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On the Newton–Kleinman method for strongly stabilizable infinite-dimensional systems☆

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ABSTRACT

We consider the Newton–Kleinman method for strongly stabilizable infinite-dimensional systems. Under certain assumptions, the maximal self-adjoint solution to the associated control algebraic Riccati equation is constructed. The constructed solution is also the maximal solution to the corresponding control algebraic Riccati inequality.

1. Introduction

We consider the control algebraic Riccati equation (CARE)

$$A^*X + XA - XBB^*X + C^*C = 0,$$

which has been widely studied in the area of systems and control for both finite-dimensional and infinite-dimensional systems, together with the corresponding control algebraic inequality (CARI)

$$A^*X + XA - XBB^*X + C^*C \geq 0.$$

There are many approaches to the study of the CARE and we will mention only some of them which are most related to this note. One approach is in the line with Willems (1971), where comparison results and a classification of all solutions of the CARE for finite-dimensional systems have been provided. Moreover, the solvability of the CARI implies the solvability of the CARE for finite-dimensional controllable systems (also in Willems, 1971). Necessary and sufficient conditions for the existence of a maximal/minimal solution of the CARE or CARI for sign-controllable systems can be found in Scherer (1991). A comparison of the solutions of two CARE associated to different systems was provided for the first time in Wimmer (1985). A second approach is in terms of the Hamiltonian (see e.g. Martensson, 1971). A third approach is based on Kleinman (1968), and provides iterative procedures for constructing hermitian solutions of the CARE as proposed in Gohberg, Lancaster, and Rodman (1986) and Ran and Vreugdenhil (1988). Moreover, it has been shown in Gohberg et al. (1986) and Ran and Vreugdenhil (1988) that solvability of the CARI implies the existence of a maximal solution of the CARI provided that (A, B) is stabilizable. Furthermore, the maximal solution of the CARI also satisfies the CARE.

Some of the results mentioned above for finite-dimensional systems have been extended to infinite-dimensional systems $\Sigma(A, B, C)$. Following the approach in Willems (1971), a classification of all non-negative self-adjoint solution of the CARE for exponentially stabilizable systems has been proposed in Callier, Dumortier, and Winkin (1995) and for discrete-time infinite-dimensional systems in Malinen (2000). In Iftime, Curtain, and Zwart (2005) a representation of all self-adjoint solutions to the CARE has been obtained under the following assumptions: A generates a strongly continuous semigroup, output stabilizability, strong detectability and the invertibility of the minimal self-adjoint solution of the filter algebraic Riccati equation. Following the approach in Martensson (1971), the relation between the CARE and the eigenvectors of the Hamiltonian for Riesz spectral systems has been studied in Kuiper and Zwart (1993).

In this note the third approach is followed, known in the literature as the Newton–Kleinman method. The finite-dimensional case has been initiated in Kleinman (1968). Since then, this problem has been widely studied for finite-dimensional case (see for example Ran & Vreugdenhil, 1988 and the references therein) and

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also for infinite-dimensional systems which are exponentially stabilizable (see for example Burns, Sachs, & Zietsman, 2008; Curtain & Rodman, 1990). However, there are many infinite-dimensional systems which are not exponentially stabilizable but have nice stability properties (see for example Oostveen, 2000). We provide a Newton–Kleinman type result for strongly stabilizable infinite-dimensional systems. More precisely, provided that $\Sigma(A, B, C)$ is strongly stabilizable and additional assumptions are satisfied, an iterative procedure for constructing the maximal self-adjoint solution to the CARE as the strong limit of Newton–Kleinman iterates. Moreover, it is shown that the maximal solution of the CARI also satisfies CARE.

This note is structured as follows. In Section 2 the necessary concepts are introduced and two preliminary results (Theorem 2.2 and Lemma 2.5) are stated. The main result is Theorem 3.4 and it is presented in Section 3.

Notation: $\mathcal{L}(U, Z)$ denotes the set of bounded operators from the Hilbert space U to the Hilbert space Z. Let B^* be the adjoint operator of $B \in \mathcal{L}(U, Z)$. For self-adjoint operators $X_1, X_2 \in \mathcal{L}(Z)$, by $X_1 \geq X_2$ we understand $X_1 - X_2 \geq 0$.

2. Preliminaries

Let Z, Y and U be separable Hilbert spaces. Consider a state linear system $\Sigma(A, B, C)$ given by

$$\begin{cases} \dot{z}(t) &= Az(t) + Bu(t), \qquad z(0) = z_0 \in Z \\ y(t) &= Cz(t), \end{cases} \tag{1}$$

where $B \in \mathcal{L}(U, Z)$, $C \in \mathcal{L}(Z, Y)$, $z \in Z$, $y \in Y$ and $u \in U$. The unbounded operator $A : \mathcal{D}(A) \subset Z \to Z$ generates a strongly continuous semigroup $T(t)$. If $T(t)z \to 0$ as $t \to \infty$ (i.e. $\lim_{t\to\infty} \|T(t)z\|_Z = 0$) for all $z \in Z$, then $T(t)$ is called a *strongly stable semigroup*.

Recall now the notions of output stability, output stabilizability and strong stabilizability.

Definition 2.1. The system $\Sigma(A, B, C)$ is *output stable* if there exists a constant $\gamma > 0$ such that

$$\int_0^\infty \|CT(s)z\|_Y^2 ds \leq \gamma \|z\|_Z^2, \quad \text{for all } z \in Z.$$

It is known that (see e.g. Grabowski, 1991) output stability of the system $\Sigma(A, B, C)$ is equivalent to the existence of a nonnegative self-adjoint solution $\Pi \in \mathcal{L}(Z)$ of the Lyapunov operator equation

$$\langle Az_1, \Pi z_2\rangle + \langle \Pi z_1, Az_2\rangle = -\langle Cz_1, Cz_2\rangle, \tag{2}$$

where $z_1, z_2 \in \mathcal{D}(A) \subset Z$. The following result is a dual version of Hansen and Weiss (1997, Theorem 3.1). Note that the notion of output stability for the infinite dimensional systems $\Sigma(A, B, C)$ is the same as requiring (in Hansen & Weiss, 1997) that C is an infinite-time admissible operator for $T(t)$.

Theorem 2.2 (*Hansen & Weiss, 1997*). *Consider the system $\Sigma(A, B, C)$. The following statements are equivalent:*

- (i) *The system $\Sigma(A, B, C)$ is output stable.*
- (ii) *There exists an operator $Q \in \mathcal{L}(Z)$ such that, for every $z \in Z$*

$$Qz := \int_0^\infty T^*(t)C^*CT(t)zdt. \tag{3}$$

- (iii) *There exist operators $\Pi \in \mathcal{L}(Z)$, $\Pi \geq 0$ which satisfy the Lyapunov operator equation (2).*

Moreover, if any of the above statements holds, then the following statements are true:

- (I) *Q defined in (3) satisfies (2) and it is the smallest nonnegative solution of (2).*
- (II) *For any $z \in Z$, $\lim_{t\to\infty} Q^{\frac{1}{2}}T(t)z = 0$. In particular, if $Q > 0$ then $T(t)$ is strongly stable.*
- (III) *If $T(t)$ is strongly stable, then Q is the unique self-adjoint solution of (2).*

Definition 2.3. The system $\Sigma(A, B, C)$ is *output stabilizable* if there exists an $F \in \mathcal{L}(Z, U)$ such that $\Sigma\left(A + BF, B, \begin{bmatrix} F \\ C \end{bmatrix}\right)$ is output stable.

Definition 2.4. The system $\Sigma(A, B, C)$ is *strongly stabilizable* if the following conditions are satisfied

- it is output stabilizable by some $F \in \mathcal{L}(Z, U)$, and
- $A_F := A + BF$ generates a strongly stable semigroup $T_F(t)$.

To the system (1) one associates the classical cost to be minimized on infinite-time

$$J(z_0, u) := \int_0^\infty (\langle Cz(s), Cz(s)\rangle + \langle u(s), Ru(s)\rangle)\, ds$$

where R is a self-adjoint coercive operator in $\mathcal{L}(U)$. For the sake of simplicity of the exposition we shall take $R = I$ in the sequel. The cost $J(z_0, u)$ is closely related to the (control) algebraic Riccati equation (see e.g. Curtain & Zwart, 1995)

$$\langle Az_1, Xz_2\rangle + \langle Xz_1, Az_2\rangle - \langle B^*Xz_1, B^*Xz_2\rangle + \langle Cz_1, Cz_2\rangle = 0 \tag{4}$$

for $z_1, z_2 \in \mathcal{D}(A) \subset Z$. Define $\mathcal{R}(X)$ weakly by $\langle z_1, \mathcal{R}(X)z_2\rangle$ being equal to the left hand side of (5). Then (4) can be written as

$$\langle z_1, \mathcal{R}(X)z_2\rangle = 0, \quad z_1, z_2 \in \mathcal{D}(A) \subset Z. \tag{5}$$

Consider also the control algebraic Riccati inequality (CARI)

$$\langle z, \mathcal{R}(X)z\rangle \geq 0, \quad z \in \mathcal{D}(A) \subset Z. \tag{6}$$

Define the sets of self-adjoint solutions of the CARE (5) and the CARI (6) as

$$\mathcal{S}_{\mathcal{E}} := \left\{X \mid X = X^*,\ \langle z_1, \mathcal{R}(X)z_2\rangle = 0,\ z_1, z_2 \in \mathcal{D}(A)\right\} \tag{7}$$

$$\mathcal{S}_{\mathcal{I}} := \left\{X \mid X = X^*,\ \langle z, \mathcal{R}(X)z\rangle \geq 0,\ z \in \mathcal{D}(A)\right\}. \tag{8}$$

One would also need the following result.

Lemma 2.5. *Consider the system $\Sigma(A, B, C)$ such that A generates a strongly stable continuous semigroup $T(t)$. If $L = L^* \in \mathcal{L}(Z)$ satisfies*

$$\langle Az, Lz\rangle + \langle Lz, Az\rangle \leq 0, \quad z \in \mathcal{D}(A) \subset Z, \tag{9}$$

then $L \geq 0$.

Proof. Let $z \in \mathcal{D}(A) \in Z$, $t \geq 0$ and define

$$f(t) := \langle T(t)z, LT(t)z\rangle,$$

which is continuous differentiable. By taking the derivative with respect to t, then integrating on $[0, \tau]$, letting $\tau \to \infty$ and using strong stability of $T(t)$ one has $\langle z, Lz\rangle \geq 0$ on $\mathcal{D}(A)$. Using the density of $\mathcal{D}(A)$ in Z the proof is completed. ■

3. Main results

Consider a strongly stabilizable system $\Sigma(A, B, C)$. Let $F \in \mathcal{L}(Z, U)$ be such that $A_F := A + BF$ generates a strongly continuous semigroup $T_F(t)$. Then the system $\Sigma\left(A_F, B, \begin{bmatrix} F \\ C \end{bmatrix}\right)$ is output stable (see Oostveen, 2000, Definition 2.1.7), and the Lyapunov equation

$$\langle A_F z_1, X_F z_2\rangle + \langle X_F z_1, A_F z_2\rangle = -\langle F z_1, F z_2\rangle - \langle C z_1, C z_2\rangle, \quad (10)$$

for $z_1, z_2 \in \mathcal{D}(A) \subset Z$, has a unique self-adjoint solution $X_F \in \mathcal{L}(Z)$ (see Theorem 2.2).

Since $\Sigma(A, B, C)$ is strongly stabilizable, then there exists X_+, a minimal self-adjoint, nonnegative solution of the CARE (5), and the minimizing control for the cost is given by $u(t) = -B^* X_+ z(t)$ (see Oostveen, 2000, Theorem 3.3.2). Obviously X_+ also satisfies CARI (6). Therefore the sets $\mathcal{S}_{\mathcal{E}}$ and $\mathcal{S}_{\mathcal{I}}$ are non-empty.

Theorem 3.1. *Consider a strongly stabilizable system $\Sigma(A, B, C)$. Let $X_F \in \mathcal{L}(Z)$ be the unique self-adjoint solution of (10) and $X \in \mathcal{S}_{\mathcal{I}}$ a self-adjoint solution of the CARI (6). Then*

$$X \le X_F.$$

Proof. Let $z_1, z_2 \in \mathcal{D}(A)$. One can write the following sequence of equalities:

$$\begin{aligned}
&\langle A_F z_1, (X_F - X) z_2\rangle + \langle (X_F - X) z_1, A_F z_2\rangle \\
&\quad = \langle A_F z_1, X_F z_2\rangle + \langle X_F z_1, A_F z_2\rangle \\
&\qquad - \langle (A + BF) z_1, X z_2\rangle - \langle X z_1, (A + BF) z_2\rangle \\
&\quad = -\langle F z_1, F z_2\rangle - \langle C z_1, C z_2\rangle \\
&\qquad - (\langle A z_1, X z_2\rangle + \langle BF z_1, X z_2\rangle) \\
&\qquad - (\langle X z_1, A z_2\rangle + \langle X z_1, BF z_2\rangle) \\
&\quad = -(\langle A z_1, X z_2\rangle + \langle X z_1, A z_2\rangle - \langle B^* X z_1, B^* X z_2\rangle \\
&\qquad + \langle C z_1, C z_2\rangle) - \langle B^* X z_1, B^* X z_2\rangle \\
&\qquad - \langle F z_1, F z_2\rangle - \langle FB z_1, X z_2\rangle - \langle X z_1, BF z_2\rangle \\
&\quad = -\langle z_1, \mathcal{R}(X) z_2\rangle - \langle (F + B^* X) z_1, (F + B^* X) z_2\rangle.
\end{aligned}$$

If one denotes $\hat{F} := F + B^* X$ and takes $z_1 = z_2 = z$ then

$$\begin{aligned}
&\langle A_F z, (X_F - X) z\rangle + \langle (X_F - X) z, A_F z\rangle \\
&\quad = -\langle z, \mathcal{R}(X) z\rangle - \langle \hat{F} z, \hat{F} z\rangle \le 0.
\end{aligned}$$

Since A_F is strongly stable, it follows from Lemma 2.5 that

$$X_F - X \ge 0$$

and the proof is completed. ■

The next result is a consequence of Theorems 3.1 and 2.2.

Lemma 3.2. *Consider a strongly stabilizable system $\Sigma(A, B, C)$. Let $X_F \in \mathcal{L}(Z)$ be the unique self-adjoint solution of (10) and $X \in \mathcal{S}_{\mathcal{I}}$ a self-adjoint solution of the CARI (6). Define $A_1 := A - BB^* X_F$ with $\mathcal{D}(A_1) = \mathcal{D}(A)$ and*

$$\begin{aligned}
Q := \int_0^\infty & T_1(t)^* (\mathcal{R}(\mathcal{X}) + (X_F - X) BB^* (X_F - X) \\
&+ (B^* X_F - F)^* (B^* X_F - F)) T_1(t) dt \quad (11)
\end{aligned}$$

where $T_1(t)$ is the strongly continuous semigroup generated by A_1. Note that one must interpret the integral in (11) as a quadratic form on $\mathcal{D}(A_1) \times \mathcal{D}(A_1)$. Let $z_1, z_2 \in \mathcal{D}(A)$. The following statements hold:

(1) *$X_F - X$ satisfies*

$$\begin{aligned}
&\langle A_1 z_1, (X_F - X) z_2\rangle + \langle (X_F - X) z_1, A_1 z_2\rangle \\
&\quad = -\langle z_1, \mathcal{R}(X) z_2\rangle - \langle B^*(X_F - X), B^*(X_F - X)\rangle \\
&\qquad - \langle (B^* X_F - F) z_1, (B^* X_F - F) z_2\rangle \quad (12)
\end{aligned}$$

(2) *$Q \in \mathcal{L}(Z)$ is the smallest positive solution of (12). Moreover, if $Q > 0$ then $T_1(t)$ is a strongly stable semigroup and $Q = X_F - X$ is the unique self-adjoint solution of (12).*

Proof. The fact that A_1 generates a strongly continuous semigroup $T_1(t)$ follows from Curtain and Zwart (1995, Theorem 3.2.1).

(1): Similar to the sequence of equalities in the proof of Theorem 3.1, one obtains that $X_F - X \in \mathcal{L}(Z)$ satisfies Eq. (12).

(2): Since $X \in \mathcal{S}_{\mathcal{I}}$, then

$$\begin{aligned}
&\langle z_1, \mathcal{R}(\mathcal{X}) z_2\rangle + \langle z_1, (X_F - X) BB^* (X_F - X) z_2\rangle \\
&\quad + \langle z_1, (B^* X_F - F)^* (B^* X_F - F) z_2\rangle \ge 0. \quad (13)
\end{aligned}$$

From Theorem 3.1 one has $X_F - X \ge 0$. Then $X_F - X$ is nonnegative and it satisfies Eq. (12). Let $z_1 = z_2 = z$ and note that $\langle z, \mathcal{R}(\mathcal{X}) z\rangle$ is defined on $\mathcal{D}(A)$ which is dense in Z. Since the left-hand side of (12) is integrable, self-adjoint and nonnegative (see (13)), one can extend it (and also the right-hand side of (12)) to a bounded (self-adjoint) operator on Z. But every nonnegative, self-adjoint bounded operator has a unique self-adjoint square root and the right-hand side can be factorized as $C_1^* C_1$. Consequently, Eq. (12) can be extended to a Lyapunov equation of form (2). Then the statement (iii) in Theorem 2.2 corresponding to (12) is satisfied. One can now apply Theorem 2.2 part (II) and (III) to draw the conclusion. To complete the proof, one can use a standard technique to show that the operator Q is bounded (the quadratic form in (11) is induced by a bounded operator). More precisely, define the continuous differentiable function

$$E_t(z) := \langle (X_F - X) T_1(t) z, T_1(t) z\rangle.$$

Note that $E_t(z)$ is nonnegative. Taking the derivative with respect to t one has

$$\begin{aligned}
\frac{d}{dt} E_t(z) &= \langle (X_F - X) A_1 T_1(t) z, T_1(t) z\rangle \\
&\quad + \langle (X_F - X) T_1(t) z, A_1 T_1(t) z\rangle \le 0, \quad (14)
\end{aligned}$$

so $E_t(z)$ is nonincreasing. Since $E_t(z)$ is continuous in z and $\mathcal{D}(A)$ is dense in Z, $E_t(z)$ is nonincreasing for any $z \in Z$. So for $t \ge s \ge 0$,

$$T_1^*(t)(X_F - X) T_1(t) \le T_1^*(s)(X_F - X) T_1(s).$$

So we have a nonincreasing sequence of positive operator valued functions and consequently a strong limit exists as $t \to \infty$. Integrating (14) on $[0, \infty)$ and using (12) it follows that Q is bounded. ■

Let $A_0 = A - BF$ and $X_0 = X_F$. Then

$$A_1 = A - BB^* X_0. \quad (15)$$

Let also $Q_0 = Q$ from (11) and assume that $Q_0 > 0$. From Lemma 3.2(2), A_1 generates a strongly stable semigroup $T_1(t)$, and assume that the system $\Sigma\left(A_1, B, \begin{bmatrix} B^* X_0 \\ C \end{bmatrix}\right)$ is output stable. Therefore (see Theorem 2.2)

$$X_1 := \int_0^\infty T_1^*(t)(X_0 BB^* X_0 + C^* C) T_1(t) dt \quad (16)$$

is the unique self-adjoint solution of the Lyapunov equation

$$\begin{aligned}
\langle A_1 z_1, X_1 z_2\rangle + \langle X_1 z_1, A_1 z_2\rangle &= -\langle B X_0 z_1, B X_0 z_2\rangle \\
&\quad - \langle C z_1, C z_2\rangle \quad (17)
\end{aligned}$$

Define inductively the sequences of operators

$$A_i = A - BB^* X_{i-1}, \quad i = 1, \ldots, n \quad (18)$$

$$X_i = \int_0^\infty T_i(t)^* (X_{i-1} BB^* X_{i-1} + C^* C) T_i(t) dt$$

$$\begin{aligned}
Q_i = \int_0^\infty & T_i(t)^* (\mathcal{R}(\mathcal{X}) + (X_i - X) BB^* (X_i - X) \\
&+ (X_i - X_{i-1}) BB^* (X_i - X_{i-1})) T_i(t) dt
\end{aligned}$$

where $\mathcal{D}(A_i) = \mathcal{D}(A) \subset Z$. Note that one must interpret the integral from the definition of Q_i as a quadratic form on the appropriate domain, similarly to (11) as it was done in Lemma 3.2.

Lemma 3.3. *Consider the sequences of operators defined in (18) and assume that $Q_i > 0$ for $i = 0, \ldots, n-1$. The following statements hold:*

(1) *A_i, $i = 1, \ldots, n$, generates a strongly stable semigroup $T_i(t)$;*
(2) *X_i, $i = 1, \ldots, n$, is the unique self-adjoint solution of the Lyapunov equation*

$$\langle A_i z_1, X_i z_2\rangle + \langle X_i z_1, A_i z_2\rangle = -\langle B^* X_{i-1}, B^* X_{i-1}\rangle - \langle Cz_1, Cz_2\rangle \quad (19)$$

(3) *Let $X \in \mathcal{S}_{\mathcal{I}}$ be a self-adjoint solution of the CARI (6). Then*

$$X \leq X_i, \quad i = 0, \ldots, n.$$

Proof. (1): From Lemma 3.2, A_1 generates a strongly stable semigroup $T_1(t)$. For $i = 2, \ldots, n, X_{i-1} - X$, $i = 2, \ldots, n$, satisfies the equality

$$\langle A_i z_1, (X_{i-1} - X)z_2\rangle + \langle (X_{i-1} - X)z_1, A_1 z_2\rangle = -\langle z_1, \mathcal{R}(X)z_2\rangle - \langle B^*(X_i - X), B^*(X_i - X)\rangle - \langle B^*(X_i - X_{i-1})z_1, B^*(X_i - X_{i-1})z_2\rangle. \quad (20)$$

Then, using similar reasoning as in the proof of Lemma 3.2(2), $Q_{i-1} \in \mathcal{L}(Z)$ is the smallest nonnegative solution of (20). Using $Q_{i-1} > 0$, one obtains that $T_i(t)$ is a strongly stable semigroup.

(2): Using that $T_i(t)$ is a strongly stable semigroup and that the system $\Sigma\left(A_i, B, \begin{bmatrix} -B^*X_{i-1} \\ C \end{bmatrix}\right)$ is output stable, we can apply Theorem 2.2 to obtain that X_i is the unique self-adjoint solution of (19).

(3): Since $X_0 = X_F$, $X \leq X_0$ is stated in Theorem 3.1. The inequality $X \leq X_i$, $i = 1, \ldots, n$, follows in the same way as in Theorem 3.1 using the equality

$$\langle A_i z_1, (X_i - X)z_2\rangle + \langle (X_i - X)z_1, A_i z_2\rangle = -\langle z_1, \mathcal{R}(X)z_2\rangle - \langle B^*(X_i - X)z_1, B^*(X_i - X)z_2\rangle$$

and the fact that

$$-\langle z, \mathcal{R}(X)z\rangle - \langle B^*(X_i - X)z, B^*(X_i - X)z\rangle \leq 0.$$

The above inequality is true because $X \in \mathcal{S}_{\mathcal{I}}$. ■

One can now state the main result.

Theorem 3.4. *Consider a strongly stabilizable system $\Sigma(A, B, C)$. Consider the sequences of operators $(X_i)_{i=0}^{n}$ and $(Q_i)_{i=0}^{n-1}$ defined in (18), and assume that $Q_i > 0$, $i = 0, \ldots, n-1$. Then the sequence $(X_i)_{i=0}^{n}$ converges strongly to a bounded operator $X_+ \in \mathcal{L}(Z)$. Moreover,*

(1) *For any $X \in \mathcal{S}_{\mathcal{I}}$ a self-adjoint solution of the CARI (6)*

$$X \leq X_+.$$

(2) *$X_+ \in \mathcal{S}_{\mathcal{E}}$, i.e. X_+ is a self-adjoint solution of the CARE (4).*

Proof. (1): From Lemma 3.3, X_i is the unique self-adjoint solution of (19). Then the following equality holds for $z_1, z_2 \in \mathcal{D}(A)$

$$\langle A_i z_1, (X_{i-1} - X_i)z_2\rangle + \langle (X_{i-1} - X_i)z_1, A_i z_2\rangle = -\langle B^*(X_{i-1} - X_{i-2})z_1, B^*(X_{i-1} - X_{i-2})z_2\rangle$$

and so,

$$\langle A_i z_1, (X_{i-1} - X_i)z_2\rangle + \langle (X_{i-1} - X_i)z_1, A_i z_2\rangle \leq 0.$$

Take $z_1 = z_2 = z$ and use Lemma 2.5 to obtain

$$X_{i-1} - X_i \geq 0.$$

The sequence $(X_i)_{i=0}^{n}$ of bounded operators in $\mathcal{L}(Z)$ is decreasing and has a lower bound therefore converges strongly. Define $X_+ \in \mathcal{L}(Z)$ its limit as $n \to \infty$. Let $X \in \mathcal{S}_{\mathcal{I}}$ be a self-adjoint solution of the CARI (6). Since X_i are self-adjoint operators and $X_i \geq X$, taking the limit for $n \to \infty$ it follows that X_+ is also self-adjoint and

$$X_+ \geq X.$$

(2): Note that X_i satisfies

$$\langle (A - BB^*X_{i-1})z_1, X_i z_2\rangle + \langle X_i z_1, (A - BB^*X_{i-1})z_2\rangle = -\langle B^*X_{i-1}, B^*X_{i-1}\rangle - \langle Cz_1, Cz_2\rangle.$$

The proof is completed by taking the strong limit for $n \to \infty$ in the above equality and using that $(X_i)_{i=0}^{n}$ converges strongly to X_+. ■

Remark 3.5. The sequence $(X_i)_{i=0}^{n}$ is known as Newton–Kleinman iterates. Note that Theorem 3.4 is closely related to Burns et al. (2008, Theorem 6.3) which deals with the exponentially stabilizable and detectable case. As a consequence of the proof of Theorem 3.4, the Newton iterates satisfy

$$0 \leq X_+ \leq \cdots \leq X_{k+1} \leq X_k \leq \cdots \leq X_1.$$

Remark 3.6. If one considers a system $\Sigma(A, B, C)$ which satisfies also the strong detectability assumption, then X_+ constructed in Theorem 3.4 is the unique nonnegative self-adjoint solution of the CARE (4) (see Iftime et al., 2005, Theorem 3.5). So the maximal and the minimal nonnegative self-adjoint solutions of the CARE (4) coincide.

Remark 3.7. Note that Q_i, $i = 0, \ldots, n-1$, are computed iteratively in the algorithm. Once Q_i is computed, then the condition $Q_i > 0$ can be checked.

4. Conclusions

For strongly stabilizable systems it has been proved that, under certain conditions, there exists a maximal solution of the CARI. The maximal solution of the CARI also satisfies the CARE. Furthermore, the solution of the CARE has been constructed using Newton–Kleinman iterates. These results are obtained for strongly stabilizable infinite-dimensional systems and are in line with similar results presented for finite-dimensional systems in Ran and Vreugdenhil (1988) and for exponentially stabilizable infinite-dimensional systems in Curtain and Rodman (1990) or Burns et al. (2008, Section 6). The main result proven in this paper (Theorem 3.4) is closely related to Burns et al. (2008, Theorem 6.3) which deals with the exponentially stabilizable and detectable case. Therefore, our new result makes a step forward to finite-dimensional approximations and mesh-independence for strongly stabilizable systems following the Kleinman–Newton approach as presented in Burns et al. (2008) for exponentially stabilizable systems.

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