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## Robust Synchronization and Model Reduction of Multi-Agent Systems

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## ROBUST SYNCHRONIZATION OF UNCERTAIN MULTI-AGENT SYSTEMS

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### 2.1 INTRODUCTION

The foundations of the state space approach to  $\mathcal{H}_\infty$  and robust control theory for linear systems were developed in the eighties of the twentieth century. Important results on optimal robust stabilization were obtained by Glover in [19] and Glover and MacFarlane in [20], see also [64]. In the present chapter we will adopt ideas from [20] to formulate and resolve a control synthesis problem in the more recent context of robust synchronization of networked multi-agent systems. In the last decade, extensive effort has been invested in the theory of distributed control of networked multi-agent systems. A networked multi-agent system is a dynamical system that consists of a group of input-output systems and an interaction topology that dictates the allowed exchange of information between these systems and their neighbours. These input-output systems are called the *agents* of the network. The possible interactions between the agents are modeled by a graph, called the *network graph*, in which the nodes represent the agents, and the edges represent the interaction topology. This network graph is unweighted and can either be directed or undirected. Well-known problems in the theory of networked systems are the problems of *consensus* and *synchronization*, see [49, 50, 54, 55] and [66] or, more recently, [38] and [37]. Consensus and synchronization of networks with nonlinear dynamics were recently investigated in [75] and [76]. In these problems, the goal is to reach a state of agreement on certain quantities of interest which depend on the states of each agent. This is to be achieved by means of local information exchange only. A communication protocol that achieves this goal is said to achieve consensus or synchronization within the network.

Recently, in [65], results on synchronization of linear multi-agent systems have been extended to accommodate the presence of *uncertainty* in the agent dynamics. While the agents in the network have identical nominal dynamics, the actual dynamics of each agent is uncertain in the sense that the transfer matrix of each agent is a perturbation of the common nominal dynamics. In [65], for both directed and undirected networks, *additively perturbed* agent dynamics was considered and conditions for the existence of dynamic protocols that achieve robust synchronization and methods to obtain such protocols were established. In the current chapter, we extend these results to directed and undirected networked multi-agent systems with *coprime factor perturbed* agent dynamics. We provide explicit equations for dynamic protocols that achieve robust synchronization for this kind of perturbations.

The outline of this chapter is as follows. In Sections 2.2 and 2.3 we introduce some notation and review some basic facts in the theory of networked systems. Next, in Section 2.4, the theory of synchronization of unperturbed linear multi-agent systems is briefly reviewed. Then, in Section 2.5 we provide a formulation of the problem of robust synchronization of coprime factor perturbed multi-agent systems. In Section 2.6 we formulate the main results of this chapter. Section 2.7 gives a numerical example that illustrates the obtained results. Finally, some concluding remarks are presented in Section 2.8.

## 2.2 PRELIMINARIES

In this chapter, we denote the set of all proper and stable real rational matrices by  $\mathcal{RH}_\infty$ . If  $G \in \mathcal{RH}_\infty$ , then  $\|G\|_\infty$  denotes its  $\mathcal{H}_\infty$ -norm,  $\|G\|_\infty = \sup_{\text{Re}(\lambda) \geq 0} \sigma_1(G(\lambda))$ . For a given square complex matrix  $M$  we denote its spectral radius by  $\rho(M)$ . A square matrix  $M$  is called Hurwitz if all its eigenvalues have strictly negative real parts.

This chapter will use ideas and results from  $\mathcal{H}_\infty$ -control. The  $\mathcal{H}_\infty$ -control problem dates back to work by G. Zames in [72] and the first full solution in a state space setting was provided in [12]. A result that is instrumental in  $\mathcal{H}_\infty$ -control is the *bounded real lemma*, see also [64, Section

12.6.2]. In this chapter we will use a version of this lemma that has been adapted to our purposes.

**LEMMA 2.1:** *Consider the system  $\dot{x} = Ax + Bu$ ,  $y = Cx + Du$  with transfer matrix  $G(s) = C(sI - A)^{-1}B + D$ . Assume  $D^T D = I$  and  $A$  is Hurwitz. Let  $\tau > 1$ . The  $\mathcal{H}_\infty$ -norm  $\|G\|_\infty$  of the transfer matrix from  $u$  to  $y$  satisfies  $\|G\|_\infty < \tau$  if there exists  $\epsilon > 0$  and a real symmetric positive semi-definite solution  $P$  to the Riccati inequality*

$$A^T P + PA + C^T C + \frac{1}{\tau^2 - 1} (PB + C^T D)(B^T P + D^T C) \leq -\epsilon (PB + C^T D)(B^T P + D^T C). \quad (2.1)$$

*Proof.* Assume that  $P$  is a solution to (2.1). Let  $\delta = \tau - \sqrt{\frac{1}{\epsilon + 1/(\tau^2 - 1)} + 1}$ . It is easily verified that  $0 < \delta < \tau$  and  $\tau - \delta > 1$ . Inequality (2.1) is then equivalent to

$$A^T P + PA + C^T C + \frac{1}{(\tau - \delta)^2 - 1} (PB + C^T D)(B^T P + D^T C) \leq 0. \quad (2.2)$$

Next, for any solution  $x(t)$  and at any time  $t$ , we have

$$\begin{aligned} \frac{d}{dt} x^T P x &= x^T (A^T P + PA)x + u^T B^T P x + x^T P B u, \\ &\leq -\frac{1}{(\tau - \delta)^2 - 1} x^T (PB + C^T D)(B^T P + D^T C)x \\ &\quad + u^T B^T P x + x^T P B u - x^T C^T C x, \\ &= -\left\| \sqrt{(\tau - \delta)^2 - 1} u - \frac{1}{\sqrt{(\tau - \delta)^2 - 1}} (B^T P + D^T C)x \right\|^2 \\ &\quad + (\tau - \delta)^2 \|u\|^2 - \|y\|^2, \\ &\leq (\tau - \delta)^2 \|u\|^2 - \|y\|^2, \end{aligned}$$

where the second inequality follows directly from (2.2). Now we take  $x(0) = 0$ ,  $u \in \mathcal{L}_2(\mathbb{R}^+)$  and integrate from 0 to  $\infty$ , which yields  $0 \leq (\tau - \delta)^2 \|u\|_2^2 - \|y\|_2^2$ . We obtain that  $\|y\|_2^2 \leq (\tau - \delta)^2 \|u\|_2^2$  for all  $u \in$

$\mathcal{L}_2(\mathbb{R}^+)$ . This implies that the operator norm  $\|G\|_\infty$  satisfies  $\|G\|_\infty \leq \tau - \delta < \tau$ . The converse statement, that  $\|G\|_\infty < \tau$  implies the existence of  $P$  and  $\epsilon$ , is not needed in this chapter.  $\square$

### 2.3 GRAPHS

In this chapter, we consider networks whose interaction topologies are represented by directed or undirected graphs, see [39, 68]. A directed graph consists of a pair  $G = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V} = \{1, 2, \dots, N\}$  is the set of *nodes*, and where  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$  is the set of *edges*. Given two nodes  $i, j \in \mathcal{V}$  with  $i \neq j$ , then an edge from  $i$  to  $j$  is represented by the pair  $(i, j) \in \mathcal{E}$ . A graph with the property that  $(i, j) \in \mathcal{E}$  implies  $(j, i) \in \mathcal{E}$  is called undirected. The *neighboring set*  $\mathcal{N}_i$  of vertex  $i$  is defined as  $\mathcal{N}_i := \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$ . For a graph  $G$ , its adjacency matrix  $A$  is given by  $A = (a_{ij})$ , with  $a_{ii} = 0$  and  $a_{ij} = 1$  if  $(j, i) \in \mathcal{E}$ , and  $a_{ij} = 0$  otherwise. The *Laplacian matrix* of  $G$  is defined as  $L = (l_{ij})$ , where we have  $l_{ii} = \sum_{j \neq i} a_{ij}$ ,  $l_{ij} = -a_{ij}$ ,  $i \neq j$ . Since all row-sums of  $L$  are zero, i.e.  $\sum_j l_{ij} = 0 \forall i$ , zero is an eigenvalue of  $L$  with eigenvector  $\mathbf{1} := \text{col}(1, \dots, 1)$ . Consequently,  $L$  has at most rank  $N - 1$ .

In the case that  $G$  is undirected, the Laplacian  $L$  is a positive semi-definite real symmetric matrix. The Laplacian matrix of an undirected graph has rank  $N - 1$  if and only if the graph is connected. Under this condition, the zero eigenvalue of  $L$  has multiplicity one. The  $N - 1$  nonzero eigenvalues of  $L$  can be ordered increasingly as

$$0 < \lambda_2 \leq \dots \leq \lambda_N.$$

Furthermore,  $L$  can be *diagonalized* by an orthogonal transformation  $U$  that brings it to the form  $\Lambda := U^T L U = \text{diag}(0, \lambda_2, \dots, \lambda_N)$ , which is denoted by  $\Lambda$ .

For a general directed graph  $G$ ,  $L$  is not necessarily symmetric, and the eigenvalues of  $L$  are not guaranteed to be real. In this case, still, all eigenvalues of  $L$  have nonnegative real part. A directed graph contains a spanning tree if and only if its Laplacian has rank  $N - 1$ . In this case, the set of nonzero eigenvalues of  $L$  is denoted in arbitrary order

by  $\{\lambda_2, \lambda_3, \dots, \lambda_N\}$ , and  $L$  can be brought to *upper triangular form* by a complex unitary transformation  $U$ :  $U^*LU = \Lambda_u$ , where  $\Lambda_u$  is a complex upper triangular matrix with  $0, \lambda_2, \dots, \lambda_N$  on the diagonal.

## 2.4 MULTI-AGENT SYSTEMS

In this section, the problem of synchronization of multi-agent systems is briefly reviewed. We consider multi-agent systems with  $N$  agents, where the communication topology of the system is represented by a directed or undirected graph  $G$  with Laplacian matrix  $L$ . For each agent  $i$  of the network, the nominal agent dynamics is given by one and the same finite-dimensional linear time-invariant system

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i. \quad (2.3)$$

For each  $i$ , the state  $x_i$  takes its values in  $\mathbb{R}^n$ , and the input signal  $u_i$  and output signal  $y_i$  take values in  $\mathbb{R}^m$  and  $\mathbb{R}^p$ , respectively. It is a standing assumption in this chapter that  $(A, B)$  is stabilizable and  $(C, A)$  is detectable.

Following [37, 65], these agents are then interconnected using an observer-based *dynamic* protocol of the form

$$\begin{aligned} \dot{w}_i &= Aw_i + B \sum_{j \in \mathcal{N}_i} (u_i - u_j) + G \sum_{j \in \mathcal{N}_i} \left( (y_i - y_j) - Cw_i \right), \\ u_i &= Fw_i. \end{aligned} \quad (2.4)$$

for  $i = 1, 2, \dots, N$ . The structure of this protocol is as follows. Each controller is able to observe the disagreement output signal  $\sum_{j \in \mathcal{N}_i} (y_i - y_j)$  and the relative input  $\sum_{j \in \mathcal{N}_i} (u_i - u_j)$  of its corresponding agent. The differential equation in (2.4) acts as an *observer* for the relative state  $\sum_{j \in \mathcal{N}_i} (x_i - x_j)$  of agent  $i$ . The protocol state  $w_i$  is an estimate of this quantity. It is easily verified that the error  $e_i := w_i - \sum_{j \in \mathcal{N}_i} (x_i - x_j)$  has error dynamics  $\dot{e}_i = (A - GC)e_i$ , which is asymptotically stable if  $A - GC$  is Hurwitz. This estimate is then fed back to the agent by means of a static feedback.

By interconnecting the agents (2.3) using the above protocol, we obtain the closed-loop dynamics of the entire network. Denote

$$x = \text{col}(x_1, x_2, \dots, x_N), \quad u = \text{col}(u_1, u_2, \dots, u_N),$$

and similarly

$$y = \text{col}(y_1, y_2, \dots, y_N), \quad w = \text{col}(w_1, w_2, \dots, w_N).$$

The network dynamics is now

$$\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} I \otimes A & I \otimes BF \\ L \otimes GC & I \otimes (A - GC) + L \otimes BF \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}. \quad (2.5)$$

Next, we state the prevalent definition of synchronization of such a network.

**DEFINITION 2.2:** The network with agent dynamics (2.3) is said to be synchronized by protocol (2.4) if for all  $i, j = 1, 2, \dots, N$  we have that  $x_i(t) - x_j(t) \rightarrow 0$  and  $w_i(t) - w_j(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

In [65], it is shown that synchronization of the network with agent dynamics (2.3) by protocol (2.4) is equivalent to the stabilization of a single linear system by each controller from a given set of  $N - 1$  related controllers. These controllers depend on the matrices  $F$  and  $G$  of the protocol, and the nonzero eigenvalues of the Laplacian. We will use a similar argument in the next section, where we examine the synchronizability of multi-agent system in which the agent dynamics is a coprime factor perturbation of the nominal agent dynamics.

## 2.5 ROBUST SYNCHRONIZATION

While the nominal agent dynamics is still given by the unperturbed dynamics (2.3), we now allow uncertainty in the form of coprime factor perturbations of the nominal agent dynamics. In [20], this paradigm for model uncertainty was used in the context of optimal robust stabilization, see also [64]. The agents have identical nominal transfer matrices

given by  $G(s) = C(sI - A)^{-1}B$ . It is well known that there exists a coprime factorization of  $G$  of the form  $G = M^{-1}N$  with  $M, N \in \mathcal{RH}_\infty$  such that  $NN^* + MM^* = I$ , where  $N^*(s) := N^T(-s)$ , see e.g. [20, 64]. Such a factorization is called a normalized coprime factorization over  $\mathcal{RH}_\infty$ . Such factorization can be obtained by means of the algebraic Riccati equation

$$AQ + QA^T - QC^T CQ + BB^T = 0. \quad (2.6)$$

By detectability it has a unique real symmetric positive semi-definite solution such that  $A - QC^T C$  is Hurwitz. This matrix  $Q$  is called the stabilizing solution of (2.6). A normalized coprime factorization  $G = M^{-1}N$  is then obtained by taking

$$M(s) := I - C(sI - A + QC^T C)^{-1}QC^T$$

and

$$N(s) := C(sI - A + QC^T C)^{-1}B.$$

In this chapter we consider the situation that the transfer matrices of the agents are coprime factor perturbations of the nominal transfer matrix, i.e. the transfer matrix  $G = M^{-1}N$  of agent  $i$  is perturbed to

$$G_{(\Delta_M^i, \Delta_N^i)} := (M + \Delta_M^i)^{-1}(N + \Delta_N^i),$$

where  $\Delta_M^i, \Delta_N^i \in \mathcal{RH}_\infty$ ,  $i = 1, 2, \dots, N$ , and where  $\|\Delta_M^i, \Delta_N^i\|_\infty \leq \gamma$ , with  $\gamma > 0$  a desired uncertainty tolerance. It is easily verified that the perturbed transfer matrix  $G_{(\Delta_M^i, \Delta_N^i)}$  can be represented as the feedback interconnection of the system

$$y_i = M^{-1}(s)N(s)u_i + M^{-1}(s)d_i$$

with feedback loop

$$d_i = \begin{pmatrix} -\Delta_M^i(s) & \Delta_N^i(s) \end{pmatrix} \begin{pmatrix} y_i \\ u_i \end{pmatrix},$$



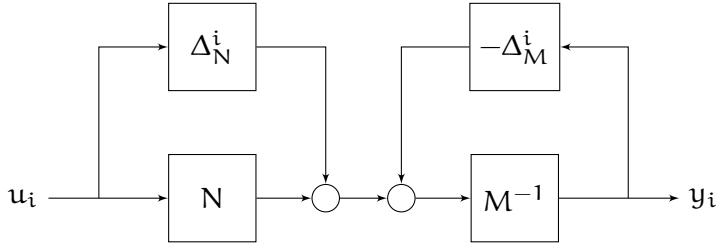


Figure 2.1: Block diagram of the coprime factor perturbed transfer matrix of each agent.

see Figure 2.1. Since  $M^{-1}(s) = C(sI - A)^{-1}QC^T + I$ , a state space representation of this feedback interconnection is given by

$$\begin{aligned} \dot{x}_i &= Ax_i + Bu_i + QC^T d_i, & y_i &= Cx_i + d_i, \\ z_i &= \begin{pmatrix} C \\ 0 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ I \end{pmatrix} u_i + \begin{pmatrix} I \\ 0 \end{pmatrix} d_i \end{aligned} \quad (2.7)$$

with the feedback loop

$$d_i = \begin{pmatrix} -\Delta_M^i & \Delta_N^i \end{pmatrix} z_i. \quad (2.8)$$

Figure 2.2 shows a block diagram of this interconnection. Here,  $\Sigma_{\text{com}}$  represents the common dynamics (2.7). Next, we give a definition of the robust synchronization problem.

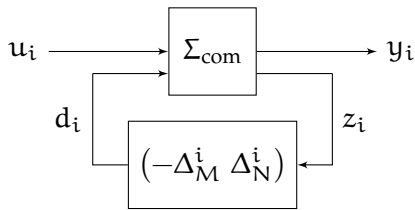


Figure 2.2: Block diagram of a coprime factor perturbed agent.

DEFINITION 2.3: Given a desired tolerance  $\gamma > 0$ , the protocol (2.4) is said to *robustly synchronize* the network (2.5) if for all  $\Delta_M^i, \Delta_N^i \in \mathcal{RH}_\infty$  with  $\|(\Delta_M^i \ \Delta_N^i)\|_\infty \leq \gamma$  we have that for all  $i, j = 1, 2, \dots, N$

$$x_i(t) - x_j(t) \rightarrow 0, \quad w_i(t) - w_j(t) \rightarrow 0$$

as  $t \rightarrow \infty$ .

In this chapter we will treat the case that the  $\Delta_M$ -parts of the perturbations are identical for all agents  $i$ , that is, we will assume that  $\Delta_M^i = \Delta_M^j$  for all  $i, j = 1, 2, \dots, N$ . The  $\Delta_N$ -parts may differ from agent to agent. In other words, we assume that the perturbations are of the form  $(-\Delta_M \ \Delta_N^i)$ , with  $\|(\Delta_M \ \Delta_N^i)\|_\infty \leq \gamma$ ,  $i = 1, 2, \dots, N$ . Thus, the nominal transfer matrix  $G(s)$  of each of the agents is allowed to be perturbed by distinct perturbations in a somewhat restricted sense. This is different from the situation in [65], where the (additive) perturbations were allowed to be completely distinct for each of the agents. The more restrictive perturbation structure used in this chapter is a technical assumption needed in order to prove that our robust synchronization problem is equivalent to a problem of robust stabilization of a single system by all controllers from a set of  $N - 1$  feedback controllers. In the additive perturbation case, this equivalence holds for arbitrary distinct perturbations.

For notational convenience we will sometimes denote  $\Delta_i = (-\Delta_M \ \Delta_N^i)$  and write the uncertain feedback loop in the form  $d_i = \Delta_i z_i$ .

Following [65], for the robust synchronization problem, we consider a modified version of protocol (2.4). A *weighting factor* on the Laplacian matrix  $L$  of the network graph is used, denoted by  $\kappa$ .

$$\dot{w}_i = Aw_i + B \sum_{j \in \mathcal{N}_i} \frac{1}{\kappa} (u_i - u_j) + G \sum_{j \in \mathcal{N}_i} \left( \frac{1}{\kappa} (y_i - y_j) - Cw_i \right), \quad (2.9)$$

$$u_i = Fw_i.$$

The positive real valued parameter  $\kappa$  is introduced as an extra design parameter for which a value has to be determined, next to the gain matrices  $F$  and  $G$ .

To start off, we now first examine conditions under which, for a given uncertainty tolerance  $\gamma > 0$ , there exists a robustly synchronizing protocol.

After interconnecting the agents with protocol (2.9), the overall network dynamics can be conveniently represented by again denoting the aggregate agent and controller state vectors by  $x = \text{col}(x_1, x_2, \dots, x_N)$  and  $w = \text{col}(w_1, w_2, \dots, w_N)$ , and the aggregate output and input vectors  $y = \text{col}(y_1, y_2, \dots, y_N)$  and  $u = \text{col}(u_1, u_2, \dots, u_N)$ , respectively. The aggregate output and input vectors of the feedback-loop are denoted  $d = \text{col}(d_1, d_2, \dots, d_N)$  and  $z = \text{col}(z_1, z_2, \dots, z_N)$ , respectively. Then by interconnecting the perturbed network (2.7), (2.8) with the protocol (2.9) we obtain that the dynamics of the overall perturbed network is given by

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} I \otimes A & I \otimes BF \\ \frac{1}{\kappa} L \otimes GC & I \otimes (A - GC) + \frac{1}{\kappa} L \otimes BF \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \\ &\quad + \begin{pmatrix} I \otimes QC^T \\ \frac{1}{\kappa} L \otimes G \end{pmatrix} d, \\ z &= \begin{pmatrix} I \otimes \begin{pmatrix} C \\ 0 \end{pmatrix} & I \otimes \begin{pmatrix} 0 \\ F \end{pmatrix} \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} I \otimes \begin{pmatrix} I \\ 0 \end{pmatrix} \end{pmatrix} d. \end{aligned}$$

and

$$d = \begin{pmatrix} \Delta_1 & 0 & \dots & 0 \\ 0 & \Delta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta_N \end{pmatrix} z,$$

where we recall that  $\Delta_i := (-\Delta_M \Delta_N^i)$ . Also recall that if the network graph is undirected, then there exists a orthogonal transformation  $U$  that diagonalizes  $L$ . As before, let  $\Lambda = \text{diag}(0, \lambda_2, \dots, \lambda_N)$ . By applying the transformations

$$\begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} := \begin{pmatrix} U^T \otimes I & 0 \\ 0 & U^T \otimes I \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}, \quad \tilde{z} := (U^T \otimes I)z,$$

and  $\tilde{\mathbf{d}} := (\mathbf{U}^\top \otimes \mathbf{I})\mathbf{d}$ , we obtain

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{\mathbf{x}}} \\ \dot{\tilde{\mathbf{w}}} \end{pmatrix} &= \begin{pmatrix} \mathbf{I} \otimes \mathbf{A} & \mathbf{I} \otimes \mathbf{B}\mathbf{F} \\ \frac{1}{\kappa} \boldsymbol{\Lambda} \otimes \mathbf{G}\mathbf{C} & \mathbf{I} \otimes (\mathbf{A} - \mathbf{G}\mathbf{C}) + \frac{1}{\kappa} \boldsymbol{\Lambda} \otimes \mathbf{B}\mathbf{F} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} + \begin{pmatrix} \mathbf{I} \otimes \mathbf{Q}\mathbf{C}^\top \\ \frac{1}{\kappa} \boldsymbol{\Lambda} \otimes \mathbf{G} \end{pmatrix} \tilde{\mathbf{d}}, \\ \tilde{\mathbf{z}} &= \begin{pmatrix} \mathbf{I} \otimes \begin{pmatrix} \mathbf{C} \\ 0 \end{pmatrix} & \mathbf{I} \otimes \begin{pmatrix} 0 \\ \mathbf{F} \end{pmatrix} \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{x}} \\ \tilde{\mathbf{w}} \end{pmatrix} + \begin{pmatrix} \mathbf{I} \otimes \begin{pmatrix} \mathbf{I} \\ 0 \end{pmatrix} \end{pmatrix} \tilde{\mathbf{d}}, \\ \tilde{\mathbf{d}} &= (\mathbf{U}^\top \otimes \mathbf{I}) \begin{pmatrix} \Delta_1 & 0 & \dots & 0 \\ 0 & \Delta_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta_N \end{pmatrix} (\mathbf{U} \otimes \mathbf{I}) \tilde{\mathbf{z}}. \end{aligned} \tag{2.10}$$

The following theorem reduces the problem of robust synchronization to a problem of simultaneous robust stabilization, and gives *necessary and sufficient* conditions on the weighting parameter  $\kappa$  and gains  $\mathbf{F}$  and  $\mathbf{G}$  such that protocol (2.9) robustly synchronizes the network. We first consider the case that the network topology is given by an undirected graph.

**THEOREM 2.4:** *Consider the network with nominal agent dynamics given by (2.3). Assume that the network graph is a connected, undirected graph. Let  $\gamma > 0$ . Then the following statements are equivalent:*

1. *The dynamic protocol (2.9) synchronizes the network with coprime factor perturbed agent dynamics*

$$\begin{aligned} \dot{\mathbf{x}}_i &= \mathbf{A}\mathbf{x}_i + \mathbf{B}\mathbf{u}_i + \mathbf{Q}\mathbf{C}^\top \mathbf{d}_i, \\ \mathbf{y}_i &= \mathbf{C}\mathbf{x}_i + \mathbf{d}_i, \\ \mathbf{z}_i &= \begin{pmatrix} \mathbf{C} \\ 0 \end{pmatrix} \mathbf{x}_i + \begin{pmatrix} 0 \\ \mathbf{I} \end{pmatrix} \mathbf{u}_i + \begin{pmatrix} \mathbf{I} \\ 0 \end{pmatrix} \mathbf{d}_i, \\ \mathbf{d}_i &= \begin{pmatrix} -\Delta_M & \Delta_N^i \end{pmatrix} \mathbf{z}_i, \end{aligned} \tag{2.11}$$

for all  $\Delta_M, \Delta_N^i \in \mathcal{RH}_\infty$  with  $\|(\Delta_M \ \Delta_N^i)\|_\infty \leq \gamma$ ,  $i = 1, 2, \dots, N$ .

2. *The single coprime factor perturbed system*

$$\begin{aligned}
 \dot{x} &= Ax + Bu + QC^T d, \\
 y &= Cx + d, \\
 z &= \begin{pmatrix} C \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u + \begin{pmatrix} I \\ 0 \end{pmatrix} d, \\
 d &= \begin{pmatrix} -\Delta_M & \Delta_N \end{pmatrix} z,
 \end{aligned} \tag{2.12}$$

is internally stabilized for all  $\Delta_M, \Delta_N \in \mathcal{RH}_\infty$  with  $\|(\Delta_M \ \Delta_N)\|_\infty \leq \gamma$  by all  $N - 1$  controllers

$$\dot{w} = Aw + Bu + G(y - Cw), \quad u = \frac{1}{\kappa} \lambda_i Fw, \tag{2.13}$$

where  $i = 2, 3, \dots, N$  and  $\lambda_i$  is the  $i$ th eigenvalue of the Laplacian  $L$ .

*Proof.* In this proof, we use the shorthand notation  $\Delta_i$  for  $(-\Delta_M \ \Delta_N^i)$ . Let  $H$  be any  $(N - 1) \times N$  matrix such that  $\ker(H) = \text{im}(\mathbf{1})$ . Note that the first column of the orthogonal transformation  $U$  that diagonalizes  $L$  is equal to  $u_1 = \frac{1}{\sqrt{N}} \mathbf{1}$ , the normalized vector of ones. Let  $U_2$  be such that  $U = (u_1 \ U_2)$ . Now we have  $HU = (0 \ HU_2)$ , where  $HU_2$  has full column rank. It is clear that  $x_i(t) - x_j(t) \rightarrow 0$  as  $t \rightarrow \infty$  for all  $i, j$  if and only if  $(H \otimes I)x \rightarrow 0$ . Similarly,  $w_i(t) - w_j(t) \rightarrow 0$  for all  $i, j$  if and only if  $(H \otimes I)w \rightarrow 0$ . This is equivalent with  $\tilde{x}_i(t) \rightarrow 0$  and  $\tilde{w}_i(t) \rightarrow 0$  for  $i = 2, 3, \dots, N$ .

(only if) First, we show that if dynamic protocol (2.9) robustly synchronizes the network, then the interconnection of the plant (2.12) with each controller (2.13) is robustly stabilized. Assume that the network with perturbed agent dynamics (2.7) is synchronized by protocol (2.9) for all perturbations  $\Delta_i = (-\Delta_M \ \Delta_N^i)$  with  $\|\Delta_i\|_\infty \leq \gamma$ . Let  $\Delta = (-\Delta_M \ \Delta_N) \in \mathcal{RH}_\infty$  with  $\|\Delta\|_\infty \leq \gamma$  and take  $\Delta_i = \Delta$  for all  $i = 1, 2, \dots, N$ , i.e., we perturb each agent in the network (2.11) with the same  $\Delta$ . Since the network is robustly synchronized by the protocol, we have that  $\tilde{x}_i \rightarrow 0$ ,  $\tilde{w}_i \rightarrow 0$

as  $t \rightarrow \infty$  for  $i = 2, 3, \dots, N$  in (2.10). This implies that for  $i = 2, 3, \dots, N$  the following systems are internally stable:

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{x}}_i \\ \dot{\tilde{w}}_i \end{pmatrix} &= \begin{pmatrix} A & BF \\ \frac{1}{\kappa}\lambda_i GC & A - GC + \frac{1}{\kappa}\lambda_i BF \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} + \begin{pmatrix} QC^T \\ \frac{1}{\kappa}\lambda_i G \end{pmatrix} \tilde{d}_i, \\ \tilde{z}_i &= \begin{pmatrix} C & 0 \\ 0 & F \end{pmatrix} \begin{pmatrix} \tilde{x}_i \\ \tilde{w}_i \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} \tilde{d}_i, \\ \tilde{d}_i &= \Delta \tilde{z}_i. \end{aligned} \quad (2.14)$$

Interconnecting (2.12) and (2.13) yields

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} &= \begin{pmatrix} A & \frac{1}{\kappa}\lambda_i BF \\ GC & A - GC + \frac{1}{\kappa}\lambda_i BF \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} QC^T \\ G \end{pmatrix} d, \\ z &= \begin{pmatrix} C & 0 \\ 0 & \frac{1}{\kappa}\lambda_i F \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} d, \\ d &= \Delta z. \end{aligned} \quad (2.15)$$

By the simple state transformation  $\tilde{w}_i = \frac{1}{\kappa}\lambda_i \bar{w}_i$ , we see that this system is equivalent with (2.14). Therefore, the system (2.15) is internally stable for  $i = 2, 3, \dots, N$ .

(if) Next, assume that the  $N - 1$  controllers (2.13) stabilize system (2.12) for all  $\Delta \in \mathcal{RH}_\infty$  with  $\|\Delta\|_\infty \leq \gamma$ . From the small gain theorem, it then follows that for  $i = 2, 3, \dots, N$  the closed-loop systems (2.15) are internally stable and the transfer matrices  $G_i$  from  $d$  to  $z$  satisfy  $\|G_i\|_\infty < \frac{1}{\gamma}$ . We show that the perturbed network is synchronized by protocol (2.9) for all perturbations  $\Delta_i$  of the form  $(\Delta_M \ \Delta_N^i)$  that satisfy  $\|\Delta_i\|_\infty \leq \gamma$ . This is done by showing that for  $i = 2, 3, \dots, N$  we have  $\tilde{x}_i(t) \rightarrow 0$  and  $\tilde{w}_i(t) \rightarrow 0$  as  $t \rightarrow \infty$ , where  $\tilde{x}_i$  and  $\tilde{w}_i$  satisfy (2.10). Denote  $\bar{x} = \text{col}(\tilde{x}_2, \tilde{x}_3, \dots, \tilde{x}_N)$ ,  $\bar{w} = \text{col}(\tilde{w}_2, \tilde{w}_3, \dots, \tilde{w}_N)$ ,  $\bar{z} = \text{col}(\tilde{z}_2, \tilde{z}_3, \dots, \tilde{z}_N)$ ,

and  $\bar{d} = \text{col}(\tilde{d}_2, \tilde{d}_3, \dots, \tilde{d}_N)$ . Let  $\Lambda_1 = \text{diag}(\lambda_2, \lambda_3, \dots, \lambda_N)$ . From (2.10) we obtain

$$\begin{aligned} \begin{pmatrix} \dot{\bar{x}} \\ \dot{\bar{w}} \end{pmatrix} &= \begin{pmatrix} I_{N-1} \otimes A & I_{N-1} \otimes BF \\ \frac{1}{\kappa} \Lambda_1 \otimes GC & I_{N-1} \otimes (A - GC) + \frac{1}{\kappa} \Lambda_1 \otimes BF \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{w} \end{pmatrix} \\ &\quad + \begin{pmatrix} I_{N-1} \otimes QC^T \\ \frac{1}{\kappa} \Lambda_1 \otimes G \end{pmatrix} \bar{d}, \\ \bar{z} &= \begin{pmatrix} I_{N-1} \otimes \begin{pmatrix} C \\ 0 \end{pmatrix} & I_{N-1} \otimes \begin{pmatrix} 0 \\ F \end{pmatrix} \end{pmatrix} \begin{pmatrix} \bar{x} \\ \bar{w} \end{pmatrix} + (I_{N-1} \otimes \begin{pmatrix} I \\ 0 \end{pmatrix}) \bar{d}. \end{aligned} \quad (2.16)$$

The transfer matrix  $G$  of this system from  $\bar{d}$  to  $\bar{z}$  is a block diagonal matrix with  $G_2, G_3, \dots, G_N$  on the diagonal. We obtain that the transfer matrix  $G$  of the network from  $\bar{d}$  to  $\bar{z}$  satisfies  $\|G\|_\infty < \frac{1}{\gamma}$ . We now look at the feedback loop from  $\bar{z}$  to  $\bar{d}$ . Using the fact that the  $\Delta_M^i$  parts of the perturbations  $\Delta_i$  are identical, say  $\Delta_M$ , it can be verified immediately that

$$\begin{aligned} &(\mathbf{U}^T \otimes \mathbf{I}) \begin{pmatrix} -\Delta_M & \Delta_N^1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & -\Delta_M & \Delta_N^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\Delta_M & \Delta_N^N \end{pmatrix} (\mathbf{U} \otimes \mathbf{I}) \\ &= \left( \begin{array}{c|cccccc} -\Delta_M & \Delta_{11} & 0 & \Delta_{12} & \dots & 0 & \Delta_{1p} \\ 0 & \Delta_{21} & -\Delta_M & \Delta_{22} & \dots & 0 & \Delta_{2p} \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & \Delta_{p1} & 0 & \Delta_{p2} & \dots & -\Delta_M & \Delta_{pp} \end{array} \right) \end{aligned}$$

for certain  $\Delta_{ij} \in \mathcal{RH}_\infty$ . Denote the block matrices in the lower right corner by

$$\bar{\Delta}_1 := \begin{pmatrix} \Delta_{21} \\ \vdots \\ \Delta_{p1} \end{pmatrix}, \quad \bar{\Delta}_2 := \begin{pmatrix} -\Delta_M & \Delta_{22} & \dots & 0 & \Delta_{2p} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & \Delta_{p2} & \dots & -\Delta_M & \Delta_{pp} \end{pmatrix}.$$

Using (2.10) note that

$$\tilde{z}_i = \begin{pmatrix} \tilde{y}_i \\ \tilde{u}_i \end{pmatrix} := \begin{pmatrix} C\tilde{x}_i + \tilde{d}_i \\ F\tilde{w}_i \end{pmatrix} \quad (i = 1, 2, \dots, N).$$

Also, (2.10) implies that  $\dot{\tilde{w}}_1 = (A - GC)\tilde{w}_1$ , where  $A - GC$  has all its eigenvalues in  $\mathbb{C}^-$ . Finally, by (2.10) we have

$$\bar{d} = \bar{\Delta}_1 F\tilde{w}_1 + \bar{\Delta}_2 \bar{z} \quad (2.17)$$

By observing that the  $\mathcal{H}_\infty$ -norm of  $\bar{\Delta}_2$  is less than or equal to  $\gamma$  and applying the small gain theorem, it follows that in the system (2.16) with feedback loop (2.17) and external input  $\tilde{w}_1$  we have  $\bar{x}(t) \rightarrow 0$  and  $\bar{w}(t) \rightarrow 0$  as  $t \rightarrow \infty$ . So indeed, we have  $\tilde{x}_i \rightarrow 0$  and  $\tilde{w}_i \rightarrow 0$  in (2.10) for  $i = 2, 3, \dots, N$ .  $\square$

In the case that the network graph is directed and contains a spanning tree, robust synchronization of the plant (2.12) by the  $N - 1$  controllers (2.13) *implies* robust synchronization of (2.10) against perturbations  $(\Delta_M \ \Delta_N^i)$  with the restriction that also  $\Delta_N^i = \Delta_N^j$  for all  $i$  and  $j$ , in other words, also the  $\Delta_N$ -parts of the perturbations are restricted to be identical. A similar situation occurred in [65], where in the directed network case also all (additive) perturbations were restricted to be identical.

In this case, there exists a complex unitary transformation  $U$  that brings the Laplacian  $L$  to upper triangular form  $U^*LU = \Lambda_u$ , with  $\Lambda_u$  complex and  $0, \lambda_2, \dots, \lambda_N$  on the diagonal. The following lemma can be proven by letting the unitary transformation  $U$  take over the role of the orthogonal transformation and replacing  $\Lambda$  with  $\Lambda_u$  in the proof of Theorem 2.4.

**LEMMA 2.5:** *Consider the network with perturbed agent dynamics (2.7). Assume that the network graph is directed and contains a spanning tree. Let  $\gamma > 0$ . Consider the statements (2) as formulated in Theorem 2.4. Then statement (2)*



implies that the dynamic protocol (2.9) synchronizes the network with homogeneously perturbed agents

$$\begin{aligned}
 \dot{x}_i &= Ax_i + Bu_i + QC^T d_i, \\
 y_i &= Cx_i + d_i, \\
 z_i &= \begin{pmatrix} C \\ 0 \end{pmatrix} x_i + \begin{pmatrix} 0 \\ I \end{pmatrix} u_i + \begin{pmatrix} I \\ 0 \end{pmatrix} d_i, \\
 d_i &= \begin{pmatrix} -\Delta_M & \Delta_N \end{pmatrix} z_i,
 \end{aligned} \tag{2.18}$$

for all  $\Delta_M, \Delta_N \in \mathcal{RH}_\infty$  with  $\|(\Delta_M \ \Delta_N)\|_\infty \leq \gamma$ .

## 2.6 ROBUSTLY SYNCHRONIZING PROTOCOLS

In this section, we examine which values  $\gamma > 0$  of the uncertainty tolerance can be achieved, given the nominal agent dynamics and the Laplacian matrix of the network graph of either a directed network containing a spanning tree or a connected undirected network. We characterize an *achievable interval*, such that for every  $\gamma$  within this interval, there exist a dynamic protocol of the form (2.9) that robustly synchronizes the network.

In addition to the Riccati equation (2.6), we consider the Riccati equation

$$A^T P + PA - PBB^T P + C^T C = 0. \tag{2.19}$$

Next to the stabilizing solution  $Q$  of (2.6), let  $P$  be the unique real symmetric positive semi-definite solution of (2.19) such that  $A - BB^T P$  is Hurwitz. The existence and uniqueness of  $P$  follow from the fact that  $(A, B)$  is stabilizable and  $(C, A)$  is detectable. In the remainder of this chapter, we denote any eigenvalue of the Laplacian with the maximal modulus by  $\lambda_M$  and any eigenvalue with minimal real part by  $\lambda_m$ , i.e.

$$\operatorname{Re}(\lambda_m) = \min_{i=2,3,\dots,N} \operatorname{Re}(\lambda_i), \quad |\lambda_M| = \max_{i=2,3,\dots,N} |\lambda_i|.$$

Now we formulate the first main result of this chapter.

**THEOREM 2.6:** *Consider the network with homogeneously perturbed agent dynamics (2.18). Assume the network is directed and contains a spanning tree. Define*

$$\gamma^* = \frac{\operatorname{Re}(\lambda_m)}{|\lambda_M|} \frac{1}{\sqrt{1 + \rho(PQ)}}. \quad (2.20)$$

*Then, for all  $\gamma$  in the open interval  $(0, \gamma^*)$ , there exists a dynamic protocol of the form (2.9) that robustly synchronizes the network with uncertainty tolerance  $\gamma$ .*

In the remainder of this section, we show how to construct such a protocol, i.e. how to choose the parameter  $\kappa$ , and matrices  $F$  and  $G$ . Keeping in mind Lemma 2.5, we will prove that for  $\gamma \in (0, \gamma^*)$ , there exist  $\kappa$ ,  $F$ , and  $G$  such that the  $N - 1$  controllers (2.13) stabilize the system (2.12) for all  $\Delta = (-\Delta_M \ \Delta_N)$  with  $\|\Delta\|_\infty \leq \gamma$ .

**REMARK 2.7:** Note that for the case that  $N = 2$ , i.e. a ‘network’ consisting of two agents, we have  $\lambda_2 = \lambda_N = 1$ . In that case the bound  $\gamma^*$  is equal to  $\frac{1}{\sqrt{1 + \rho(PQ)}}$ , which exactly coincides with the *optimal* uncertainty tolerance in the problem of optimal robust stabilization of the single system (2.12) as it was obtained in the work of Glover and MacFarlane in [20], see also [64]. For general number of nodes  $N > 2$ , it is unclear how conservative the tolerance  $\gamma^*$  given by (2.20) is. However, the fact that for  $N = 2$  the bound  $\gamma^*$  is actually optimal indicates that also for general  $N$  it is not overly conservative.

First, we provide a lemma that will be instrumental in this proof. Recall that  $Q$  denotes the stabilizing solution to (2.6) and  $P$  denotes the stabilizing solution of (2.19). Then the following holds:

**LEMMA 2.8:** *Let  $\tau > 0$  be such that  $\tau^2 < \frac{1}{1 + \rho(PQ)}$ . Then the matrix  $(\frac{1}{\tau^2} - 1)I - PQ$  is nonsingular. Define*

$$\tilde{P} := \left( \left( \frac{1}{\tau^2} - 1 \right) I - PQ \right)^{-1} P. \quad (2.21)$$

Then  $\tilde{P}$  is a real symmetric solution of the algebraic Riccati equation

$$\begin{aligned} A^T \tilde{P} + \tilde{P} A + \tau^2 C^T C - \frac{1}{\tau^2} \tilde{P} B B^T \tilde{P} \\ + \frac{1}{1 - \tau^2} (\tilde{P} Q + \tau^2 I) C^T C (Q \tilde{P} + \tau^2 I) = 0. \end{aligned} \quad (2.22)$$

*Proof.* The proof follows by combining equations (2.6) and (2.19). First, pre- and post-multiply (2.22) with

$$\left(\frac{1}{\tau^2} - 1\right)I - PQ \quad \text{and} \quad \left(\left(\frac{1}{\tau^2} - 1\right)I - PQ\right)^T$$

respectively, to obtain

$$\begin{aligned} & \left(\frac{1}{\tau^2} - 1\right)I - PQ \left) A^T P + P A \left(\frac{1}{\tau^2} - 1\right)I - Q P \right) - \frac{1}{\tau^2} P B B^T P \\ & + \tau^2 \left(\frac{1}{\tau^2} - 1\right)I - P Q \right) C^T C \left(\frac{1}{\tau^2} - 1\right)I - Q P \right) \\ & + \frac{1}{1 - \tau^2} ((1 - \tau^2)[PQ + I]) C^T C ((1 - \tau^2)[QP + I]) \\ & = \left(\frac{1}{\tau^2} - 1\right)[A^T P + P A] - P[Q A^T + A Q]P - \frac{1}{\tau^2} P B B^T P \\ & + \tau^2 \left[ \left(\frac{1}{\tau^4} - 2\frac{1}{\tau^2} + 1\right)C^T C - \left(\frac{1}{\tau^2} - 1\right)P Q C^T C \right. \\ & \quad \left. - \left(\frac{1}{\tau^2} - 1\right)C^T C Q P + P Q C^T C Q P \right] \\ & + (1 - \tau^2) (C^T C + P Q C^T C + C^T C Q P + P Q C^T C Q P) \\ & = \left(\frac{1}{\tau^2} - 1\right)[A^T P + P A + C^T C] - P[Q A^T + A Q - Q C^T C Q]P \\ & \quad - \frac{1}{\tau^2} P B B^T P \\ & = \left(\frac{1}{\tau^2} - 1\right)[A^T P + P A - P B B^T P + C^T C] \\ & \quad - P[Q A^T + A Q - Q C^T C Q + B B^T]P \\ & = 0. \end{aligned}$$

So indeed, we have that  $\tilde{P}$  given by (2.21) is a solution to the algebraic Riccati equation (2.22).  $\square$

We will use this lemma to prove our second main result, giving explicit equations for a dynamic protocol that robustly synchronizes with a desired uncertainty tolerance  $\gamma \in (0, \gamma^*)$ :

**THEOREM 2.9:** *Consider the network with homogeneously perturbed agent dynamics (2.18). Assume that the network graph is directed and contains a spanning tree. Let  $\gamma > 0$  be any uncertainty tolerance such that  $\gamma \in (0, \gamma^*)$ , with  $\gamma^*$  given by (2.20). Choose  $\kappa$  any real number such that*

$$\kappa > \frac{|\lambda_M|^2}{\operatorname{Re}(\lambda_m)} \quad \text{and} \quad \gamma^2 < \frac{\operatorname{Re}(\lambda_m)}{\kappa} \frac{1}{1 + \rho(PQ)}. \quad (2.23)$$

Finally, let  $\eta > 0$  such that

$$\eta < \frac{\operatorname{Re}(\lambda_m)}{\kappa} \quad \text{and} \quad \frac{\gamma^2}{\eta} < \frac{1}{1 + \rho(PQ)}. \quad (2.24)$$

Then  $(\frac{\eta}{\gamma^2} - 1)I - PQ$  is nonsingular. Define

$$\begin{aligned} \tilde{P} &:= \left( \left( \frac{\eta}{\gamma^2} - 1 \right) I - PQ \right)^{-1} P, \\ F &:= -\frac{1}{\gamma^2} B^T \tilde{P}, \\ G &:= QC^T. \end{aligned} \quad (2.25)$$

Then the dynamic protocol (2.9) with  $\kappa$ ,  $F$  and  $G$  as chosen above synchronizes the network for all perturbations  $\Delta_M, \Delta_N \in \mathcal{RH}_\infty$  with  $\|(\Delta_M \ \Delta_N)\|_\infty \leq \gamma$ .

*Proof.* It is straightforward to verify that  $\kappa$  and  $\eta$  satisfying (2.23) and (2.24) indeed exist. By Lemma 2.5 it suffices to prove that the  $N - 1$  controllers (2.13) with  $\kappa$ ,  $F$  and  $G$  chosen as above, internally stabilize the system

$$\begin{aligned} \dot{x} &= Ax + Bu + QC^T d, \\ y &= Cx + d, \\ z &= \begin{pmatrix} C \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ I \end{pmatrix} u + \begin{pmatrix} I \\ 0 \end{pmatrix} d, \end{aligned}$$

while accomplishing  $\|G_i\|_\infty < \frac{1}{\gamma}$ , where  $G_i$  is the transfer matrix from  $d$  to  $z$  ( $i = 2, 3, \dots, N$ ) in the closed-loop system given by (2.15). To show that this is indeed the case, we apply the following state transformation:

$$\begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -I \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}.$$

Which results in the dynamics below:

$$\begin{aligned} \begin{pmatrix} \dot{\tilde{x}} \\ \dot{\tilde{w}} \end{pmatrix} &= \begin{pmatrix} A + \mu_i BF & -\mu_i BF \\ 0 & A - GC \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} QC^T \\ QC^T - G \end{pmatrix} d, \\ z &= \begin{pmatrix} C & 0 \\ \mu_i F & -\mu_i F \end{pmatrix} \begin{pmatrix} \tilde{x} \\ \tilde{w} \end{pmatrix} + \begin{pmatrix} I \\ 0 \end{pmatrix} d, \end{aligned} \quad (2.26)$$

where  $\mu_i := \frac{\lambda_i}{\kappa}$  for  $i = 2, 3, \dots, N$ . To proceed, we will apply Lemma 2.1 to the system (2.26). We will show that there exists a real symmetric solution to the Riccati inequality associated with the system. As before, denote

$$\tilde{P} = \left( \left( \frac{\eta}{\gamma^2} - 1 \right) I - PQ \right)^{-1} P,$$

and apply Lemma 2.8 with  $\tau = \frac{\gamma}{\sqrt{\eta}}$ . We find that  $\tilde{P}$  is a solution to the following Riccati equation

$$\begin{aligned} A^T \tilde{P} + \tilde{P} A + \frac{\gamma^2}{\eta} C^T C - \frac{\eta}{\gamma^2} \tilde{P} B B^T \tilde{P} \\ + \frac{\eta}{\eta - \gamma^2} (\tilde{P} Q + \frac{\gamma^2}{\eta} I) C^T C (Q \tilde{P} + \frac{\gamma^2}{\eta} I) = 0. \end{aligned} \quad (2.27)$$

We now return to the closed-loop system (2.26). For ease of notation, we label the system matrices as

$$\begin{aligned} \tilde{A}_i &= \begin{pmatrix} A + \mu_i BF & -\mu_i BF \\ 0 & A - GC \end{pmatrix}, & \tilde{B}_i &= \begin{pmatrix} QC^T \\ QC^T - G \end{pmatrix}, \\ \tilde{C}_i &= \begin{pmatrix} C & 0 \\ \mu_i F & -\mu_i F \end{pmatrix}, & \tilde{D} &= \begin{pmatrix} I \\ 0 \end{pmatrix}. \end{aligned}$$

Take the controller and observer gains  $F$  and  $G$  as defined in (2.25). Now we will apply Lemma 2.1 to show that  $\|G_i\|_\infty < \frac{1}{\gamma}$ , where  $G_i$  is the transfer function from  $d$  to  $z$  ( $i = 2, 3, \dots, N$ ) in (2.26). We will show that there exists a suitable choice for  $Z_i$  such that  $Z_i$  is a solution to the complex version of algebraic Riccati inequality (2.1) with  $\tau = \frac{1}{\gamma}$ , and  $\epsilon$  sufficiently small:

$$\begin{aligned} \tilde{A}_i^* Z_i + Z \tilde{A}_i + \tilde{C}_i^* \tilde{C}_i + \frac{\gamma^2}{1-\gamma^2} (Z_i \tilde{B}_i + \tilde{C}_i^* \tilde{D})(\tilde{B}_i^* Z_i + \tilde{D}^* \tilde{C}_i) \\ \leq -\epsilon (Z_i \tilde{B}_i + \tilde{C}_i^* \tilde{D})(\tilde{B}_i^* Z_i + \tilde{D}^* \tilde{C}_i) \end{aligned} \quad (2.28)$$

Lemma 2.1 requires that the matrices  $\tilde{A}_i$  ( $i = 2, 3, \dots, N$ ) are Hurwitz. The set of eigenvalues of  $\tilde{A}_i$  is the union of those of  $A + \mu_i BF$  and  $A - GC$ . We immediately see that  $A - GC = A - QC^T C$  is Hurwitz. The fact that  $A + \mu_i BF$  is Hurwitz for  $i = 2, 3, \dots, N$  is also easily verified. Next, for  $i = 2, 3, \dots, N$ , let  $Y_i$  be the unique positive semi-definite solution to the Lyapunov equation

$$Y_i(A - QC^T C) + (A - QC^T C)^T Y_i + \left( \frac{|\mu_i|^2}{\gamma^4} + \frac{\alpha_i}{\gamma^4} \right) \tilde{P} B B^T \tilde{P} = 0, \quad (2.29)$$

for a yet to be determined  $\alpha_i \geq 0$ . Since  $A - QC^T C$  is Hurwitz, we can find a solution  $Y_i$  for any such  $\alpha_i$ . Next, take

$$Z_i := \begin{pmatrix} \frac{k_i}{\gamma^2} \tilde{P} & 0 \\ 0 & Y_i \end{pmatrix}, \quad (2.30)$$

where  $k_i = \frac{|\mu_i|^2}{\operatorname{Re}(\mu_i)}$ . With  $Z_i$  given by (2.30), the inequality (2.28) is equivalent to

$$\begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{12}^* & \Phi_{22} \end{pmatrix} \leq \begin{pmatrix} -\epsilon \left( \frac{k_i}{\gamma^2} \tilde{P} Q + I \right) C^T C \left( \frac{k_i}{\gamma^2} Q \tilde{P} + I \right) & 0 \\ 0 & 0 \end{pmatrix}, \quad (2.31)$$

where we have

$$\begin{aligned}
\Phi_{11} &= \frac{k_i}{\gamma^2} \tilde{P}A + \frac{k_i}{\gamma^2} A^T \tilde{P} - \frac{k_i \mu_i}{\gamma^4} \tilde{P}BB^T \tilde{P} - \frac{k_i \mu_i^*}{\gamma^4} \tilde{P}BB^T \tilde{P} \\
&\quad + C^T C + \frac{|\mu_i|^2}{\gamma^4} \tilde{P}BB^T \tilde{P} \\
&\quad + \frac{\gamma^2}{1-\gamma^2} \left( \frac{k_i}{\gamma^2} \tilde{P}Q + I \right) C^T C \left( \frac{k_i}{\gamma^2} Q \tilde{P} + I \right), \\
\Phi_{12} &= \frac{k_i \mu_i}{\gamma^4} \tilde{P}BB^T \tilde{P} - \frac{|\mu_i|^2}{\gamma^4} \tilde{P}BB^T \tilde{P}, \\
\Phi_{22} &= Y_i (A - QC^T C) + (A - QC^T C)^T Y_i + \frac{|\mu_i|^2}{\gamma^4} \tilde{P}BB^T \tilde{P}.
\end{aligned}$$

We will now first consider the upper left corner of the left hand side of (2.31). By adding and subtracting  $\frac{k_i}{\gamma^2} \frac{\eta}{\eta-\gamma^2} (\tilde{P}Q + \frac{\gamma^2}{\eta} I) C^T C (Q \tilde{P} + \frac{\gamma^2}{\eta} I)$ , and using algebraic Riccati equation (2.27), it follows that  $\Phi_{11}$  is equal to

$$\begin{aligned}
\Phi_{11} &= \frac{k_i}{\gamma^2} \left[ \gamma^2 \left( \frac{1}{k_i} - \frac{1}{\eta} \right) C^T C \right. \\
&\quad + \frac{1}{\gamma^2} (\eta - \operatorname{Re}(\mu_i)) \tilde{P}BB^T \tilde{P} \\
&\quad + \left( \frac{k_i}{1-\gamma^2} \frac{\gamma^4}{k_i^2} - \frac{\eta}{\eta-\gamma^2} \frac{\gamma^4}{\eta^2} \right) C^T C \\
&\quad + \left( \frac{k_i}{1-\gamma^2} \frac{\gamma^2}{k_i} - \frac{\eta}{\eta-\gamma^2} \frac{\gamma^2}{\eta} \right) (\tilde{P}QC^T C + C^T CQ \tilde{P}) \\
&\quad \left. + \left( \frac{k_i}{1-\gamma^2} - \frac{\eta}{\eta-\gamma^2} \right) \tilde{P}QC^T CQ \tilde{P} \right]. \tag{2.32}
\end{aligned}$$

This expression can be greatly simplified by using the following notation:

$$\begin{aligned}
\alpha_i &:= \frac{k_i}{1-\gamma^2} \frac{\gamma^4}{k_i^2} - \frac{\eta}{\eta-\gamma^2} \frac{\gamma^4}{\eta^2} - \gamma^2 \left( \frac{1}{\eta} - \frac{1}{k_i} \right), \\
\beta_i &:= \frac{k_i}{1-\gamma^2} \frac{\gamma^2}{k_i} - \frac{\eta}{\eta-\gamma^2} \frac{\gamma^2}{\eta}, \\
\delta_i &:= \frac{k_i}{1-\gamma^2} - \frac{\eta}{\eta-\gamma^2}.
\end{aligned}$$

Observe that we can now express (2.32) in terms of this notation as

$$\frac{k_i}{\gamma^2} \left[ \frac{1}{\gamma^2} (\eta - \operatorname{Re}(\mu_i)) \tilde{P} B B^T \tilde{P} + \begin{pmatrix} I & \tilde{P} Q \end{pmatrix} \begin{pmatrix} \alpha_i C^T C & \beta_i C^T C \\ \beta_i C^T C & \delta_i C^T C \end{pmatrix} \begin{pmatrix} I \\ Q \tilde{P} \end{pmatrix} \right]. \quad (2.33)$$

We will now prove that for  $\epsilon$  sufficiently small, it holds that

$$\begin{aligned} \frac{k_i}{\gamma^2} \begin{pmatrix} I & \tilde{P} Q \end{pmatrix} \begin{pmatrix} \alpha_i C^T C & \beta_i C^T C \\ \beta_i C^T C & \delta_i C^T C \end{pmatrix} \begin{pmatrix} I \\ Q \tilde{P} \end{pmatrix} \\ \leq -\epsilon \frac{k_i}{\gamma^2} \begin{pmatrix} I & \tilde{P} Q \end{pmatrix} \begin{pmatrix} \frac{\gamma^2}{k_i} C^T C & C^T C \\ C^T C & \frac{k_i}{\gamma^2} C^T C \end{pmatrix} \begin{pmatrix} I \\ Q \tilde{P} \end{pmatrix}. \end{aligned} \quad (2.34)$$

Here, the right hand side is rewritten from the nonzero corner of the right hand side of (2.31). Note that (2.34) is satisfied if

$$\begin{pmatrix} \alpha_i + \epsilon \frac{\gamma^2}{k_i} & \beta_i + \epsilon \\ \beta_i + \epsilon & \delta_i + \epsilon \frac{k_i}{\gamma^2} \end{pmatrix} \leq 0. \quad (2.35)$$

We will show this is indeed the case for  $\epsilon > 0$  sufficiently small. First, we show that for all  $i = 2, 3, \dots, N$  we have  $\delta_i < 0$ . Indeed, we have

$$\begin{aligned} \delta_i < 0 &\Leftrightarrow \frac{k_i}{1 - \gamma^2} < \frac{\eta}{\eta - \gamma^2} \\ &\Leftrightarrow k_i(\eta - \gamma^2) < \eta(1 - \gamma^2) \\ &\Leftrightarrow k_i\eta - k_i\gamma^2 < \eta - \eta\gamma^2. \end{aligned}$$

Since  $\eta < \operatorname{Re}(\mu_i) \leq k_i$  we already have  $-k_i\gamma^2 \leq -\eta\gamma^2$ . Furthermore,  $k_i < 1$ , which proves  $\delta_i < 0$ . If we now take  $\epsilon$  sufficiently small such that  $\delta_i + \frac{k_i}{\gamma^2}\epsilon < 0$ , then it is straightforward to check that also  $\alpha_i + \epsilon \frac{\gamma^2}{k_i} < 0$ . It then follows that (2.35) holds if and only if  $(\alpha_i + \epsilon \frac{\gamma^2}{k_i})(\delta_i + \epsilon \frac{k_i}{\gamma^2}) -$



$(\beta_i + \epsilon)^2 \geq 0$ . Below, we show that this is the case for  $\epsilon$  sufficiently small.

$$\begin{aligned}
& (\alpha_i + \epsilon \frac{\gamma^2}{k_i})(\delta_i + \epsilon \frac{k_i}{\gamma^2}) - (\beta_i + \epsilon)^2 \\
&= -\gamma^2 \left( \frac{1}{\eta} - \frac{1}{k_i} \right) \left( \frac{k_i}{1-\gamma^2} - \frac{\eta}{\eta-\gamma^2} \right) \\
&\quad - \frac{k_i}{1-\gamma^2} \frac{\eta}{\eta-\gamma^2} \gamma^4 \left( \frac{1}{\eta} - \frac{1}{k_i} \right)^2 \\
&\quad - \epsilon k_i \left[ \gamma^2 \frac{\eta}{\eta-\gamma^2} \left( \frac{1}{\eta} - \frac{1}{k_i} \right)^2 + \left( \frac{1}{\eta} - \frac{1}{k_i} \right) \right], \\
&= -\gamma^2 \left( \frac{1}{\eta} - \frac{1}{k_i} \right) \frac{\eta}{(1-\gamma^2)(\eta-\gamma^2)} \left[ k_i - 1 + \epsilon \left( \frac{k_i}{\gamma^2} - 1 \right) (1-\gamma^2) \right].
\end{aligned}$$

Since  $-\gamma^2 \left( \frac{1}{\eta} - \frac{1}{k_i} \right) \leq 0$  and  $(1-\gamma^2)(\eta-\gamma^2) > 0$ , it follows that  $(\alpha_i + \epsilon \frac{\gamma^2}{k_i})(\delta_i + \epsilon \frac{k_i}{\gamma^2}) - (\beta_i + \epsilon)^2 \geq 0$  if  $k_i - 1 + \epsilon \left( \frac{k_i}{\gamma^2} - 1 \right) (1-\gamma^2) \leq 0$ . We have that  $k_i - 1 < 0$ , and  $\left( \frac{k_i}{\gamma^2} - 1 \right) (1-\gamma^2) > 0$ . It follows that there exists  $\epsilon > 0$  such that (2.35) is satisfied, and consequently (2.34) is satisfied.

Next, recall that  $Y_i$  is a solution to (2.29). From (2.29), (2.31), and (2.34), we have that the algebraic Riccati inequality (2.28) holds if

$$\frac{1}{\gamma^4} \begin{pmatrix} k_i(\eta - \operatorname{Re}(\mu_i)) & k_i\mu_i - |\mu_i|^2 \\ k_i\mu_i^* - |\mu_i|^2 & -\alpha_i \end{pmatrix} \otimes \tilde{P} B B^T \tilde{P} \leq 0. \quad (2.36)$$

For this it suffices that

$$k_i(\operatorname{Re}(\mu_i) - \eta)\alpha_i - (k_i\mu_i - |\mu_i|^2)^*(k_i\mu_i - |\mu_i|^2) \geq 0, \quad (2.37)$$

for some  $\alpha_i \geq 0$ . Since  $\operatorname{Re}(\mu_i) - \eta$  is strictly greater than zero, for each  $i$  there exists  $\alpha_i \geq 0$  such that (2.37) holds. Since (2.34) and (2.37) hold, it follows that (2.28) is satisfied by our choice of  $Z_i$  in (2.30) for  $\epsilon$  sufficiently small. By Lemma 2.1 we obtain that for  $i = 2, 3, \dots, N$  the transfer matrix  $G_i$  of (2.26) satisfies  $\|G_i\|_\infty < \frac{1}{\gamma}$ . Finally, we apply Lemma 2.5 and obtain that the dynamic protocol robustly synchronizes the network.  $\square$

Theorem 2.6 and Theorem 2.9 both deal with the case that the network graph is directed and contains a spanning tree. To conclude this chapter, we will focus on the case that network graph is undirected. In that case the homogeneity restriction on the perturbations can be lifted, and results on robust synchronization against heterogeneous perturbations of the form  $(\Delta_M \Delta_N^i)$  can be established. Also, we can improve the upper bound in (2.20) if we assume that the network graph is undirected. Recall that in that case all eigenvalues of the Laplacian matrix  $L$  are real. We then have that  $\text{Re}(\lambda_i) = |\lambda_i| = \lambda_i$  for  $i = 2, 3, \dots, N$  and  $k_i$  in (2.30) reduces to  $\mu_i = \frac{\lambda_i}{\kappa}$ . Furthermore, the left hand side of (2.31) becomes a block diagonal matrix, which can then be further simplified by taking  $\alpha_i = 0$  in (2.29) so that the bottom right corner also becomes zero. It then follows that the network is synchronized if  $(\eta - \mu_i) \tilde{P} B B^T \tilde{P} \leq 0$  and for  $\epsilon > 0$  sufficiently small inequality (2.34) is satisfied. In the proof of Theorem 2.9, it was shown that there exists  $\epsilon > 0$  sufficiently small such that (2.34) holds, if  $k_i < 1$  and  $\eta \leq k_i$  for  $i = 2, 3, \dots, N$ , where  $\eta$  satisfies (2.24). Then it follows that there exists  $\epsilon > 0$  such that (2.34) is satisfied if  $\mu_i < 1$  and  $\eta \leq \mu_i$  for  $i = 2, 3, \dots, p$ , and  $\eta$  satisfies (2.24). Hence, in (2.23), we can simply take  $\kappa > \lambda_N$  so that  $\gamma^2 < \frac{\lambda_2}{\kappa} \frac{1}{1 + \rho(PQ)}$ , and take  $\eta = \frac{\lambda_2}{\kappa} = \mu_2$  so that (2.24) is satisfied. From this observation, we obtain the following corollary:

**COROLLARY 2.10:** Consider the network with perturbed agent dynamics (2.11). Assume the network graph is undirected and connected. Define

$$\gamma^* = \sqrt{\frac{\lambda_2}{\lambda_N} \frac{1}{\sqrt{1 + \rho(PQ)}}}. \quad (2.38)$$

Then for all  $\gamma \in (0, \gamma^*)$  there exists a dynamic protocol of the form (2.9) that robustly synchronizes the network with tolerance  $\gamma$ . In fact, for any

such  $\gamma$ , choose  $\kappa$  any real number such that  $\kappa > \lambda_N$  and  $(\frac{\lambda_2}{\kappa\gamma^2} - 1)I - PQ$  is nonsingular. Define

$$\begin{aligned}\tilde{P} &:= \left( \left( \frac{\lambda_2}{\kappa\gamma^2} - 1 \right) I - PQ \right)^{-1} P, \\ F &:= -\frac{1}{\gamma^2} B^T \tilde{P}, \\ G &:= QC^T.\end{aligned}\tag{2.39}$$

Then the dynamic protocol (2.9) with  $\kappa$ ,  $F$  and  $G$  as chosen above synchronizes the network for all perturbations  $\Delta_M, \Delta_N^i \in \mathcal{RH}_\infty$  satisfying  $\|(\Delta_M \Delta_N^i)\|_\infty \leq \gamma$ ,  $i = 1, 2, \dots, N$ .

It is interesting to note that in [65], for the case of additive perturbations, an analogous guaranteed interval  $(0, \gamma^*)$  was found, with  $\gamma^* = \frac{\lambda_2}{\lambda_N} \frac{1}{\sqrt{\rho(\tilde{P}\tilde{Q})}}$ , with  $\tilde{P}$  and  $\tilde{Q}$  the maximal real symmetric solutions of the ARE'S  $AQ + QA^T - QC^T CQ = 0$  and  $PA + A^T P - PBB^T P = 0$ . A direct comparison of this guaranteed interval with the one obtained in this chapter is not relevant since the perturbation structure used in [65] is completely different. In both cases however the guaranteed uncertainty tolerance heavily depends on the quotient  $\frac{\lambda_2}{\lambda_N}$ .

As noted before, in contrast to [65] (where completely heterogeneous perturbations were allowed), for undirected networks we were only able to deal with heterogeneity of the perturbations in a restricted sense: the  $\Delta_M$ -parts of the perturbations are assumed to be identical for all agents. Similar as in [65], for the directed network case we were only able to deal with the case of homogeneous perturbations.

## 2.7 NUMERICAL EXAMPLE

In this section, we consider a numerical example to illustrate the results obtained in this chapter.

**EXAMPLE 2.11:** We consider a network where the nominal agent dynamics is given by the mass-spring system in Figure 2.3. This system consists of two masses and two dampers. We take  $m_1 = 1$  and  $m_2 = 2$

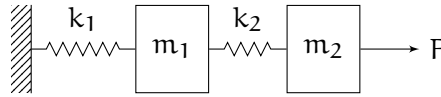


Figure 2.3: Mass-spring system with two masses.

as the weight of the masses, and  $k_1 = 1$ ,  $k_2 = 0.5$  as the spring constants. The input to the system is the force  $F$  exerted on the mass  $m_2$  and the output of the system is the position of the mass  $m_1$ . A state space representation of this system is then given by

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

where the matrices  $A$ ,  $B$ , and  $C$  are given by

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1.5 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \\ 0.25 & 0 & -0.25 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is easily checked that  $(A, B)$  is stabilizable and  $(C, A)$  is detectable. In fact, the system is both controllable and observable. Computing the stabilizing solutions  $Q$  and  $P$  to the algebraic Riccati equations (2.6) and (2.19), we obtain

$$Q = \begin{pmatrix} 0.5352 & 0.1432 & 1.0943 & 0.1355 \\ 0.1432 & 0.3324 & 0.4502 & 0.2123 \\ 1.0943 & 0.4502 & 3.1719 & 0.5988 \\ 0.1355 & 0.2123 & 0.5988 & 0.6677 \end{pmatrix}$$

and

$$P = \begin{pmatrix} 6.7339 & 0.1188 & -2.4745 & -0.8905 \\ 0.1188 & 4.1821 & 0.3686 & -0.9751 \\ -2.4745 & 0.3686 & 1.3295 & 0.5730 \\ -0.8905 & -0.9751 & 0.5730 & 2.1410 \end{pmatrix}.$$

As the network graph, we take the undirected graph depicted in Figure 2.4. The smallest and largest nonzero eigenvalues of the Laplacian matrix of the network graph are  $\lambda_2 = 0.5674$  and  $\lambda_N = 5.8634$ . Finally, using  $\rho(PQ) = 3.2062$ , we can compute  $\gamma^*$  in (2.38) as

$$\gamma^* = \sqrt{\frac{0.5674}{5.8634}} \frac{1}{\sqrt{1 + 3.2062}} = 0.1517.$$

We take  $\gamma = 0.1 < \gamma^*$  and  $\kappa = 6$ , such that  $\kappa > \lambda_N$  and  $(\frac{\lambda_2}{\kappa\gamma^2} - 1)I - PQ$  is nonsingular. Finally, we compute the gains

$$F = \begin{pmatrix} 8.7843 & 4.7366 & -6.0489 & -16.1689 \end{pmatrix},$$

$$G = \begin{pmatrix} 0.5352 & 0.1432 & 1.0943 & 0.1355 \end{pmatrix}^T.$$

As perturbations, we take

$$\Delta_M(s) = \frac{\frac{1}{2}(s+1)}{(s+3) * (s+4)},$$

$$\Delta_N^i(s) = \frac{\frac{1}{2}(s-i)}{(s+3+i) * (s+4+i)}.$$

We then have  $\|(\Delta_M \ \Delta_N^i)\|_\infty \leq \gamma$  for  $i = 1, 2, \dots, N$ . After connecting the agents with the disturbances and the controllers, we simulate the network for random initial conditions between  $-5$  and  $5$ . The resulting output (i.e. the position of  $m_1$ ) of the agents is plotted in Figure 2.5. For clarity, we have only plotted the first five agents. Indeed, the network reaches consensus.

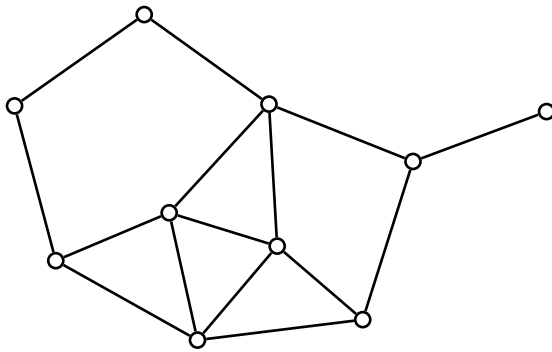


Figure 2.4: Undirected graph with  $N = 10$  nodes.

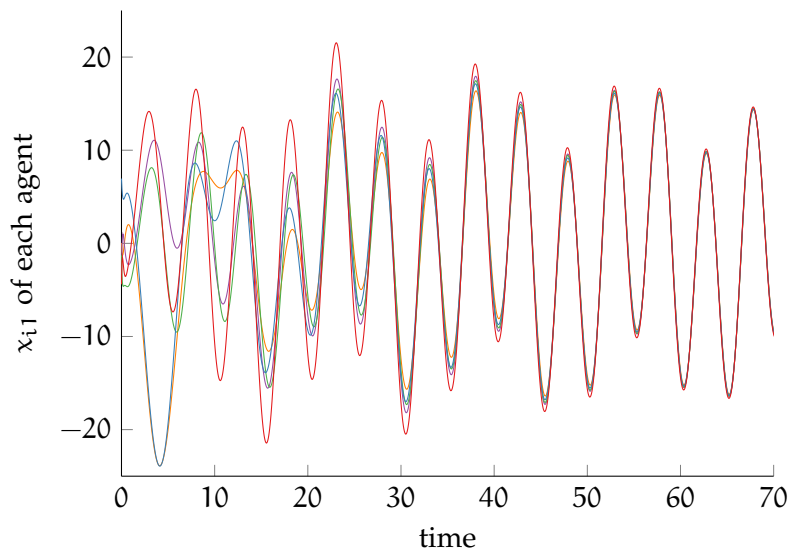


Figure 2.5: Output of the first five agents.

## 2.8 CONCLUSIONS

In this chapter, we have considered the problem of robust synchronization of multi-agent networks with uncertain agent dynamics. For a given network with identical nominal agent dynamics for each agent, coprime factor perturbations have been considered. We have provided an achievable interval for the values of the uncertainty tolerance. For the class of protocols in this chapter and for a directed network, the supremum of the achievable interval is proportional to the quotient of the smallest real part and the largest modulus of the eigenvalues of the graph Laplacian. For undirected graphs, these eigenvalues are real and the supremum of the achievable interval is proportional to the square root of the quotient of the smallest and largest eigenvalues of the Laplacian.

Part II

MODEL REDUCTION



