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van Waarde, Henk J.; Camlibel, M. Kanat; Trentelman, Harry L.

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Comments on “On the Necessity of Diffusive Couplings in Linear Synchronization Problems With Quadratic Cost”

Henk J. van Waarde, M. Kanat Camlibel, *Member, IEEE*, and Harry L. Trentelman, *Fellow, IEEE*

Abstract—In this note, we want to comment on the recent paper “On the necessity of diffusive couplings in linear synchronization problems with quadratic cost” (*IEEE Transactions on Automatic Control*, vol. 60, pp. 3029–3034, 2015) by Montenbruck *et al.* on the necessity of diffusiveness of optimal control laws in linear quadratic control problems in the context of synchronization. In the above paper, the authors concentrate on the optimal feedback law associated with the maximal solution of the underlying algebraic Riccati equation (ARE). In this comment, we argue that it is more natural to use the smallest positive semidefinite solution of the ARE. Our approach generalizes the results in the above paper to the case in which the agent dynamics is allowed to have eigenvalues in the open right half plane. Moreover, our proof considerably simplifies the one in the above paper, as it avoids the analysis of the Riccati differential equation. In addition, we propose and solve the zero endpoint version of the linear quadratic problem studied in the above paper.

Index Terms—Linear systems, networked control systems, optimal control.

I. PROBLEM STATEMENT

The authors of [1] consider N identical linear systems of the form

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t) \quad (1)$$

where $x_i \in \mathbb{R}^n$ is the state of the i th system and $u_i \in \mathbb{R}^m$ is its input, for $i = 1, 2, \dots, N$. The overall system is then given by

$$\dot{x}(t) = (I_N \otimes A)x(t) + (I_N \otimes B)u(t) \quad (2)$$

where $x(t) = \text{col}(x_1(t), x_2(t), \dots, x_N(t))$, \otimes denotes the Kronecker product, and $u(t) = \text{col}(u_1(t), u_2(t), \dots, u_N(t))$. Associated with system (2), Montenbruck *et al.* consider the cost functional

$$J(x_0, u) = \int_0^\infty x(t)^T P Q P x(t) + u(t)^T R u(t) dt \quad (3)$$

where Q and R are symmetric positive-definite matrices, and $P = (I_N - \frac{1}{N} \mathbb{1}_N \otimes \mathbb{1}_N^T) \otimes I_n$ is the orthogonal projection onto the orthogonal complement S^\perp of the synchronization manifold

$$S = \{x \in \mathbb{R}^{Nn} \mid \exists z \in \mathbb{R}^n : x = \mathbb{1}_N \otimes z\} \subset \mathbb{R}^{Nn}. \quad (4)$$

The control law that minimizes (3), hence, implicitly synchronizes the systems (1), in the sense that it makes the Q -weighted L_2 -norm of the distance of the global state trajectory $x(t)$ to the synchronization manifold small. In the remainder of this note, we shall work with the

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The authors are with the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9700, AV Groningen, The Netherlands (e-mail: h.j.van.waarde@rug.nl; h.l.trentelman@rug.nl; m.k.camlibel@rug.nl).

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shorthand notations $\bar{A} = I_N \otimes A$, $\bar{B} = I_N \otimes B$, and $\bar{Q} = P Q P$. The authors of [1] are interested in the question whether the optimal control input $u(t) = Kx(t)$ that minimizes (3) is diffusive, i.e., whether K belongs to the vector space of matrices given by

$$\mathcal{M}_{m \times n}^N = \{M \in \mathbb{R}^{Nn \times Nn} \mid M(\mathbb{1}_N \otimes I_n) = 0\}. \quad (5)$$

The inclusion $K \in \mathcal{M}_{m \times n}^N$ implies that each control input $u_i(t)$ only uses relative states and requires no absolute information about the global state $x(t)$.

Montenbruck *et al.* consider the feedback matrix $K^+ = -R^{-1} \bar{B}^T X^+$, where X^+ is the largest positive semidefinite solution to the algebraic Riccati equation

$$X \bar{A} + \bar{A}^T X - X \bar{B} R^{-1} \bar{B}^T X + \bar{Q} = 0 \quad (6)$$

and study the problem under which conditions, $u(t) = K^+ x(t)$ is diffusive. Note, however, that $u(t) = K^+ x(t)$ only minimizes (3) in the special case that $X^+ = X^-$, where X^- is the smallest positive semidefinite solution to (6). This is the case if and only if \bar{A} has all eigenvalues in the closed left half plane. In fact, in general, the optimal control law that minimizes (3) is given by $u(t) = K^- x(t)$, where $K^- = -R^{-1} \bar{B}^T X^-$ (see, e.g., [2, Ch. 10.3]). Hence, in our opinion, it is more natural to investigate under which conditions, the control law $u(t) = K^- x(t)$ is diffusive.

On the other hand, if the goal is to minimize (3) under the zero endpoint constraint $x_i(t) \rightarrow 0$ ($t \rightarrow \infty$) for $i = 1, 2, \dots, N$, then the largest positive semidefinite solution to the algebraic Riccati equation does come into play, and we should consider the feedback law $u(t) = K^+ x(t)$ (see [2, Ch. 10.4]). In other words, we propose to distinguish between the following two problems.

Problem 1: Consider system (2) equipped with the cost functional (3). Assume that the optimal control law $u(t) = K^- x(t)$ that minimizes (3) exists. Determine necessary and sufficient conditions under which $u(t) = K^- x(t)$ is diffusive.

Problem 2: Consider system (2) and the associated cost functional (3). Assume that the optimal control law $u(t) = K^+ x(t)$ that minimizes (3) under the constraint $x(t) \rightarrow 0$ ($t \rightarrow \infty$) exists. Determine necessary and sufficient conditions under which $u(t) = K^+ x(t)$ is diffusive.

In the following section, the solutions to both problems are presented. The approach used to prove these results has an advantage over the framework discussed in [1] in its simplicity and does not require the analysis of the Riccati differential equation. Furthermore, we will show that the results obtained in Section II generalize the results obtained in [1].

II. CONTRIBUTION

Before giving the solutions to Problems 1 and 2, we want to explain the approach. The idea is to find the relation between the kernels of X^- and X^+ and the image of $\mathbb{1}_N \otimes I_n$. This relation is found via

the unobservable subspace of the pair (\bar{Q}, \bar{A}) , which we will denote by $\langle \ker \bar{Q} \mid \bar{A} \rangle$. More explicitly, the idea is as follows: we first prove that $\langle \ker \bar{Q} \mid \bar{A} \rangle = \text{im}(\mathbb{1}_N \otimes I_n)$. Subsequently, we find conditions under which the kernels of X^- and X^+ are equal to $\langle \ker \bar{Q} \mid \bar{A} \rangle$. This immediately yields conditions under which the kernels of X^- and X^+ are equal to $\text{im}(\mathbb{1}_N \otimes I_n)$. Note that this approach is attractive as it does not require difficult and lengthy analysis of the Riccati differential equation.

First note that

$$\ker P = \text{im}(\mathbb{1}_N \otimes I_n). \quad (7)$$

Indeed, we easily obtain

$$\begin{aligned} P(\mathbb{1}_N \otimes I_n) &= \left(\left(I_N - \frac{1}{N} \mathbb{1}_N \otimes \mathbb{1}_N^T \right) \otimes I_n \right) (\mathbb{1}_N \otimes I_n) \\ &= \left(\left(I_N - \frac{1}{N} \mathbb{1}_N \otimes \mathbb{1}_N^T \right) \mathbb{1}_N \right) \otimes I_n \\ &= 0. \end{aligned} \quad (8)$$

Moreover, as $\text{rank } P = \text{rank} \left(I_N - \frac{1}{N} \mathbb{1}_N \otimes \mathbb{1}_N^T \right) \otimes I_n = (N-1)n$, we conclude that (7) holds. Second, we have

$$\begin{aligned} P\bar{A} &= P(I_N \otimes A) \\ &= \left(\left(I_N - \frac{1}{N} \mathbb{1}_N \otimes \mathbb{1}_N^T \right) \otimes I_n \right) (I_N \otimes A) \\ &= \left(I_N - \frac{1}{N} \mathbb{1}_N \otimes \mathbb{1}_N^T \right) \otimes A \end{aligned} \quad (9)$$

and consequently

$$P\bar{A}(\mathbb{1}_N \otimes I_n) = 0. \quad (10)$$

Hence, $\ker P$ is \bar{A} -invariant. Furthermore, note that $\ker \bar{Q} = \ker P$ by the symmetry of P and positive definiteness of \bar{Q} . This immediately allows us to write

$$\langle \ker \bar{Q} \mid \bar{A} \rangle = \langle \ker P \mid \bar{A} \rangle = \ker P = \text{im}(\mathbb{1}_N \otimes I_n) \quad (11)$$

where the second identity follows from \bar{A} -invariance of $\ker P$, and the final equality is given by (7). Now that we have proven the equality of $\langle \ker \bar{Q} \mid \bar{A} \rangle$ and $\text{im}(\mathbb{1}_N \otimes I_n)$, we are ready to tackle the first problem.

A. Diffusive Couplings in the Free Endpoint Problem

In this section, we will consider the optimal control law $u(t) = K^-x(t)$ that minimizes (3). We will see that this control law is always diffusive.

Theorem 1: Consider system (2) and assume (A, B) is stabilizable. The optimal control law $u(t) = K^-x(t)$ that minimizes (3) exists and is diffusive.

Proof: The stabilizability of (A, B) implies (\bar{A}, \bar{B}) is stabilizable. Therefore, X^- exists and $u(t) = K^-x(t)$ minimizes (3) by [2, Th. 10.13]. It remains to be shown that the inclusion $K^- \in \mathcal{M}_{m \times n}^N$ holds. By [3, Th. 8], we have $\ker X^- = \langle \ker \bar{Q} \mid \bar{A} \rangle$. Subsequently, by (11), we conclude that $\ker X^- = \text{im}(\mathbb{1}_N \otimes I_n)$ and consequently

$$K^-(\mathbb{1}_N \otimes I_n) = -R^{-1}(I_N \otimes B^T)X^-(\mathbb{1}_N \otimes I_n) = 0. \quad (12)$$

Thus, $K^- \in \mathcal{M}_{m \times n}^N$, which completes the proof. ■

Before we continue with the solution to Problem 2, a few remarks are in place. Note that [1, Ths. 1 and 2] immediately follow from Theorem 1 of this note. Indeed, in the case that the matrix A has

only eigenvalues in the closed left half plane, the identity $X^- = X^+$ holds (see [3, Corollary (a), Th. 8]). Hence, in this case, K^+ is indeed diffusive. However, note that Theorem 1 generalizes [1, Ths. 1 and 2] to the case A has eigenvalues in the open right half plane.

Second, note that the proof of Theorem 1 is only based on the relation (11) and the fact that $\ker X^- = \langle \ker \bar{Q} \mid \bar{A} \rangle$. Hence, our proof simplifies the one provided in [1] and does not use any information about the Riccati differential equation.

Third, our approach can also be applied to prove that the optimal control law is diffusive if we deal with N nonidentical systems, which would benefit [1, proof of Corollary 1].

Finally, we remark that a result similar to Theorem 1 was independently found by the authors of [4] (see [4, Th. 2]). However, note that the controllability assumption in [4, Th. 2] can be replaced by the (weaker) stabilizability assumption. In fact, the stabilizability assumption in Theorem 1 of this note is not even needed for the existence of X^- (a weaker, necessary, and sufficient condition is given in [2, Th. 10.13]).

B. Diffusive Couplings in the Zero Endpoint Problem

In this section, we consider the optimal control law $u(t) = K^+x(t)$ that minimizes (3) under the constraint $x(t) \rightarrow 0$ ($t \rightarrow \infty$). We will see that this control law is diffusive under a condition on the eigenvalues of A . More explicitly, we have the following result.

Theorem 2: Consider system (2). The optimal control law $u(t) = K^+x(t)$ that minimizes (3) under the constraint $x(t) \rightarrow 0$ ($t \rightarrow \infty$) exists and is diffusive if and only if the matrix A is Hurwitz.

Proof: We first assume that A is Hurwitz. This implies that the stabilizability of (A, B) and, thus (\bar{A}, \bar{B}) , is stabilizable. Furthermore, \bar{A} is Hurwitz, so \bar{A} has no (\bar{Q}, \bar{A}) -unobservable eigenvalues on the imaginary axis. We conclude by [2, Th. 10.18] that the optimal control law $u(t) = K^+x(t)$ exists. Subsequently, we want to prove that K^+ is diffusive. As \bar{A} is Hurwitz, by [3, Th. 7], the equality $\ker X^+ = \langle \ker \bar{Q} \mid \bar{A} \rangle$ holds. Consequently, we conclude by (11) that $\ker X^+ = \text{im}(\mathbb{1}_N \otimes I_n)$, and thus, $K^+ \in \mathcal{M}_{m \times n}^N$.

Conversely, suppose the optimal control law $u(t) = K^+x(t)$ that minimizes (3) under the constraint $x(t) \rightarrow 0$ ($t \rightarrow \infty$) exists and is diffusive. This implies that the closed-loop system matrix $\bar{A} + \bar{B}K^+$ is Hurwitz, and $K^+(\mathbb{1}_N \otimes I_n) = 0$. Let λ be an eigenvalue of A with eigenvector v . This yields

$$(I_N \otimes A)(\mathbb{1}_N \otimes v) = \mathbb{1}_N \otimes Av = \lambda(\mathbb{1}_N \otimes v) \quad (13)$$

and thus λ is an eigenvalue of \bar{A} with eigenvector $(\mathbb{1}_N \otimes v)$. Moreover, we have

$$(\bar{A} + \bar{B}K^+)(\mathbb{1}_N \otimes v) = \bar{A}(\mathbb{1}_N \otimes v) = \lambda(\mathbb{1}_N \otimes v). \quad (14)$$

Therefore, λ is an eigenvalue of the closed-loop system matrix $(\bar{A} + \bar{B}K^+)$, and as $(\bar{A} + \bar{B}K^+)$ is Hurwitz, λ has negative real part. We conclude that the matrix A is Hurwitz, which proves the theorem. ■

III. CONCLUSION

In this note, we have commented on [1]. We have argued that one should distinguish two problems, instead of (only) considering the feedback matrix associated with the maximal solution to the ARE. The first problem is to find conditions under which the optimal control law that minimizes the cost functional is diffusive. The second problem is to determine conditions under which the optimal stabilizing control law is diffusive.

We have provided solutions to both problems and have shown how these results extend [1, Ths. 1 and 2]. Furthermore, we have argued that the proofs in this note simplify those in [1] in the sense that they do not require the analysis of the Riccati differential equation.

Finally, we have remarked that the approach described in this note can also be applied to simplify [1, proof of Corollary 1] to prove that the optimal control law is diffusive in the case of nonidentical linear systems.

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