Lower Network Degrees Promote Cooperation in the Prisoner's Dilemma With Environmental Feedback

Stella, Leonardo; Baar, Wouter; Bauso, Dario

Published in:
IEEE Control Systems Letters

DOI:
10.1109/LCSYS.2022.3175402

Lower Network Degrees Promote Cooperation in the Prisoner's Dilemma With Environmental Feedback

Leonardo Stella, Wouter Baar, and Dario Bauso

***Abstract*—Cooperation is a fundamental aspect in nature, as it determines many levels of biological organization. Examples include single cells, but also social insects, such as ants and honeybees, and groups of animals, such as vampire bats and bird flocks. In unstructured populations, where individuals interact with each other with equal probability, the dynamics have been thoroughly investigated and results indicate that the predominant strategy to be favored by natural selection is defection. The focus of this research is to study these evolutionary dynamics in structured population, where the structure is captured by a regular graph. A recent line of research investigated the impact of the population dynamics onto an environmental resource and the mutual effects that the changes in the quantity of this resource have on the game dynamics. In this framework the impact takes the form of game-environment feedback on the population dynamics. The contributions of this paper are as in the following. Firstly, we study the impact of a regular network in the prisoner's dilemma (PD) game and provide a threshold on the degree of the network below which cooperation is favored. Secondly, we derive the corresponding structured model with environmental feedback. Lastly, we carry out the stability analysis of this system and discuss the impact of the network on the environmental resource.**

***Index Terms*—Game theory, networked control systems, Prisoner's dilemma, feedback-evolving games.**

I. Introduction

EVOLUTIONARY game theory investigates the evolution of strategic interactions in a population of rational decision makers. Each decision obtains an incentive based on strategy-dependent payoffs [1], [2]. The frequencies of the strategies are influenced by the payoffs associated with each strategy, thus resulting in selfish behaviors. One of the most popular game where this happens is the well-known prisoner's dilemma (PD), in which the dominant strategy is defection [3]. In the setting of finite games, many evolutionary dynamics have been used, including payoff comparison dynamics [4] and replicator dynamics. The latter are recognised to be the ones that are most widely used [5]. However, one of the main limitations of the traditional PD game is the assumption that the population is well-mixed. This limitation does not consider the role of a structured population in affecting the game dynamics and therefore the frequency of the strategies in the population.

The study of evolutionary dynamics on structured populations was initially introduced by Lieberman *et al.* [6]. The main difference with the classical formulation of the game dynamics is that players are represented as nodes of a network and they play with their neighbors. The player with the highest fitness among the neighbors replaces the one with the lowest fitness, as this aims to capture the evolutionary aspects of the framework. This approach was extensively used to study cooperation in structured populations, where analytical expressions are derived for three update rules: Birth-Death (BD), Death-Birth (DB) and Imitation (IM) [7], [8]. A recent work has extended the replicator equation to regular communities, namely introducing heterogeneity in the game dynamics [9], and providing numerical results in terms of stability.

A prominent line of research, initiated in the work by Weitz *et al.* [10], has gained increasing popularity over the past years because of its original approach in which one considers a bi-directional dependence between the frequencies of the strategies and an environmental resource. This approach takes into account enhancement and degradation effects on this resource. The main contribution of this research consists in system dynamics characterized by oscillatory behaviours and equilibria on the boundary of the phase space. The resulting oscillations correspond to closed periodic orbits. Limit cycles have also been observed when a time-scale difference between the game and the environment dynamics is considered [11].

Manuscript received March 3, 2022; revised May 2, 2022; accepted May 4, 2022. Date of publication May 16, 2022; date of current version May 25, 2022. Recommended by Senior Editor S. Dey. *(Corresponding author: Leonardo Stella.)*

Leonardo Stella is with the School of Computer Science, College of Engineering and Physical Sciences, University of Birmingham, Birmingham B15 2TT, U.K. (e-mail: l.stella@bham.ac.uk).

Wouter Baar is with the Jan C. Willems Center for Systems and Control, ENTEG, Faculty of Science and Engineering, University of Groningen, 9747 AG Groningen, The Netherlands (e-mail: w.baar@rug.nl).

Dario Bauso is with the Jan C. Willems Center for Systems and Control, ENTEG, Faculty of Science and Engineering, University of Groningen, 9747 AG Groningen, The Netherlands, and also with the Dipartimento di Ingegneria, University of Palermo, 90133 Palermo, Italy (e-mail: d.bauso@rug.nl).

Digital Object Identifier 10.1109/LCSYS.2022.3175402

The key point of this feedback mechanism is its ability to explain the complexity of real-world systems [11]. Interactions of this kind can be found in a wide range of disciplines, such as sociology, economics and animal behavior [12]–[15], including social insects. Indeed, in [16], the authors discuss the impact of game-environment feedback in the context of the collective decision-making process originating in honeybee swarms through the parameters of the model. Recently, the collective decision-making process in honeybees has been studied through the development of the optimal control problem, where the parameters of the corresponding mean-field game model act as an implicit environmental feedback [17].

Highlights of contributions: Motivated by our initial work on the impact of irrational behaviors induced by different game dynamics [4], the contribution of this paper focuses on the interplay between the level of cooperation and the node degree in a network and is threefold. First, we introduce the networked PD over a regular network with given node degree. We study the stability of the networked system under three main update rules, Birth-Death (BD), Death-Birth (DB) and Imitation (IM). Second, we extend the game-environment framework to account for a structured population. This system is obtained by the replicator equation resulting from pair approximation and weak selection over a regular network. Lastly, we carry out the stability analysis of the proposed model and discuss the main differences with the traditional framework involving game-environment feedback.

This paper is organized as follows. In Section II, we introduce the model for the structured PD game resulting from replicator dynamics where the structure is captured by a regular graph with given node degree. We carry out the stability analysis of this model and discuss the impact of the connectivity. In Section III, we introduce the game with environmental feedback. In this framework the frequencies of the strategies depend on the state of an environmental resource. Players interact by means of a regular network, which is novel in our formulation of the model. Finally, in Section IV, we draw conclusions and present our future research directions.

II. Networked PD Game Model

In this section, we present the formulation of the PD game with environmental feedback resulting from replicator dynamics. First, we introduce the model where players interact by means of a regular network. Second, we study the stability of the system equilibria and provide a discussion on the impact of the introduced structure.

A. Networked Model

We consider the traditional formulation of the PD where the two available strategies are cooperate (C) or defect (D), henceforth referred to as strategy 1 and strategy 2, respectively. The corresponding payoff matrix is:

$$A = \begin{bmatrix} R & S \\ T & P \end{bmatrix}. \tag{1}$$

In the above, R is the reward for cooperating, S is the sucker's payoff, T is the payoff associated with the temptation to cheat and P is the punishment for cheating as in the classical formulation of the game. Furthermore, it is assumed that $T > R > P > S$, resulting in mutual defection being the only stable Nash equilibrium for this game. We define the fitness of strategy 1 as $f_1(x)$ and the fitness of strategy 2 as $f_2(x)$ in the following [5]:

$$f_1(x) = \sum_j a_{1j}x_j = (Ax)_1 = Rx + S(1-x), \tag{2}$$

$$f_2(x) = \sum_j a_{2j}x_j = (Ax)_2 = Tx + P(1-x), \tag{3}$$

where the frequencies of strategies x_1 and x_2 have been replaced by x and $1-x$, respectively, because of the conservation of mass law, namely, $x_1 + x_2 = 1$. For the sake of brevity, the dependence on time is omitted throughout the paper, i.e., we use x in place of $x(t)$. The average fitness of the population is defined as $\phi = \sum_i x_i f_i$, leading to the replicator equation:

$$\begin{aligned}\dot{x} &= x(f_1(x) - \phi) \\ &= x(1-x)(f_1(x) - f_2(x)).\end{aligned} \tag{4}$$

We can now specialize the above equation to the case of the PD. Indeed, the model resulting from replicator dynamics for the PD is [4]:

$$\dot{x} = x(1-x)((\delta_{PS} - \delta_{TR})x - \delta_{PS}), \tag{5}$$

where $\delta_{TR} = T - R$ and $\delta_{PS} = P - S$.

The above model assumes well-mixed populations without any structure. We can now introduce a structure in the form of a regular network. To this end, let us consider a regular network of degree k. By regular networks, we mean networks where all nodes have the same number of neighbors, or same node degree. Under weak selection, the replicator equation obtained with pair approximation (see [7], [20]) is given by [8]:

$$\dot{x}_i = x_i \left[\sum_{j=1}^{n} (a_{ij} + b_{ij}(k, A)) - \phi \right], \tag{6}$$

where the parameter $b_{ij}(k, A)$ depends on the degree of the network k and the payoff matrix A. In simple terms, the evolutionary game dynamics on a regular graph of degree k can be described by the following transformation of the payoff matrix A: $[a_{ij}] \to [a_{ij} + b_{ij}]$. Furthermore, parameter $b_{ij}(k, A)$ depends on the update rule chosen. In [7], the authors derive three update rules (see also [18]):

- In the Birth-Death (BD) rule, a node is selected with a probability proportional to its fitness and one of its k neighbors at random is replaced by the offspring:

$$b_{ij}(k, A) = \frac{a_{ii} + a_{ij} - a_{ji} - a_{jj}}{k-2}.$$

- In the Death-Birth (DB) rule, a node is randomly selected and one of its k neighbors replaces it with its offspring with a probability proportional to their fitness:

$$b_{ij}(k, A) = \frac{(k+1)a_{ii} + a_{ij} - a_{ji} - (k+1)a_{jj}}{(k+1)(k-2)}.$$

- The Imitation (IM) rule is similar to the DB rule, but the difference is that a node is randomly chosen to update its strategy by imitating one of its k neighbours proportionally to their fitness:

$$b_{ij}(k, A) = \frac{(k+3)a_{ii} + 3a_{ij} - 3a_{ji} - (k+3)a_{jj}}{(k+3)(k-2)}.$$

Note that regardless of the update rule chosen, $b_{ii} = 0$. The replicator equation on regular graph with degree k for the PD is given by:

$$\begin{aligned}\dot{x} &= x(1-x)(f_1(x,k) - f_2(x,k)) \\ &= x(1-x)((\delta_{PS} - \delta_{TR})x - \delta_{PS} + b_{12}(k,A)),\end{aligned} \tag{7}$$

where $f_i(x,k) = \sum_j x_j(a_{ij} + b_{ij}(k,A))$ is the equivalent of the fitness over a regular network.

B. Stability Analysis

Now, we carry out the stability analysis of system (7). In general, the PD game resulting from the replicator dynamics has been extensively studied in the case where the payoff matrix is of the form [8]:

$$A = \begin{bmatrix} b-c & -c \\ b & 0 \end{bmatrix}.$$

In the above, a player choosing to cooperate pays a cost c and receives a benefit b, and it is assumed that $b > c$. In the case of the PD game on regular networks, it is shown that when $b/c > k$, then cooperators win over defectors [8]. Our contribution in this section is to provide the corresponding threshold on k for the general payoff matrix (1).

Lemma 1: Consider system (7). This system has two boundary fixed points, namely $x^* = 0$ and $x^* = 1$, and an internal fixed point $x^* = -(b_{12}(k,A) - \delta_{PS})/(\delta_{PS} - \delta_{TR})$.

Proof: The existence of the equilibria for system (7) can be proved by setting $\dot{x} = 0$. Trivially, we obtain $x^* = 0$, $x^* = 1$ and $x^* = -(b_{12}(k,A) - \delta_{PS})/(\delta_{PS} - \delta_{TR})$. ■

Remark. It is worth noting that the internal fixed point exists only if $0 \leq -(b_{12}(k,A) - \delta_{PS})/(\delta_{PS} - \delta_{TR}) \leq 1$. When this condition is not satisfied, the only equilibria that the system admits are the boundary fixed points.

In the following theorem, we carry out the stability analysis of the system equilibria via the formulation of the threshold for the three update rules discussed in the previous section when the payoff matrix is (1). To this end, let the threshold for the BD, DB and IM rules be, respectively:

$$\mu_{BD} := 1 - \delta_{TR}/\delta_{PS}, \tag{8}$$

$$\mu_{DB} := \frac{R - S + \sqrt{(S-R)^2 - 4(\delta_{PS})(\delta_{TR} - \delta_{PS})}}{2\delta_{PS}}, \tag{9}$$

$$\mu_{IM} := \frac{-(\delta_{PS} + P - R) + \sqrt{\beta}}{2\delta_{PS}}, \tag{10}$$

where $\beta = (\delta_{PS} + P - R)^2 - 12\delta_{PS}(\delta_{TR} - \delta_{PS})$.

We are now in a position to carry out the stability analysis of the equilibrium points of system (7) and use the above thresholds to analyze the change in stability properties.

Theorem 1: Consider system (7) and μ_{BD}, μ_{DB} and μ_{IM} as in (8)-(10). Depending on the update rule, the following stability properties hold.

- For the BD update rule, the only stable equilibrium point is $x^* = 0$, regardless of the network degree k.
- For the DB update rule, the equilibrium point $x^* = 0$ is asymptotically stable when $\mu_{DB} < k$. When $\mu_{DB} > k$, the equilibrium point $x^* = 0$ becomes unstable and the trajectories converge to the interior fixed point $x^* = -(b_{12}(k,A) - \delta_{PS})/(\delta_{PS} - \delta_{TR})$.
- For the IM update rule, the equilibrium point $x^* = 0$ is asymptotically stable when $\mu_{IM} < k$. When $\mu_{DB} > k$, the equilibrium point $x^* = 0$ becomes unstable and the trajectories converge to the interior fixed point $x^* = -(b_{12}(k,A) - \delta_{PS})/(\delta_{PS} - \delta_{TR})$.

Proof: To prove the stability of the system, we take the partial derivative w.r.t. x of the right-hand side of (7) and linearize in $x^* = 0$, yielding $-\delta_{PS} + b_{12}$.

- For the BD updating, the threshold defined in equation (8) is obtained by substituting the value of b_{12} into the above equation. In particular, the fixed point $x^* = 0$ is asymptotically stable when:

$$\begin{aligned}&-k\delta_{PS} + \delta_{PS} - \delta_{TR} < 0, \\ &k - 1 + \delta_{TR}/\delta_{PS} > 0, \\ &k > 1 - \delta_{TR}/\delta_{PS} = \mu_{BD},\end{aligned}$$

  which is always true for $k \geq 2$ since $T > R > P > S$ holds in a PD game.
- DB update rule: asymptotic stability is ensured when $\mu_{DB} > k$. This is obtained from

$$\begin{aligned}-\delta_{PS}(k^2 - k - 2) + (R-P)k - \delta_{TR} - \delta_{PS} < 0, \\ \delta_{PS}k^2 + (S-R)k + \delta_{TR} - \delta_{PS} > 0,\end{aligned}$$

  which can be solved to obtain equation (9), and the solution with the minus sign is either negative or less than 2.
- Similar to the previous case, for the IM update rule, asymptotic stability of $x^* = 0$ is ensured when $\mu_{IM} > k$. The calculation is the following:

$$\begin{aligned}-\delta_{PS}(k^2 + k - 6) + (R-P)(k+3) + 3(S-T) < 0, \\ \delta_{PS}k^2 + (\delta_{PS} + P - R)k + 3\delta_{TR} - 3\delta_{PS} > 0,\end{aligned}$$

  which can be solved to obtain equation (10), and comments similar to the DB updating apply.

This concludes the proof. ■

Remark. The above theorem provides the conditions for stability of the equilibrium point $x^* = 0$. It is worth noting that the other boundary equilibrium point, namely $x^* = 1$, is always unstable, regardless of the network degree k.

Example 1: Consider the following payoff matrix

$$A = \begin{bmatrix} 5 & 0 \\ 8 & 1 \end{bmatrix}.$$

For the sake of this example and the following one, among the three update rules described in the previous section, we consider the DB update rule to show some interesting phenomena. The fitness of strategy 1 and the fitness of strategy 2 of the PD game over a regular network with degree $k = 3$ resulting from equation (7) can be calculated as:

$$\begin{aligned}f_1(x,k) &= 5x + 2(1-x) = 3x + 2, \\ f_2(x,k) &= 6x + 1 - x = 5x + 1,\end{aligned}$$

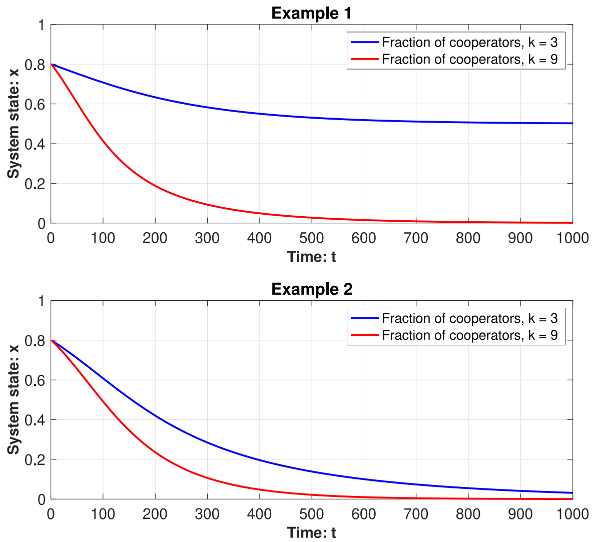

Fig. 1. Time evolution of system (7) over a regular network with degree $k = 3$ and $k = 9$. The first plot (top) shows the evolution of the cooperation strategy for *Example 1*, while the second one (bottom) for *Example 2*.

and the resulting model via replicator dynamics is

$$\dot{x} = x(1-x)(-2x+1).$$

In accordance with Theorem 1, the internal fixed point is asymptotically stable as $\mu_{DB} = 4 > k = 3$. We now study the impact of the network degree on the evolution of the cooperators in the PD game with same payoff matrix. To this end, let us consider a higher network degree, e.g., $k = 9$. The corresponding model can be obtained by analogous calculations (omitted for brevity) as:

$$\dot{x} = x(1-x)\left(-2x - \frac{19}{35}\right).$$

It is important to note that the internal fixed point is not within the feasible set in the state space and therefore the system has only the two boundary equilibria. Therefore, the system converges to the stable equilibrium $x^* = 0$. This shows that smaller connectivity promotes cooperation as clustering is easier. The trajectories corresponding to the time evolution of the system for $k = 3$ and $k = 9$ with initial condition $x_0 = 0.8$ for this example are depicted in Fig. 1 (top).

Example 2: In this example, we consider the case where the same update rule and network degrees are used, namely the DB rule and $k = 3$ and $k = 9$, respectively, but the payoff matrix is different

$$A = \begin{bmatrix} 3 & 0 \\ 5 & 1 \end{bmatrix}.$$

The model resulting from the replicator dynamics over a network with degree $k = 3$ is

$$\dot{x} = x(1-x)\left(-x - \frac{5}{8}\right),$$

while the model over a network with degree $k = 9$ is

$$\dot{x} = x(1-x)\left(-x - \frac{11}{14}\right).$$

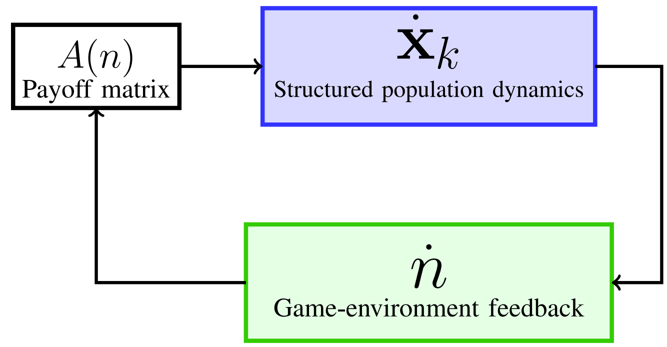

Fig. 2. Diagram representation of the evolutionary game dynamics with environmental feedback. The corresponding structured population dynamics are determined by the payoff matrix $A(n)$ and by the interactions among the players in the population. We study the case on a regular network with node degree k. Finally, there is a mutual dependence between the environmental resource and the frequencies of the strategy through the game-environment feedback.

Differently from the previous example, in this example the internal equilibrium point is outside the state space and therefore the only stable equilibrium point is $x^* = 0$ in either case, meaning that defection is the only stable strategy. Larger connectivity speeds up the convergence to the equilibrium point, namely the transient response is faster (similar to what happens in a similar case where the structure is captured by a complex network [19]). The dynamics for this example with starting condition $x_0 = 0.8$ are depicted in Fig. 1 (bottom).

III. Networked PD With Game-Environment Feedback

In this section, we extend the previous model by considering the game dynamics with environmental feedback. In this framework, the evolution of the frequencies of the strategies x_i for all i are dependent on an environmental resource n and vice versa. This mutual dependence is captured by the dependence of the payoff matrix $A(n)$ on the environmental resource n. Figure 2 shows the diagram representation of the structured framework with game-environment feedback.

A. Networked Model

Because of the interest that feedback-evolving games have sparked in the research community, we extend the framework proposed by Weitz *et al.* [10] to structured populations. In this framework, the payoff matrix depends on an environmental resource n which in turns has an impact on the frequencies of the strategies. To calculate the game-environment feedback dynamics, let the environment-dependent payoff matrix be defined as:

$$\begin{aligned} A(n) &= (1-n)\begin{bmatrix} T & P \\ R & S \end{bmatrix} + n\begin{bmatrix} R & S \\ T & P \end{bmatrix} \\ &= \begin{bmatrix} T - n\delta_{TR} & P - n\delta_{PS} \\ R + n\delta_{TR} & S + n\delta_{PS} \end{bmatrix}. \end{aligned} \tag{11}$$

The model resulting from replicator dynamics can be obtained via:

$$\begin{aligned} \epsilon\dot{x} &= x(1-x)[f_1(x, k, (A(n)) - f_2(x, k, (A(n))], \\ \dot{n} &= n(1-n)[(1+\lambda)x - 1], \end{aligned} \tag{12}$$

TABLE I
LIST OF ALL THE FIXED POINTS FOR SYSTEM (13)

#	x	n
1	0	0
2	1	0
3	0	1
4	1	1
5	$(b_{12}+\delta_{PS})/(\delta_{PS}-\delta_{TR})$	0
6	$-(b_{12}+\delta_{PS})/(\delta_{PS}-\delta_{TR})$	1
7	$1/(\lambda+1)$	$(b_{12}+\delta_{TR}+b_{12}\lambda+\delta_{PS}\lambda)/2(\delta_{TR}-\delta_{PS}\lambda)$

where the term $n(1-n)$ is used to ensure that the state of the environment is within the domain [0, 1] and the rate at which the population dynamics change the environment is denoted by ϵ. Parameter $\lambda > 0$ represents the ratio between the enhancement and degradation effects, resulting from cooperation and defection, respectively. When $\lambda < 1$, the degradation effect is stronger than the enhancement effect; when $\lambda = 1$, the two effects are in balance; when $\lambda > 1$ the enhancement effect is stronger than the degradation effect. As before, the fitness of a strategy i depends on the interactions of the players in the population but now parameters $b_{ij}(k, A(n))$ depend on the environmental resource n as well. Therefore, the structured population dynamics for the PD game with environmental feedback are:

$$\epsilon\dot{x} = x(1-x)[(\delta_{PS} + (\delta_{TR} - \delta_{PS})x)(1-2n) + b_{12}],$$
$$\dot{n} = n(1-n)[(1+\lambda)x - 1], \quad (13)$$

where b_{12} has been used in place of $b_{12}(k, A(n))$ for brevity.

B. Stability Analysis

The focus of this section is to study the stability of the networked system with environmental feedback. Specifically, in the next lemma we prove the existence of the fixed points of system (13) and later we study their stability properties in relation to the network degree k.

Lemma 2: Consider system (13). This system has 7 fixed points, as listed in Table I. The first four represent boundary fixed points, namely $(x^*, n^*) = (0, 0)$ defectors in a depleted environment, (1, 0) cooperators in a depleted environment, (0, 1) defectors in a replete environment, and (1, 1) cooperators in a replete environment. The fifth and sixth fixed points represent a mixed population in a depleted and replete environment, respectively. Finally, the last one is the only internal fixed point, namely a mixed population in an intermediate environment.

Proof: To prove the existence of the equilibria listed in Table I, we set $\dot{x} = 0, \dot{n} = 0$. The four boundary fixed points can be trivially calculated when the outer products are set to zero, while the fifth and sixth fixed points by setting the content of the square bracket in the first equation to zero:

$$(\delta_{PS} + (\delta_{TR} - \delta_{PS})x)(1-2n) + b_{12} = 0,$$

and substituting $n^* = 0$ and $n^* = 1$, respectively. Finally, the internal fixed point can be calculated by setting to zero the content of the square bracket in the second equation as

$$(1+\lambda)x - 1 = 0,$$

and then substituting the value $x^* = 1/(\lambda+1)$ into the first equation. ■

Now, we turn our attention to the stability of the fixed point stated in the above lemma. We consider the impact of the connectivity onto the stability properties of the system.

Theorem 2: Consider system (13). The only stable equilibrium point is the internal fixed point $(x^*, n^*) = 1/(\lambda + 1), (b_{12} + \delta_{TR} + b_{12}\lambda + \delta_{PS}\lambda)/2(\delta_{TR} - \delta_{PS}\lambda)$. With increasing connectivity k, the internal fixed point turns neutrally stable and the system exhibits an oscillatory behavior.

Proof: To study the stability of this system, we derive the Jacobian:

$$J(x, n) = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \quad (14)$$

where $J_{11} = (1-x)[(\delta_{PS} + (\delta_{TR} - \delta_{PS})x)(1-2n) + b_{12}]$, $J_{12} = -2x(1-x)[\delta_{PS} + (\delta_{TR} - \delta_{PS})x]$, $J_{21} = n(1-n)(1+\lambda)$ and $J_{22} = (1-n)[(1+\lambda)x-1]$. The above Jacobian, linearized about the first of the boundary fixed points yields:

$$J(x, n) = \begin{bmatrix} \delta_{PS} + b_{12} & 0 \\ 0 & -1 \end{bmatrix},$$

which is asymptotically stable when $\delta_{PS}+b_{12} < 0$ and unstable otherwise as one of the two eigenvalues would be positive. A similar calculation can be done for all the other fixed points, except for the internal one, which is asymptotically stable. With increasing connectivity, applying the separability of the factors for the internal fixed point yields a constant of motion, meaning that the internal fixed point exhibits periodic closed orbits [4]. ■

Example 3 and 4: Given the payoff matrix in (11), consider system (13) under the DB update rule with the following parameters:

$$\epsilon = 0.3, \quad \lambda = 2,$$
$$R = 3, \quad S = 0, \quad T = 5, \quad P = 1.$$

For the sake of this example, we consider a node degree $k = 3$, first. The corresponding model is:

$$0.3\dot{x} = x(1-x)[(1-2n)(1+x) + b_{12}],$$
$$n = n(1-n)[3x-1].$$

Among the seven fixed points, the only stable equilibrium point is the internal fixed point $(x^*, n^*) = (0.33, 0.91)$, in accordance with Theorem 2. With increasing connectivity, e.g., $k = 9$, the interior fixed point turns neutrally stable, since the corresponding Jacobian matrix has both eigenvalues with

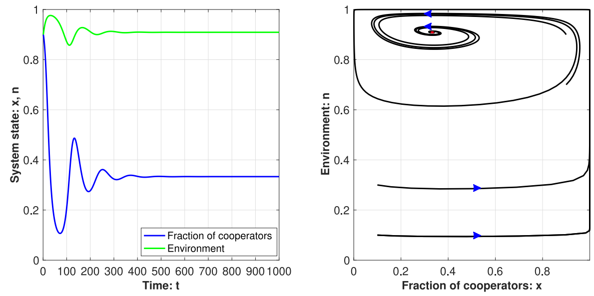

Fig. 3. Feedback-evolving game model with $k = 3$ (*Example 3*). The first plot (left) shows the evolution of cooperators and the environment over time. The second plot (right) shows the phase plane dynamics in the x-n plane for system (13): the blue arrows indicate the direction of dynamics while the red dot denotes the interior fixed point.

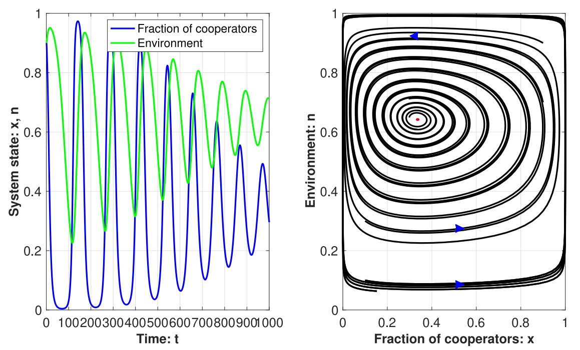

Fig. 4. Feedback-evolving game model with $k = 9$ (*Example 4*). The first plot (left) shows the evolution of cooperators and the environment over time. The second plot (right) shows the phase plane dynamics in the x-n plane for system (13): the blue arrows indicate the direction of dynamics while the red dot denotes the interior fixed point.

purely imaginary components. Differently from the results with mixed populations, introducing a structure in the form of a regular network into the game dynamics changes the behavior of the dynamical system by turning the oscillating tragedy of the commons into a situation where cooperation is favored. However, increasing the connectivity brings the system back to periodic orbits. Figures 3 and 4 show the time evolution and phase plane dynamics for these examples.

IV. Conclusion

In this work, we have studied the impact of a structured population onto the prisoner's dilemma game resulting from replicator dynamics. We have modeled the structure via a regular network with given node degree and have carried out the stability analysis for three update rules. Motivated by the recent interest towards feedback-evolving games, we have extended our results to this framework. We have discussed the differences between our model and prior works and, interestingly, the behaviour of these two systems under game-environment feedback is notably different. Future directions of research will focus on the characterization of the threshold for the network degree in terms of the change of stability from periodic orbits and of the relative speed of environmental versus strategic change.

References

[1] J. M. Smith and G. R. Price, "The logic of animal conflict," *Nature*, vol. 246, no. 5427, pp. 15–18, 1973, doi: 10.1038/246015a0.

[2] J. M. Smith, *Evolution and the Theory of Games*. Cambridge, U.K.: Cambridge Univ. Press, 1982.

[3] R. Axelrod and W. D. Hamilton, "The evolution of cooperation," *Science*, vol. 211, no. 4489, pp. 1390–1396, 1981, doi: 10.1126/science.7466396.

[4] L. Stella and D. Bauso, "The impact of irrational behaviors in the optional Prisoner's dilemma with game-environment feedback," *Int. J. Robust Nonlinear Control*, to be published. [Online]. Available: https://doi.org/10.1002/rnc.5935

[5] J. Hofbauer, "Deterministic evolutionary game dynamics," in *Proc. Symposia Appl. Math.*, vol. 69, 2011, pp. 61–79, doi: 10.1090/psapm/069.

[6] E. Lieberman, C. Hauert, and M. A. Nowak, "Evolutionary dynamics on graphs," *Nature*, vol. 433, pp. 312–316, Jan. 2005, doi: 10.1038/nature03204.

[7] H. Ohtsuki, C. Hauert, E. Lieberman, and M. A. Nowak, "A simple rule for the evolution of cooperation on graphs and social networks," *Nature*, vol. 441, no. 7092, pp. 502–505, May 2006, doi: 10.1038/nature04605.

[8] H. Ohtsuki and M. A. Nowak, "The replicator equation on graphs," *J. Theor. Biol.*, vol. 243, no. 1, pp. 86–97, 2006, doi: 10.1016/j.jtbi.2006.06.004.

[9] D. Cassese, "Replicator equation on networks with degree regular communities," *Appl. Netw. Sci.*, vol. 3, no. 1, p. 29, 2018, doi: 10.1007/s41109-018-0083-2.

[10] J. Weitz, C. Eksin, K. Paarporn, S. Brown, and W. Ratcliff, "An oscillating tragedy of the commons in replicator dynamics with game-environment feedback," *Proc. Nat. Acad. Sci.*, vol. 113, no. 47, pp. E7518–E7525, 2016, doi: 10.1073/pnas.1604096113.

[11] A. Tilman, J. Plotkin, and E. Akçay, "Evolutionary games with environmental feedbacks," *Nat. Commun.*, vol. 11, no. 1, p. 915, 2020, doi: 10.1038/s41467-020-14531-6.

[12] D. Pais, C. Caicedo-Núñez, and N. Leonard, "Hopf bifurcations and limit cycles in evolutionary network dynamics," *SIAM J. Appl. Dyn. Syst.*, vol. 11, no. 4, pp. 1754–1784, 2012, doi: 10.1137/120878537.

[13] C. Mullon, L. Keller, and L. Lehmann, "Social polymorphism is favoured by the co-evolution of dispersal with social behaviour," *Nat. Ecol. Evol.*, vol. 2, no. 1, pp. 132–140, 2018, doi: 10.1038/s41559-017-0397-y.

[14] J. Lee, Y. Iwasa, U. Dieckmann, and K. Sigmund, "Social evolution leads to persistent corruption," *Proc. Nat. Acad. Sci.*, vol. 116, no. 27, pp. 13276–13281, 2019, doi: 10.1073/pnas.1900078116.

[15] S. Estrela *et al.*, "Environmentally mediated social dilemmas," *Trends Ecol. Evol.*, vol. 34, no. 1, pp. 6–18, 2019, doi: 10.1016/j.tree.2018.10.004.

[16] W. Baar and D. Bauso, "Environmental feedback incorporated on a collective decision making model," in *Proc. 21st IFAC World Congr.*, 2020, vol. 53, pp. 2832–2837, doi: 10.1016/j.ifacol.2020.12.953.

[17] L. Stella, D. Bauso, and P. Colaneri, "Mean-field game for collective decision-making in honeybees via switched systems," *IEEE Trans. Autom. Control*, early access, Sep. 3, 2021, doi: 10.1109/TAC.2021.3110166.

[18] S. Tan, J. Lü, G. Chen, and D. J. Hill, "When structure meets function in evolutionary dynamics on complex networks," *IEEE Circuits Syst. Mag.*, vol. 14, no. 4, pp. 36–50, 4th Quart., 2014, doi: 10.1109/MCAS.2014.2360790.

[19] L. Stella and D. Bauso, "Bio-inspired evolutionary dynamics on complex networks under uncertain cross-inhibitory signals," *Automatica*, vol. 100, pp. 61–66, Feb. 2019, doi: 10.1016/j.automatica.2018.11.005.

[20] H. Matsuda, N. Ogita, A. Sasaki, and K. Satō, "Statistical mechanics of population: The lattice Lotka-Volterra model," *Progr. Theor. Phys.*, vol. 88, no. 6, pp. 1035–1049, 1992, doi: 10.1143/ptp/88.6.1035.