Robust Synchronization and Model Reduction of Multi-Agent Systems

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3 MODEL REDUCTION BY CLUSTERING

3.1 INTRODUCTION

In the last few decades, the world has become increasingly connected. This has brought a significant interest to fields such as complex networks, smart-grids, distributed systems, transportation networks, biological networks, and networked multi-agent systems, see e.g. [7, 14, 46]. Widely studied problems in networked systems are the problems of consensus and synchronization, see [37, 38, 45, 49]. In the consensus problem, the goal is to have the agents in the network reach agreement on certain physical or measured quantities depending on the states of all the agents, where the agents use only locally available information. Other important subjects in the theory of networked systems are flocking, formation control, sensor placement, and controllability of networks, see e.g. [10, 13, 15, 18, 41, 47].

Behavioral analysis and controller design for large-scale complex networks can potentially become extremely expensive from a computational point of view, especially for problems where the complexity of the network scales as a power of the number of nodes it contains. In order to tackle this problem, there is a need for methods and procedures to approximate the original networks by smaller, less complex ones.

Direct application of established model reduction techniques, such as balanced truncation, Hankel-norm approximation, and Krylov subspace methods, see e.g. [1, 6], to the dynamical models of networked systems generally leads to a collapse of the network structure, as well as the loss of important properties such as consensus. Furthermore, the resulting reduced models often cannot even be interpreted as networked systems anymore.

While there do exist structure-preserving techniques which preserve certain properties such as the Lagrangian structure [35], the second order structure [36], and the interconnection structure of interconnected subsystems [53, 57, 69], multi-agent systems possess their own specific internal structure: the topology of the network. In the past, model reduction techniques specifically for networked multi-agent systems have been proposed in [8, 24, 25]. These methods are based on clustering nodes in the network. With clustering, the idea is to partition the set of nodes in the network graph into disjoint sets called clusters, and to associate with each cluster a single, new, node in the reduced network, thus reducing the number of nodes and connections and the complexity of the network topology. Other techniques instead reduce the network topology in a different manner, for instance by removing connections in the network graph that are of lesser importance, see e.g. [30].

In [43] a model reduction technique was introduced that harnesses a specific class of graph partitions called *almost equitable partitions*. The results in [43] provide explicit expressions for the $\mathcal{H}_2$ model reduction error if a *leader-follower network* with single integrator agent dynamics is clustered according to an almost equitable partition of the network graph. In a leader-follower network, a subset of the nodes receive an external input. These nodes are called the *leaders* of the network. The other nodes only receive relative information from their neighbors in the graph, these are called the *followers*. In the present chapter, we extend the results in [43] to networks where the agent dynamics is given by an arbitrary multivariable input-state-output system. We provide a priori upper bounds on both the $\mathcal{H}_2$ and the $\mathcal{H}_\infty$ model reduction errors if the agents are clustered according to almost equitable partitions. Compared to [43], we use a slightly different output equation to measure the disagreement between the agents in the network, which enables us to also consider the problem of clustering a network according to arbitrary, not necessarily almost equitable, graph partitions.

The outline of this chapter is as follows. In Section 3.2 we introduce some notation and review the theory needed for computing the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ model reduction error bounds in the remainder of the chapter. In Section 3.3 we precisely formulate the problem of model reduction of leader-follower networks with arbitrary agent dynamics. Section 3.4 reviews the needed theory on graph partitions and introduces the reduced network, obtained by applying a Petrov-Galerkin projection to the dynamical system of the original network. In Section 3.5 we provide a priori error bounds on the $\mathcal{H}_2$ model reduction error for networks with arbitrary agent dynamics, clustered according to almost equitable partitions. In Section 3.6, we complement these results by providing upper bounds on the $\mathcal{H}_\infty$ model reduction error. In Section 3.7 the problem of clustering networks according to general partitions is considered and the first steps towards a priori error bounds on both the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ model reduction errors are made. Finally, Section 3.8 provides some conclusions.

3.2 PRELIMINARIES

Consider the input-state-output system

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{3.1}$$

with $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, and transfer matrix $S(s) = C(sI - A)^{-1}B$. It is well known that if A is Hurwitz, then the $\mathcal{H}_2$-norm of S can be computed as

$$\|S\|_2^2 = \operatorname{tr} B^T X B,$$

where X is the unique positive semi-definite solution of the Lyapunov equation

$$A^T X + XA + C^T C = 0. \tag{3.2}$$

For the purposes of this chapter, we also need to deal with the situation when A is not Hurwitz. Let $\mathcal{X}_+(A)$ denote the unstable subspace of A, i.e., the direct sum of the generalized eigenspaces of A corresponding to its eigenvalues in the closed right half plane.

PROPOSITION 3.1: Assume that $\mathcal{X}_+(A) \subset \ker C$. Then the Lyapunov equation (3.2) has at least one positive semi-definite solution. Among all positive semi-definite solutions, there is exactly one solution, say X, with the property $\mathcal{X}_+(A) \subset \ker X$. For this particular solution X we have $\|S\|_2^2 = \operatorname{tr} B^T X B$.

Proof. Without loss of generality, assume that

$$A = \begin{pmatrix} A_- & 0 \\ 0 & A_+ \end{pmatrix}, \quad B = \begin{pmatrix} B_- \\ B_+ \end{pmatrix}, \quad C = \begin{pmatrix} C_- & 0 \end{pmatrix},$$

where A_- is Hurwitz, and A_+ has all its eigenvalues in the closed right half plane. Let X_- be the unique solution to the reduced Lyapunov equation

$$A_-^T X_- + X_- A_- + C_-^T C_- = 0. \tag{3.3}$$

Then $X_- = \int_0^\infty e^{A_-^T t} C_-^T C_- e^{A_- t}\, dt \geqslant 0$. Obviously then, $X = \operatorname{diag}(X_-, 0)$ is a positive semi-definite solution of (3.2). Now let X be a positive semi-definite solution to (3.2) with the property that $\mathcal{X}_+(A) \subset \ker X$. Then X must be of the form $X = \operatorname{diag}(X_1, 0)$, and X_1 must satisfy the reduced Lyapunov equation (3.3). Thus $X = \operatorname{diag}(X_-, 0)$. Finally, S is stable since $\mathcal{X}_+(A) \subset \ker C$. Moreover,

$$\begin{aligned} \|S\|_2^2 &= \operatorname{tr}\left(B^T \int_0^\infty e^{A^T t} C^T C e^{At}\, dt\ B \right) \\ &= \operatorname{tr}\left(B_-^T \int_0^\infty e^{A_-^T t} C_-^T C_- e^{A_- t}\, dt\ B_- \right) \\ &= \operatorname{tr}\left(B_-^T X_- B_-\right) \\ &= \operatorname{tr}\left(B^T X B\right). \end{aligned}$$

□

We will now deal with computing the $\mathcal{H}_\infty$-norm of a stable transfer function. The result is a generalization of Lemma 4 in [26].

LEMMA 3.2: *Consider the system (3.1). Assume that its transfer function S has all its poles only in the open left half plane. If there exists $X \in \mathbb{R}^{p \times p}$ such that $X = X^T$ and $CA = XC$, then $\|S\|_\infty = \sigma_1(S(0))$.*

Proof. For the first part of the proof, let us assume that (A, B, C) is minimal. Then, in particular, A is a Hurwitz matrix and (A, B) is controllable.

Clearly, the inequality $\|S\|_\infty \geqslant \sigma_1(S(0))$ is always satisfied. We will prove that $\|S\|_\infty \leqslant \sigma_1(S(0))$ using the bounded real lemma [52], which states that $\|S\|_\infty \leqslant \gamma$ if and only if there exists $P \in \mathbb{R}^{n\times n}$ such that $P = P^T$ and

$$A^T P + PA + C^T C + \frac{1}{\gamma^2} PBB^T P \leqslant 0.$$

Let us take $\gamma = \sigma_1(S(0)) = \sigma_1(CA^{-1}B)$. This implies that

$$CA^{-1}BB^TA^{-T}C^T \leqslant \gamma^2 I_p. \tag{3.4}$$

Defining $P := -A^{-T}C^TXCA^{-1}$ and using (3.4) gives us

$$\begin{aligned} A^T P + PA + C^T C + \frac{1}{\gamma^2} PBB^T P \\ &= -C^TXCA^{-1} - A^{-T}C^TXC + C^TC \\ &\quad + \frac{1}{\gamma^2} A^{-T}C^TXCA^{-1}BB^TA^{-T}C^TXCA^{-1} \\ &\leqslant -C^TXCA^{-1} - A^{-T}C^TXC + C^TC + A^{-T}C^TXXCA^{-1} \\ &= (XCA^{-1} - C)^T(XCA^{-1} - C) \\ &= 0. \end{aligned}$$

From the bounded real lemma, we conclude that $\|S\|_\infty \leqslant \sigma_1(S(0))$.

For a non-minimal representation (A, B, C), applying the Kalman decomposition, let T be a nonsingular matrix such that

$$T^{-1}AT = \begin{pmatrix} A_1 & 0 & A_6 & 0 \\ A_2 & A_3 & A_4 & A_5 \\ 0 & 0 & A_7 & 0 \\ 0 & 0 & A_8 & A_9 \end{pmatrix}, \quad T^{-1}B = \begin{pmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{pmatrix},$$
$$CT = \begin{pmatrix} C_1 & 0 & C_2 & 0 \end{pmatrix},$$

where (A_1, B_1, C_1) is a minimal representation of (A, B, C) with A_1 Hurwitz. Obviously,

$$(CT)(T^{-1}AT) = CAT = XCT = X(CT),$$

thus the condition is preserved under system transformation. From this, it follows that $C_1A_1 = XC_1$. Therefore, the minimal representation satisfies the sufficient condition and using the result obtained above the proof is completed. □

Continuing our effort to compute the $\mathcal{H}_\infty$-norm, we formulate a lemma that will be instrumental in evaluating a transfer function at the origin. Recall that for a given matrix A, its Moore–Penrose inverse is denoted by A^+.

LEMMA 3.3: *Consider the system* (3.1). *If* A *is symmetric and* $\ker A \subset \ker C$, *then* 0 *is not a pole of the transfer function* S *and we have* $S(0) = -CA^+B$.

Proof. If A is nonsingular, then the conclusion follows immediately. Otherwise, let $A = U\Lambda U^T$ be an eigenvalue decomposition with orthogonal U and $\Lambda = \operatorname{diag}(0, \Lambda_2)$, where $\Lambda_2 \in \mathbb{R}^{r\times r}$ and r is the rank of A. We denote $U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}$, with $U_2 \in \mathbb{R}^{n\times r}$. Then

$$A^+ = U\Lambda^+U^T = \begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_2^{-1} \end{pmatrix} \begin{pmatrix} U_1^T \\ U_2^T \end{pmatrix} = U_2\Lambda_2^{-1}U_2^T.$$

Note that $CU_1 = 0$. We have

$$\begin{aligned} S(s) &= CU(sI - \Lambda)^{-1}U^TB \\ &= C\begin{pmatrix} U_1 & U_2 \end{pmatrix} \begin{pmatrix} s^{-1}I & 0 \\ 0 & (sI - \Lambda_2)^{-1} \end{pmatrix} \begin{pmatrix} U_1^T \\ U_2^T \end{pmatrix} B \\ &= CU_2(sI - \Lambda_2)^{-1}U_2^TB. \end{aligned}$$

Hence, $S(s)$ is defined at $s = 0$ and $S(0) = -CU_2\Lambda_2^{-1}U_2^TB = -CA^+B$. □

Finally we discuss the model reduction technique known as Petrov-Galerkin projection.

DEFINITION 3.4: Consider the system (3.1). Let $W, V \in \mathbb{R}^{n \times r}$, with $r < n$, such that $W^T V = I$. The matrix VW^T is then a projector, called a *Petrov-Galerkin projector*. The reduced order system

$$\dot{\hat{x}} = W^T A V \hat{x} + W^T B u,$$
$$\hat{y} = C V \hat{x},$$

with $\hat{x} \in \mathbb{R}^r$ is called *the Petrov-Galerkin projection* of the original system (3.1).

3.3 PROBLEM FORMULATION

In this chapter, we consider networks of diffusively coupled linear subsystems. These subsystems, called *agents*, have identical dynamics, however a selected subset of the agents, called the *leaders*, also receive an input from outside the network. The remaining agents are called *followers*. The network consists of N agents, indexed by i, so $i \in \mathcal{V} := \{1, 2, \dots, N\}$. The subset $\mathcal{V}_L \subset \mathcal{V}$ is the index set of the leaders, more explicitly $\mathcal{V}_L = \{v_1, v_2, \dots, v_m\}$. The followers are indexed by $\mathcal{V}_F := \mathcal{V} \setminus \mathcal{V}_L$. More specifically, the leaders are represented by the finite dimensional linear system

$$\dot{x}_i = A x_i + B \sum_{j=1}^{N} a_{ij}(x_j - x_i) + E u_j, \quad i \in \mathcal{V}_L, \ i = v_j,$$

whereas the followers have dynamics

$$\dot{x}_i = A x_i + B \sum_{j=1}^{N} a_{ij}(x_j - x_i), \quad i \in \mathcal{V}_F.$$

The weights $a_{ij} \geqslant 0$ represent the coupling strengths of the diffusive coupling between the agents. In this chapter, we assume that $a_{ij} = a_{ji}$ for all $i, j \in \mathcal{V}$. Also, $a_{ii} = 0$ for all $i \in \mathcal{V}$. Furthermore, $x_i \in \mathbb{R}^n$ is the state of agent i, and $u_j \in \mathbb{R}^r$ is the external input to the leader v_j. Finally,

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times n}$ and $E \in \mathbb{R}^{n \times r}$ are real matrices. It is customary to represent the interaction between the agents by the graph G with node set $\mathcal{V} = \{1, 2, \ldots, N\}$ and adjacency matrix $\mathcal{A} = (a_{ij})$. In the set up of this chapter, this graph is undirected, reflecting the assumption that $\mathcal{A}$ is symmetric. The *Laplacian matrix* of the graph G is denoted by L and defined as

$$L_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -a_{ij} & \text{if } i \neq j. \end{cases}$$

with $d_i = \sum_{j=1}^{N} a_{ij}$.

Recall that the set of leader nodes is $\mathcal{V}_L = \{v_1, v_2, \ldots, v_m\}$, and define the matrix $M \in \mathbb{R}^{N \times m}$ as

$$M_{ij} = \begin{cases} 1 & \text{if } i = v_j, \\ 0 & \text{otherwise.} \end{cases}$$

Denote $x = \text{col}(x_1, x_2, \ldots, x_N)$ and $u = \text{col}(u_1, u_2, \ldots, u_m)$. The total network is then represented by

$$\dot{x} = (I \otimes A - L \otimes B)x + (M \otimes E)u. \tag{3.5}$$

The goal of this chapter is to find a reduced order networked system, whose dynamics is a good approximation of the networked system (3.5). In this chapter, the idea to obtain such approximation is to *cluster* groups of agents in the network, and to treat each of the resulting clusters as a node in a new, reduced order, network. The reduced order network will again be a leader-follower network, and by the clustering procedure essential interconnection features of the network will be preserved. We will require that the *synchronization* properties of the network are preserved after reduction. We will assume that the original network is synchronized, meaning that if the external inputs $u_j = 0$ for $j = 1, 2, \ldots, m$, the network reaches synchronization, that is, for all $i, j \in \mathcal{V}$, we have

$$x_i(t) - x_j(t) \to 0$$

as $t \to \infty$. We will impose that the reduction procedure preserves this property. In this chapter, a standing assumption will be that the graph G of the original network is *connected*. This is equivalent to the condition that 0 is a simple eigenvalue of the Laplacian L, see [39, Theorem 2.8]. In this case, the network reaches synchronization if and only if $(L \otimes I)x(t) \to 0$ as $t \to \infty$.

In order to be able to compare the original network (3.5) with its reduced order approximation and to make statements about the approximation error, we need a notion of *distance* between the networks. One way to obtain such notion is to introduce an *output* associated with the network (3.5). By doing this, both the original network and its approximation become input-output systems, and we can compare them by looking at the difference of their transfer functions. Being a measure for the disagreement between the states of the agents in (3.5), we choose $y = (L \otimes I)x$ as the output of the original network. Indeed, this output y can be considered a measure of the disagreement in the network, in the sense that $y(t)$ is small if and only if the network is close to being synchronized. Thus, with the original system (3.5) we now identify the input-state-output system:

$$\begin{aligned} \dot{x} &= (I \otimes A - L \otimes B)x + (M \otimes E)u, \\ y &= (L \otimes I)x. \end{aligned} \tag{3.6}$$

The state space dimension of (3.6) is equal to nN, its number of inputs equals to mr, and the number of outputs is nN.

In this chapter, we will use clustering to obtain a reduced order network, i.e. a network with a reduced number of agents, as an approximation of the original network (3.6). We also aim at deriving upper bounds for the approximation error. We will obtain upper bounds both for the $\mathcal{H}_2$-norm as well as the $\mathcal{H}_\infty$-norm of the difference of the transfer functions of the original network and its approximation.

3.4 GRAPH PARTITIONS AND REDUCTION BY CLUSTERING

We consider networks whose interaction topologies are represented by weighted graphs G with node set $\mathcal{V}$. The graph of the original network (3.5) is undirected, however, our reduction procedure will lead to networks on directed graphs. As before, the adjacency matrix of the graph G is the matrix $\mathcal{A} = (a_{ij})$, where $a_{ij} \geqslant 0$ is the weight of the arc from node j to node i. As noted before, the graph is undirected if and only if $\mathcal{A}$ is symmetric.

A nonempty subset $C \subset \mathcal{V}$ is called a *cell* or *cluster* of $\mathcal{V}$. A *partition* of a graph is defined as follows.

DEFINITION 3.5: Let G be an undirected graph. A partition π of $\mathcal{V}$, with $\pi = \{C_1, C_2, \dots, C_k\}$, is a collection of cells such that $\mathcal{V} = \bigcup_{i=1}^{k} C_i$ and $C_i \cap C_j = \emptyset$ whenever $i \neq j$. When we say that π is a partition of G, we mean that π is a partition of the vertex set $\mathcal{V}$ of G. Nodes i and j are called *cellmates* in π if they belong to the same cell of π. The *characteristic vector* of a cell $C \subset \mathcal{V}$ is the N-dimensional column vector $p(C)$ defined as

$$p_i(C) = \begin{cases} 1 & \text{if } i \in C, \\ 0 & \text{otherwise.} \end{cases}$$

The *characteristic matrix* of the partition $\pi = \{C_1, C_2, \dots, C_k\}$ is defined as the $N \times k$ matrix

$$P(\pi) = \begin{pmatrix} p(C_1) & p(C_2) & \cdots & p(C_k) \end{pmatrix}.$$

For a given partition $\pi = \{C_1, C_2, \dots, C_k\}$, consider the cells C_p and C_q with $p \neq q$. For any given node $j \in C_q$, we define its *degree with respect to* C_p as the sum the weights of all arcs from j to $i \in C_p$, i.e. the number

$$d_{pq}(j) := \sum_{i \in C_p} a_{ij}.$$

Next, we will construct a reduced order approximation of (3.6) by clustering the agents in the network according to a partition of G. Let π be a partition of G, and let $P := P(\pi)$ be its characteristic matrix. Extending the main idea in [43], we take as reduced order system the Petrov-

Galerkin projection of the original system (3.6), with the following choice for the matrices V and W:

$$W = P\big(P^T P\big)^{-1} \otimes I, \quad V = P \otimes I.$$

The dynamics of the resulting reduced order model is then given by

$$\begin{aligned} \dot{\hat{x}} &= (I \otimes A - \hat{L} \otimes B)\hat{x} + (\hat{M} \otimes E)u \\ \hat{y} &= (LP \otimes I)\hat{x}. \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} \hat{L} &= \big(P^T P\big)^{-1} P^T L P \\ \hat{M} &= \big(P^T P\big)^{-1} P^T M, \end{aligned}$$

We claim that the matrix $\hat{L}$ is the Laplacian of a weighted *directed* graph with node set $\{1, 2, \ldots, k\}$, with k equal to the number of clusters in the partition π. Indeed, by inspection it can be seen that the adjacency matrix of this reduced graph is $\hat{\mathcal{A}} = (\hat{a}_{pq})$, with

$$\hat{a}_{pq} = \frac{1}{|C_p|} \sum_{j \in C_q} d_{pq}(j),$$

where $d_{pq}(j)$ is the degree of $j \in C_q$ with respect to C_p, and $|C_p|$ the cardinality of C_p. Note also that the row sums of $\hat{L}$ are equal to zero since $\hat{L}\mathbb{1}_k = 0$. The matrix $\hat{M} \in \mathbb{R}^{k \times m}$ satisfies

$$\hat{M}_{pj} = \begin{cases} \frac{1}{|C_p|} & \text{if } v_j \in C_p, \\ 0 & \text{otherwise}, \end{cases}$$

where $v_1, v_2, \ldots, v_m$ are the leader nodes, and we have $p = 1, 2, \ldots, k$, and $j = 1, 2, \ldots, m$.

Clearly, the state space dimension of the reduced order network (3.7) is equal to nk, whereas the dimensions mr and nN of the input and output have remained unchanged. Thus we can investigate the error between the original and reduced order network by looking at the difference of their transfer functions. In the sequel we will both investigate the $\mathcal{H}_2$-norm as well as the $\mathcal{H}_\infty$-norm of this difference.

Before doing this however, we will first study the question whether our reduction procedure preserves synchronization. It is important to note that since, by assumption, the original undirected graph is connected, it has a directed spanning tree. It is easily verified that this property is preserved by our clustering procedure. Then, since the property of having a directed spanning tree is equivalent with 0 being a simple eigenvalue of the Laplacian (see [39, Proposition 3.8]), the reduced order Laplacian $\hat{L}$ has again 0 as a simple eigenvalue.

Now assume that the original network (3.6) is synchronized. It is well known, see e.g. [65], that this is equivalent with the condition that for each nonzero eigenvalue λ of the Laplacian L the matrix $A - \lambda B$ is Hurwitz. Thus, synchronization is preserved if and only if for each nonzero eigenvalue $\hat{\lambda}$ of the reduced order Laplacian $\hat{L}$ the matrix $A - \hat{\lambda} B$ is Hurwitz.

Unfortunately, in general, the fact that $A - \lambda B$ is Hurwitz for every nonzero $\lambda \in \sigma(L)$ does *not* imply that also $A - \hat{\lambda} B$ is Hurwitz for every nonzero $\hat{\lambda} \in \sigma(\hat{L})$. An exception is the 'single integrator' case $A = 0$ and $B = 1$, where this condition is trivially satisfied, so in this special case synchronization is preserved. Also if we restrict ourselves to a special type graph partitions, namely *almost equitable partitions*, then synchronization turns out to be preserved. We will review this type of partition now.

Again let G be a weighted, undirected graph, and let $\pi = \{C_1, C_2, \ldots, C_k\}$ be a partition of G. Given two cells C_p and C_q with $p \neq q$, and a given node $j \in C_q$, recall that $d_{pq}(j)$ denotes its degree with respect to C_p. We call the partition π an *almost equitable partition (AEP)* if for each p, q with $p \neq q$, the degree $d_{pq}(j)$ is independent of $j \in C_q$, i.e. $d_{pq}(j_1) = d_{pq}(j_2)$ for all $j_1, j_2 \in C_q$.

It is easily verified that every graph with $N > 1$ nodes has at least two trivial almost equitable partitions: a partition in which every node is in a cell by itself and a partition where all the nodes are in a single cell. An example of a graph with a nontrivial almost equitable partition is shown in Figure 3.1.

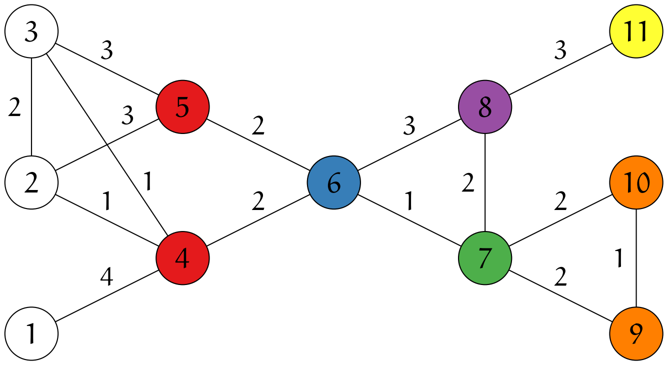

Figure 3.1: Graph G with N = 11 nodes. It has a nontrivial almost equitable partition $\pi = \{\{1,2,3\},\{4,5\},\{6\},\{7\},\{8\},\{9,10\},\{11\}\}$.

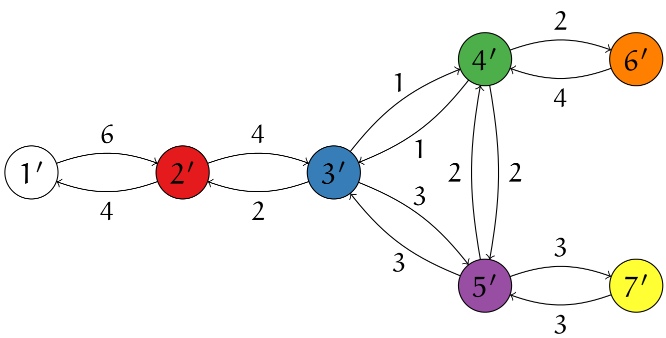

Figure 3.2: Reduced order graph $\hat{G}$ with N = 7 nodes. The colored nodes represent the clusters in the original graph.

It is a well known fact (see [9, Prop. 1]) that π is an AEP if and only if the image of its characteristic matrix is invariant under the Laplacian.

LEMMA 3.6: *Consider the weighed undirected graph* G *with Laplacian matrix* L*. Let* π *be a partition of* G *with characteristic matrix* $P := P(\pi)$*. Then* π *is an almost equitable partition if and only if* $L \operatorname{im} P \subset \operatorname{im} P$*.*

As an immediate consequence, the reduced Laplacian $\hat{L}$ obtained using an AEP satisfies $LP = P\hat{L}$. Indeed, since $\operatorname{im} P$ is L-invariant we have $LP = PX$ for some matrix X. Obviously we must then have

$$X = \left(P^T P\right)^{-1} P^T L P = \hat{L}.$$

From this, it follows that $\sigma(\hat{L}) \subset \sigma(L)$. It then readily follows that synchronization is preserved if we cluster according to an AEP:

THEOREM 3.7: *Assume that the network* (3.6) *is synchronized. Let* π *be an almost equitable partition. Then the reduced order network* (3.7) *obtained by clustering according to* π *is synchronized.*

EXAMPLE 3.8: Consider the graph G in Figure 3.1. The Laplacian L of G is given by

$$L = \begin{pmatrix} 4 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & -2 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 6 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & -1 & -1 & 8 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 & 8 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & 8 & -1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 7 & -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 & -2 & 8 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 3 \end{pmatrix}.$$

Let $\pi = \{\{1,2,3\},\{4,5\},\{6\},\{7\},\{8\},\{9,10\},\{11\}\}$. It is easily verified that π is an almost equitable partition of G. Next, let $P(\pi)$ be the characteristic matrix of π. We then have

$$P(\pi) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^T.$$

If G is clustered according to π, then the reduced graph is given by $\hat{G}$ in Figure 3.2. The reduced Laplacian $\hat{L}$ is then given by

$$\hat{L} = \begin{pmatrix} 4 & -4 & 0 & 0 & 0 & 0 & 0 \\ -6 & 8 & -2 & 0 & 0 & 0 & 0 \\ 0 & -4 & 8 & -1 & -3 & 0 & 0 \\ 0 & 0 & -1 & 7 & -2 & -4 & 0 \\ 0 & 0 & -3 & -2 & 8 & 0 & -3 \\ 0 & 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 & 3 \end{pmatrix}.$$

3.5 $\mathcal{H}_2$-ERROR BOUNDS

In this section, we investigate the $\mathcal{H}_2$-norm of the error system mapping the input u to the difference $y - \hat{y}$ in the case that the original network is clustered according to an AEP. Let S and $\hat{S}$ denote the transfer functions of the original (3.6) and reduced order network (3.7), respectively. We have the following lemma:

LEMMA 3.9: *Let π be an almost equitable partition of the graph G. The approximation error when clustering G according to π then satisfies*

$$\|S - \hat{S}\|_2^2 = \|S\|_2^2 - \|\hat{S}\|_2^2.$$

Proof. First, note that the columns of $P(\pi)$ are orthogonal. We construct a matrix $T = \begin{pmatrix} P & Q \end{pmatrix}$, where $P := P(\pi)$, and where the $N \times (N-k)$ matrix Q is chosen such that the columns of T form an orthogonal basis for $\mathbb{R}^N$. In this case, we have $P^T Q = 0$. Next, we apply the state space transformation $x = T\tilde{x}$ to system (3.6). We obtain

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{x}}_1 \\ \dot{\tilde{x}}_2 \end{pmatrix} &= A_e \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + B_e u \\ y &= C_e \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}, \end{aligned} \tag{3.8}$$

where the matrices A_e, B_e, and C_e are given by

$$A_e = \begin{pmatrix} I \otimes A - (P^T P)^{-1} P^T L P \otimes B & -(P^T P)^{-1} P^T L Q \otimes B \\ -(Q^T Q)^{-1} Q^T L P \otimes B & I \otimes A - (Q^T Q)^{-1} Q^T L Q \otimes B \end{pmatrix},$$

$$B_e = \begin{pmatrix} (P^T P)^{-1} P^T M \otimes E \\ (Q^T Q)^{-1} Q^T M \otimes E \end{pmatrix}, \quad C_e = \begin{pmatrix} LP \otimes I & LQ \otimes I \end{pmatrix}.$$

Obviously, in (3.8) the transfer function from u to y is equal to S. Furthermore, if the state component $\tilde{x}_2$ is truncated from (3.8), what we are left with is the reduced order model (3.7). Since π is an AEP of G, by Lemma 3.6, $\operatorname{im} P$ is invariant under L. From this, it follows that not only $Q^T P = 0$, but also

$$Q^T L P = 0 \text{ and } Q^T L^2 P = 0. \tag{3.9}$$

It is easily checked that

$$S(s) = \hat{S}(s) + \Delta(s),$$

where $\Delta(s)$ is given by

$$\begin{aligned} \Delta(s) = (LQ \otimes I)\Big(sI - \Big(I \otimes A - (Q^T Q)^{-1} Q^T L Q \otimes B\Big)\Big)^{-1} \\ \times\Big((Q^T Q)^{-1} Q^T M \otimes E\Big). \end{aligned} \tag{3.10}$$

From (3.9) and (3.10), we have $\hat{S}(-s)^T \Delta(s) = 0$. Thus we find that

$$\|S\|_2^2 = \|\hat{S}\|_2^2 + \|\Delta\|_2^2,$$

which concludes the proof. □

We will now formulate the main theorem of this section, which establishes an a priori upper bound for the $\mathcal{H}_2$-norm of the approximation error in the case that we cluster according to an AEP. Before formulating the theorem, we discuss some important ingredients. An important role is played by the $N-1$ auxiliary input-state-output systems

$$\begin{aligned} \dot{x} &= (A - \lambda B)x + Ed, \\ z &= \lambda x, \end{aligned} \tag{3.11}$$

where λ ranges over the nonzero eigenvalues of the Laplacian L. Let $S_\lambda(s) = \lambda(sI - A + \lambda B)^{-1}E$ be the transfer functions of these systems. We assume that the original network (3.6) is synchronized, so that all of the $A - \lambda B$ are Hurwitz. Let $\|S_\lambda\|_2$ denote the $\mathcal{H}_2$-norm of S_λ. Recall that the set of leader nodes is $\mathcal{V}_L = \{v_1, v_2, \ldots, v_m\}$. Node v_i will be called leader i. This leader is an element of cluster C_{k_i} for some $k_i \in \{1, 2, \ldots, k\}$. We now have the following theorem:

THEOREM 3.10: *Assume that the network* (3.6) *is synchronized. Let* π *be an almost equitable partition of the graph* G. *The absolute approximation error when clustering* G *according to* π *then satisfies*

$$\|S - \hat{S}\|_2^2 \leqslant S_{\max,\mathcal{H}_2}^2 \sum_{i=1}^{m} \left(1 - \frac{1}{|C_{k_i}|}\right),$$

where C_{k_i} *is the set of cellmates of leader* i, *and*

$$S_{\max,\mathcal{H}_2} := \max_{\lambda \in \sigma(L) \setminus \sigma(\hat{L})} \|S_\lambda\|_2.$$

Furthermore, the relative approximation error satisfies

$$\frac{\|S - \hat{S}\|_2^2}{\|S\|_2^2} \leqslant \frac{S_{\max,\mathcal{H}_2}^2}{S_{\min,\mathcal{H}_2}^2} \frac{\sum_{i=1}^{m}\left(1 - \frac{1}{|C_{k_i}|}\right)}{m\left(1 - \frac{1}{N}\right)},$$

where

$$S_{\min,\mathcal{H}_2} := \min_{\lambda \in \sigma(L)\setminus\{0\}} \|S_\lambda\|_2.$$

REMARK 3.11: We see that with a fixed number of agents and a fixed number of leaders, the approximation error is equal to 0 if in each cluster that contains a leader, the leader is the only node in that cluster. In general, the upper bound increases if the number of cellmates of the leaders increases.

Proof. Recall that $\sigma(\hat{L}) \subset \sigma(L)$. Label the eigenvalues of L as $0, \lambda_2, \lambda_3, \ldots, \lambda_N$ in such a way that $0, \lambda_2, \lambda_3, \ldots, \lambda_k$ are the eigenvalues of $\hat{L}$. Also, without loss of generality, we assume that π is *regularly formed*, i.e. all ones in each of the columns of $P(\pi)$ are consecutive. One can always relabel the agents in the graph in such a way that this is achieved. For simplicity, we again denote $P(\pi)$ by P. Recall that the reduced Laplacian matrix is given by $\hat{L} = \left(P^T P\right)^{-1} P^T L P$. From Lemma 3.9 we have that the approximation error satisfies

$$\|S - \hat{S}\|_2^2 = \|S\|_2^2 - \|\hat{S}\|_2^2.$$

We will first compute the $\mathcal{H}_2$-norms of S and $\hat{S}$ separately and then give an upper bound for the difference.

Consider the symmetric matrix

$$\bar{L} := \left(P^T P\right)^{\frac{1}{2}} \hat{L} \left(P^T P\right)^{-\frac{1}{2}} = \left(P^T P\right)^{-\frac{1}{2}} P^T L P \left(P^T P\right)^{-\frac{1}{2}}. \tag{3.12}$$

Note that the eigenvalues of $\bar{L}$ and $\hat{L}$ coincide. Let $\hat{U}$ be an orthogonal matrix that diagonalizes $\bar{L}$. We then have

$$\hat{U}^T \left(P^T P\right)^{-\frac{1}{2}} P^T L P \left(P^T P\right)^{-\frac{1}{2}} \hat{U} = \operatorname{diag}(0, \lambda_2, \ldots, \lambda_k) =: \hat{\Lambda}. \tag{3.13}$$

Next, take $U_1 = P\left(P^T P\right)^{-\frac{1}{2}} \hat{U}$. The columns of U_1 have unit length and are orthogonal:

$$U_1^T U_1 = \hat{U}^T \left(P^T P\right)^{-\frac{1}{2}} P^T P \left(P^T P\right)^{-\frac{1}{2}} \hat{U} = \hat{U}^T \hat{U} = I.$$

Furthermore, we have that

$$U_1^T L U_1 = \hat{U}^T (P^T P)^{-\frac{1}{2}} P^T L P (P^T P)^{-\frac{1}{2}} \hat{U} = \hat{\Lambda}.$$

Now choose U_2 such that $U = \begin{pmatrix} U_1 & U_2 \end{pmatrix}$ is an orthogonal matrix and

$$\Lambda := U^T L U = \begin{pmatrix} \hat{\Lambda} & 0 \\ 0 & \bar{\Lambda} \end{pmatrix},$$

where $\bar{\Lambda} = \mathrm{diag}(\lambda_{k+1}, \ldots, \lambda_N)$. It is easily verified that the first column of U_1, and thus the first column of U, is given by $\frac{1}{\sqrt{N}} \mathbb{1}_N$, where $\mathbb{1}_N$ is the N-vector of 1's, a fact that we will use in the remainder of this chapter. To compute the $\mathcal{H}_2$-norm of S we can use the result of Proposition 3.1. It can be verified, using the fact that $A - \lambda_i B$ is Hurwitz for $i = 2, 3 \ldots, N$, that

$$\mathcal{X}_+(I \otimes A - L \otimes B) = \mathbb{1}_N \otimes \mathcal{X}_+(A).$$

This immediately implies that $\mathcal{X}_+(I \otimes A - L \otimes B) \subset \ker(L \otimes I)$. As a consequence, we have

$$\|S\|_2^2 = \mathrm{tr}(M^T \otimes E^T) X (M \otimes E),$$

where X is the unique positive semi-definite solution to the Lyapunov equation

$$(I \otimes A^T - L \otimes B^T) X + X (I \otimes A - L \otimes B) + L^2 \otimes I = 0 \tag{3.14}$$

with the property that $\mathcal{X}_+(I \otimes A - L \otimes B) \subset \ker X$. In order to compute this solution X, premultiply (3.14) by $U^T \otimes I$ and postmultiply by $U \otimes I$, and substitute $Z = (U^T \otimes I) X (U \otimes I)$ to obtain

$$(I \otimes A^T - \Lambda \otimes B^T) Z + Z (I \otimes A - \Lambda \otimes B) + \Lambda^2 \otimes I = 0. \tag{3.15}$$

Solving (3.15) we take Z as

$$Z = \mathrm{diag}(0, X_2, \ldots, X_N),$$

where X_i, for $i = 2, \dots, N$, is the observability Gramian of the auxiliary system $(A - \lambda_i B, E, \lambda_i I)$ in (3.11). Next, $X := (U \otimes I) Z (U^T \otimes I)$ is a solution of the original Lyapunov equation, and it is easily verified that indeed $\mathcal{X}_+(I \otimes A - L \otimes B) \subset \ker X$. Thus we obtain the following expression for the $\mathcal{H}_2$-norm of S:

$$\begin{aligned} \|S\|_2^2 &= \text{tr}\, \left(M^T U \otimes E^T\right) \text{diag}(0, X_2, \dots, X_N) \left(U^T M \otimes E\right), \\ &= \text{tr}\, \left(U^T M M^T U \otimes I\right) \text{diag}(0, E^T X_2 E, \dots, E^T X_N E). \end{aligned} \tag{3.16}$$

Next, we compute the $\mathcal{H}_2$-norm for the reduced system. Firstly, it can be verified that

$$\mathcal{X}_+(I \otimes A - \hat{L} \otimes B) = \mathbb{1}_k \otimes \mathcal{X}_+(A)$$

This implies that $\mathcal{X}_+(I \otimes A - \hat{L} \otimes B) \subset \ker(LP \otimes I)$. By Proposition 3.1 we then have

$$\|\hat{S}\|_2^2 = \text{tr}\left(\hat{M}^T \otimes E^T\right) \hat{X} \left(\hat{M} \otimes E\right),$$

where $\hat{X}$ is the unique positive semi-definite solution to the Lyapunov equation

$$\left(I \otimes A^T - \hat{L}^T \otimes B^T\right)\hat{X} + \hat{X}(I \otimes A - \hat{L} \otimes B) + P^T L^2 P \otimes I = 0. \tag{3.17}$$

with the property that $\mathcal{X}_+(I \otimes A - \hat{L} \otimes B) \subset \ker \hat{X}$. In order to compute this solution, pre- and postmultiply (3.17) by $\left(P^T P\right)^{-\frac{1}{2}} \otimes I$ and substitute

$$\hat{Y} = \left(\left(P^T P\right)^{-\frac{1}{2}} \otimes I\right) \hat{X} \left(\left(P^T P\right)^{-\frac{1}{2}} \otimes I\right)$$

to obtain

$$\begin{aligned} &\left(I \otimes A^T - \bar{L} \otimes B^T\right)\hat{Y} + \hat{Y}\left(I \otimes A - \bar{L} \otimes B\right) \\ &\quad + \left(P^T P\right)^{-\frac{1}{2}} P^T L^2 P \left(P^T P\right)^{-\frac{1}{2}} \otimes I = 0. \end{aligned} \tag{3.18}$$

Recall from Section 3.4 that $LP = P\hat{L}$. From this it follows that

$$\left(P^T P\right)^{-\frac{1}{2}} P^T L^2 P (P^T P)^{-\frac{1}{2}} = \bar{L}^2.$$

Consequently, we can diagonalize the corresponding term in (3.18) by premultiplying by $\hat{U}^T \otimes I$ and postmultiplying by $\hat{U} \otimes I$, where $\hat{U}$ is as in (3.13). Next, we denote $\hat{Z} = (\hat{U}^T \otimes I)\hat{Y}(\hat{U} \otimes I)$ so that (3.18) reduces to

$$\left(I \otimes A^T - \hat{\Lambda} \otimes B^T\right)\hat{Z} + \hat{Z}\left(I \otimes A - \hat{\Lambda} \otimes B\right) + \hat{\Lambda}^2 \otimes I = 0,$$

which can be solved by taking

$$\hat{Z} = \operatorname{diag}(0, X_2, \dots, X_k),$$

where again X_i, for $i = 2, \dots, k$, is the observability Gramian of the auxiliary system $(A - \lambda_i B, E, \lambda_i I)$ in (3.11). Next,

$$\hat{X} = \left(\left(P^T P\right)^{\frac{1}{2}} \hat{U} \otimes I\right)\hat{Z}\left(\hat{U}^T \left(P^T P\right)^{\frac{1}{2}} \otimes I\right)$$

then satisfies (3.17), and it can be verified that $\mathcal{X}_+(I \otimes A - \hat{L} \otimes B) \subset \ker \hat{X}$. Thus, the $\mathcal{H}_2$-norm of $\hat{S}$ is given by:

$$\begin{aligned}\|\hat{S}\|_2^2 &= \operatorname{tr}\left(\hat{M}^T \left(P^T P\right)^{\frac{1}{2}} \hat{U} \otimes E^T\right) \operatorname{diag}(0, X_2, \dots, X_k) \\ &\quad \times \left(\hat{U}^T \left(P^T P\right)^{\frac{1}{2}} \hat{M} \otimes E\right), \\ &= \operatorname{tr}\left(\hat{U}^T \left(P^T P\right)^{\frac{1}{2}} \hat{M}\hat{M}^T \left(P^T P\right)^{\frac{1}{2}} \hat{U} \otimes I\right) \\ &\quad \times \operatorname{diag}\left(0, E^T X_2 E, \dots, E^T X_k E\right).\end{aligned} \tag{3.19}$$

Using Lemma 3.9, and formulas (3.16) and (3.19), we compute

$$\begin{aligned}\|S - \hat{S}\|_2^2 &= \operatorname{tr}\left(U^T M M^T U \otimes I\right) \operatorname{diag}\left(0, E^T X_2 E, \dots, E^T X_N E\right) \\ &\quad - \operatorname{tr}\left(\hat{U}^T \left(P^T P\right)^{\frac{1}{2}} \hat{M}\hat{M}^T \left(P^T P\right)^{\frac{1}{2}} \hat{U} \otimes I\right) \\ &\qquad \times \operatorname{diag}\left(0, E^T X_2 E, \dots, E^T X_k E\right) \\ &= \operatorname{tr}\left(\begin{pmatrix} U_1^T M M^T U_1 & U_1^T M M^T U_2 \\ U_2^T M M^T U_1 & U_2^T M M^T U_2 \end{pmatrix} \otimes I\right) \\ &\qquad \times \operatorname{diag}\left(0, E^T X_2 E, \dots, E^T X_N E\right) \\ &\quad - \operatorname{tr}\left(U_1^T M M^T U_1 \otimes I\right) \operatorname{diag}\left(0, E^T X_2 E, \dots, E^T X_k E\right) \\ &= \operatorname{tr}\left(U_2^T M M^T U_2 \otimes I\right) \operatorname{diag}\left(E^T X_{k+1} E, \dots, E^T X_N E\right),\end{aligned} \tag{3.20}$$

where the second equality follows from the fact that

$$\begin{aligned}\hat{M}^{\mathsf{T}}(P^{\mathsf{T}}P)^{\frac{1}{2}}\hat{U} &= M^{\mathsf{T}}P(P^{\mathsf{T}}P)^{-1}(P^{\mathsf{T}}P)^{\frac{1}{2}}\hat{U} \\ &= M^{\mathsf{T}}P(P^{\mathsf{T}}P)^{-\frac{1}{2}}\hat{U} \\ &= M^{\mathsf{T}}U_1.\end{aligned}$$

Next, observe that (3.20) can be rewritten as

$$\begin{aligned}\|S-\hat{S}\|_2^2 &= \operatorname{tr}\,(U_2^{\mathsf{T}}MM^{\mathsf{T}}U_2 \otimes I)\operatorname{diag}(E^{\mathsf{T}}X_{k+1}E,\dots,E^{\mathsf{T}}X_NE) \\ &= \operatorname{tr}\,(U_2^{\mathsf{T}}MM^{\mathsf{T}}U_2) \\ &\quad \times \operatorname{diag}(\operatorname{tr}E^{\mathsf{T}}X_{k+1}E,\dots,\operatorname{tr}E^{\mathsf{T}}X_NE) \\ &= \operatorname{tr}\,(U_2^{\mathsf{T}}MM^{\mathsf{T}}U_2)\operatorname{diag}\Big(\|S_{\lambda_{k+1}}\|_2^2,\dots,\|S_{\lambda_N}\|_2^2\Big),\end{aligned}$$

where S_{λ_j} for $j = k+1,\dots,N$ is the transfer function of the auxiliary system (3.11). An upper bound for this expression is given by

$$\operatorname{tr}(U_2^{\mathsf{T}}MM^{\mathsf{T}}U_2)\operatorname{diag}\Big(\|S_{\lambda_{k+1}}\|_2^2,\dots,\|S_{\lambda_N}\|_2^2\Big) \leqslant S^2_{\max,\mathcal{H}_2}\operatorname{tr}U_2^{\mathsf{T}}MM^{\mathsf{T}}U_2,$$

where $S^2_{\max,\mathcal{H}_2} = \max_{k+1\leqslant j\leqslant N}\|S_{\lambda_j}\|_2^2$. Furthermore, we have

$$\begin{aligned}\operatorname{tr}U_2^{\mathsf{T}}MM^{\mathsf{T}}U_2 &= \operatorname{tr}U^{\mathsf{T}}MM^{\mathsf{T}}U - \operatorname{tr}U_1^{\mathsf{T}}MM^{\mathsf{T}}U_1 \\ &= m - \operatorname{tr}P(P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}MM^{\mathsf{T}}.\end{aligned}$$

Since, by assumption, the partition π is regularly formed, the matrix $P(P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}$ is a block diagonal matrix of the form

$$P(P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}} = \operatorname{diag}(P_1, P_2, \dots, P_k).$$

It is easily verified that each P_i is a $|C_i| \times |C_i|$ matrix whose elements are all equal to $\frac{1}{|C_i|}$. The matrix MM^{T} is a diagonal matrix whose diagonal entries are either 0 or 1. We then have that the ith column of $P(P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}MM^{\mathsf{T}}$ is either equal to the ith column of $P(P^{\mathsf{T}}P)^{-1}P^{\mathsf{T}}$ if agent i is a leader, or zero otherwise. It then follows that the diagonal elements of $P(P^TP)^{-1}P^TMM^T$ are either zero or $\frac{1}{|C_{k_i}|}$ if i is part of the leader set, where C_{k_i} is the cell containing agent i. Hence, we have

$$\operatorname{tr} U_1^T MM^T U_1 = \sum_{i=1}^{m} \frac{1}{|C_{k_i}|},$$

and consequently

$$\operatorname{tr} U_2^T MM^T U_2 = m - \sum_{i=1}^{m} \frac{1}{|C_{k_i}|}.$$

In conclusion, we have

$$\|S - \hat{S}\|_2^2 \leqslant S^2_{\max,\mathcal{H}_2} \sum_{i=1}^{m} \left(1 - \frac{1}{|C_{k_i}|}\right),$$

which completes the proof of the first part of the theorem.

We now prove the statement about the relative error. For this, we will establish a lower bound for $\|S\|_2^2$. By (3.16) we have

$$\begin{aligned} \|S\|_2^2 &= \operatorname{tr}\left(M^T U \otimes E^T\right) \operatorname{diag}(0, X_2, \dots, X_N)\left(U^T M \otimes E\right) \\ &= \operatorname{tr}\left(U^T MM^T U \otimes I\right) \operatorname{diag}\left(0, E^T X_2 E, \dots, E^T X_N E\right) \\ &= \operatorname{tr}\left(U^T MM^T U\right) \operatorname{diag}\left(0, \operatorname{tr} E^T X_2 E, \dots, \operatorname{tr} E^T X_N E\right) \end{aligned} \tag{3.21}$$

The first column of U spans the eigenspace corresponding to the eigenvalue 0 of L and hence must be equal to $u_1 = \frac{1}{\sqrt{N}}\mathbb{1}_N$. Let $\bar{U}$ be such that $U = \begin{pmatrix} u_1 & \bar{U} \end{pmatrix}$. It is then easily verified using (3.21) that

$$\begin{aligned} \|S\|_2^2 &= \operatorname{tr}\left(\bar{U}^T MM^T \bar{U}\right) \operatorname{diag}\left(\operatorname{tr} E^T X_2 E, \dots, \operatorname{tr} E^T X_N E\right) \\ &= \operatorname{tr}\left(\bar{U}^T MM^T \bar{U}\right) \operatorname{diag}\left(\|S_{\lambda_2}\|_2^2, \dots, \|S_{\lambda_N}\|_2^2\right) \end{aligned}$$

Finally, since

$$\operatorname{tr} \bar{U}^T MM^T \bar{U} = \operatorname{tr} M^T \bar{U}\bar{U}^T M = \operatorname{tr} M^T \left(UU^T - u_1 u_1^T\right) M = m - \frac{m}{N},$$

we obtain that $\|S\|_2^2 \geqslant m\left(1 - \frac{1}{N}\right) S^2_{\min,\mathcal{H}_2}$. This then yields the upper bound for the relative error as claimed. □

REMARK 3.12: Note that by our labeling of the eigenvalues of L, in the formulation of Theorem 3.10, we have that $\sigma(L) \setminus \sigma(\hat{L})$ is equal to $\{\lambda_{k+1}, \dots, \lambda_N\}$ used in the proof. We stress that this should not be confused with the notation often used in the literature, where the λ_i's are labeled in increasing order.

REMARK 3.13: For the special case that the agents are single integrators (so $n = 1$, $A = 0$, $B = 1$, and $E = 1$) it is easily seen that $S_{\max,\mathcal{H}_2} = \frac{1}{2}\max\{\lambda \mid \lambda \in \sigma(L) \setminus \sigma(\hat{L})\}$ and $S_{\min,\mathcal{H}_2} = \frac{1}{2}\min\{\lambda \mid \lambda \in \sigma(L),\ \lambda \neq 0\}$. Thus, in the single integrator case the corresponding a priori upper bounds explicitly involve the Laplacian eigenvalues.

As noted in the Introduction, the single integrator case was also studied in [43] for the slightly different set up that the output equation in the original network (3.6) is taken as $y = (W^{\frac{1}{2}} R^T \otimes I)x$ instead of $y = (L \otimes I)x$. Here, R is the incidence matrix of the graph and W the diagonal matrix with the edge weights on the diagonal (in other words, $L = RWR^T$). It was shown in [43] that in that case the absolute and relative approximation errors admit the explicit expressions

$$\|S - \hat{S}\|_2^2 = \frac{1}{2}\sum_{i=1}^{m}\left(1 - \frac{1}{|C_{k_i}|}\right),$$

and

$$\frac{\|S - \hat{S}\|_2^2}{\|S\|_2^2} = \frac{\sum_{i=1}^{m}\left(1 - \frac{1}{|C_{k_i}|}\right)}{m\left(1 - \frac{1}{N}\right)}.$$

3.6 $\mathcal{H}_\infty$-ERROR BOUNDS

In the previous section, we obtained a priori upper bounds for the approximation error in terms of the $\mathcal{H}_2$-norm of the difference between the transfer functions of the original network and its reduced order approximation. In the present section, we express the error in terms of the $\mathcal{H}_\infty$-norm.

3.6.1 *The single integrator case*

In this first subsection, we consider the special case that the agent dynamics is a single integrator system. In this case, we have $A = 0$, $B = 1$, and $E = 1$ and the original system (3.6) then reduces to

$$\begin{aligned} \dot{x} &= -Lx + Mu, \\ y &= Lx. \end{aligned} \tag{3.22}$$

The state space dimension of (3.22) is then simply N, the number of agents. For a given partition $\pi = \{C_1, C_2, \ldots, C_k\}$, the reduced system (3.7) is now given by

$$\begin{aligned} \dot{\hat{x}} &= -\hat{L}\hat{x} + \hat{M}u, \\ \hat{y} &= LP\hat{x}, \end{aligned}$$

where $P = P(\pi)$ is again the characteristic matrix of π and $\hat{x} \in \mathbb{R}^k$. The transfer functions S and $\hat{S}$, of the original and reduced system respectively, are given by

$$\begin{aligned} S(s) &= L(sI_N + L)^{-1}M, \\ \hat{S}(s) &= LP\big(sI_k + \hat{L}\big)^{-1}\hat{M}. \end{aligned}$$

We then have the following explicit expressions for the $\mathcal{H}_\infty$ model reduction error:

THEOREM 3.14: *Let π be an AEP of the graph G. If the network with single integrator agent dynamics is clustered according to π, then the $\mathcal{H}_\infty$-error is given by*

$$\big\|S - \hat{S}\big\|_\infty^2 = \begin{cases} \max\limits_{1 \leqslant i \leqslant m} \left(1 - \frac{1}{|C_{k_i}|}\right) & \text{if the leaders are in different cells,} \\ 1 & \text{otherwise,} \end{cases}$$

where C_{k_i} is the set of cellmates of leader i for some $k_i \in \{1, 2, \ldots, k\}$. Furthermore, since $\|S\|_\infty = 1$, the relative and absolute $\mathcal{H}_\infty$-errors coincide.

REMARK 3.15: We see that the $\mathcal{H}_\infty$-error lies in the interval $[0, 1]$. The error is maximal ($= 1$) if and only if two or more leader nodes occupy one and the same cell. The error is minimal ($= 0$) if and only if each leader node occupies a different cell, and is the only node in this cell. In general, the error decreases if the number of cellmates of the leaders decreases.

Proof. To simplify notation, denote $\Delta(s) = S(s) - \hat{S}(s)$. Note that both S and $\hat{S}$ have all poles in the open left half plane. We now first show that since π is an AEP we have

$$\|\Delta\|_\infty = \sigma_1(\Delta(0)). \tag{3.23}$$

First note that $\hat{S}(s) = LP\left(P^T P\right)^{-\frac{1}{2}}(sI_k + \bar{L})^{-1}\left(P^T P\right)^{\frac{1}{2}}\hat{M}$, where the symmetric matrix $\bar{L}$ is given by (3.12). Thus, a state space representation for the error system is given by

$$\begin{aligned} \dot{x}_e &= \begin{pmatrix} -L & 0 \\ 0 & -\bar{L} \end{pmatrix} x_e + \begin{pmatrix} M \\ (P^T P)^{\frac{1}{2}}\hat{M} \end{pmatrix} u \\ e &= \begin{pmatrix} L & -LP\left(P^T P\right)^{-\frac{1}{2}} \end{pmatrix} x_e. \end{aligned} \tag{3.24}$$

Next, we show that (3.23) holds by applying Lemma 3.2 to system (3.24). Indeed, with $X = -L$ we have

$$\begin{aligned} &\begin{pmatrix} L & -LP\left(P^T P\right)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} -L & 0 \\ 0 & -\bar{L} \end{pmatrix} \\ &\quad = \begin{pmatrix} -L^2 & LP\left(P^T P\right)^{-\frac{1}{2}}\bar{L} \end{pmatrix} \\ &\quad = \begin{pmatrix} -L^2 & LP\hat{L}\left(P^T P\right)^{-\frac{1}{2}} \end{pmatrix} \\ &\quad = \begin{pmatrix} -L^2 & L^2 P\left(P^T P\right)^{-\frac{1}{2}} \end{pmatrix} = X \begin{pmatrix} L & -LP\left(P^T P\right)^{-\frac{1}{2}} \end{pmatrix}, \end{aligned}$$

and from Lemma 3.2 it then immediately follows that $\|\Delta\|_\infty = \sigma_1(\Delta(0))$. To compute $\sigma_1(\Delta(0))$ we apply Lemma 3.3 to system (3.24). First, it is easily verified that

$$\ker \begin{pmatrix} -L & 0 \\ 0 & -\bar{L} \end{pmatrix} \subset \ker \begin{pmatrix} L & -LP\left(P^T P\right)^{-\frac{1}{2}} \end{pmatrix}.$$

By applying Lemma 3.3 we then obtain

$$\begin{aligned}\Delta(0) &= \begin{pmatrix} L & -LP(P^\mathsf{T}P)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} L & 0 \\ 0 & \bar{L} \end{pmatrix}^+ \begin{pmatrix} M \\ (P^\mathsf{T}P)^{\frac{1}{2}}\hat{M} \end{pmatrix} \\ &= L\Big(L^+ - P(P^\mathsf{T}P)^{-\frac{1}{2}}\bar{L}^+(P^\mathsf{T}P)^{-\frac{1}{2}}P^\mathsf{T}\Big)M.\end{aligned} \tag{3.25}$$

Recall that $\hat{U}$ in (3.13) is an orthogonal matrix that diagonalizes $\bar{L}$ and that $U_1 = P(P^\mathsf{T}P)^{-\frac{1}{2}}\hat{U}$. Then $\bar{L}^+ = \hat{U}\hat{\Lambda}^+\hat{U}^\mathsf{T}$. Thus we have

$$P(P^\mathsf{T}P)^{-\frac{1}{2}}\bar{L}^+(P^\mathsf{T}P)^{-\frac{1}{2}}P^\mathsf{T} = U_1\hat{\Lambda}^+U_1^\mathsf{T}.$$

Next, we compute

$$\begin{aligned}LL^+ &= U\Lambda U^\mathsf{T}U\Lambda^+U^\mathsf{T} \\ &= U\Lambda\Lambda^+U^\mathsf{T} \\ &= I_N - \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\mathsf{T},\end{aligned} \tag{3.26}$$

where the last equality follows from the fact that the first column of U is $\frac{1}{\sqrt{N}}\mathbb{1}_N$. Next, observe that

$$\begin{aligned}LU_1\hat{\Lambda}^+U_1^\mathsf{T} &= U\Lambda U^\mathsf{T}U_1\hat{\Lambda}^+U_1^\mathsf{T} \\ &= U_1\hat{\Lambda}\hat{\Lambda}^+U_1^\mathsf{T} \\ &= U_1U_1^\mathsf{T} - \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\mathsf{T} \\ &= P(P^\mathsf{T}P)^{-1}P^\mathsf{T} - \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^\mathsf{T}.\end{aligned} \tag{3.27}$$

Combining (3.26) and (3.27) with (3.25), we obtain

$$\Delta(0) = \Big(I_N - P(P^\mathsf{T}P)^{-1}P^\mathsf{T}\Big)M.$$

From (3.23) then, we have that the $\mathcal{H}_\infty$-error is given by

$$\begin{aligned}\left\|S-\hat{S}\right\|_\infty^2 &= \lambda_{\max}\Big(\Delta(0)^T\Delta(0)\Big)\\ &= \lambda_{\max}\left(M^T\Big(I_N - P\big(P^TP\big)^{-1}P^T\Big)^2 M\right)\\ &= \lambda_{\max}\Big(I_m - M^TP\big(P^TP\big)^{-1}P^TM\Big)\\ &= 1-\lambda_{\min}\Big(M^TP\big(P^TP\big)^{-1}P^TM\Big).\end{aligned} \tag{3.28}$$

All that is left now is to compute the minimal eigenvalue of the matrix $M^TP\big(P^TP\big)^{-1}P^TM$. Again let $\{v_1, v_2, \dots, v_m\}$ be the set of leaders and note that M satisfies

$$M = \begin{pmatrix} e_{v_1} & e_{v_2} & \cdots & e_{v_m}\end{pmatrix}.$$

Again, without loss of generality, assume that π is regularly formed. Then the matrix $P\big(P^TP\big)^{-1}P^T$ is block diagonal where each diagonal block P_i is a $|C_i| \times |C_i|$ matrix whose entries are all $\frac{1}{|C_i|}$. Next, let $k_i \in \{1, 2, \dots, k\}$ be such that $v_i \in C_{k_i}$. If all the leaders are in different cells, then

$$M^TP\big(P^TP\big)^{-1}P^TM = \operatorname{diag}\left(\frac{1}{|C_{k_1}|}, \frac{1}{|C_{k_2}|}, \dots, \frac{1}{|C_{k_m}|}\right),$$

and so

$$\lambda_{\min}\Big(M^TP\big(P^TP\big)^{-1}P^TM\Big) = \min_{1\leqslant i\leqslant m}\frac{1}{|C_{k_i}|}. \tag{3.29}$$

Now suppose that two leaders v_i and v_j are cellmates. Then we have

$$M^TP\big(P^TP\big)^{-1}P^TM(e_i - e_j) = M^TP\big(P^TP\big)^{-1}P^T(e_{v_i} - e_{v_j}) = 0.$$

which together with $M^TP\big(P^TP\big)^{-1}P^TM \geqslant 0$ implies

$$\lambda_{\min}\Big(M^TP\big(P^TP\big)^{-1}P^TM\Big) = 0. \tag{3.30}$$

From (3.28), (3.29), and (3.30), we find the absolute $\mathcal{H}_\infty$-error. To find the relative $\mathcal{H}_\infty$-error, we compute $\|S\|_\infty$ by applying Lemma 3.2 and Lemma 3.3 to the original system (3.22). Combined with (3.26), this results in the $\mathcal{H}_\infty$-norm of the original system:

$$\|S\|_\infty^2 = \lambda_{\max}\Big(S(0)^T S(0)\Big) = \lambda_{\max}\left(M^T\left(I_N - \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^T\right)M\right) = 1.$$

□

3.6.2 *The general case with symmetric agent dynamics*

In this subsection, we deal with the case that the agent dynamics is given by an arbitrary multivariable system. The original and the reduced network are again given by (3.6) and (3.7), respectively. As in the proof of Theorem 3.14 we will rely heavily on Lemma 3.3 to compute the $\mathcal{H}_\infty$-error. Since Lemma 3.3 relies on a symmetry argument, we will need to assume that the matrices A and B are both symmetric, which will be a standing assumption in the remainder of this section.

The main theorem of this section establishes an a priori upper bound for the $\mathcal{H}_\infty$-norm of the approximation error in the case that we cluster according to an AEP. Again, an important role is played by the $N-1$ auxiliary systems (3.11) with λ ranging over the nonzero eigenvalues of the Laplacian L. Again, let $S_\lambda(s) = \lambda(sI - A + \lambda B)^{-1}E$ be their transfer functions We assume that the original network (3.6) is synchronized, so that all of the $A - \lambda B$ are Hurwitz. We again use S, $\hat{S}$, and Δ to denote the relevant transfer functions.

We have the following theorem:

THEOREM 3.16: *Consider the network* (3.6) *and assume that* A *and* B *are symmetric matrices. Assume the network is synchronized. Let* π *be an almost equitable partition of the graph* G. *The* $\mathcal{H}_\infty$*-error when clustering* G *according to* π *then satisfies*

$$\left\|S - \hat{S}\right\|_\infty^2 \leqslant \begin{cases} S_{\max,\mathcal{H}_\infty}^2 \max\limits_{1\leqslant i\leqslant m}\left(1 - \frac{1}{|C_{k_i}|}\right) & \textit{if the leaders are in different cells,} \\ S_{\max,\mathcal{H}_\infty}^2 & \textit{otherwise} \end{cases}$$

and

$$\frac{\|S-\hat{S}\|_\infty^2}{\|S\|_\infty^2} \leqslant \begin{cases} \frac{S^2_{\max,\mathcal{H}_\infty}}{S^2_{\min,\mathcal{H}_\infty}} \max\limits_{1\leqslant i\leqslant m}\left(1-\frac{1}{|C_{k_i}|}\right) & \text{if the leaders are in different cells,} \\ \frac{S^2_{\max,\mathcal{H}_\infty}}{S^2_{\min,\mathcal{H}_\infty}} & \text{otherwise,} \end{cases}$$

where

$$S_{\max,\mathcal{H}_\infty} := \max_{\lambda\in\sigma(L)\setminus\sigma(\hat{L})} \|S_\lambda\|_\infty \tag{3.31}$$

and

$$S_{\min,\mathcal{H}_\infty} := \min_{\lambda\in\sigma(L)\setminus\{0\}} \sigma_{\min}(S_\lambda(0)), \tag{3.32}$$

with $S_\lambda(s)$ *the transfer function of the auxiliary system* (3.11).

REMARK 3.17: The absolute $\mathcal{H}_\infty$-error thus lies in the closed interval $[0, S_{\max,\mathcal{H}_\infty}]$ with $S_{\max,\mathcal{H}_\infty}$ the maximum over the $\mathcal{H}_\infty$-norms of the transfer functions S_λ with $\lambda \in \sigma(L) \setminus \sigma(\hat{L})$. The error is minimal (equal to 0) if each leader node occupies a different cell, and is the only node in this cell. In general, the upper bound decreases if the number of cellmates of the leaders decreases.

Proof. First note that the transfer function $\hat{S}$ of the reduced network (3.7) is equal to

$$\hat{S}(s) = \left(LP(P^TP)^{-\frac{1}{2}} \otimes I_n\right)\left(sI - I_k \otimes A + \bar{L} \otimes B\right)^{-1}\left((P^TP)^{\frac{1}{2}}\hat{M} \otimes E\right), \tag{3.33}$$

with the symmetric matrix $\bar{L}$ given by (3.12). Analogous to the proof of Theorem 3.14, we first apply Lemma 3.2 to the error system

$$\begin{aligned} \dot{x}_e &= \begin{pmatrix} I_N \otimes A - L \otimes B & 0 \\ 0 & I_k \otimes A - \bar{L} \otimes B \end{pmatrix} x_e + \begin{pmatrix} M \otimes E \\ (P^TP)^{\frac{1}{2}}\hat{M} \otimes E \end{pmatrix} u \\ e &= \begin{pmatrix} L \otimes I_n & -LP(P^TP)^{-\frac{1}{2}} \otimes I_n \end{pmatrix} x_e, \end{aligned}$$

with transfer function $\Delta(s)$. Take $X = I_N \otimes A - L \otimes B$. We then have

$$\begin{aligned}&\left(L \otimes I_n \quad -LP\left(P^TP\right)^{-\frac{1}{2}} \otimes I_n\right) \begin{pmatrix} I_N \otimes A - L \otimes B & 0 \\ 0 & I_k \otimes A - \bar{L} \otimes B \end{pmatrix} \\ &\qquad = X \left(L \otimes I_n \quad -LP\left(P^TP\right)^{-\frac{1}{2}} \otimes I_n\right).\end{aligned}$$

From Lemma 3.2, we then obtain that

$$\|\Delta\|_\infty = \sigma_1(\Delta(0)) = \lambda_{\max}\left(\Delta(0)^T\Delta(0)\right)^{\frac{1}{2}}.$$

In the proof of Lemma 3.9, it was shown that

$$\hat{S}(-s)^T\Delta(s) = \hat{S}(-s)^T(S(s) - \hat{S}(s)) = 0.$$

Since all transfer functions involved are stable, in particular this holds for $s = 0$. We then have that $\hat{S}(0)^T(S(0) - \hat{S}(0)) = 0$, i.e. $\hat{S}(0)^TS(0) = \hat{S}(0)^T\hat{S}(0)$. By transposing, we also have $S(0)^T\hat{S}(0) = \hat{S}(0)^T\hat{S}(0)$. Therefore,

$$\begin{aligned}\Delta(0)^T\Delta(0) &= \left(S(0) - \hat{S}(0)\right)^T(S(0) - \hat{S}(0)) \\ &= S(0)^TS(0) - S(0)^T\hat{S}(0) - \hat{S}(0)^TS(0) + \hat{S}(0)^T\hat{S}(0) \\ &= S(0)^TS(0) - \hat{S}(0)^T\hat{S}(0).\end{aligned}$$

By applying Lemma 3.3 to system (3.6), we obtain

$$\begin{aligned}S(0)^TS(0) &= \left(M^T \otimes E^T\right)(I_N \otimes A - L \otimes B)^+\left(L^2 \otimes I_n\right) \\ &\qquad \times (I_N \otimes A - L \otimes B)^+(M \otimes E) \\ &= \left(M^T \otimes E^T\right)(U \otimes I_n)(I_N \otimes A - \Lambda \otimes B)^+\left(\Lambda^2 \otimes I_n\right) \\ &\qquad \times (I_N \otimes A - \Lambda \otimes B)^+\left(U^T \otimes I_n\right)(M \otimes E) \\ &= \left(M^TU \otimes E^T\right) \\ &\qquad \times \operatorname{diag}\left(0, \lambda_2^2(A - \lambda_2 B)^{-2}, \dots, \lambda_N^2(A - \lambda_N B)^{-2}\right) \\ &\qquad \times (U^TM \otimes E) \\ &= \left(M^TU \otimes I_r\right)\operatorname{diag}\left(0, S_{\lambda_2}(0)^TS_{\lambda_2}(0), \dots, S_{\lambda_N}(0)^TS_{\lambda_N}(0)\right) \\ &\qquad \times \left(U^TM \otimes I_r\right),\end{aligned}$$

(3.34)

where S_λ is again the transfer function of the auxiliary system (3.11). Recall that $\hat{M} = \left(P^T P\right)^{-1} P^T M$ and $U_1 = P\left(P^T P\right)^{-\frac{1}{2}} \hat{U}$. We now apply Lemma 3.3 to the transfer function (3.33) of the system (3.7):

$$
\begin{aligned}
\hat{S}(0)^T \hat{S}(0) &= \left(M^T P\left(P^T P\right)^{-\frac{1}{2}} \otimes E^T\right)\left(I_N \otimes A - \bar{L} \otimes B\right)^{+} \\
&\quad \times \left(\left(P^T P\right)^{-\frac{1}{2}} P^T L^2 P\left(P^T P\right)^{-\frac{1}{2}} \otimes I_n\right) \\
&\quad \times \left(I_N \otimes A - \bar{L} \otimes B\right)^{+}\left(\left(P^T P\right)^{-\frac{1}{2}} P^T M \otimes E\right) \\
&= \left(M^T P\left(P^T P\right)^{-\frac{1}{2}} \otimes E^T\right)\left(\hat{U} \otimes I_n\right)\left(I_N \otimes A - \hat{\Lambda} \otimes B\right)^{+} \\
&\quad \times \left(\hat{\Lambda}^2 \otimes I_n\right)\left(I_N \otimes A - \hat{\Lambda} \otimes B\right)^{+} \\
&\quad \times \left(\hat{U}^T \otimes I_n\right)\left(\left(P^T P\right)^{-\frac{1}{2}} P^T M \otimes E\right) \\
&= \left(M^T U_1 \otimes E^T\right) \\
&\quad \times \operatorname{diag}\left(0, \lambda_2^2 (A - \lambda_2 B)^{-2}, \ldots, \lambda_k^2 (A - \lambda_k B)^{-2}\right) \\
&\quad \times \left(U_1^T M \otimes E\right) \\
&= \left(M^T U_1 \otimes I_r\right) \\
&\quad \times \operatorname{diag}\left(0, S_{\lambda_2}(0)^T S_{\lambda_2}(0), \ldots, S_{\lambda_k}(0)^T S_{\lambda_k}(0)\right) \\
&\quad \times \left(U_1^T M \otimes I_r\right).
\end{aligned}
$$

Combining the two expression above, it immediately follows that

$$
\begin{aligned}
\Delta(0)^T \Delta(0) &= S(0)^T S(0) - \hat{S}(0)^T \hat{S}(0) \\
&= \left(M^T U_2 \otimes I_r\right) \\
&\quad \times \operatorname{diag}\left(S_{\lambda_{k+1}}(0)^T S_{\lambda_{k+1}}(0), \ldots, S_{\lambda_N}(0)^T S_{\lambda_N}(0)\right) \\
&\quad \times \left(U_2^T M \otimes I_r\right).
\end{aligned}
$$

By taking $S_{\max,\mathcal{H}_\infty}$ as defined by (3.31) it then holds that

$$\begin{aligned}\Delta(0)^T\Delta(0) &\leqslant \left(M^T U_2 \otimes I_r\right) \operatorname{diag}(S^2_{\max,\mathcal{H}_\infty} I_r, \dots, S^2_{\max,\mathcal{H}_\infty} I_r) \\ &\quad \times \left(U_2^T M \otimes I_r\right) \\ &= S^2_{\max,\mathcal{H}_\infty}\left(M^T U_2 U_2^T M \otimes I_r\right) \\ &= S^2_{\max,\mathcal{H}_\infty}\left(M^T (I_N - U_1 U_1^T) M \otimes I_r\right) \\ &= S^2_{\max,\mathcal{H}_\infty}\left(\left(I_m - M^T P\left(P^T P\right)^{-1} P^T M\right) \otimes I_r\right).\end{aligned}$$

Continuing as in the proof of Theorem 3.14, we find an upper bound for the $\mathcal{H}_\infty$-error:

$$\|\Delta\|^2_\infty \leqslant S^2_{\max,\mathcal{H}_\infty}\lambda_{\max}\left(I_m - M^T P\left(P^T P\right)^{-1} P^T M\right).$$

To compute an upper bound for the relative $\mathcal{H}_\infty$-error, we bound the $\mathcal{H}_\infty$-norm of system (3.6) from below. Again, let $\bar{U}$ be such that $U = \begin{pmatrix} u_1 & \bar{U} \end{pmatrix}$ and let $S_{\min,\mathcal{H}_\infty}$ be as defined by (3.32). From (3.34) it now follows that

$$\begin{aligned}S(0)^T S(0) &= \left(M^T \bar{U} \otimes I_r\right) \operatorname{diag}\left(S_{\lambda_2}(0)^T S_{\lambda_2}(0), \dots, S_{\lambda_N}(0)^T S_{\lambda_N}(0)\right) \\ &\quad \times \left(\bar{U}^T M \otimes I_r\right) \\ &\geqslant \left(M^T \bar{U} \otimes I_r\right) \operatorname{diag}(S^2_{\min,\mathcal{H}_\infty} I_r, \dots, S^2_{\min,\mathcal{H}_\infty} I_r)\left(\bar{U}^T M \otimes I_r\right) \\ &= S^2_{\min,\mathcal{H}_\infty}\left(M^T \bar{U}\bar{U} M \otimes I_r\right) \\ &= S^2_{\min,\mathcal{H}_\infty}\left(M^T\left(I_N - \frac{1}{N}\mathbb{1}_N\mathbb{1}_N^T\right) M \otimes I_r\right).\end{aligned}$$

Again using Lemma 3.3, we find a lower bound to the $\mathcal{H}_\infty$-norm of S:

$$\|S\|^2_\infty = \lambda_{\max}\left(S(0)^T S(0)\right) \geqslant S^2_{\min,\mathcal{H}_\infty},$$

which concludes the proof of our theorem. □

3.7 TOWARDS A PRIORI ERROR BOUNDS FOR GENERAL GRAPH PARTITIONS

Up to now, in this chapter we have dealt with establishing a priori error bounds for network reduction by clustering using AEPs of the network graph. Of course, an important problem is to find error bounds for *arbitrary*, possibly non almost equitable, partitions. In this section, we address this more general problem. We restrict ourselves to the special case that the agents have single integrator dynamics. Thus, we consider the multi-agent network

$$\begin{aligned} \dot{x} &= -Lx + Mu, \\ y &= Lx. \end{aligned} \tag{3.35}$$

As before, we assume that the underlying (undirected) graph G is connected, so that the network is synchronized. Next, assume that $\pi = \{C_1, C_2, \dots, C_k\}$ is a graph partition, not necessarily an AEP, and let $P = P(\pi) \in \mathbb{R}^{N \times k}$ be its characteristic matrix. As before, the reduced order network is taken to be the Petrov-Galerkin projection of (3.35), and is represented by

$$\begin{aligned} \dot{\hat{x}} &= -\hat{L}\hat{x} + \hat{M}u, \\ \hat{y} &= LP\hat{x}, \end{aligned} \tag{3.36}$$

Again, let S and $\hat{S}$ be the transfer functions of (3.35) and (3.36), respectively. We address the problem of obtaining a priori upper bounds for $\|S - \hat{S}\|_2$ and $\|S - \hat{S}\|_\infty$.

The idea for establishing such upper bounds is as follows: as a first step we will approximate the original Laplacian matrix L (of the original network graph G) by a new Laplacian matrix, denoted by L_{AEP} (corresponding to a 'nearby' graph G_{AEP}) such that the given partition π is an AEP with respect to this new graph G_{AEP}. This new graph G_{AEP} defines a new multi-agent system with transfer function

$$S_{AEP}(s) = L_{AEP}(sI + L_{AEP})^{-1}M.$$

The reduced order network of S_{AEP} (using the AEP π) has transfer function $\hat{S}_{AEP}(s) = L_{AEP}P\big(sI + \hat{L}_{AEP}\big)^{-1}\hat{M}$. Then using the triangle inequality both for $p = 2$ and $p = \infty$ we have

$$\begin{aligned} \|S - \hat{S}\|_p &= \|S - S_{AEP} + S_{AEP} - \hat{S}_{AEP} + \hat{S}_{AEP} - \hat{S}\|_p \\ &\leqslant \|S - S_{AEP}\|_p + \|S_{AEP} - \hat{S}_{AEP}\|_p + \|\hat{S}_{AEP} - \hat{S}\|_p. \end{aligned} \tag{3.37}$$

The idea is to obtain a priori upper bounds for all three terms in (3.37). We first propose an approximating Laplacian matrix L_{AEP}, and subsequently study the problems of establishing upper bounds for the three terms in (3.37) separately.

For a given matrix M, let $\|M\|_F := (\operatorname{tr} M^T M)^{\frac{1}{2}}$ denote its Frobenius norm. In the following, denote $\mathcal{P} := P\left(P^T P\right)^{-1} P^T$. Note that $\mathcal{P}$ is the orthogonal projector onto $\operatorname{im} P$. As approximation for L, we compute the unique solution to the convex optimization problem

$$\begin{aligned} \underset{L_{AEP}}{\text{minimize}} \quad & \|L - L_{AEP}\|_F^2, \\ \text{subject to} \quad & (I_N - \mathcal{P}) L_{AEP} P = 0, \\ & L_{AEP} = L_{AEP}^T, \\ & L_{AEP} \geqslant 0, \\ & L_{AEP} \mathbb{1}_N = 0. \end{aligned} \tag{3.38}$$

In other words, we want to compute a positive semi-definite matrix L_{AEP} with row sums equal to zero, and with the property that $\operatorname{im} P$ is invariant under L_{AEP} (equivalently, the given partition π is an AEP for the new graph). We will show that such L_{AEP} may correspond to an undirected graph *with negative weights*. However, it is constrained to be positive semi-definite, so the results of Sections 3.4, 3.5, and 3.6 in this chapter will remain valid.

THEOREM 3.18: *The matrix* $L_{AEP} := \mathcal{P} L \mathcal{P} + (I_N - \mathcal{P}) L (I_N - \mathcal{P})$ *is the unique solution to the convex optimization problem* (3.38). *If* L *corresponds to a connected graph, then, in fact,* $\ker L_{AEP} = \operatorname{im} \mathbb{1}_N$.

Proof. Clearly, L_{AEP} is symmetric and positive semi-definite since L is. Also, $(I_N - \mathcal{P}) L_{AEP} P = 0$ since $(I_N - \mathcal{P}) P = 0$. It is also obvious that $L_{AEP} \mathbb{1}_N = 0$ since $\mathcal{P} \mathbb{1}_N = \mathbb{1}_N$. We now show that L_{AEP} uniquely minimizes the distance to L. Let X satisfy the constraints and define $\Delta = L_{AEP} - X$. Then we have

$$\|L - X\|_F^2 = \|L - L_{AEP}\|_F^2 + \|\Delta\|_F^2 + 2 \operatorname{tr}(L - L_{AEP}) \Delta.$$

It can be verified that $L - L_{AEP} = (I_N - \mathcal{P})L\mathcal{P} + \mathcal{P}L(I_N - \mathcal{P})$. Thus,

$$\operatorname{tr}(L - L_{AEP})\Delta = \operatorname{tr}(I_N - \mathcal{P})L\mathcal{P}\Delta + \operatorname{tr}\mathcal{P}L(I_N - \mathcal{P})\Delta.$$

Now, since both X and L_{AEP} satisfy the first constraint, we have $(I_N - \mathcal{P})\Delta\mathcal{P} = 0$. Using this we have

$$\operatorname{tr}(I_N - \mathcal{P})L\mathcal{P}\Delta = \operatorname{tr}\mathcal{P}\Delta(I_N - \mathcal{P})L = \operatorname{tr}L(I_N - \mathcal{P})\Delta\mathcal{P} = 0.$$

Also,

$$\operatorname{tr}\mathcal{P}L(I_N - \mathcal{P})\Delta = \operatorname{tr}L(I_N - \mathcal{P})\Delta\mathcal{P} = 0.$$

Thus, we obtain

$$\|L - X\|_F^2 = \|L - L_{AEP}\|_F^2 + \|\Delta\|_F^2,$$

from which it follows that $\|L - X\|_F$ is minimal if and only if $\Delta = 0$, equivalently $X = L_{AEP}$.

To prove the second statement, let $x \in \ker L_{AEP}$, so $x^T L_{AEP} x = 0$. Then both $x^T \mathcal{P}L\mathcal{P}x = 0$ and $x^T(I_N - \mathcal{P})L(I_N - \mathcal{P})x = 0$. This clearly implies $L\mathcal{P}x = 0$ and $L(I_N - \mathcal{P})x = 0$. Since L corresponds to a connected graph we must have $\mathcal{P}x \in \operatorname{im}\mathbb{1}_N$ and $(I_N - \mathcal{P})x \in \operatorname{im}\mathbb{1}_N$. We conclude that $x \in \operatorname{im}\mathbb{1}_N$ as desired. □

As announced above, L_{AEP} may have positive off-diagonal elements, corresponding to a graph with some of its edge weights being negative. For example, for

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

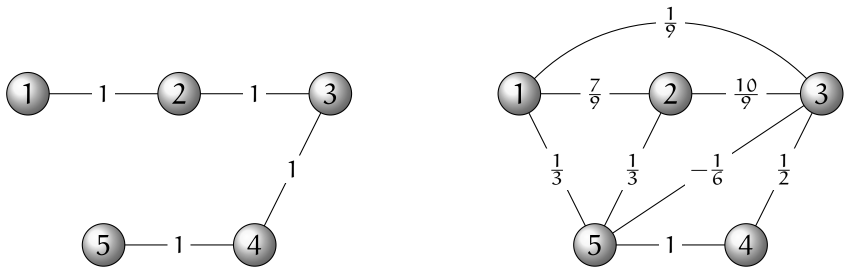

Figure 3.3: Modifying the line graph on 5 nodes to make the partition $\{\{1,2,3\},\{4,5\}\}$ almost equitable.

we have

$$L_{AEP} = \begin{pmatrix} \frac{11}{9} & -\frac{7}{9} & -\frac{1}{9} & 0 & -\frac{1}{3} \\ -\frac{7}{9} & \frac{20}{9} & -\frac{10}{9} & 0 & -\frac{1}{3} \\ -\frac{1}{9} & -\frac{10}{9} & \frac{14}{9} & -\frac{1}{2} & \frac{1}{6} \\ 0 & 0 & -\frac{1}{2} & \frac{3}{2} & -1 \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{6} & -1 & \frac{3}{2} \end{pmatrix},$$

so the edge between nodes 3 and 5 has a negative weight. Figure 3.3 shows the graphs corresponding to L and L_{AEP}. Although L_{AEP} is not necessarily a Laplacian matrix with only nonpositive off-diagonal elements, it has all the properties we associate with a Laplacian matrix. Specifically, it can be checked that all results in this chapter remain valid, since they only depend on the symmetric positive semi-definiteness of the Laplacian matrix.

Using the approximating Laplacian $L_{AEP} = \mathcal{P}L\mathcal{P} + (I_N - \mathcal{P})L(I_N - \mathcal{P})$ as above, we will now deal with establishing upper bounds for the three terms in (3.37). We start off with the middle term $\|S_{AEP} - \hat{S}_{AEP}\|_p$ in (3.37).

According to Remark 3.13, for $p = 2$ this term has an upper bound depending on the maximal eigenvalue of L_{AEP} that is not an eigenvalue of $\hat{L}_{AEP}$, on the minimal nonzero eigenvalue of L_{AEP}, and on the number of cellmates of the leaders with respect to the partitioning π.

For $p = \infty$, in Theorem 3.14 this term was expressed in terms of the maximal number of cellmates with respect to the partitioning π (noting that it is equal to 1 in case two or more leaders share the same cell).

Next, we will take a look at the first and third term in (3.37), i.e. $\|S - S_{AEP}\|_p$ and $\|\hat{S} - \hat{S}_{AEP}\|_p$. Let us denote $\Delta L = L - L_{AEP}$. We find

$$\begin{aligned} S(s) - S_{AEP}(s) &= L(sI + L)^{-1}M - L_{AEP}(sI + L_{AEP})^{-1}M \\ &= L(sI + L)^{-1}M \\ &\quad - L_{AEP}\Big[(sI + L)^{-1} + (sI + L_{AEP})^{-1}\Delta L(sI + L)^{-1}\Big]M \\ &= L(sI + L)^{-1}M - L_{AEP}(sI + L)^{-1}M \\ &\quad - L_{AEP}(sI + L_{AEP})^{-1}\Delta L(sI + L)^{-1}M \\ &= \Delta L(sI + L)^{-1}M - L_{AEP}(sI + L_{AEP})^{-1}\Delta L(sI + L)^{-1}M \\ &= \Big[I_N - L_{AEP}(sI + L_{AEP})^{-1}\Big]\Delta L(sI + L)^{-1}M. \end{aligned}$$

Thus, for $p = 2$ and $p = \infty$ we have

$$\begin{aligned} \|S - S_{AEP}\|_p &\leqslant \Big\|I_N - L_{AEP}(sI + L_{AEP})^{-1}\Big\|_\infty \Big\|\Delta L(sI + L)^{-1}M\Big\|_p \\ &\leqslant 2\Big\|\Delta L(sI + L)^{-1}M\Big\|_p. \end{aligned} \tag{3.39}$$

It is also easily seen that $\hat{L}_{AEP} = \big(P^T P\big)^{-1} P^T L_{AEP} P = \big(P^T P\big)^{-1} P^T L P = \hat{L}$ and $L_{AEP}P = P\big(P^T P\big)^{-1} P^T L P = P\hat{L}$. Therefore,

$$\begin{aligned} \hat{S}(s) - \hat{S}_{AEP}(s) &= LP\big(sI + \hat{L}\big)^{-1}\hat{M} - L_{AEP}P\big(sI + \hat{L}_{AEP}\big)^{-1}\hat{M} \\ &= LP\big(sI + \hat{L}\big)^{-1}\hat{M} - P\hat{L}\big(sI + \hat{L}\big)^{-1}\hat{M} \\ &= \big(LP - P\hat{L}\big)\big(sI + \hat{L}\big)^{-1}\hat{M}. \end{aligned}$$

Since, finally, $(LP - P\hat{L})^T (LP - P\hat{L}) = P^T (\Delta L)^2 P$, for $p = 2$ and $p = \infty$ we obtain

$$\Big\|\hat{S} - \hat{S}_{AEP}\Big\|_p \leqslant \Big\|\Delta LP(sI + \hat{L})^{-1}\hat{M}\Big\|_p. \tag{3.40}$$

Thus, both in (3.39) and (3.40) the upper bound involves the difference $\Delta L = L - L_{AEP}$ between the original Laplacian and its optimal approximation in the set of Laplacian matrices for which the given partition π is an AEP. In a sense, the difference ΔL measures how far π is away from being an AEP for the original graph G. Obviously, $\Delta L = 0$ if and only if π is an AEP for G. In that case only the middle term in (3.37) is present.

3.8 CONCLUSIONS

In this chapter, we have extended results on model reduction of leader-follower networks with single integrator agent dynamics to leader-follower networks with arbitrary linear multivariable agent dynamics. The proposed model reduction technique reduces the complexity of the network topology by clustering the agents according to a special class of graph partitions called almost equitable partitions. We have shown that if the original undirected network is reduced by means of a specific Petrov-Galerkin projection associated with such graph partition, then the resulting reduced order model can be interpreted as a networked multi-agent system with a weighted, directed network graph. If the original network is clustered according to an almost equitable partition, then its consensus properties are preserved. We have provided a priori upper bounds on the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ model reduction errors in this case. These error bounds depend on an auxiliary system closely related to the agent dynamics, the eigenvalues of the Laplacian matrices of the original and the reduced network, and on the number of cellmates of the leaders in the network. Finally, we have provided some insight into the general case of clustering according to arbitrary, not necessarily almost equitable, partitions. Here, direct computation of a priori upper bounds on the error is not as straightforward as in the case of almost equitable partitions. We have shown that in this more general case one can bound the model reduction errors by first optimally approximating the original network by a new network for which the chosen partition is almost equitable, and then bounding the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ errors using the triangle inequality.