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Analysis and applications of spectral properties of grounded Laplacian matrices for directed networks ⋆

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Abstract

In-depth understanding of the spectral properties of grounded Laplacian matrices is critical for the analysis of convergence speeds of dynamical processes over complex networks, such as opinion dynamics in social networks with stubborn agents. We focus on grounded Laplacian matrices for directed graphs and show that their eigenvalues with the smallest real part must be real. Lower and upper bounds for such eigenvalues are provided utilizing tools from nonnegative matrix theory. For those eigenvectors corresponding to such eigenvalues, we discuss two cases when we can identify the vertex that corresponds to the smallest eigenvector component. We then discuss an application in leader-follower social networks where the grounded Laplacian matrices arise naturally. With the knowledge of the vertex corresponding to the smallest eigenvector component for the smallest eigenvalue, we prove that by removing or weakening specific *directed* couplings pointing to the vertex having the smallest eigenvector component, all the states of the other vertices converge faster to that of the leading vertex. This result is in sharp contrast to the well-known fact that when the vertices are connected together through *undirected* links, removing or weakening links does not accelerate and in general decelerates the converging process.

Key words: grounded Laplacian matrix, convergence speed, essentially nonnegative matrices, accelerating consensus

1 Introduction

The spectral properties of certain matrices of a given network topology graph reveal ample information on the structures of the corresponding network. The study on those spectral properties plays an important role in the analysis of the convergence and convergence speed of the dynamical process evolving on such networks. In the study of multi-agent networks (Jadbabaie et al. [2003], Ren and Beard [2005], Cao et al. [2008], Scardovi and Sepulchre [2009], Ni and Cheng [2010], Xia and Cao [2011, 2014]), researchers have been especially interested in the process of aligning followers with the leaders when some agents are taking the role of leaders that guide the followers to reach consensus (Jadbabaie et al. [2003], Cao et al. [2008], Scardovi and Sepulchre [2009], Ni and Cheng [2010]); similarly, in the study of social networks (Blondel et al. [2009], Yildiz et al. [2011], Ghaderi and Srikant [2012], Acemoglu et al. [2013], Xia et al. [2016]), people have also studied the process of opinion forming in the presence of stubborn agents that keep their opinions unchanged over time (Yildiz et al. [2011], Ghaderi and Srikant [2012], Acemoglu et al. [2013]). In such cases, the grounded Laplacian matrices (Miekkala [1993], Bollobas [1998]) obtained by removing the rows and columns corresponding to the leaders or stubborn agents in the Laplacian matrices become critical in determining the convergence and the convergence rate of the system. The spectral properties of grounded Laplacian matrices are especially useful for the stability analysis of multi-agent formations (Barooah and Hespanha [2006]).

For undirected graphs, the spectral properties of grounded Laplacian matrices have been investigated, where upper and lower bounds have been established for their smallest eigenvalues; in particular, a special class of graphs, i.e., random graphs, have been discussed (Pirani and Sundaram [2014, 2016]). In the study of synchronization of complex networks, great efforts have been devoted to identifying which vertices in a network should be controlled and what kinds of controllers should be designed to achieve synchronization and to optimize the convergence speed (Yu et al. [2009], Shi et al. [2014]).

Although the study on the spectral properties of Laplacian matrices and grounded Laplacian matrices for undirected

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graphs is fruitful, the counterpart for directed graphs is limited (Agaev and Chebotarev [2005], Hao and Barooah [2011]) and some of the established results for undirected graphs do not carry over to the directed case. In this paper, we study the spectral properties of the grounded Laplacian matrices for directed graphs and look into their applications. Since the graphs are directed, the results, such as Rayleigh quotient inequality and the interlacing theorem for deriving some bounds for symmetric Laplacian matrices of undirected graphs in Pirani and Sundaram [2014, 2016], do not apply. We resort to nonnegative matrix theory and show that the eigenvalue with the smallest real part of the directed Laplacian matrix is real and the bounds established in Pirani and Sundaram [2014] still hold for this eigenvalue. The properties of the eigenvector corresponding to this eigenvalue of the directed Laplacian matrix are also discussed. In addition, two specific cases are identified when one can tell which vertex corresponds to the smallest eigenvector component.

We then discuss an application to leader-follower networks in multi-agent systems. With the knowledge of the vertex whose eigenvector component for the smallest eigenvalue is the smallest, we study the problem of accelerating the process of reaching consensus in a network with leaders. We propose a new strategy based on weakening the weights of or removing some specific edges. Although in *undirected* multi-agent networks, stronger or more links between followers often accelerate convergence (Xiao and Boyd [2004]), in *directed* networks, the convergence speed changes in more complicated fashions (Cao et al. [2008]). We claim that if we cut or weaken the links that point from the other followers to that follower corresponding to the smallest eigenvector component, the convergence process of *all* the followers may get accelerated.

The rest of the paper is organized as follows. In Section 2, we introduce grounded Laplacian matrices and give some preliminaries on nonnegative matrices. In Section 3, we establish the bounds for the eigenvalue with the smallest real part of the grounded Laplacian matrix and discuss the properties of its corresponding eigenvector. Section 4 identifies two cases when we can tell which vertex corresponds to the smallest eigenvector component. Section 5 discusses the applications of grounded Laplacian matrices in leader-follower networks.

2 Grounded Laplacian matrices for directed networks

Consider a directed network consisting of $N > 1$ vertices whose topology is described by a directed, positively weighted graph $\mathbb{G}$. Let $A = (a_{ij})_{N\times N}$ be the adjacency matrix for $\mathbb{G}$, and then a_{ij}, $1 \le i, j \le N$, is nonzero if and only if there is a directed edge from vertex j to i in $\mathbb{G}$ in which case a_{ij} is exactly the positive weight of the edge (j, i). Let $d_i = \sum_{j=1, j\neq i}^{N} a_{ij}$ be the *in-degree* of each vertex i and associate $\mathbb{G}$ with the diagonal *degree matrix* $D = \text{diag}\,\{d_1, d_2, \ldots, d_N\}$. Then the *Laplacian matrix* for the positively weighted, direct graph $\mathbb{G}$ is defined by $L = D - A$. It is well known that the spectral properties of the Laplacian matrix L can be conveniently studied when taking the network to be an N-vertex electrical network where each a_{ij} corresponds to the resistance from vertex j to i and some vertices are taken to be the source and some others the sink of the electrical current flowing in the network ([Bollobas, 1998, Chap 2]). In this context, it is of particular interest to study the case when some vertices are grounded. Let $\mathcal{V} = \{1, \ldots, N\}$ denote the set of indices of all the vertices and $\mathcal{S} = \{n + 1, \ldots, N\}$ for some $1 < n < N$ be the set of indices of all the grounded vertices. Then the Laplacian matrix can be partitioned into

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} L_{g} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, \tag{1}$$

where the rows and columns of L_{22} correspond to the vertices in $\mathcal{S}$ and the $n \times n$ submatrix L_{11} is called the *grounded Laplacian matrix* (Miekkala [1993]) and we denote it in the rest of the paper by L_g.

The grounded Laplacian matrices have some special properties and it is the main goal of this paper to study their spectral properties. But before doing that, we first summarize and prove some useful general results for matrix analysis.

Let $M = (m_{ij})_{N\times N}$ be a real matrix. We write $M \ge 0$ if $m_{ij} \ge 0$, $i, j = 1, \ldots, N$, and such a matrix M is called a *nonnegative* matrix. It is straightforward to check that the grounded Laplacian matrices are *not* nonnegative, but later we will show how to transform a grounded Laplacian matrix into a nonnegative matrix. We denote the spectral radius of M by $\rho(M)$. It follows from the Perron-Frobenius theorem (Horn and Johnson [1985]) that for a non-negative matrix M, $\rho(M)$ is an eigenvalue of M and there is a nonnegative vector $x \ge 0$, $x \neq 0$, such that $Mx = \rho(M)x$. In addition, if M is irreducible, then $\rho(M)$ is a simple eigenvalue of M and there is a positive vector $x > 0$ such that $Mx = \rho(M)x$.

Lemma 1 *Suppose that $M \in \mathbb{R}^{N\times N}$ is an irreducible nonnegative matrix and $\min_{1\le i\le N} \sum_{j=1}^{N} m_{ij} < \max_{1\le i\le N} \sum_{j=1}^{N} m_{ij}$. Then*

$$\min_{1\le i\le N} \sum_{j=1}^{N} m_{ij} < \rho(M) < \max_{1\le i\le N} \sum_{j=1}^{N} m_{ij}. \tag{2}$$

Proof. Let $\alpha = \max_{1\le i\le N} \sum_{j=1}^{N} m_{ij}$ and construct a new matrix B with $b_{ij} = \alpha \frac{m_{ij}}{\sum_{j=1}^{N} m_{ij}}$. Then $B \ge M$, and $\sum_{j=1}^{N} b_{ij} = \alpha$ for all $i = 1, \ldots, N$, implying $\rho(B) = \alpha$. Since $B - M \ge 0$, $B - M \neq 0$, and M is irreducible, from Problem 15 in pp. 515 in Horn and Johnson [1985], one knows $\rho(M) < \rho(B) = \alpha$. The lower bound can be established in a similar manner. □

Lemma 2 *Let $M \in \mathbb{R}^{N\times N}$ be an irreducible nonnegative matrix. Then for any positive vector x we have*

$$\min_{1\le i\le N} \frac{(Mx)_i}{x_i} \le \rho(M) \le \max_{1\le i\le N} \frac{(Mx)_i}{x_i}, \tag{3}$$

where $(Mx)_i$ is the ith element of the vector Mx. There is a unique vector $x^* \in \{x | x > 0,\ x^T x = 1\}$ such that $\rho(M) = \frac{(Mx^*)_i}{x^*_i}$, $i = 1, \ldots, N$, and for any $y \in \{x | x > 0,\ x^T x = 1\}$, $y \neq x^*$,

$$\min_{1\le i\le N} \frac{(My)_i}{y_i} < \rho(M) < \max_{1\le i\le N} \frac{(My)_i}{y_i}. \tag{4}$$

Proof. Inequality (3) is Theorem 8.1.26 in Horn and Johnson [1985]. Since M is nonnegative and irreducible, there is a unique vector $x^* \in \{x|x > 0,\ x^T x = 1\}$ such that $Mx^* = \rho(M)x^*$, which implies $\rho(M) = \frac{(Mx^*)_i}{x_i^*},\ i = 1, \ldots, N$.

Since M^T is nonnegative and irreducible, there is a positive vector $z > 0$ such that $M^T z = \rho(M)z$. Now we prove (4) by contradiction. Suppose there is another vector $y \in \{x|x > 0,\ x^T x = 1\},\ y \neq x^*$, such that $\rho(M) = \min_{1 \leq i \leq N} \frac{(My)_i}{y_i}$. Thus $\rho(M)y_i \leq (My)_i$ for all $i = 1, \ldots, N$, namely $My - \rho(M)y \geq 0$. Then

$$z^T(My - \rho(M)y) = \rho(M)z^T y - \rho(M)z^T y = 0.$$

Thus $My = \rho(M)y$ and it follows that $y = x^*$, which is a contradiction. We have proved $\rho(M) > \min_{1 \leq i \leq N} \frac{(My)_i}{y_i}$; $\rho(M) < \max_{1 \leq i \leq N} \frac{(My)_i}{y_i}$ can be proved in a similar manner. □

An $N \times N$ real matrix M with nonnegative off-diagonal elements $m_{ij},\ i \neq j$, is called *essentially nonnegative* (Cohen [1981], also called a *Metzler matrix* in Siljak [1978]). The *dominant eigenvalues* of such an M are defined as those eigenvalues with the largest real parts.

Lemma 3 *Let $M \in \mathbb{R}^{N \times N}$ be an essentially nonnegative matrix. Then its dominant eigenvalue, denoted by $r(M)$, is real. There is a nonnegative vector x, $x \neq 0$, such that $Mx = r(M)x$.*

Proof. Since M is essentially nonnegative, $M + \alpha I$ is nonnegative when α is a constant satisfying $\alpha \geq -\min_{1 \leq i \leq N} m_{ii}$. Obviously, $r(M) + \alpha$ is an eigenvalue of $M + \alpha I$ with the largest real part. Since $\rho(M + \alpha I)$ is a real eigenvalue of $M + \alpha I$ with the largest real part and there is a nonnegative eigenvector x corresponding to $\rho(M + \alpha I)$, $r(M) + \alpha = \rho(M + \alpha I)$ must be real and hence $r(M) = \rho(M + \alpha I) - \alpha$ is real and $Mx = r(M)x$. □

In the next two sections, we present our main results on s-tudying the spectral properties of grounded Laplacian matrices. Since the network graphs are directed, the tools such as Rayleigh quotient inequality used in Pirani and Sundaram [2016] for undirected graphs do not apply. We propose to transform the grounded Laplacian matrices into nonnegative matrices and utilize tools from nonnegative matrix theory to carry out spectral analysis.

3 New spectral properties

Let $\lambda(L_g)$ denote that eigenvalue of the grounded Laplacian matrix L_g that has the smallest real part. If such a $\lambda(L_g)$ is not unique, we take any of them and the conclusions to be drawn will apply. We first show that $\lambda(L_g)$ has to be real and then provide bounds for it. We impose the following assumption on the connectivity of the network graph.

Assumption 1 *In the directed graph $\mathbb{G}$, every vertex in $\mathcal{V} \backslash \mathcal{S}$ can be reached through a directed path from some vertex in $\mathcal{S}$.*

For a subset $\mathcal{V}'$ of $\mathcal{V}$, a subgraph of $\mathbb{G}$ induced by $\mathcal{V}'$ is the graph whose vertex set is $\mathcal{V}'$ and whose edge set consists of all the edges of $\mathbb{G}$ that have both associated vertices in $\mathcal{V}'$ (Bondy and Murty [1976]). Rewrite L_g as

$$L_g = L' + E, \tag{5}$$

where L' is the Laplacian matrix of the subgraph $\mathbb{G}'$ of $\mathbb{G}$ induced by $\mathcal{V} \backslash \mathcal{S}$ and $E = \text{diag}\{\epsilon_1, \epsilon_2, \ldots, \epsilon_n\}$ is the corresponding unique diagonal nonnegative matrix. It is easy to check that $\epsilon_i = \sum_{j \in \mathcal{S}} a_{ij}$ for $i = 1, \ldots, n$. For example, if there is only one vertex in $\mathcal{S}$, then $\mathcal{S} = \{N\}$, $n = N - 1$, and $\epsilon_i = a_{iN}$. Let $\bar{\epsilon} = \max_{1 \leq i \leq n} \epsilon_i$. Obviously when Assumption 1 holds, $\epsilon_i > 0$ for some i and thus $\bar{\epsilon} > 0$.

Theorem 1 *For a grounded Laplacian matrix L_g, it always holds that $\lambda(L_g)$ is real satisfying $0 \leq \lambda(L_g) \leq \bar{\epsilon}$ and there is a nonnegative eigenvector corresponding to $\lambda(L_g)$. If Assumption 1 holds, then $\lambda(L_g) > 0$, and if furthermore L_g is irreducible, then the corresponding nonnegative eigenvector is strictly positive.*

Proof. Since $-L_g$ is essentially nonnegative, from Lemma 3 we know that its dominant eigenvalue $r(-L_g)$ is real and has a nonnegative eigenvector. So $\lambda(L_g) = -r(-L_g)$ is real and has a corresponding nonnegative eigenvector.

Let α be a sufficiently large positive constant such that $P = -L_g + \alpha I \geq 0$. Then one can easily check that $\lambda(L_g) = \alpha - \rho(P)$, which implies that to prove $0 \leq \lambda(L_g) \leq \bar{\epsilon}$, it suffices to prove $\alpha - \bar{\epsilon} \leq \rho(P) \leq \alpha$. Since

$$-L' - \bar{\epsilon}I + \alpha I \leq P \leq P + E = -L' + \alpha I,$$

it follows from Theorem 8.1.18 in Horn and Johnson [1985] that $\rho(P) \leq \rho(P + E) = \rho(-L' + \alpha I) = \alpha$, and $\alpha - \bar{\epsilon} = \rho(-L' - \bar{\epsilon}I + \alpha I) \leq \rho(P)$.

Under Assumption 1, it has been proved in Lemma 4 in Hu and Hong [2007] that all the eigenvalues of L_g have positive real parts. It follows that $\lambda(L_g) > 0$. When in addition L_g is irreducible, $P = -L_g + \alpha I$ is irreducible and nonnegative. Hence, there exists a positive eigenvector of P corresponding to $\rho(P)$, and this eigenvector is exactly a positive eigenvector of L_g corresponding to $\lambda(L_g)$. □

In fact all the grounded vertices can merge as a single vertex, which agrees with the common practice in computations for electrical networks. Then L_g can be regarded as a matrix derived from the Laplacian matrix $L^\dagger$

$$L^\dagger = \begin{bmatrix} & & -\epsilon_1 \\ & L_g & \vdots \\ & & -\epsilon_n \\ 0 & \cdots 0 & 0 \end{bmatrix} \tag{6}$$

by grounding the vertex N. In the rest of the paper, for the purpose of spectral analysis of grounded Laplacian matrices, we assume without loss of generality that $\mathcal{S} = \{N\}$. Then we can classify the vertices $1, \ldots, n$ according to their topological distances to the grounded vertex N. In a directed graph

$\mathbb{G}$, for two vertices i and j, if $(i_0, i_1),\ (i_1, i_2), \ldots, (i_{k-1}, i_k)$ is a directed path from $i_0 = i$ to $i_k = j$ with the smallest number of edges, then the distance from i to j is defined as this smallest number of edges, k. Let s be the longest distance from N to any ungrounded vertex. We say a vertex is an α_i-vertex if the distance from N to this vertex is i with $1 \le i \le s$. In the rest of the paper, we relabel the set of vertices $\mathcal{V}\backslash\mathcal{S} = \{1, \ldots, n\}$ such that α_1-vertices are followed by α_2-vertices, then by α_3-vertices, until finally by α_s-vertices.

Using Theorem 1, in the following proposition, we identify a scenario where one can give a necessary and sufficient condition for $\lambda(L_g)$ to reach its upper bound.

Proposition 1 *Suppose L_g is irreducible and $a_{iN} = \epsilon$ whenever $a_{iN} \neq 0$ for $i = 1, \ldots n$. Then $\lambda(L_g) = \epsilon$ if and only if $a_{iN} \neq 0$ for all $i \in \mathcal{V}\backslash\mathcal{S}$.*

Proof. (Sufficiency) Now $a_{iN} \neq 0$ for all $i \in \mathcal{V}\backslash\mathcal{S}$. Then $E = \epsilon I$. Since $L_g = L' + \epsilon I$ and L' is a Laplacian matrix whose eigenvalue with the minimum real part is a real number 0, it follows that $\lambda(L_g) = \epsilon$.

(Necessity) Now $\lambda(L_g) = \epsilon$. It is easy to see that there must exist some i such that $a_{iN} = \epsilon$ and hence Assmption 1 holds. Let $P = -L_g + \alpha I \geq 0$, where α is a sufficiently large positive constant. From Theorem 1, we know that $\lambda(L_g) \le \epsilon$. We prove by contradiction. Assume that there exists some $i \in \mathcal{V}\backslash\mathcal{S}$ such that $a_{iN} = 0$. Then from (2) in Lemma 1, it follows that $\min_{1 \le i \le n} \sum_{j=1}^{n} p_{ij} = \alpha - \epsilon < \rho(P) < \alpha$, since L_g is irreducible. This implies that $\lambda(L_g) < \epsilon$, which contradicts the fact that $\lambda(L_g) = \epsilon$. □

In what follows, we look more carefully at the nonnegative eigenvector for $\lambda(L_g)$. We further assume that for every α_1-vertex i, it holds that $a_{iN} = \epsilon$.

Proposition 2 *Suppose that L_g is irreducible and there is at least one α_1-vertex. Let x be a positive eigenvector corresponding to $\lambda(L_g)$. Then $x_i \le \frac{\sum_{j=1, j\neq i}^{n} a_{ij} x_j}{\sum_{j=1, j\neq i}^{n} a_{ij}}$ when i is an α_1-vertex and $x_i > \frac{\sum_{j=1, j\neq i}^{n} a_{ij} x_j}{\sum_{j=1, j\neq i}^{n} a_{ij}}$ when i is an α_j-vertex, $2 \le j \le s$.*

Proof. Since $L_g x = (L' + E)x = \lambda(L_g)x$, one has that for all i, $1 \le i \le n$,

$$-\sum_{j=1, j\neq i}^{n} a_{ij} x_j + \left(\sum_{j=1, j\neq i}^{n} a_{ij} + \epsilon_i - \lambda(L_g) \right) x_i = 0. \quad (7)$$

From Theorem 1, one knows that $\lambda(L_g) \le \epsilon$. When i is an α_1-vertex, it follows from the fact that $\epsilon_i = \epsilon$ and (7) that $0 \ge -\sum_{j=1, j\neq i}^{n} a_{ij} x_j + \sum_{j=1, j\neq i}^{n} a_{ij} x_i$, implying that $x_i \le \frac{\sum_{j=1, j\neq i}^{n} a_{ij} x_j}{\sum_{j=1, j\neq i}^{n} a_{ij}}$.

When i is an α_j-vertex, $2 \le j \le s$, one has $\epsilon_i = 0$. In view of the fact that $\lambda(L_g) > 0$ and (7), one has $0 < -\sum_{j=1, j\neq i}^{n} a_{ij} x_j + \sum_{j=1, j\neq i}^{n} a_{ij} x_i$, implying that $x_i > \frac{\sum_{j=1, j\neq i}^{n} a_{ij} x_j}{\sum_{j=1, j\neq i}^{n} a_{ij}}$. □

Proposition 2 implies that for any vertex i that is not an α_1-vertex, there is always an adjacent vertex j such that the corresponding eigenvector component satisfies $x_i > x_j$. This idea of the decreasing order in magnitude for some eigenvector components is formalized in the following corollary.

Corollary 1 *Suppose that L_g is irreducible and there is at least one α_1-vertex. Let x be a positive eigenvector corresponding to $\lambda(L_g)$. For any α_j-vertex i, $1 < j \le s$, one can always find a sequence of eigenvector components $x_i > x_{i_1} > \cdots > x_k$ in which vertex k is an α_1-vertex. If node 1 is the only α_1-vertex, then $x_1 < x_i,\ 2 \le i \le n$.*

Eigenvector components of L' in (5) can be used to give bounds for $\lambda(L_g)$.

Proposition 3 *Suppose L_g is irreducible and there is only one α_1 vertex. Then $\lambda(L_g) < \epsilon\xi_1$, where ξ_1 is the first component of the nonnegative vector ξ satisfying $\xi^T L' = 0$ and $\xi^T \mathbf{1} = 1$, $\mathbf{1}$ is the all-one vector and L' is defined in (5).*

Proof. Since L_g is irreducible, there exists a positive left eigenvector ξ of L' such that $\xi^T L' = 0$ and $\xi^T \mathbf{1} = 1$. Let x be a positive eigenvector of L_g corresponding to $\lambda(L_g)$, i.e., $(L' + E)x = \lambda(L_g)x$. Multiplying the row vector ξ^T from left on both sides leads to $\xi^T (L' + E)x = \lambda(L_g)\xi^T x$. One has that $\xi^T E x = \lambda(L_g)\xi^T x$, which gives

$$\xi_1(\epsilon - \lambda(L_g))x_1 = \lambda(L_g) \sum_{i=2}^{n} \xi_i x_i > \lambda(L_g)(1 - \xi_1)x_1,$$

where in the last inequality we have used the fact that $x_1 < x_i,\ 2 \le i \le n$ from Corollary 1. It follows that $\lambda(L_g) < \epsilon\xi_1$. □

Remark 1 *Propositions 2, 3, and Corollary 1 are derived under the key assumption that L_g is irreducible. If only Assumption 1 is assumed to hold, then these results need to be reexamined to see whether they still hold.*

Remark 2 *The eigenvalue $\lambda(L_g)$ and its corresponding eigenvector can be calculated in a distributed way making use of power iteration methods. Note that $P = -L_g + \alpha I$ is a nonnegative matrix if $\alpha > \max_{1 \le i \le n} d_i$. Such an α can be identified by a max-consensus algorithm (Tahbaz-Salehi and Jadbabaie [2006]). Distributed asynchronous iteration algorithms with gossip based normalization have been reported in the literature to compute a nonnegative eigenvector of P corresponding to $\rho(P)$ (Jelasity et al. [2007]), which is also an eigenvector of L_g corresponding to $\lambda(L_g)$. Then $\rho(P)$ can be calculated as well and so does $\lambda(L_g)$.*

Next we compare the derived results with their counterparts for undirected graphs. It can be seen that those results for grounded Laplacian matrices of undirected graphs derived in Pirani and Sundaram [2014] carry over to directed graphs. We have employed tools from nonnegative matrix theory to establish the bounds for the eigenvalue $\lambda(L_g)$ in Theorem 1 and Proposition 1 which allows us to deal with the symmetric and asymmetric grounded Laplacian matrix in a unified way. However, some bounds established in Pirani and Sundaram [2016] do not hold anymore.

Remark 3 *For the grounded Laplacian matrix of an undirected graph, a tighter upper bound,* $\frac{w(\partial\mathcal{S})}{|\mathcal{V}\backslash\mathcal{S}|}$*, on* $\lambda(L_g)$ *has been established in Pirani and Sundaram [2016], where* $w(\partial\mathcal{S})$ *is the total weight of the edges from grounded vertices to the remaining vertices and* $|\mathcal{V}\backslash\mathcal{S}|$ *is the cardinality of* $\mathcal{V}\backslash\mathcal{S}$*. However, for an unweighted directed graph, the inequality* $\lambda(L_g) \leq \frac{w(\partial\mathcal{S})}{|\mathcal{V}\backslash\mathcal{S}|}$ *does not hold in general. For example, consider*

$$L_g = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

obtained by grounding one vertex which has one unweighted edge connecting with vertex 1. In this case $\frac{w(\partial\mathcal{S})}{|\mathcal{V}\backslash\mathcal{S}|} = \frac{1}{3}$*. However, the eigenvalue* $\lambda(L_g)$ *is* 0.382*, greater than* $\frac{1}{3}$*.*

Remark 4 *The eigenvalue* $\lambda(L_g)$ *of the grounded Laplacian matrix of a directed graph is different in general from that of its corresponding undirected graph. It highly depends on the assignment of the directions to the edges. For example, let* L_{g1}*,* L_{g2}*, and* L_{g3} *be given by*

$$L_{g1} = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}, \, L_{g2} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \, L_{g3} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \\ -1 & -1 & 2 \end{bmatrix}.$$

L_{g2} *and* L_{g3} *are both grounded Laplacian matrices with different assignments of the directions to the edges in the undirected graph corresponding to* L_{g1}*. We find that* $\lambda(L_{g2}) < \lambda(L_{g1}) < \lambda(L_{g3})$*.*

Although we have so far given several results on the components of the positive eigenvector of grounded Laplacian matrices corresponding to $\lambda(L_g)$, more can be said when additional conditions are stipulated. Since $-L_g$ is essentially nonnegative, this relates to the study on the components of dominant eigenvectors, which is an important topic in spectral matrix analysis. We will show in Section 5 when applying the spectral properties how to use such information about the eigenvector components to change network dynamics.

4 Further discussion on the smallest component of the nonnegative eigenvector for $\lambda(L_g)$

Corollary 1 only states that one of the α_1-vertices corresponds to the minimum eigenvector component, but does not indicate how to identify it. It is the aim of this section to identify for two cases.

4.1 Case I

The following lemma gives a criterion to determine when vertex 1 corresponds to the smallest eigenvector component than all the other vertices except for N corresponding to the eigenvalue of a Laplacian matrix with the second smallest real part.

Lemma 4 *Let* $A = (a_{ij})_{N\times N} \in \mathbb{R}^{N\times N}$ *and* $L^\dagger \in \mathbb{R}^{N\times N}$ *be the adjacency matrix and Laplacian matrix of a directed graph, respectively. Suppose that*

$$\begin{aligned} &a_{Nj} \leq a_{1j},\ 1 < j \leq N-1, \\ &a_{iN} \leq a_{1N},\ 1 < i \leq N-1 \\ &a_{1j} \leq a_{ij},\ 1 < i, j \leq N-1,\ i \neq j. \end{aligned} \tag{8}$$

Let $\lambda_2(L^\dagger)$ *be the eigenvalue of* $L^\dagger$ *with the second smallest real part. Then* $\lambda_2(L^\dagger)$ *is real and there exists a vector* $x \neq 0$ *satisfying that* $L^\dagger x = \lambda_2(L^\dagger)x$ *and* $x_N \leq x_1 \leq x_i,\ 2 \leq i \leq N-1$*.*

Proof. Define two matrices $S \in \mathbb{R}^{(N-1)\times N}$ and $T \in \mathbb{R}^{N\times(N-1)}$ as follows

$$S = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ -1 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \ T = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Note that $ST = I$ and $TS = I - \mathbf{1}e_N^T$, where $e_N^T = [0, 0, \ldots, 1]$. Let $M = SL^\dagger T$. Since $L^\dagger\mathbf{1} = 0$, one has that

$$SL^\dagger = SL^\dagger(I - \mathbf{1}e_N^T) = SL^\dagger TS = MS. \tag{9}$$

It can be shown that $\sigma\{L^\dagger\} = \{0\} \cup \sigma\{M\}$, where $\sigma\{L^\dagger\}$ is the spectrum of $L^\dagger$.

Now we show that
$(*)$ if $y \in \mathbb{R}^{N-1}$ is an eigenvector of M corresponding to λ, then there exists a vector $x \in \mathbb{R}^N$ such that $y = Sx$ and x is an eigenvector of $L^\dagger$ corresponding to λ.

Since rank $(S) = N-1$, for the eigenvector $y \in \mathbb{R}^{N-1}$, there exists a vector $\bar{x} \in \mathbb{R}^N$, $\bar{x} \neq c\mathbf{1}$ such that $y = S\bar{x}$. Plugging $y = S\bar{x}$ into $My = \lambda y$ leads to

$$MS\bar{x} = SL^\dagger\bar{x} = \lambda S\bar{x}. \tag{10}$$

It follows that $S(L^\dagger\bar{x} - \lambda\bar{x}) = 0$. Since $\ker(S) = span\{\mathbf{1}\}$, $L^\dagger\bar{x} - \lambda\bar{x} = a\mathbf{1}$ for some constant a and therefore $L^\dagger(L^\dagger\bar{x} - \lambda\bar{x}) = 0$.

If $\lambda \neq 0$, then $L^\dagger\bar{x} \neq 0$ from (10). Let $x = \frac{1}{\lambda}L^\dagger\bar{x}$. One has $x \neq 0$ and it follows from (10) that $Sx = \frac{1}{\lambda}SL^\dagger\bar{x} = y$. In addition $L^\dagger x = \frac{1}{\lambda}L^{\dagger 2}\bar{x} = L^\dagger\bar{x} = \lambda x$.

If $\lambda = 0$, then from (10), it follows that $SL^\dagger\bar{x} = 0$, implying that $L^\dagger\bar{x} = a\mathbf{1}$ for some constant a. If $a \neq 0$, then $\bar{x}$ is a generalized eigenvector of $L^\dagger$ corresponding to 0. Hence for the eigenvalue 0, its algebraic multiplicity is larger than the geometric multiplicity. However these two quantities should be equal for a Laplacian matrix (Agaev and Chebotarev [2005]). We conclude that $a = 0$ and $L^\dagger\bar{x} = 0$. Letting $x = \bar{x}$, the desired conclusion follows.

Next we calculate matrix $M = (m_{ij})_{(N-1)\times(N-1)}$. It is easy to see that

$$(SL^\dagger)_{ik} = \begin{cases} l_{1k} - l_{Nk}, & i = 1, \\ l_{ik} - l_{1k}, & 2 \le i \le N-1. \end{cases}$$

Thus for $2 \le j \le N-1$, $m_{1j} = \sum_{k=1}^{N} (SL^\dagger)_{1k} t_{kj} = a_{Nj} - a_{1j}$; for $2 \le i \le N-1$, $m_{i1} = \sum_{k=1}^{N} (l_{ik} - l_{1k}) t_{k1} = a_{iN} - a_{1N}$; for $2 \le i, j \le N-1,\ i \ne j$, one has $m_{ij} = \sum_{k=1}^{N} (l_{ik} - l_{1k}) t_{kj} = a_{1j} - a_{ij}$.

From equation (8), we know that the off-diagonal elements of M are non-positive, i.e., $m_{ij} \le 0$ for $1 \le i, j \le N-1,\ i \ne j$ and therefore $-M$ is an essentially nonnegative matrix. $-r(-M)$ is an eigenvalue of M having the smallest real part and there exists a nonnegative eigenvector $y \in \mathbb{R}^{N-1}$ corresponding to the eigenvalue $-r(-M)$. Note that $-r(-M) = \lambda_2(L^\dagger)$ since $\sigma\{L^\dagger\} = \{0\} \cup \sigma\{M\}$. From $(*)$ proved above, there exists a vector $x \in \mathbb{R}^N$ such that $Sx = y$ and $L^\dagger x = \lambda_2(L^\dagger)x$. In view of the structure of S and $y = Sx \ge 0$, it follows that $x_N \le x_1 \le x_i,\ 2 \le i \le N-1$. □

The proof technique of Lemma 4 relates to a spectral algorithm proposed to deal with the seriation problem (Atkins et al. [1998]) that makes use of properties of the second smallest eigenvalue of a symmetric Laplacian matrix and its corresponding eigenvector. Applying the above lemma to the matrix L_g, we can immediately establish a scenario when vertex 1 corresponds to the smallest eigenvector component.

Proposition 4 *Assume that Assumption 1 holds, $S = \{N\}$ and $a_{iN} = \epsilon$ if i is an α_1-vertex. Suppose that*

$$a_{1j} \le a_{ij},\ 2 \le i, j \le N-1,\ i \ne j. \tag{11}$$

There exists a nonnegative eigenvector $[x_1, x_2, \ldots, x_{N-1}]^T$ of L_g corresponding to $\lambda(L_g)$ and $x_1 \le x_i,\ 2 \le i \le N-1$.

Proof. Consider $L^\dagger$ given in (6) and note that 0 is a simple eigenvalue of $L^\dagger$. It can be verified that the assumptions in Lemma 4 are satisfied and therefore there exists an eigenvector $x \in \mathbb{R}^N$ such that $L^\dagger x = \lambda(L_g)x$ and $x_N \le x_1 \le x_i,\ 2 \le i \le N-1$. Since $\lambda(L_g) \ne 0$, $x_N = 0$. Hence $[x_1, x_2, \ldots, x_{N-1}]^T$ is an eigenvector of L_g corresponding to $\lambda(L_g)$ and $x_1 \le x_i,\ 2 \le i \le N-1$. Vertex 1 corresponds to the minimum eigenvector component. □

Remark 5 *For an unweighted directed graph, to satisfy the condition (11) in Proposition 4, the graph should have the property that whenever there is a directed edge $(j, 1)$ in the graph $\mathbb{G}'$ induced by $\mathcal{V}\backslash\mathcal{S}$, (j, i), $2 \le i \le N-1,\ i \ne j$, is a directed edge of $\mathbb{G}'$.*

4.2 Case II

Since the vertices have been labeled such that vertices $1, \ldots, l_1$ are α_1-vertices and vertices $l_1 + 1, \ldots, l_1 + l_2$ are α_2-vertices, and so on, the grounded Laplacian matrix can be written in the form

$$L_g = L' + E = \begin{bmatrix} L'_{11} & L'_{12} & L'_{13} & \cdots & L'_{1s} \\ L'_{21} & L'_{22} & L'_{23} & \cdots & L'_{2s} \\ 0 & L'_{32} & L'_{33} & \cdots & L'_{3s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & L'_{ss} \end{bmatrix} + E, \tag{12}$$

where $L'_{ij} \in \mathbb{R}^{l_i \times l_j}$, which is zero for $3 \le i \le s, 1 \le j \le i-2$, and $E = \mathrm{diag}\{\epsilon, \ldots, \epsilon, 0 \ldots, 0\}$ with l_1 nonzero elements. Thus $L'_{i+1,i}$ $i = 1, \ldots, s-1$ has at least one nonzero entry in each row. We give the following assumption.

Assumption 2 *L'_{ij} has equal-row-sum c_{ij} for $i \ne j$, $1 \le i, j \le s$, where c_{ij} are constants with $c_{ij} = 0$ for $3 \le i \le s, 1 \le j \le i-2$.*

Remark 6 *For an unweighted directed graph, to satisfy Assumption 2, the graph should have the property that each α_i-vertex, $1 \le i \le s$, has the same total number of incoming edges from all the α_{i-1}-vertices and has the same total number of incoming edges from all the α_j-vertices, $j > i$, where the α_0-vertex can be regarded as the grounded vertex and the set of α_{s+1}-vertices is an empty set.*

Proposition 5 *Assuming that Assumption 2 holds, L_g is irreducible and there is at least one α_1-vertex. Then L_g has a positive eigenvector $[x_1\mathbf{1}_{l_1}^T, x_2\mathbf{1}_{l_2}^T, \cdots, x_s\mathbf{1}_{l_s}^T]^T$ corresponding to $\lambda(L_g)$ satisfying that $0 < x_1 < x_2 < \ldots < x_s$.*

Proof. Let $P = -L_g + \alpha I \ge 0$, where α is a sufficiently large positive constant. Then, similar to L' in (12), the nonnegative matrix P can be partitioned as an s-by-s block matrix $P = (P_{ij})_{s\times s}$ and P_{ij} is the ij-th block. Thus P_{ij} has equal-row-sum r_{ij}, where $r_{ij} = -c_{ij}$ for $i \ne j$, $r_{11} = \alpha - c_{11} - \epsilon$ and $r_{ii} = \alpha - c_{ii}$ for $i = 2, \ldots, s$. Consider the nonnegative matrix $R = (r_{ij})_{s\times s}$. P is irreducible and hence R is irreducible since L_g is irreducible. R has a positive eigenvector $x = [x_1, \ldots, x_s]^T$ corresponding to $\rho(R)$. Simple calculation shows that $P[x_1\mathbf{1}_{l_1}^T, \cdots, x_s\mathbf{1}_{l_s}^T]^T = \rho(R)[x_1\mathbf{1}_{l_1}^T, \cdots, x_s\mathbf{1}_{l_s}^T]^T$. Hence $\rho(R) = \rho(P)$ (Horn and Johnson [1985]). In addition, it follows from Theorem 1 that $\rho(P) < \alpha$, implying $\rho(R) < \alpha$.

Assume that $x_{s-1} \ge x_s$. Then from $Rx = \rho(R)x$, one has

$$\rho(R)x_s = r_{s,s-1}x_{s-1} + r_{ss}x_s \ge (r_{s,s-1} + r_{ss})x_s = \alpha x_s.$$

This implies that $(\rho(R) - \alpha)x_s \ge 0$, which contradicts the fact that $(\rho(R) - \alpha)x_s < 0$. Hence $x_{s-1} < x_s$. Similarly one can use a proof of contradiction to prove that $x_{s-2} < x_{s-1}$. Continuing this process, one has that $x_1 < x_2 < \ldots < x_s$. □

5 Applications

In this section, we discuss the leader-follower network of multi-agent systems where grounded Laplacian matrices arise and their properties become applicable.

Consider a leader-follower network consisting of N agents whose topology is described by a directed graph $\mathbb{G}$. Let $A =$

$(a_{ij})_{N\times N}$ be the adjacency matrix for $\mathbb{G}$. Let the agent in the set $\mathcal{S}=\{N\}$ plays the role of leader or stubborn agent whose state is kept constant and the agents in the set $\mathcal{V}\backslash\mathcal{S}$ are the followers whose dynamics are described by the following equation

$$\dot{z}_i=\sum_{j=1,j\neq i}^{N}a_{ij}(z_j-z_i), \tag{13}$$

where $i\in\mathcal{V}\backslash\mathcal{S}$ and $z_i\in\mathbb{R}$ is the state of agent i. If we decompose the system state $z=[z_1,\ldots,z_N]^T$ into the followers' state z_F and the leader's state z_L, then the dynamics of the leader-follower network can be described by

$$\begin{bmatrix}\dot{z}_F\\ \dot{z}_L\end{bmatrix}=-\begin{bmatrix}L_{11} & L_{12}\\ 0 & 0\end{bmatrix}\begin{bmatrix}z_F\\ z_L\end{bmatrix}=-\begin{bmatrix}L_g & L_{12}\\ 0 & 0\end{bmatrix}\begin{bmatrix}z_F\\ z_L\end{bmatrix}. \tag{14}$$

The state of every follower converges to that of the leader under mild connectivity conditions. If there are multiple leaders, then the state of every follower converges to a weighted average of the leaders' states. $-\lambda(L_g)$ is the slowest pole and the magnitude of $\lambda(L_g)$ is a measure of the convergence speed of the follower network (Barooah and Hespanha [2006]). A larger value of $\lambda(L_g)$ indicates that system (14) has a faster convergence rate.

The knowledge of the sorting of the nonnegative eigenvector components corresponding to $\lambda(L_g)$ is useful, especially in accelerating the convergence speed of system (14) by changing network dynamics. We can improve the convergence speed of the network by weakening the coupling strength from the other vertices to some α_1-vertex. Suppose that the grounded Laplacian matrix L_g is irreducible and vertex 1 is an α_1-vertex. Let $(k,1)$ be an edge with weight a_{1k} that points from some vertex k to 1. For a positive eigenvector corresponding to $\lambda(L_g)$, suppose that vertex 1 corresponds to the smallest eigenvector component and vertex k corresponds to a larger eigenvector component than that of vertex 1. If we decrease the weight a_{1k} of $(k,1)$ by δ to $\bar{a}_{1k}$ such that $0<\delta\leq a_{1k}$ and keep the weights of the other edges unchanged, then the induced new grounded Laplacian matrix, denoted by $\bar{L}_g$, has a larger smallest eigenvalue compared to L_g. We formalize the idea and prove the following result.

Theorem 2 *Assume that the grounded Laplacian matrix L_g is irreducible and vertex 1 is an α_1-vertex. $\bar{L}_g$ is obtained by weakening the weight a_{1k} of some edge $(k,1)$ by δ to $\bar{a}_{1k}$, where $\delta\in(0,a_{1k}]$ is a constant. For a positive eigenvector x corresponding to $\lambda(L_g)$, if $x_1<x_k$, then*

$$0<\lambda(L_g)<\lambda(\bar{L}_g)\leq\epsilon. \tag{15}$$

Proof. Suppose that the weight a_{1k} of $(k,1)$ is not decreased to 0. If a_{1k} is decreased to 0, a continuity argument can be used to show that (15) still holds.

Let $P=-L_g+\alpha I$ and $\bar{P}=-\bar{L}_g+\alpha I$ where α is a sufficiently large positive constant. We have the following relationship between the elements of P and $\bar{P}$:

$$\begin{aligned}&\bar{p}_{11}=p_{11}+\delta,\ \bar{p}_{1k}=p_{1k}-\delta,\\ &p_{1j}=\bar{p}_{1j},\ j\neq 1,k,\\ &\bar{p}_{ij}=p_{ij},\ 2\leq i\leq n,\ 1\leq j\leq n.\end{aligned} \tag{16}$$

We prove the inequality $\alpha-\epsilon\leq\rho(\bar{P})<\rho(P)<\alpha$, which is equivalent to (15). From Theorem 1, we know that $\rho(P)<\alpha$ and $\rho(\bar{P})\geq\alpha-\epsilon$. It suffices to show that $\rho(\bar{P})<\rho(P)$.

Since x is a positive eigenvector of L_g corresponding to $\lambda(L_g)$, it is also an eigenvector of P and hence $\rho(P)=\frac{(Px)_i}{x_i},\ 1\leq i\leq n$. We compare $\frac{(Px)_i}{x_i}$ with $\frac{(\bar{P}x)_i}{x_i}$ for $i=1,\ldots,n$. Since $p_{ij}=\bar{p}_{ij}$ for $i=2,\ldots,n,\ j=1,\ldots,n$, one has $\frac{(Px)_i}{x_i}=\frac{(\bar{P}x)_i}{x_i},\ i=2,\ldots,n$. In view of equation (16) and the fact that $x_1<x_k$, simple calculation shows that

$$\begin{aligned}\frac{(Px)_1}{x_1}=&p_{11}+p_{1k}\frac{x_k}{x_1}+\frac{\sum_{j\neq 1,k}^{n}p_{1j}x_j}{x_1}\\ >&\bar{p}_{11}+\bar{p}_{1k}\frac{x_k}{x_1}+\frac{\sum_{j\neq 1,k}^{n}p_{1j}x_j}{x_1}=\frac{(\bar{P}x)_1}{x_1}.\end{aligned} \tag{17}$$

Then it follows from (3) in Lemma 2 that

$$\rho(\bar{P})\leq\max_{1\leq j\leq n}\frac{(\bar{P}x)_j}{x_j}=\frac{(Px)_i}{x_i}=\rho(P),\ i=2,\ldots n.$$

Since $\bar{P}$ is irreducible, $\bar{P}$ has a positive eigenvector satisfying $\bar{x}^T\bar{x}=1$. x is not an eigenvector of $\bar{P}$ since $\frac{(\bar{P}x)_2}{x_2}>\frac{(\bar{P}x)_1}{x_1}$. One knows that $\bar{x}\neq x$ and hence it follows from (4) in Lemma 2 that $\rho(\bar{P})<\max_{1\leq i\leq n}\frac{(\bar{P}x)_i}{x_i}=\rho(P)$. □

Remark 7 *A close look at the proof of Theorem 2 shows that if $x_1=x_k$ and other conditions in Theorem 2 keep unchanged, then $\lambda(L_g)=\lambda(\bar{L}_g)$; if $x_1>x_k$, $\lambda(L_g)>\lambda(\bar{L}_g)$. In general, if $x_{i_1}<x_{i_2}$ for some $1\leq i_1,i_2\leq n$ and (i_2,i_1) is an edge of $\mathbb{G}$, then $\lambda(L_g)$ is monotonically increasing when the weight $a_{i_1i_2}$ is decreasing.*

Remark 8 *A direct implication of Theorem 2 is that if the graph that describes the communication topology between agents is directed, then stronger connectivity of the graph might actually slow down the convergence. This is in sharp contrast with the case when the graph is undirected and unweighted, for which adding edges between vertices or increasing edge weights does not decelerate and in general accelerates the convergence.*

Theorem 2 has investigated the variation of the eigenvalue $\lambda(L_g)$ in the process of weakening the weights of the edges from the other vertices to the α_1-vertex. For the variation of the other eigenvalues, they may not monotonically decrease or increase.

6 Conclusion

We have provided upper and lower bounds for the real smallest eigenvalue of the grounded Laplacian matrices for directed graphs and explored the property of the eigenvector corresponding to that eigenvalue. A new strategy has been proposed to accelerate the convergence to consensus in leader-follower networks by making the follower who corresponds to the smallest eigenvector component more focused on its information about the leader. It has been shown that the dominant eigenvalue of the system matrix decreases in the process of removing the links pointing to that follower corresponding to the smallest eigenvector component from the other followers. For future work, we are interested in investigating the spectral properties of grounded Laplacian matrices for typical classes of directed graphs of large-scale complex networks and checking whether our proposed acceleration strategy still works for more complicated agent models.

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