C_ϵ and infinity. This was done in practice by rewriting the inequality (4.12) as a negative cubic form, and by determining numerically, for increasing values of b , larger than C_ϵ , the real positive root. We then determined the largest value of ρ that was consistent with (4.19).

The angle made by the locus of Fig. 3 to the real axis at the origin is zero. Equation (3.9), and thus also the original equation (3.2), has an analytic solution inside the cardioid of Fig. 3. Of course, the solution has a continuation outside the cardioid; but it will in general have singularities, the location of which we shall ascertain numerically.

V. NUMERICAL ANALYSIS

In order to set up a computer program to effect the continuation out of the contractive region of Sec. 4, we return to the differential equation (3.5), which can be rewritten in the form

$$(1 - x^2 G)(x^4 G'' + 9x^3 G' + 15x^2 G) = 3 - (2 + 3x^2)G. \quad (5.1)$$

On substituting a formal power series solution,

$$G(x) = \sum_{n=0}^{\infty} G_n x^n, \quad (5.2)$$

we find that the odd terms vanish, $G_0 = \frac{3}{2}$, $G_2 = -\frac{27}{2}$ and, for $n \geq 4$,

$$G_n = \frac{1}{2} \sum_{m=2}^{n-2} (m^2 + 4m + 3) G_{m-2} G_{n-m-2} - \frac{1}{2} (n^2 + 4n + 6) G_{n-2}. \quad (5.3)$$

This relation allows G_n to be determined recursively in terms of $G_m, m < n$. However, it is easy to show by estimates

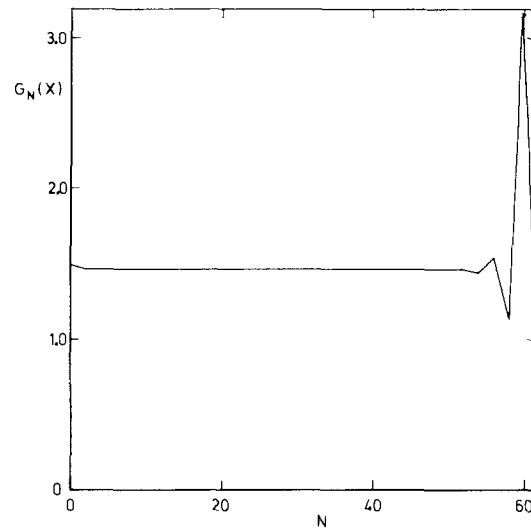


FIG. 4. The partial sums $G_N(x)$ for $x = 0.05$ and N values up to 60 [see Eq. (5.4a)].

that $(G_n)^{-1/n} \rightarrow 0$ as $n \rightarrow \infty$. This means that the series (5.2) has a zero radius of convergence, and therefore that $G(x)$ has a singularity at the origin. This is the expected branch point caused by soft-gluon effects. Although (5.2) is divergent for any x , it can be used as an asymptotic series for small x . This is important, since the Runge-Kutta routine cannot be used at the origin, due to the existence of the singularity.

In Fig. 4, we show how the partial sum,

$$G_N(x) = \sum_{n=0}^N G_n x^n, \quad (5.4a)$$

depends on N for the typical value, $x = 0.05$. Any value of N between 15 and 52 gives a stable value of $G_N(x)$ to five decimal places, although the series blows up as N is further increased (see Fig. 4). The derivative of (5.4a),

$$G_N'(x) = \sum_{n=1}^N n G_n x^{n-1}, \quad (5.4b)$$

was also computed. At $x = 0.05$, this gave four-decimal stability for N between 15 and 35. In fact, we finally used the program for the still smaller value, $x = 0.01$, since this gives G_N and G_N' , stable to more than ten significant figures, for N up to and beyond 50. We find

$$G(0.01) = 1.498\ 652\ 724, \quad (5.5a)$$

$$G'(0.01) = -0.268\ 912\ 765, \quad (5.5b)$$

and this was used as the initial point for the production runs of the Runge-Kutta routine.

As a check on the reliability of the method, we have recalculated $G(x)$ from the formula

$$G(x) = \frac{3}{2} + 2^{-1/2} x^{-7/2} \int_0^x dy y^{7/2} \sin \left[2^{1/2} \left(\frac{1}{x} - \frac{1}{y} \right) \right] \times \left(\frac{189}{8} - \frac{3}{4} G(y) + 2 \frac{G^2(y)}{1 - y^2 G(y)} \right), \quad (5.6)$$

which is obtained from (3.9) by two partial integrations. The expression (5.6) was iterated 10 times, the initial value for

$G(y)$ on the right-hand side being $3/2$. The integral was performed at 5000 equally spaced points between 0 and x ; and this then allowed the integrand to be determined at the same points for the next iterate. A four-point finite-difference formula was used to effect the quadrature. Convergence of the iteration was rapid, and agreement with the result of the asymptotic series, for small x , was good. For example, the routine yielded

$$G(0.01) = 1.498\ 658\ 026, \quad (5.7)$$

which agrees to five places of decimals with (5.5a). The value (5.5a), from the series, was stable to beyond the ninth decimal, and is the more accurate value.

In order to apply the Runge-Kutta integration procedure, the differential equation (3.5) was rewritten in the form

$$G''(x) = f(x, G, G') \\ \equiv \frac{3}{x^4} - \frac{15x^2 + 2}{x^4} G(x) - \frac{2G^2(x)}{x^2[1 - x^2G(x)]} - \frac{9}{x} G'(x). \quad (5.8)$$

Given $G(x)$ and $G'(x)$ at $x = x_n$, a standard formula⁶ estimates these functions at the next point, $x = x_{n+1}$. It was found that satisfactory accuracy was obtained (to at least 7 significant figures) if 5000 steps were taken between points of interest. Close to the origin, large cancellations occur, due to the denominators in (5.8); and so the first few steps must be cautious. In practice, we found that 5000 steps from $x = 0.01$, as given by (5.5), to $x = 0.1$, followed by 5000 to $x = 1.0$, preserved good accuracy. Complex excursions could then be made without danger from rounding errors; and in particular, a subroutine called LOOP continued $G(x)$ around a rectangle. The initial and final values of $G(x)$ and $G'(x)$ were the same, typically to an accuracy of about 10^{-10} , unless a branch point was enclosed in the rectangle in question, in which case the mismatch was of order 1 or more. Thus it was easy to distinguish between cases in which a branch point was, or was not, enclosed within a given rectangle; and an automatic routine, in which a rectangle containing a branch point was systematically halved and subjected to the test, enabled branch points to be located to good accuracy. The method had earlier been developed⁴ to handle a similar problem in QED.

As a check on the reliability of this program, we repeated the procedure with different real and complex values of x in the asymptotic series (5.4), which defined the starting point for the Runge-Kutta routine. For stability, the starting point could not be too far from the origin, for then the asymptotic series was useless, nor too close to the origin, for then the cancellations between the various terms in (5.8) were too fierce. Fortunately, an intermediate region exists, in which stable results were obtained. An oval region around the origin, defined roughly by ± 0.03 in the real, and ± 0.06 in the imaginary direction, is inaccessible to the Runge-Kutta routine, because of large cancellations.

In Table I, and in Fig. 3, we show the locations of six branch points that we have discovered in the second quadrant. We expect that an infinite number of them exists, accumulating at the origin; but we were unable to approach the origin more closely, to pick up more branch points, because of numerical instabilities. Since the cardioid domain, in

TABLE I. Location of the first six branch points of $G(x)$.

n	Re x	Im x
1	-0.1355	0.1110
2	-0.0903	0.0428
3	-0.0658	0.0226
4	-0.0514	0.0141
5	-0.0421	0.0097
6	-0.0356	0.0071

which the existence theorem of Sec. 4 works, has a horizontal tangent at the origin it follows that, if an infinite number of branch points do accumulate at the origin, they must approach it along a curve that is asymptotic to the negative real axis (see Fig. 3).

It is interesting to compare the above results with those of the averaged case that is treated in Appendix C. There, it is proved that an infinite number of branch points also accumulate at the origin, but along the lines $\arg x = \pm 3\pi/4$. Apparently the averaging procedure has aggravated the situation, for first-sheet complex singularities should not be present in a healthy theory. They are inconsistent with causality, and they spoil the Wick rotation that relates the Lorentz and Euclidean equations. Their occurrence must be regarded as a sickness of the Mandelstam approximation; and it is of importance to improve the treatment of the Dyson-Schwinger equation, in order to cure the disease. A hopeful sign is the fact that the equation of Sec. 3 appears to be less sick than that of Appendix C. Perhaps a relatively minor modification of the Mandelstam approximation will cause the complex branch points to be pushed on to a secondary Riemann sheet, where they can be tolerated.

ACKNOWLEDGMENTS

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APPENDIX A

In this appendix, we shall prove Eq. (1.5). In polar coordinates, the left-hand side of (1.5) can be written

$$-\frac{1}{4\pi} \lim_{\lambda \rightarrow \infty} \int_0^\lambda r^3 dr \int d^3k \Omega(\mathbf{k}) \int_{-1}^1 \cos\theta \int_0^{2\pi} d\phi e^{i|\mathbf{k}|r \cos\theta} \\ = \lim_{\lambda \rightarrow \infty} \int d^3k \Omega(\mathbf{k}) \left(\frac{2}{k^4} \right. \\ \left. - \frac{2 \cos k\lambda}{k^4} - \frac{2\lambda \sin k\lambda}{k^3} + \frac{\lambda^2 \cos k\lambda}{k^2} \right), \quad (A1)$$

where $k = |\mathbf{k}|$, and where the r , θ , and ϕ integrals have been performed inside the k integral, this being permissible for finite λ . To show that the last three terms vanish in the limit $\lambda \rightarrow \infty$, we use polar coordinates in the k variable and perform integrations by parts: three for the fourth term, two for the third term, and one for the second term. For example,

$$\int d\Omega_k \int_0^\infty dk \Omega(\mathbf{k}) \frac{\cos k\lambda}{k^2} \\ = \frac{1}{\lambda} \int d\Omega_k \int_0^\infty dk \sin k\lambda \frac{\partial}{\partial k} \left(\frac{\Omega(\mathbf{k})}{k^2} \right). \quad (A2)$$

The vanishing of $\Omega(\mathbf{k})$ and its first three derivatives at the origin is essential.

APPENDIX B

In this appendix we shall study the function $f(\xi)$, defined in Eq. (3.13). We shall show that, for any ϵ lying in $(0, \pi)$, $|f(\xi)|$ is bounded by a constant C_ϵ when $|\arg \xi| \leq \pi - \epsilon$. However, as $\epsilon \rightarrow 0$, so $C_\epsilon \rightarrow \infty$. Thus $|f(\xi)|$ is bounded in $D(\rho, \epsilon)$, for any nonzero ϵ .

Integrating (3.13) by parts, we find

$$f(\xi) = \frac{3}{2} - \frac{21}{4} \xi^{7/2} \int_0^\infty \frac{d\zeta}{(\zeta + \xi)^{9/2}} \cos \zeta. \quad (\text{B1})$$

We change the integration variable to

$$\omega = \zeta / |\xi|, \quad (\text{B2})$$

so that

$$f(\xi) = \frac{3}{2} - \frac{21}{4} e^{7i\theta/2} \int_0^\infty \frac{d\omega}{(\omega + e^{i\theta})^{9/2}} \cos(\omega|\xi|), \quad (\text{B3})$$

where $\theta = \arg \xi$. Hence, if $|\theta| \leq \pi - \epsilon$,

$$|f(\xi)| \leq \frac{3}{2} + \frac{21}{4} \int_0^\infty \frac{d\omega}{|\omega - e^{i\epsilon}|^{9/2}} \equiv C_\epsilon, \quad (\text{B4})$$

which is finite if $\epsilon > 0$. It may easily be shown that $C_\pi = 3$ and $C_{\pi/2} \leq \frac{3}{2} + 21\pi/16 \approx 5.6$, i.e., $|f(\xi)|$ is bounded by 3 on the positive real axis and by 5.6 in the right half-plane.

In order to examine the behavior of $f(\xi)$ on the negative real axis, we need to modify the expression (3.13), since this diverges if ξ is real and negative, and it then no longer represents the continuation of $f(\xi)$. Integrate twice by parts the other way:

$$f(\xi) = -\frac{2}{5} \xi^2 + \frac{2}{5} \xi^{7/2} \int_0^\infty \frac{d\zeta}{(\zeta + \xi)^{3/2}} \sin \zeta. \quad (\text{B5})$$

This still fails to exist for ξ real and negative; but we dare not integrate once more by parts, for this would cause the integral to diverge at infinity. Instead add and subtract $\sin \zeta$ to $\sin \zeta$ in (B5). The result can be written

$$f(\xi) = -\frac{2}{5} \xi^2 - \frac{4}{5} \xi^3 \sin \xi + \frac{2}{5} \xi^3 g(\xi), \quad (\text{B6})$$

where

$$g(\xi) = \xi^{1/2} \int_0^\infty \frac{d\zeta}{(\zeta + \xi)^{3/2}} [\sin \zeta + \sin \xi]. \quad (\text{B7})$$

The integral (B7) exists for ξ real and negative, and it effects the analytic continuation of $f(\xi)$. Substitute $\eta = \zeta + \xi$, for the moment keeping ξ real and positive:

$$g(\xi) = \xi^{1/2} \int_\xi^\infty \frac{d\eta}{\eta^{3/2}} [\sin \xi (1 - \cos \eta) + \cos \xi \sin \eta]. \quad (\text{B8})$$

Now since

$$\int_0^\infty \frac{d\eta}{\eta^{3/2}} (1 - \cos \eta) = (2\pi)^{1/2} = \int_0^\infty \frac{d\eta}{\eta^{3/2}} \sin \eta, \quad (\text{B9})$$

we may write

$$g(\xi) = (2\pi)^{1/2} \xi^{1/2} [\sin \xi + \cos \xi] - g_1(\xi) \sin \xi - g_2(\xi) \cos \xi, \quad (\text{B10})$$

where

$$g_1(\xi) = \xi^{1/2} \int_0^\xi \frac{d\eta}{\eta^{3/2}} [1 - \cos \eta], \quad (\text{B11a})$$

$$g_2(\xi) = \xi^{1/2} \int_0^\xi \frac{d\eta}{\eta^{3/2}} \sin \eta, \quad (\text{B11b})$$

which are closely related to the Fresnel integrals. Scale η by setting $\omega = \eta/\xi$:

$$g_1(\xi) = \int_0^1 \frac{d\omega}{\omega^{3/2}} [1 - \cos(\omega\xi)], \quad (\text{B12a})$$

$$g_2(\xi) = \int_0^1 \frac{d\omega}{\omega^{3/2}} \sin(\omega\xi). \quad (\text{B12b})$$

Now we continue ξ away from the real, positive axis, and discover that g_1 and g_2 are entire functions! This means that the only contribution to $\text{Im} f(\xi)$ for ξ real and negative comes from the $\xi^{1/2}$ term in (B10). Thus $f(\xi)$ has a cut $-\infty < \xi < 0$, and on it

$$\text{Im} f(\xi \pm i0) = \mp \frac{3}{2} (2\pi)^{1/2} |\xi|^{7/2} [\sin \xi + \cos \xi]. \quad (\text{B13})$$

For ξ real and negative, it is easy to prove from (B12) that

$$|g_1(\xi)| \leq 6\xi^{1/2}, \quad (\text{B14a})$$

$$|g_2(\xi)| \leq 4\xi^{1/2}. \quad (\text{B14b})$$

Combining these results together, we conclude that, for ξ real and negative,

$$|\text{Re} f(\xi)| \leq \frac{3}{2} |\xi|^2 + \frac{3}{2} |\xi|^3 + 4|\xi|^{7/2}, \quad (\text{B15a})$$

$$|\text{Im} f(\xi)| \leq \frac{4}{3} \pi^{1/2} |\xi|^{7/2}. \quad (\text{B15b})$$

Hence there exists a constant C such that

$$|f(\xi)| \leq C |\xi|^{7/2}, \quad (\text{B16})$$

when ξ is real and negative.

In (B4) we showed that $|f(\xi)|$ is bounded by a constant when $|\arg \xi| \leq \pi - \epsilon$; and in (B16) we have a power bound on the negative real axis. In fact we can use the Phragmén–Lindelöf theorem to extend the bound (B16) to the wedge $\pi - \epsilon < |\arg \xi| \leq \pi$. In order to do this, we need to be sure that $|f(\xi)|$ does not behave too badly as $|\xi| \rightarrow \infty$ in this wedge. At first sight the trigonometric functions in (B6) and (B12) look sick for large, complex ξ , and indeed they are, but they must cancel, because if we go back to (B5) and substitute

$$\frac{1}{\zeta + \xi} = \frac{1}{\zeta + 1} + \frac{1 - \xi}{(\zeta + \xi)(\zeta + 1)}, \quad (\text{B17})$$

we find

$$f(\xi) = -\frac{2}{5} \xi^2 + \frac{2}{5} \xi^{7/2} \int_0^\infty \frac{d\zeta}{(\zeta + \xi)^{1/2}} \times \left[\frac{\sin \zeta}{\zeta + 1} + 2(1 - \xi) \left(\frac{\cos \zeta}{\zeta + 1} - \frac{\sin \zeta}{(\zeta + 1)^2} \right) \right], \quad (\text{B18})$$

after an integration by parts. This integral exists in the whole cut ξ plane, and it is easy to obtain the crude upper bound

$$|f(\xi)| \leq C |\xi|^5, \quad (\text{B19})$$

where C is some constant. This is a poor bound, and it is only needed to make sure that the Phragmén–Lindelöf theorem

really applies in the wedge $\pi - \epsilon < |\arg \xi| < \pi$, so that actually (B19) can be tightened to (B16) in the wedge.

The above fulsome treatment of $f(\xi)$ was motivated by the fact that this function may be regarded as the zeroth approximation to $\tilde{G}(\xi)$, and thus to the Mandelstam gluon propagator itself.

APPENDIX C

Here we shall approximate the integral equation (3.2) still further by replacing the kernel $(1 - y^2/x^2)$ by its average value, namely $\frac{2}{3}$. By a simultaneous rescaling of x, y , and G , this multiplicative factor can be removed [cf. Eq. (2.13) *et seq.*]. The result is

$$G(x) = \frac{1}{x^2} - \frac{1}{x^3} \int_0^x dy \frac{G(y)}{1 - y^2 G(y)}, \quad (C1)$$

and it is interesting to study this averaged equation, since it turns out to be explicitly soluble. A similar averaging in the QED case⁴ was shown to have minor quantitative, but not qualitative, effects on the solution.

From (C1) we deduce the differential equation

$$\frac{d}{dx} [x^3 G(x)] = 1 - \frac{G(x)}{1 - x^2 G(x)}, \quad (C2)$$

which is greatly simplified if we substitute

$$\psi(x) = x - 1/x - x^3 G(x), \quad (C3)$$

for then

$$\frac{dx}{d\psi} = 1 + x\psi, \quad (C4)$$

which is a linear equation for x as a function of ψ , with the solution

$$x(\psi) = \int_0^\infty d\omega \exp[\omega\psi - \frac{1}{2}\omega^2]. \quad (C5)$$

This is the relevant solution of (C4), since it corresponds to the boundary condition $x \rightarrow 0$ through positive values as $\psi \rightarrow -\infty$, which is consistent with (C3) and (C1). Evidently x is an entire function of ψ , and hence the only singularities of ψ as a function of x are the points where

$$\frac{dx}{d\psi} = 0. \quad (C6)$$

These points can be evaluated numerically, and the work has already been done, for

$$x(\psi) = (\pi/2)^{1/2} \omega(-i2^{-1/2}\psi), \quad (C7)$$

where

$$w(z) = e^{-z^2} \left(1 + 2\pi^{-1/2} i \int_0^z dt e^{t^2} \right), \quad (C8)$$

and its derivative have been tabulated in the literature.⁷ Using these results, we find that $\psi(x)$, and therefore $G(x)$, have an infinite number of first-sheet branch points that accumulate at the origin along the asymptotes $\arg x = \pm 3\pi/4$ (see Fig. 5).

The differential equation (C2), together with an asymptotic series like that of Sec. 5, have been used for testing the Runge-Kutta program: the first few of the branch points

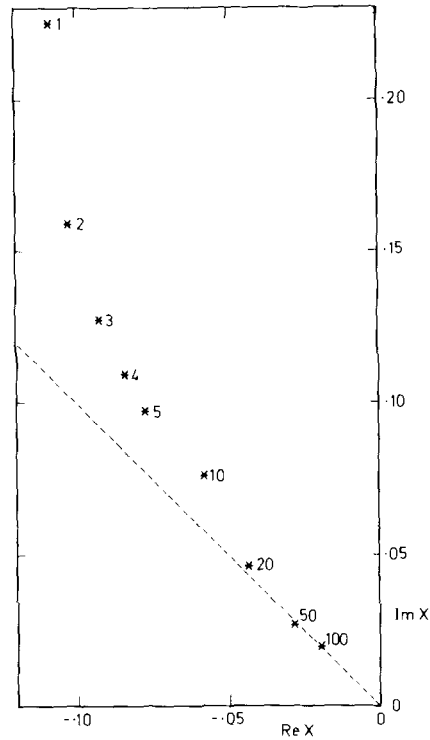


FIG. 5. Some of the branch points in the upper half-plane for the averaged equation (C1), which have been numbered, starting from the one most distant from the origin. The branch points accumulate at the origin along the line $\arg x = 3\pi/4$.

shown in Fig. 5 are picked up without difficulty.⁸

Although an existence proof along the lines of that given in Sec. 4 is strictly superfluous, since (C5) gives a representation of the solution, albeit in inverse form, it is nevertheless instructive. Without giving all the details,⁹ we quote the analogs of Eqs. (3.12)–(3.14):

$$\tilde{G}(\xi) = f(\xi) - \xi^3 \int_0^\infty \frac{d\xi'}{(\xi + \xi')^2} e^{-\xi\xi' - (1/2)\xi'^2} \Sigma(\xi + \xi'), \quad (C9)$$

where

$$f(\xi) = \xi^3 \int_0^\infty \frac{d\xi'}{(\xi + \xi')^2} e^{-\xi\xi' - (1/2)\xi'^2}, \quad (C10)$$

and

$$\Sigma(\xi) = \frac{\tilde{G}^2(\xi)}{\xi^2 - \tilde{G}(\xi)}, \quad (C11)$$

where $\xi = 1/x$, etc. An existence proof can be completed for $|x|$ small (i.e., $|\xi|$ large) only if $|\arg x| < 3\pi/4 - \epsilon$, instead of $|\arg x| < \pi - \epsilon$, as was possible in Sec. 4. The reason is that $f(\xi)$ blows up as $|\xi| \rightarrow \infty$ when $\pi > |\arg \xi| > 3\pi/4$. In fact, one can show easily that $f(\xi)$ is bounded if $\text{Re} \xi \geq 0$, and also that

$$f(-\xi) = f(\xi) - (\pi/2)^{1/2} \xi^3 e^{(1/2)\xi^2}, \quad (C12)$$

from which the results follow. The failure of the proof for $|\arg \xi| > 3\pi/4 - \epsilon$ is of course due to the existence of the branch points of Fig. 5.

It is interesting that Eq. (3.2) is "softer" than is Eq. (C1), in the sense that the accumulation of branch points has been pushed in the former, as compared with the latter equation, from $|\arg \xi| = 3\pi/4$ to $|\arg \xi| = \pi$.

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