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Control-oriented modeling and passivity analysis of thermal dynamics in a multi-producer district heating system ⋆

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Abstract: Climate change and geopolitics have led to the conception of plans for reducing greenhouse gas emissions and improving the sustainability of existing fossil-based energy systems. In this respect, district heating has been identified as an indispensable player for its potential to integrate seamlessly environmentally-friendly heat sources. To improve the efficiency of these district heating systems, optimal operation schemes can be devised and enforced through control systems. To this end, we present a control-oriented nonlinear ODE-based model of temperature dynamics in a multi-producer district heating system. The model features a modular design and comprises the thermal dynamics of heat exchangers of producers and consumers interconnected by a distribution network of meshed topology. Then, we establish passivity properties and zero-state detectability for the modeled temperature dynamics that could be exploited for controller design and solving constrained optimization problems.

Keywords: District heating, thermal dynamics, Krasovskii passivity, shifted passivity, zero-state detectability.

1. INTRODUCTION

Climate change and recent geopolitical implications are impelling many countries towards improving energy sustainability and transitioning from a fossil-based energy system towards a more sustainable and renewables-based energy system. District heating systems (DHSs) have the potential to include seamlessly environmentally friendly energy sources, thereby reducing greenhouse emissions and improving the sustainability of existing fossil-based energy systems. DHSs distribute heat from heat production units towards clusters of consumers through a network of underground insulated pipes; see Lund et al. (2014) for more details on DHSs. Since a considerable proportion of the total energy use is for heating (and cooling), it is then crucial to develop control-oriented models describing the thermal dynamics of DHSs that can be used to optimize—and thus improve the efficiency—of these systems.

Passivity is a powerful theoretical tool for analyzing multi-domain complex dynamical systems. It is strongly related to Lyapunov stability and provides valuable insights for control design. Among the various passivity concepts (e.g., incremental passivity and differential passivity), in this paper, we focus on shifted passivity and the recently developed concept of Krasovskii passivity, which have proven to be instrumental in designing stabilizing controllers and solving constrained optimization problems; see Kawano et al. (2020) and Cucuzzella et al. (2021).

Modeling of thermal dynamics of DHSs with a *single heat producer* has been discussed in Scholten et al. (2015), Krug et al. (2021), Hauschild et al. (2020), and Rein et al. (2021). Modeling of DHSs with multiple heat producers has been studied in Vesterlund et al. (2017), Alisic et al. (2019), and Machado et al. (2022a). In Vesterlund et al. (2017), *static* thermo-hydraulic modeling is considered for solving operational optimization problems. In Alisic et al. (2019), room and reservoir (storage) temperature dynamics of end-users (viewed as prosumers) is considered, and in Machado et al. (2022a), dynamic models of simplified heat exchangers (HXs) and distribution pipes (without losses) are considered with simplified consumer dynamics. In the context of heating networks, passivity analysis has been discussed in Mukherjee et al. (2012), Dong et al. (2019), and Machado et al. (2022a). In Mukherjee et al. (2012), a *linear* model describing temperature dynamics of a multi-zone building is shown to be passive via a storage function quadratic in the rooms' temperature. In Dong et al. (2019), a model of a network of HXs is shown to be shifted passive via a storage function based on the notion of ectropy. In Machado et al. (2022a), shifted passivity properties of an ODE-based thermo-hydraulic model are investigated.

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Contributions. In contrast to the works mentioned above, the contributions of this study are as follows. (i) We present a modular model of DHSs with multiple heat production units. (ii) We provide complete and scalable models of HXs comprising primary and secondary sides. (iii) We consider a graph-theoretic approach for modeling the temperature dynamics of the distribution network (DN) with heat losses, making it scalable. (iv) The consumer module includes the dynamics of the end-user, i.e., radiators and the rooms, allowing control of indoor temperature and quantifying comfort of the end-users; see Grassi et al. (2020). (v) We study shifted and Krasovskii passivity properties, and zero-state detectability of the overall thermal dynamics that seem beneficial for controller design and solving constrained optimization problems.

Notation. The set of real numbers is denoted by $\mathbb{R}$. Let $\mathbf{0}$ denote a null matrix, where the dimension of the matrix is understood from context. Let $\mathbf{1}$ denote a vector of an appropriate dimension whose all elements are 1. For any vector $x = [x_1, x_2, \ldots, x_n]^\top \in \mathbb{R}^n$, $\mathbf{sign}(x) = [\text{sign}(x_1), \text{sign}(x_2), \ldots, \text{sign}(x_n)]^\top$ with $\text{sign}(0) = 0$, and $\text{col}\,(x_i)|_{i=1}^{n} = [x_1, x_2, \ldots, x_n]^\top$. For any vector $x \in \mathbb{R}^n$, the operator diag is used to construct a diagonal matrix $\text{diag}(x)$ with the elements of vector x on its main diagonal. The same operator is used for representing block diagonal matrices. We also enlist relevant abbreviations and notation in Tables 1 and 2, respectively.

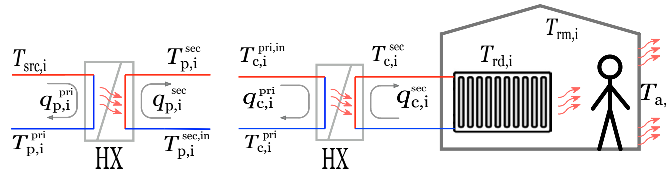
Table 1. List of abbreviations

Abbreviation	Definition
DHS	district heating system
DN	distribution network
HX	heat exchanger
pri	identifier for the primary side of a HX
sec	identifier for the secondary side of a HX
src	identifier for a heat source
p	identifier for a producer
c	identifier for a consumer
rd	identifier for a radiator
rm	identifier for a room
a	identifier for the ambience
s	identifier for the DN's supply layer
r	identifier for the DN's return layer

2. MODULAR MODELING OF A DHS

We consider a DHS with n_p distributed heat producers and n_c consumers interconnected through a DN that has a supply (hot) layer and a return (cold) layer. The specific composition of producers and consumers is shown in Fig. 1. Note that each producer can continuously drain water from the DN's return layer, heats it through a HX and injects the heated stream into the DN's supply layer. A converse operation follows for consumers. Additional modeling assumptions are the following (see e.g., Scholten et al., 2015; Machado et al., 2022a):

Assumption 1. (i) The density $\rho > 0$ and specific heat $c_{\text{s.h.}} > 0$ of water are spatially uniform and constant in time (for ease of notation, we assume $\rho = c_{\text{s.h.}} = 1$). (ii) All pipes are cylindrical. (iii) The flow through any pipe is (spatially) one-dimensional. (iv) Each device (pipe, junction) is completely filled with water all the time. (v) The internal energy of any water stream portion depends linearly on its temperature.



(a) (b)

Fig. 1. Topologies of (a) producers and (b) consumers

2.1 Temperature dynamics of the producers

The heat balance at the primary and secondary side of the HX of each producer can be, respectively, written as follows (Scholten et al., 2015; Varga et al., 1995):

$$V^{\text{pri}}_{\text{p},i}\dot{T}^{\text{pri}}_{\text{p},i} = q^{\text{pri}}_{\text{p},i}\left(T_{\text{src},i} - T^{\text{pri}}_{\text{p},i}\right) + \lambda_{\text{p},i}(T^{\text{sec}}_{\text{p},i} - T^{\text{pri}}_{\text{p},i}) \tag{1a}$$

$$V^{\text{sec}}_{\text{p},i}\dot{T}^{\text{sec}}_{\text{p},i} = q^{\text{sec}}_{\text{p},i}\left(T^{\text{sec,in}}_{\text{p},i} - T^{\text{sec}}_{\text{p},i}\right) + \lambda_{\text{p},i}(T^{\text{pri}}_{\text{p},i} - T^{\text{sec}}_{\text{p},i}), \tag{1b}$$

where $q^{\text{pri}}_{\text{p},i}$ and $q^{\text{sec}}_{\text{p},i}$ are control variables, and $T_{\text{src},i}$ is the temperature of the heat source of ith producer which is assumed to be constant. Later, when we introduce the temperature dynamics of the return layer of the DN, $T^{\text{sec,in}}_{\text{p},i}$ will be related to the temperature of a given junction in the return layer of the DN.

Table 2. List of Notation

Notation	Definition
$\mathcal{G}_\text{s}$, $\mathcal{N}_\text{s}$, $\mathcal{E}_\text{s}$	Graph, nodes (junctions), and edges (pipes) of the supply layer
$\mathcal{G}_\text{r}$, $\mathcal{N}_\text{r}$, $\mathcal{E}_\text{r}$	Graph, nodes (junctions), and edges (pipes) of the return layer
$\mathcal{B}_\text{s}/\mathcal{B}_\text{r}$	node-edge incidence matrix of $\mathcal{G}_\text{s}/\mathcal{G}_\text{r}$
$\mathcal{T}_\text{s}/\mathcal{T}_\text{r}$	positive part of $\mathcal{B}_\text{s}/\mathcal{B}_\text{r}$
$\mathcal{S}_\text{s}/\mathcal{S}_\text{r}$	negative part of $\mathcal{B}_\text{s}/\mathcal{B}_\text{r}$
$q^{\text{pri}}_{\text{p},i}/q^{\text{sec}}_{\text{p},i}$	flow rate through the primary/secondary side of the ith producer's HX, m^3/s
$q^{\text{pri}}_{\text{c},i}/q^{\text{sec}}_{\text{c},i}$	flow rate through the primary/secondary side of the ith consumer's HX, m^3/s
$q_{\text{s},i}/q_{\text{r},i}$	flow rate through $i \in \mathcal{E}_\text{s}/i \in \mathcal{E}_\text{r}$, m^3/s
$V^{\text{pri}}_{\text{p},i}/V^{\text{sec}}_{\text{p},i}$	effective volume of the primary/secondary side of the ith producer's HX, m^3
$V^{\text{pri}}_{\text{c},i}/V^{\text{sec}}_{\text{c},i}$	effective volume of the primary/secondary side of the ith consumer's HX, m^3
$V_{\text{s},i}/V_{\text{r},i}$	effective volume of $i \in \mathcal{G}_\text{s}/i \in \mathcal{G}_\text{r}$, m^3
$V_{\text{rd},i}$	effective volume of the ith radiator, m^3
$V_{\text{rm},i}$	effective volume of the ith room, m^3
$T^{\text{pri}}_{\text{p},i}/T^{\text{sec}}_{\text{p},i}$	average temp. of the primary/secondary side of the ith producer's HX, ° C
$T^{\text{pri}}_{\text{c},i}/T^{\text{sec}}_{\text{c},i}$	average temp. of the primary/secondary side of the ith consumer's HX, $°C$
$T_{\text{s},i}/T_{\text{r},i}$	average temperature of $i \in \mathcal{G}_\text{s}/i \in \mathcal{G}_\text{r}$, $°C$
T_a	average ambient temperature, $°C$
$(\cdot)^{\text{in}}_i$	quantity associated to an inlet
$(\cdot)^{\text{out}}_i$	quantity associated to an outlet
$\lambda_{\text{p},i}/\lambda_{\text{c},i}$	heat conduction coefficient of ith producer's/consumer's HX, W/°C
$\lambda_{\text{rd}-\text{rm},i}$	heat conduction coefficient between the ith radiator and the ith room, W/°C
$\lambda_{\text{rm}-\text{a},i}$	heat conduction coefficient between the ith room and the ambience, W/°C
$\lambda_{\text{s}-\text{a},i}/\ \lambda_{\text{r}-\text{a},i}$	heat conduction coefficient between $i \in \mathcal{E}_s/i \in \mathcal{E}_r$ and the ambience, W/°C

2.2 Temperature dynamics of the consumers

Analogously to producers, the heat balance at the primary and secondary side of HX of each consumer, radiator, and room is, respectively, given by

$$V_{\mathrm{c},i}^{\mathrm{pri}}\dot{T}_{\mathrm{c},i}^{\mathrm{pri}} = q_{\mathrm{c},i}^{\mathrm{pri}}\left(T_{\mathrm{c},i}^{\mathrm{pri,in}} - T_{\mathrm{c},i}^{\mathrm{pri}}\right) + \lambda_{\mathrm{c},i}(T_{\mathrm{c},i}^{\mathrm{sec}} - T_{\mathrm{c},i}^{\mathrm{pri}}) \tag{2a}$$

$$V_{\mathrm{c},i}^{\mathrm{sec}}\dot{T}_{\mathrm{c},i}^{\mathrm{sec}} = q_{\mathrm{c},i}^{\mathrm{sec}}\left(T_{\mathrm{rd},i} - T_{\mathrm{c},i}^{\mathrm{sec}}\right) + \lambda_{\mathrm{c},i}(T_{\mathrm{c},i}^{\mathrm{pri}} - T_{\mathrm{c},i}^{\mathrm{sec}}) \tag{2b}$$

$$V_{\mathrm{rd},i}\dot{T}_{\mathrm{rd},i} = q_{\mathrm{c},i}^{\mathrm{sec}}\left(T_{\mathrm{c,i}}^{\mathrm{sec}} - T_{\mathrm{rd},i}\right) + \lambda_{\mathrm{rd-rm},i}(T_{\mathrm{rm},i} - T_{\mathrm{rd},i}) \tag{2c}$$

$$V_{\mathrm{rm},i}\dot{T}_{\mathrm{rm},i} = \lambda_{\mathrm{rd-rm},i}\left(T_{\mathrm{rd},i} - T_{\mathrm{rm},i}\right) + \lambda_{\mathrm{rm-a},i}(T_{\mathrm{a}} - T_{\mathrm{rm},i}), \tag{2d}$$

where the flow rates $q_{\mathrm{c},i}^{\mathrm{pri}}$ and $q_{\mathrm{c},i}^{\mathrm{sec}}$ are control variables, while $T_{\mathrm{c},i}^{\mathrm{pri,in}}$ is an external input. Later we will relate $T_{\mathrm{c},i}^{\mathrm{pri,in}}$ with the temperature of a certain junction in the supply layer of the DN.

2.3 Temperature dynamics of the DN

The supply and return layers of the DN are represented as connected graphs with no self-loops (see, e.g., De Persis and Kallesøe, 2011; Wang et al., 2017; Hauschild et al., 2020; Machado et al., 2022a). The supply layer is represented by the graph $\mathcal{G}_{\mathrm{s}} = (\mathcal{N}_{\mathrm{s}}, \mathcal{E}_{\mathrm{s}})$, where $\mathcal{E}_{\mathrm{s}}$ is the set of edges representing the distribution pipes and $\mathcal{N}_{\mathrm{s}}$ is the set of nodes representing the pipe junctions. Similarly, the return layer of the DN is represented by the graph $\mathcal{G}_{\mathrm{r}} = (\mathcal{N}_{\mathrm{r}}, \mathcal{E}_{\mathrm{r}})$. We assume that $\mathcal{G}_{\mathrm{s}}$ and $\mathcal{G}_{\mathrm{r}}$ are isomorphic, and the bijection between $\mathcal{N}_{\mathrm{s}}$ and $\mathcal{N}_{\mathrm{r}}$ is referred to as γ_{dn}. Note that this implies that $|\mathcal{N}_{\mathrm{s}}| = |\mathcal{N}_{\mathrm{r}}|$. Without loss of generality, we assume that any two pipes $(i, j) \in \mathcal{E}_{\mathrm{s}}$ and $(\gamma_{\mathrm{dn}}(i), \gamma_{\mathrm{dn}}(j)) \in \mathcal{E}_{\mathrm{r}}$ have the same length and diameter. We identify a unique pair $(k, \gamma_{\mathrm{dn}}(k)) \in \mathcal{N}_{\mathrm{s}} \times \mathcal{N}_{\mathrm{r}}$ with the ith producer such that there is a stream with rate $q_{\mathrm{p},i}^{\mathrm{sec}}$ from the node $\gamma_{\mathrm{dn}}(k) \in \mathcal{N}_{\mathrm{r}}$ towards producer's HX reaching the node $k \in \mathcal{N}_{\mathrm{s}}$. Analogously, to the ith consumer, we associate a unique pair $(k, \gamma_{\mathrm{dn}}(k)) \in \mathcal{N}_{\mathrm{s}} \times \mathcal{N}_{\mathrm{r}}$ such that the there is a stream with rate $q_{\mathrm{c},i}^{\mathrm{pri}}$ from the node k towards the consumer's HX and eventually reaching the node $\gamma_{\mathrm{dn}}(k)$.

We introduce further notation considering the graph $\mathcal{G}_{\mathrm{s}}$ as reference. We fix an arbitrary reference orientation to every edge of $\mathcal{G}_{\mathrm{s}}$. Then, for any $i \in \mathcal{E}_{\mathrm{s}}$ with end nodes $j, k \in \mathcal{N}_{\mathrm{s}}$, $j \neq k$, we say that j is the head and k is the tail of i, or viceversa, that j is the tail and k is the head of i. Moreover, we introduce the functions $\mathcal{N}_{\mathrm{s}}^-, \mathcal{N}_{\mathrm{s}}^+ : \mathcal{E}_{\mathrm{s}} \to \mathcal{N}_{\mathrm{s}}$ and $\mathcal{E}_{\mathrm{s}}^-, \mathcal{E}_{\mathrm{s}}^+ : \mathcal{N}_{\mathrm{s}} \to \mathcal{E}_{\mathrm{s}}$ as follows. For any $i \in \mathcal{E}_{\mathrm{s}}$, $\mathcal{N}_{\mathrm{s}}^-(i)$ and $\mathcal{N}_{\mathrm{s}}^+(i)$, respectively, denote the tail and head of i; also, for any $j \in \mathcal{N}_{\mathrm{s}}$, $\mathcal{E}_{\mathrm{s}}^-(j)$ and $\mathcal{E}_{\mathrm{s}}^+(j)$ denote sets of edges with j as tail node and j as head node, respectively. For ease of exposition, we assume that the reference orientation of any edge $i \in \mathcal{E}_{\mathrm{s}}$ matches the direction of the stream through it. That is, if $j, k \in \mathcal{N}_{\mathrm{s}}$, $j \neq k$, are the tail and head of any $i \in \mathcal{E}_{\mathrm{s}}$, respectively, then the stream through i is assumed to flow from j to k and we consider that $q_{\mathrm{s},i} \geq 0$.[1]

[1] Since flow reversals may occur, or the flow through an edge may simply not match the edge's reference orientation, q_{s}-dependent functions analogous to $\mathcal{N}_{\mathrm{s}}^-$, $\mathcal{N}_{\mathrm{s}}^+$, $\mathcal{E}_{\mathrm{s}}^-$ and $\mathcal{E}_{\mathrm{s}}^+$ can be defined to identify the source and target node of the stream through any edge (see, e.g., Hauschild et al. (2020).)

Based on Machado et al. (2022a), we can now write the temperature dynamics of the supply layer of the DN as

$$\begin{aligned} V_{\mathrm{s},i}\dot{T}_{\mathrm{s},i} &= q_{\mathrm{s},i}\left(T_{\mathrm{s},j} - T_{\mathrm{s},i}\right)|_{j=\mathcal{N}_{\mathrm{s}}^-(i)} + \lambda_{\mathrm{s-a},i}(T_{\mathrm{a}} - T_{\mathrm{s},i}), \\ V_{\mathrm{s},j}\dot{T}_{\mathrm{s},j} &= \sum_{i\in\mathcal{E}_{\mathrm{s}}^+(j)} q_{\mathrm{s},i}\left(T_{\mathrm{s},i} - T_{\mathrm{s},j}\right) + \sum_{i=1}^{n_{\mathrm{p}}} \alpha_{i,j} q_{\mathrm{p},i}^{\mathrm{sec}}\left(T_{\mathrm{p},i}^{\mathrm{sec}} - T_{\mathrm{s},j}\right), \end{aligned} \tag{3}$$

for all $(i, j) \in \mathcal{E}_s \times \mathcal{N}_s$. Note that the second term in first equation of (3) incorporates the heat losses which were neglected in Machado et al. (2022a).

Similarly, the temperature dynamics of the return layer of the DN can be written as

$$\begin{aligned} V_{\mathrm{r},i}\dot{T}_{\mathrm{r},i} &= q_{\mathrm{r},i}\left(T_{\mathrm{r},j} - T_{\mathrm{r},i}\right)|_{j=\mathcal{N}_{\mathrm{r}}^-(i)} + \lambda_{\mathrm{r-a},i}(T_{\mathrm{a}} - T_{\mathrm{r},i}), \\ V_{\mathrm{r},j}\dot{T}_{\mathrm{r},j} &= \sum_{i\in\mathcal{E}_{\mathrm{r}}^+(j)} q_{\mathrm{r},i}\left(T_{\mathrm{r},i} - T_{\mathrm{r},j}\right) + \sum_{i=1}^{n_{\mathrm{c}}} \beta_{i,j} q_{\mathrm{c},i}^{\mathrm{pri}}\left(T_{\mathrm{c},i}^{\mathrm{pri}} - T_{\mathrm{r},j}\right), \end{aligned} \tag{4}$$

for all $(i, j) \in \mathcal{E}_r \times \mathcal{N}_r$. Having defined the temperature dynamics of the DN, we can define $T_{\mathrm{p},i}^{\mathrm{sec,in}}$ and $T_{\mathrm{c},i}^{\mathrm{pri,in}}$ in (1b) and (2a), respectively, as follows:

$$T_{\mathrm{p},i}^{\mathrm{sec,in}} = \alpha_{i,j} T_{\mathrm{r},k}|_{k=\gamma_{\mathrm{dn}}(j)},\ T_{\mathrm{c},i}^{\mathrm{pri,in}} = \beta_{i,j} T_{\mathrm{s},k}|_{k=\gamma_{\mathrm{dn}}^{-1}(j)}. \tag{5}$$

The overall temperature dynamics of the DHS are given by (1), (2), (3), (4), and (5).

3. DHS MODEL IN VECTOR FORM

In this section, we provide the temperature dynamics of the producers, consumers, and the DN in vector form but first we introduce the following shorthand notation:

- $\chi_\alpha^\beta := \mathrm{col}\left(\chi_{\alpha,i}^\beta\right)\Big|_{i=1}^{n_\alpha}$, $\lambda_\alpha := \mathrm{col}\left(\lambda_{\alpha,i}\right)|_{i=1}^{n_\alpha}$, where $\chi \in \{V, T, q\}$, $\beta \in \{\mathrm{pri}, \mathrm{sec}\}$, and $\alpha \in \{\mathrm{p}, \mathrm{c}\}$.
- $\Omega_\gamma^E := \mathrm{col}\left(\Omega_{\gamma,i}\right)|_{i=1}^{|\mathcal{E}_\gamma|}$, $\Omega_\gamma^N := \mathrm{col}\left(\Omega_{\gamma,i}\right)|_{i=1}^{|\mathcal{N}_\gamma|}$, $q_\gamma := \mathrm{col}\left(q_{\gamma,i}\right)|_{i=1}^{|\mathcal{E}_\gamma|}$, $\lambda_{\gamma-a} := \mathrm{col}\left(\lambda_{\gamma-a,i}\right)|_{i=1}^{|\mathcal{E}_\gamma|}$, where $\Omega \in \{V, T\}$, and $\gamma \in \{\mathrm{s}, \mathrm{r}\}$.
- $\lambda_{\mathrm{rm-a}} := \mathrm{col}\left(\lambda_{\mathrm{rm-a},i}\right)_{i=1}^{\mathrm{n_c}}$, $\lambda_{\mathrm{rd-rm}} := \mathrm{col}\left(\lambda_{\mathrm{rd-rm},i}\right)_{i=1}^{\mathrm{n_c}}$.

Now the temperature dynamics of the producers, from (1) and (5), can be written as

$$V_{\mathrm{p}}\dot{T}_{\mathrm{p}} = A_{\mathrm{p}}(q_{\mathrm{p}})T_{\mathrm{p}} + G_{\mathrm{p}}(q_{\mathrm{p}})T_{\mathrm{r}} + B_{\mathrm{p}}q_{\mathrm{p}},$$

where $T_{\mathrm{p}} = (T_{\mathrm{p}}^{\mathrm{pri}}, T_{\mathrm{p}}^{\mathrm{sec}}) \in \mathbb{R}^{2n_{\mathrm{p}}}$, $q_p = (q_{\mathrm{p}}^{\mathrm{pri}}, q_{\mathrm{p}}^{\mathrm{sec}}) \in \mathbb{R}^{2n_{\mathrm{p}}}$, $T_{\mathrm{r}} = (T_{\mathrm{r}}^{\mathrm{E}}, T_{\mathrm{r}}^{\mathrm{N}}) \in \mathbb{R}^{(|\mathcal{E}_{\mathrm{r}}|+|\mathcal{N}_{\mathrm{r}}|)}$, $V_{\mathrm{p}} = \mathrm{diag}(V_{\mathrm{p}}^{\mathrm{pri}}, V_{\mathrm{p}}^{\mathrm{sec}}) \in \mathbb{R}^{2n_{\mathrm{p}}\times 2n_{\mathrm{p}}}$

$$A_{\mathrm{p}}(q_{\mathrm{p}}) = \begin{bmatrix} -\mathrm{diag}(q_{\mathrm{p}}^{\mathrm{pri}}) - \mathrm{diag}(\lambda_{\mathrm{p}}) & \mathrm{diag}(\lambda_{\mathrm{p}}) \\ \mathrm{diag}(\lambda_{\mathrm{p}}) & -\mathrm{diag}(q_{\mathrm{p}}^{\mathrm{sec}}) - \mathrm{diag}(\lambda_{\mathrm{p}}) \end{bmatrix},$$

$$G_{\mathrm{p}}(q_{\mathrm{p}}) = \mathrm{diag}(\mathbf{0}, \mathrm{diag}(q_{\mathrm{p}}^{\mathrm{sec}})\alpha),\ \text{and}\ B_{\mathrm{p}} = \mathrm{diag}(T_{\mathrm{src}}, \mathbf{0}),$$

where $T_{\mathrm{src}} = \mathrm{col}\left(T_{\mathrm{src},i}\right)|_{i=1}^{n_{\mathrm{p}}}$ and $\alpha \in \mathbb{R}^{n_{\mathrm{p}}\times|\mathcal{N}_{\mathrm{r}}|}$ is defined as follows: $\alpha_{i,j} = 1$ if $j \in \mathcal{N}_{\mathrm{s}}$ receives a stream from the ith producer and $\alpha_{i,j} = 0$ otherwise.

For writing the temperature dynamics of the supply layer of DHS's DN, we consider the graph $\mathcal{G}_{\mathrm{s}}$ with $\mathcal{N}_{\mathrm{s}}$ nodes and $\mathcal{E}_{\mathrm{s}}$ edges, and define a flow dependent, node-edge incidence matrix $\mathcal{B}_{\mathrm{s}}(q_{\mathrm{s}})$ as follows: $(\mathcal{B}_{\mathrm{s}})_{ij}(q_{\mathrm{s}}) = 1$, if the flow q_{s} through $j \in \mathcal{E}_{\mathrm{s}}$ targets $i \in \mathcal{N}_{\mathrm{s}}$; $(\mathcal{B}_{\mathrm{s}})_{ij}(q_{\mathrm{s}}) = -1$, if the flow q_s through $j \in \mathcal{E}_s$ originates from $i \in \mathcal{N}_s$; and $(\mathcal{B}_s)_{ij}(q_s) = 0$, otherwise. Following Vladimarsson (2014), we can write $\mathcal{B}_s = \mathcal{B}_s^0 \mathrm{diag}(\mathbf{sign}(q_s))$, where $\mathcal{B}_s^0$ is a constant incidence matrix defined as follows: $(\mathcal{B}_s^0)_{ij} = 1$, if the node i is the head of the edge j; $(\mathcal{B}_s^0)_{ij} = -1$, if the node i is the tail of the edge j; and $(\mathcal{B}_s^0)_{ij} = 0$, otherwise. Since we want to identify (for each edge) its target and source nodes, therefore the positive and negative parts of $\mathcal{B}_s$ are more relevant to us. These matrices can be defined as $\mathcal{T}_s = \frac{1}{2}(\mathcal{B}_s + |\mathcal{B}_s|)$ and $\mathcal{S}_s = \frac{1}{2}(|\mathcal{B}_s| - \mathcal{B}_s)$. Then, the temperature dynamics of the supply layer of DHS's DN, from (3), can be written as

$$V_s \dot{T}_s = A_s(q_s)T_s + G_s(q_p)T_p + d_s,$$

where $V_s = \mathrm{diag}(V_s^E, V_s^N) \in \mathbb{R}^{(|\mathcal{E}_s|+|\mathcal{N}_s|)\times(|\mathcal{E}_s|+|\mathcal{N}_s|)}$, $T_s = (T_s^E, T_s^N) \in \mathbb{R}^{(|\mathcal{E}_s|+|\mathcal{N}_s|)}$, $q_s \in \mathbb{R}^{|\mathcal{E}_s|}$,

$$A_s(q_s) = \begin{bmatrix} -\mathrm{diag}(|q_s|) - \mathrm{diag}(\lambda_{s-a}) & \mathrm{diag}(|q_s|)\mathcal{S}_s{}^T \\ \mathcal{T}_s \mathrm{diag}(|q_s|) & -\mathrm{diag}(\mathcal{T}_s|q_s|) - \mathrm{diag}(\mathcal{W}_s) \end{bmatrix},$$

$G_s(q_p) = \mathrm{diag}(\mathbf{0}, \alpha^\top \mathrm{diag}(q_p^{sec}))$, $d_s = (\mathrm{diag}(\lambda_{s-a})T_a\mathbf{1}, \mathbf{0})$, with $\mathcal{W}_s = \mathrm{col}\left(\sum_{i=1}^{n_p} \alpha_{i,j} q_{p,i}^{sec}\right)\Big|_{j=1}^{|\mathcal{N}_s|}$.

The temperature dynamics of the consumers, from (2) and (5), can be written as

$$V_c \dot{T}_c = A_c(q_c)T_c + G_c(q_c)T_s + d_c,$$

where $V_c = \mathrm{diag}(V_c^{pri}, V_c^{sec}, V_{rd}, V_{rm}) \in \mathbb{R}^{4n_c \times 4n_c}$, $T_c = (T_c^{pri}, T_c^{sec}, T_{rd}, T_{rm}) \in \mathbb{R}^{4n_c}$, $q_c = (q_c^{pri}, q_c^{sec}) \in \mathbb{R}^{2n_c}$,

$$G_c(q_c) = \begin{bmatrix} \mathbf{0} & \mathrm{diag}(q_c^{pri})\beta \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \; d_c = \begin{bmatrix} \mathbf{0} \\ \mathrm{diag}(\lambda_{rm-a})T_a\mathbf{1} \end{bmatrix},$$

where $V_{rd} = \mathrm{col}\,(V_{rd,i})|_{i=1}^{n_c}$, $V_{rm} = \mathrm{col}\,(V_{rm,i})|_{i=1}^{n_c}$, and $\beta \in \mathbb{R}^{n_c \times |\mathcal{N}_s|}$ is defined as follows: $\beta_{i,j} = 1$ if from $j \in \mathcal{N}_s$, a stream is directed towards the ith consumer and $\beta_{i,j} = 0$ otherwise. The matrix $A_c(q_c)$ is given in (#).

For writing the temperature dynamics of the return layer of DHS's DN, we consider the graph $\mathcal{G}_r$, and define a node-edge incidence matrix $\mathcal{B}_r$ as we did for the supply layer above. Then the positive and negative parts of $\mathcal{B}_r$ can be, respectively, defined as $\mathcal{T}_r = \frac{1}{2}(\mathcal{B}_r + |\mathcal{B}_r|)$, $\mathcal{S}_r = \frac{1}{2}(|\mathcal{B}_r| - \mathcal{B}_r)$. Now the temperature dynamics of the return layer of DHS's DN, from (4), can be written as

$$V_r \dot{T}_r = A_r(q_r)T_s + G_r(q_c)T_c + d_r,$$

where

$$V_r = \mathrm{diag}(V_r^E, V_r^N) \in \mathbb{R}^{(|\mathcal{E}_r|+|\mathcal{N}_r|)\times(|\mathcal{E}_r|+|\mathcal{N}_r|)}, \; q_r \in \mathbb{R}^{|\mathcal{E}_r|},$$

$$A_r(q_r) = \begin{bmatrix} -\mathrm{diag}(|q_r|) - \mathrm{diag}(\lambda_{r-a}) & \mathrm{diag}(|q_r|)\mathcal{S}_r{}^T \\ \mathcal{T}_r \mathrm{diag}(|q_r|) & -\mathrm{diag}(\mathcal{T}_r|q_r|) - \mathrm{diag}(\mathcal{W}_r) \end{bmatrix},$$

$$G_r(q_c) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \beta^\top \mathrm{diag}(q_c^{pri}) & \mathbf{0} \end{bmatrix}, \text{ and } d_r = \begin{bmatrix} \mathrm{diag}(\lambda_{r-a})T_a\mathbf{1} \\ \mathbf{0} \end{bmatrix},$$

where $\mathcal{W}_r = \mathrm{col}\left(\sum_{i=1}^{n_c} \beta_{i,j} q_{c,i}^{pri}\right)\Big|_{j=1}^{|\mathcal{N}_r|}$.

Before writing the overall system dynamics in vector form, we find it convenient to represent the *hydraulic layer* of the overall interconnected DHS as a graph. To this end, let $\mathcal{G}_{p,i}^{pri}$ and $\mathcal{G}_{p,i}^{sec}$ be single-edged graphs representing the primary and secondary sides of the ith producer's HX; an analogous definition holds for $\mathcal{G}_{c,i}^{pri}$ representing the primary side of the ith consumer's HX. We consider that $\mathcal{G}_{p,i}^{pri}$ is an open hydraulic network with a source and a sink at its end nodes as in Varga et al. (1995). Let $\mathcal{G}_{c,i}^{sec}$ denote the hydraulic loop comprised of the secondary side of the ith consumer's HX and the radiator; see Fig. 1b. Considering the preceding definitions, and the description of $\mathcal{G}_s$ and $\mathcal{G}_r$ in Section 2.3 and their interconnections to the producers and consumers, the hydraulic layer of the overall DHS can be represented by the graph $\mathcal{G} = \left(\cup_{i=1}^{n_p} \mathcal{G}_{p,i}^{pri}\right) \cup \left(\cup_{i=1}^{n_p} \mathcal{G}_{p,i}^{sec}\right) \cup \mathcal{G}_s \cup \mathcal{G}_r \cup \left(\cup_{i=1}^{n_c} \mathcal{G}_{c,i}^{pri}\right) \cup \left(\cup_{i=1}^{n_c} \mathcal{G}_{c,i}^{sec}\right)$.

Note that each $\mathcal{G}_{p,i}^{pri}$ is an open, isolated, and single-edged graph and its hydraulic state is determined by the independent variable $u_{p,i}^{pri} := q_{p,i}^{pri}$. Moreover, each $\mathcal{G}_{c,i}^{sec}$ is also an isolated and closed graph whose hydraulic state is determined by the independent variable $u_c^{pri} := q_{c,i}^{pri}$. The remaining component of $\mathcal{G}$, given by $\mathcal{G}' := \left(\cup_{i=1}^{n_p} \mathcal{G}_{p,i}^{sec}\right) \cup \mathcal{G}_s \cup \mathcal{G}_r \cup \left(\cup_{i=1}^{n_c} \mathcal{G}_{c,i}^{pri}\right)$, is connected. Therefore, there exists a fundamental loop matrix, say $\mathcal{F}$, such that the hydraulic state $q' := [q_p^{sec}, q_s, q_r, q_c^{pri}]^\top$ of $\mathcal{G}'$ can be written as $q' = \mathcal{F}^\top u'$, where $u' = [u_p^{sec}, u_s, u_r, u_c^{pri}]$ with u_p^{sec} collecting all the flows $q_{p,i}^{sec}$ in the secondary of the producer's HXs, except for one, say for the mth producer. This constraint comes from Kirchhoff's current laws and implies that $q_{p,m}^{sec} = \sum_{\forall i} q_{c,i}^{pri} - \sum_{\forall j \neq m} q_{p,j}^{sec}$; see Machado et al. (2022b) for more details. Also, u_s (u_r) is composed by the flows q_s (q_r) associated to edges of $\mathcal{G}_s$ ($\mathcal{G}_r$) representing *chords* of $\mathcal{G}'$. Finally, u_c^{sec} is the collection of all the flows $q_{c,i}^{sec}$.

Then the overall temperature dynamics of the district heating system can be written as

$$V\dot{x} = A(u)x + Bu + d, \tag{6}$$

where $V = \mathrm{diag}(V_p, V_s, V_c, V_r)$, $x = \left[T_p^\top, T_s^\top, T_c^\top, T_r^\top\right]^\top \in \mathcal{X} \subseteq \mathbb{R}^n$, $u = [(u_p^{pri})^\top, (u')^\top, (u_c^{sec})^\top]^\top \in \mathcal{U} \subseteq \mathbb{R}^m$,

$$A(u) = \begin{bmatrix} A_p(u_p) & \mathbf{0} & \mathbf{0} & G_p(u_p) \\ G_s(u_p) & A_s(u_s) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & G_c(u_c) & A_c(u_c) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & G_r(u_c) & A_r(u_r) \end{bmatrix},$$

$$B = \mathrm{diag}(B_p, \mathbf{0}) \text{ and } d = [\mathbf{0}^\top, (d_s)^\top, (d_c)^\top, (d_r)^\top]^\top,$$

where $u_p = [(u_p^{pri})^\top, (u_p^{sec})^\top]^\top$, $u_c = [(u_c^{pri})^\top, (u_c^{sec})^\top]^\top$, $n = 2n_p + (|\mathcal{E}_s| + |\mathcal{N}_s|) + 4n_c + (|\mathcal{E}_r| + |\mathcal{N}_r|)$, $m = 2n_p + |\mathrm{chord}(\mathcal{G}_s)| + 2n_c + |\mathrm{chord}(\mathcal{G}_r)| - 1$.

Note that since $A(u)$ is linear, then $A(u)$ can be written as $A(u) = A_0 + u_1 A_1 + \ldots + u_m A_m$. It follows that

$$A(u)x = (A_0 + u_1 A_1 + \ldots + u_m A_m)x = A_0 x + L_A(x)u,$$

$$A_c(q_c) = \begin{bmatrix} -\mathrm{diag}(q_c^{pri}) - \mathrm{diag}(\lambda_c) & \mathrm{diag}(\lambda_c) & \mathbf{0} & \mathbf{0} \\ \mathrm{diag}(\lambda_c) & -\mathrm{diag}(q_c^{sec}) - \mathrm{diag}(\lambda_c) & \mathrm{diag}(q_c^{sec}) & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}(q_c^{sec}) & -\mathrm{diag}(q_c^{sec}) - \mathrm{diag}(\lambda_{rd-rm}) & \mathrm{diag}(\lambda_{rd-rm}) \\ \mathbf{0} & \mathbf{0} & \mathrm{diag}(\lambda_{rd-rm}) & -\mathrm{diag}(\lambda_{rd-rm}) - \mathrm{diag}(\lambda_{rm-a}) \end{bmatrix} \tag{\#}$$

where

$$L_A(x) := [A_1 x \dots A_m x] \in \mathbb{R}^{n\times m}, \quad (7)$$

and

$$\frac{\partial}{\partial u}(A(u)x) = L_A(x) \in \mathbb{R}^{n\times m}. \quad (8)$$

4. PASSIVITY AND ZERO-STATE DETECTABILITY ANALYSIS

To establish the passivity properties and zero-state detectability of system (6), let us first define the set of all feasible equilibria of (6) as follows: $E := \{(\bar{x},\bar{u}) \in \mathcal{X} \times \mathcal{U} : A(\bar{u})\bar{x}+b\bar{u}+d = 0\}$, which is assumed to be nonempty.

Assumption 2. **(I)** $|\bar{q}^{\mathrm{sec}}_{\mathrm{p},i}| > 0$ if and only if $|\bar{q}^{\mathrm{pri}}_{\mathrm{p},i}| > 0$; and **(II)** $|\bar{q}^{\mathrm{sec}}_{\mathrm{c},i}| > 0$ if and only if $|\bar{q}^{\mathrm{pri}}_{\mathrm{c},i}| > 0$.

Remark 1. The above assumption is technically motivated but it is practically reasonable: **(I)** If $\bar{q}^{\mathrm{pri}}_{\mathrm{p},i} = 0$ but $\bar{q}^{\mathrm{sec}}_{\mathrm{p},i} > 0$, then ith producer cannot deliver heat from the ith source to the DN, which could lead the producer injecting cold water (from the return layer) directly into the supply layer of the DN, which is not desirable in general. Now consider the case $\bar{q}^{\mathrm{pri}}_{\mathrm{p},i} > 0$ and $\bar{q}^{\mathrm{sec}}_{\mathrm{p},i} = 0$. Even though this scenario could of practical relevance, e.g., it could represent a warming stage for the producers' HX, it can be shown, using LaSalle's invariance principle with the positive definite function $S_{\mathrm{p}} = \frac{1}{2}V^{\mathrm{pri}}_{\mathrm{p},i}(T^{\mathrm{pri}}_{\mathrm{p},i} - \bar{T}^{\mathrm{pri}}_{\mathrm{p},i})^2 + \frac{1}{2}V^{\mathrm{sec}}_{\mathrm{p},i}(T^{\mathrm{sec}}_{\mathrm{p},i} - \bar{T}^{\mathrm{sec}}_{\mathrm{p},i})$ that $\lim_{t\to\infty} T^{\mathrm{pri}}_{\mathrm{p},i} = \lim_{t\to\infty} T^{\mathrm{sec}}_{\mathrm{p},i} = T_{\mathrm{src},i}$. **(II)** For the case $\bar{q}^{\mathrm{pri}}_{\mathrm{c},i} = 0$ and $\bar{q}^{\mathrm{sec}}_{\mathrm{c},i} > 0$, an analogous argument can be made as for the case $\bar{q}^{\mathrm{pri}}_{\mathrm{p},i} = 0$ and $\bar{q}^{\mathrm{sec}}_{\mathrm{p},i} > 0$ described above. Note that the case $\bar{q}^{\mathrm{sec}}_{\mathrm{c},i} > 0$ and $\bar{q}^{\mathrm{pri}}_{\mathrm{c},i} = 0$ is not of practical significance because it implies that no heat can be extracted from the DN by the ith consumer. Nevertheless, in this case it can be shown that $T^{\mathrm{pri}}_{\mathrm{c},i}$, $T^{\mathrm{sec}}_{\mathrm{c},i}$, $T_{\mathrm{rd},i}$ and $T_{\mathrm{rm},i}$ converge to T_{a}.

Now we establish passivity properties and zero-state detectability for system (6).

Theorem 1. System (6) satisfies the following properties:

- **(a)** It is *Krasovskii* passive with storage function $S_{\mathrm{K}} = \frac{1}{2}\dot{x}^\top V\dot{x}$ and input-output pair $u_d = \dot{u}$ and $y_{\mathrm{K}} = (B + L_A(x))^\top \dot{x}$, i.e., the following inequality holds: $\dot{S}_{\mathrm{K}} \le y_{\mathrm{K}}^\top u_d$, where $L_A(x)$ is defined in (7).
- **(b)** It is *shifted* passive with storage function $S_{\mathrm{S}}(x) = \frac{1}{2}(x - \bar{x})^\top V(x - \bar{x})$ and input-output pair u and $y_{\mathrm{S}} = (B+L_A(\bar{x}))^\top(x-\bar{x})$, i.e., it satisfies $\dot{S}_{\mathrm{S}} \le (y_{\mathrm{S}} - \bar{y}_{\mathrm{S}})^\top(u - \bar{u})$ with $\bar{y}_{\mathrm{S}} = 0$.
- **(c)** For a fixed $u = \bar{u}$, $\lim_{t\to\infty} x = \bar{x}$, where $\bar{x}$ is such that $(\bar{x},\bar{u}) \in E$.

Remark 2. The property in part **(c)** of Proposition 1 is closely related to the zero-state detectability of (6) with respect to the output y_{S}, i.e., if $u = \bar{u}$ and $y_{\mathrm{S}} = 0$, then $\lim_{t\to\infty} x = \bar{x}$; see (van der Schaft, 2017, Remark 3.2.21).

Proof. We prove each part of the proposition separately as follows.

Proof of (a): The time derivative of S_{K} along the trajectories of (6) satisfies

$$\dot{S}_{\mathrm{K}} = \dot{x}^\top A(u)\dot{x} + \dot{x}^\top (B + L_A(x))\,\dot{u}, \quad (9)$$

where $L_A(x)$ is from (8). We next show that $A(u)$ is negative semidefinite. To this end, let

$$A_{\mathrm{loss}} = \mathrm{diag}(\mathbf{0}, A^{\mathrm{loss}}_{\mathrm{s}}, A^{\mathrm{loss}}_{\mathrm{c}}, A^{\mathrm{loss}}_{\mathrm{r}}),$$

where $A^{\mathrm{loss}}_{\mathrm{s}} = \mathrm{diag}(-\lambda_{\mathrm{s-a}}, \mathbf{0})$, $A^{\mathrm{loss}}_{\mathrm{r}} = \mathrm{diag}(-\lambda_{\mathrm{r-a}}, \mathbf{0})$, and

$$A^{\mathrm{loss}}_{\mathrm{c}} = \mathrm{diag}(\mathbf{0}, \mathbf{0}, \mathbf{0}, -\lambda_{\mathrm{rm-a}}).$$

Also, let $A_{\mathrm{src}} = \mathrm{diag}(-q^{\mathrm{pri}}_{\mathrm{p}}, \mathbf{0})$ and $A_{\mathrm{cond}} = A(\mathbf{0}) - A_{\mathrm{loss}}$. Then, $A(u)$ in equation (6) can be written as

$$A(u) = A_{\mathrm{conv}}(u) + A_{\mathrm{cond}} + A_{\mathrm{loss}} + A_{\mathrm{src}}(u), \quad (10)$$

where $A_{\mathrm{conv}}(u) = A(u) - A(\mathbf{0}) - A_{\mathrm{src}}(u)$. Now we provide an argument that $A(u)$ is negative semidefinite. First, by direct computations, it can be shown that, for any $u \neq \mathbf{0}$, $A_{\mathrm{conv}}(u)$ can be decomposed as $A_{\mathrm{conv}}(u) = \mathrm{diag}(\mathcal{K}(u), \mathbf{0})$, where $\mathcal{K}$ is a Kirchhoff's Convection Matrix (KCM); see (Hangos et al., 1999, Appendix). Thus, $A_{\mathrm{conv}}(u)$ is negative semidefinite for all $u \neq \mathbf{0}$ by using (Hangos et al., 1999, Lemma 8). Using similar arguments, it can be established that A_{cond} is also negative semidefinite. Finally, it can be seen that A_{loss} and $A_{\mathrm{src}}(u)$ are negative semidefinite matrices for all u. Therefore, it can be concluded that the matrix $A(u)$ is negative semidefinite for all u. As a consequence, it follows from (9) that $\dot{S}_{\mathrm{K}} \le \dot{x}^\top(B + L_A(x))\dot{u} = y_{\mathrm{K}}^\top u_d$, concluding the proof of (a).

Proof of (b): Let $(\bar{x},\bar{u}) \in E$ be an arbitrary equilibrium pair. Then, the following identity holds: $V\dot{\bar{x}} = 0 = A(\bar{u})\bar{x} + B\bar{u} + d$. It follows that equation (6) can be written as

$$V\dot{x} = A(u)(x - \bar{x}) + (B + L_A(\bar{x}))\,(u - \bar{u}). \quad (11)$$

Note that due to linearity of $A(u)$, we have $(A(u) - A(\bar{u}))\bar{x} = L_A(\bar{x})(u - \bar{u})$. As a consequence, it follows from (11) that

$$V\dot{x} = A(u)(x - \bar{x}) + (B + L_A(\bar{x}))\,(u - \bar{u}). \quad (12)$$

Now note that the time derivative of S_{S} along the trajectories of (12) satisfies

$$\dot{S}_{\mathrm{S}} \le (x - \bar{x})^\top (B + L_A(x))\,(u - \bar{u}) = (y_{\mathrm{S}} - \bar{y}_{\mathrm{S}})^\top(u - \bar{u}).$$

with $\bar{y}_{\mathrm{S}} = 0$, concluding the proof of (b).

Proof of (c): Consider again the decomposition of $A(u)$ as in (10), i.e., $A(u) = A_{\mathrm{conv}}(u) + A_{\mathrm{cond}} + A_{\mathrm{loss}} + A_{\mathrm{src}}(u)$. Since $A_{\mathrm{conv}}(u)$ and A_{cond} are negative semidefinite, it can be established that $\dot{S}_{\mathrm{S}}$ satisfies

$$\dot{S}_{\mathrm{S}} = (x - \bar{x})^\top A(u)(x - \bar{x}) + y_{\mathrm{s}}^\top(u - \bar{u})$$
$$\le (x - \bar{x})^\top (A_{\mathrm{loss}} + A_{\mathrm{src}})\,(x - \bar{x}) + y_{\mathrm{s}}^\top(u - \bar{u}).$$

We note that $\dot{S}_{\mathrm{S}} = 0$ if $u = \bar{u}$ and x is in the following set: $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$, where $\mathcal{R}_1 = \{x : T^{\mathrm{E}}_{\mathrm{s}} = \bar{T}^{E}_{\mathrm{s}},\ T^{\mathrm{E}}_{\mathrm{r}} = \bar{T}^{\mathrm{E}}_{\mathrm{r}},\ T_{\mathrm{rm}} = \bar{T}_{\mathrm{rm}}\}$ and $\mathcal{R}_2 = \{x : T^{\mathrm{pri}}_{\mathrm{p},i} = \bar{T}^{\mathrm{pri}}_{\mathrm{p},i},$ for which, $\bar{q}^{\mathrm{pri}}_{\mathrm{p},i} > 0\}$. Next we show that the largest invariant set contained in $\mathcal{R}$ is the equilibrium $(\bar{x},\bar{u})$.

Assume that $u = \bar{u}$ and $x \in \mathcal{R}$. Consider the temperature dynamics of the producers given in (1). Since $x \in \mathcal{R}$, we have that $T^{\mathrm{pri}}_{\mathrm{p},i} = \bar{T}^{\mathrm{pri}}_{\mathrm{p},i}$ for each i for which $\bar{q}^{\mathrm{pri}}_{\mathrm{p},i} > 0$, then from (1a), we have that $T^{\mathrm{sec}}_{\mathrm{p},i} = \bar{T}^{\mathrm{sec}}_{\mathrm{p},i}$. On the other hand, for each i for which $\bar{q}^{\mathrm{pri}}_{\mathrm{p},i} = 0$, we have from Assumption 2 that $\bar{q}^{\mathrm{sec}}_{\mathrm{p},i} = 0$, then it can be shown from (1) that $\lim_{t\to\infty} T^{\mathrm{pri}}_{\mathrm{p},i} = \lim_{t\to\infty} T^{\mathrm{sec}}_{\mathrm{p},i} = k_{\mathrm{p},i}$, where $k_{\mathrm{p},i} \in \mathbb{R}$ is a constant depending on the initial conditions of $T^{\mathrm{pri}}_{\mathrm{p},i}$ and $T^{\mathrm{sec}}_{\mathrm{p},i}$.

Let us now focus on the temperature dynamics of the consumers given in (2). We show that $\lim_{t\to\infty} T_{\mathrm{c}} = \bar{T}_{\mathrm{c}}$.

Since $x \in \mathcal{R}$, it holds that $T_{\rm rm} = \bar{T}_{\rm rm}$, which implies from (2d) that $T_{\rm rd} = \bar{T}_{\rm rd}$. For each i for which $\bar{q}^{\rm sec}_{{\rm c},i} > 0$ we have from (2c) that $T^{\rm sec}_{{\rm c},i} = \bar{T}^{\rm sec}_{{\rm c},i}$. Moreover, from (2b) we get that $T^{\rm pri}_{{\rm c},i} = \bar{T}^{\rm pri}_{{\rm c},i}$. On the other hand, for each i for which $\bar{q}^{\rm pri}_{{\rm c},i} = 0$ (which from Assumption 2 implies $\bar{q}^{\rm sec}_{{\rm c},i} = 0$), it can be shown from (2a) and (2b) that $\lim_{t\to\infty} T^{\rm pri}_{{\rm c},i} = \lim_{t\to\infty} T^{\rm sec}_{{\rm c},i} = k_{{\rm c},i}$, where $k_{{\rm c},i} \in \mathbb{R}$ is a constant depending on the initial conditions of $T^{\rm pri}_{{\rm c},i}$ and $T^{\rm sec}_{{\rm c},i}$.

Now we focus on the temperature dynamics of the nodes in the supply layer given in (3). Let us recall that if $\bar{q}^{\rm sec}_{{\rm p},i} > 0$ for some i, then $T^{\rm sec}_{{\rm p},i} = \bar{T}^{\rm sec}_{{\rm p},i}$. It follows that for each $j \in \mathcal{N}_{\rm s}$ for which $\alpha_{i,j}\bar{q}^{\rm sec}_{{\rm p},i} > 0$, we have $\lim_{t\to\infty} T_{{\rm s},j} = \bar{T}_{{\rm s},j}$. Now, for each $j \in \mathcal{N}_{\rm s}$ for which $\sum_{i=1}^{n_{\rm p}} \alpha_{i,j}\bar{q}^{\rm sec}_{{\rm p},i} = 0$, we have two scenarios. If $\sum_{i\in\mathcal{E}^+_{\rm s}(j)} \bar{q}_{{\rm s},i} > 0$ for some j, then we have $\lim_{t\to\infty} T_{{\rm s},j} = \bar{T}_{{\rm s},j}$. Alternatively, if $\sum_{i\in\mathcal{E}^+_{\rm s}(j)} \bar{q}_{{\rm s},i} = 0$ for some j, then $T_{{\rm s},j}$ remains constant at its steady-state value.

Finally, consider the temperature dynamics of the nodes in the return layer given in (4). For each i for which $\bar{q}^{\rm pri}_{{\rm c},i} > 0$ we established, when analyzing the consumers' dynamics, that $T^{\rm pri}_{{\rm c},i} = \bar{T}^{\rm pri}_{{\rm c},i}$. It follows that for each $j \in \mathcal{N}_{\rm r}$ for which $\beta_{i,j}\bar{q}^{\rm pri}_{{\rm c},i} > 0$ we have $\lim_{t\to\infty} T_{{\rm r},j} = \bar{T}_{{\rm r},j}$. Now, for each $j \in \mathcal{N}_{\rm r}$ for which $\sum_{i=1}^{n_{\rm c}} \beta_{i,j}\bar{q}^{\rm pri}_{{\rm c},i} = 0$, we have two scenarios. If $\sum_{i\in\mathcal{E}^+_{\rm r}(j)} \bar{q}_{{\rm r},i} > 0$ for some j, then we have $\lim_{t\to\infty} T_{{\rm r},j} = \bar{T}_{{\rm r},j}$. Alternatively, if $\sum_{i\in\mathcal{E}^+_{\rm r}(j)} \bar{q}_{{\rm r},i} = 0$ for some j, then $T_{{\rm r},j}$ remains constant at its steady-state value. This concludes the proof of (c).

5. RESEARCH PERSPECTIVES

Using a scalable, graph-theoretic approach, we provided a multi-producer DHS model. We also established both shifted and Krasovskii passivity properties and zero-state detectability of the the modeled dynamics. Future research directions include: (i) finding conditions for the existence of a steady-state solution, (iii) finding integrability conditions of the Krasovskii passive output, and (iv) designing optimal passivity-based controllers.

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