DISTURBANCE DECOUPLING FOR CONTINUOUS PIECEWISE LINEAR BIMODAL SYSTEMS

ABSTRACT: In this chapter we tackle the disturbance decoupling problem for continuous bimodal piecewise linear systems. After establishing necessary and sufficient geometric conditions for such a system to be disturbance decoupled, we study state feedback and dynamic feedback controllers, both mode-dependent and mode-independent. For these feedback schemes, we provide necessary and sufficient conditions for the solvability of the disturbance decoupling problem. Also, we provide subspace algorithms in order to verify the presented conditions. This chapter is based on the conference paper [Everts and Camlibel, 2014b].

2.1 INTRODUCTION

One of the main problems that will be addressed in this thesis is the disturbance decoupling problem for piecewise affine systems and other linear multi-modal systems. As introduced in Chapter 1, the disturbance decoupling problem amounts to eliminating, by means of feedback, the effect of the disturbance from the output of a given input/state/output dynamical system. In this chapter, we study the disturbance decoupling problem for a simple class of piecewise affine systems, namely piecewise linear systems with only two modes.

In the context of hybrid dynamical systems, the results on the disturbance decoupling problem are limited to jumping hybrid systems [Conte et al., 2015] and switched linear systems [Conte et al., 2014; Otsuka, 2010, 2011, 2015; Yurtseven et al., 2012; Zattoni and Marro, 2013; Zattoni et al., 2016]. In this chapter, we focus on a particular class of hybrid systems exhibiting state-dependent switchings, namely continuous piecewise linear bimodal systems. The main goal of this chapter is to provide necessary and sufficient conditions for the solvability of the disturbance decoupling problem for this class of systems.

The main difference, in the context of disturbance decoupling, between the state-independent and state-dependent switchings stems from the different nature of the set of reachable states by the disturbances for these two cases. In the case of linear state-independent switching systems, the set of states that can
be reached from the origin by the disturbances constitute a subspace of the whole state space. In [Otsuka, 2010; Yurtseven et al., 2012], this leads to the solution of the disturbance decoupling problem by following the footsteps of the classical results for the linear systems. However, the same set of states is, in general, neither a subspace nor even a convex set for the case of state-dependent switchings. As such, the ideas employed in the context of linear state-independent switching systems cannot be indiscriminately applied to linear state-dependent switching systems.

To overcome this obstacle, we first investigate under which conditions a given bimodal system is disturbance decoupled. It turns out that one can still provide easily verifiable geometric necessary and sufficient conditions for disturbance decoupling (see Theorem 2.3), even though the set of reachable states does not, in general, enjoy nice geometric properties such as being convex. Based on these geometric necessary and sufficient conditions, we study the disturbance decoupling problem for both state feedback controllers and dynamic feedback controllers. For both feedback schemes, we consider mode-independent and mode-dependent controllers, and provide necessary and sufficient conditions for the solvability of the disturbance decoupling problem. These conditions amount to checking certain subspace inclusions very much analogous to linear systems and linear state-independent switching systems. To verify these conditions, we also propose subspace algorithms.

In the following section, we introduce the class of continuous piecewise linear bimodal systems as well as the disturbance decoupling problem for this class of systems. This is followed by a complete characterization of the disturbance decoupled (open-loop) bimodal systems. Based on this characterization, we first turn our attention to the disturbance decoupling problem by state feedback in Section 2.3. Subsequently, we discuss the disturbance decoupling problem by dynamic feedback in Section 2.4. In order to verify the conditions presented in these sections, we provide subspace algorithms in Section 2.5. Finally, the chapter closes with conclusions in Section 2.6.
2.2 DISTURBANCE DECOUPLED BIMODAL SYSTEMS

We consider bimodal systems of the form

\[
\begin{align*}
&\dot{x}(t) = \begin{cases} 
A_1 x(t) + E d(t) & \text{if } c^T x(t) \leq 0, \\
A_2 x(t) + E d(t) & \text{if } c^T x(t) \geq 0,
\end{cases} \\
z(t) = H x(t),
\end{align*}
\]

(2.1)

where \( x \in \mathbb{R}^{n_x} \) is the state, \( d \in \mathbb{R}^{n_d} \) is the unknown disturbance, \( z \in \mathbb{R}^{n_z} \) is the output, and the matrices \( A_1, A_2, E, H \) and the vector \( c \) are of appropriate sizes. Throughout this chapter we assume that the right-hand side of (2.1a) is continuous in \( x \). In other words, the implication

\[ c^T x = 0 \implies A_1 x = A_2 x \]

holds. Equivalently, we have

\[ A_1 - A_2 = hc^T \]

(2.2)

for a vector \( h \in \mathbb{R}^{n_x} \). As such, the right-hand side of (2.1a) is Lipschitz continuous in the variable \( x \). Therefore, for each initial condition \( x_0 \) and locally integrable disturbance \( d \) there exists a unique absolutely continuous function \( x^{x_0,d}(t) \) satisfying \( x^{x_0,d}(0) = x_0 \) and (2.1a) for almost all \( t \). We denote the corresponding output of the system by \( z^{x_0,d}(t) \).

**Example 2.1** As an example, consider the bimodal system

\[
\begin{align*}
&\dot{x}(t) = \begin{cases} 
\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} d(t) & \text{if } \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \leq 0, \\
\begin{bmatrix} 4 & 3 \\ 2 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} d(t) & \text{if } \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) \geq 0,
\end{cases} \\
z(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t).
\end{align*}
\]

We use the following definition of disturbance decoupledness.

**Definition 2.2** A continuous piecewise linear bimodal system of the form (2.1) is **disturbance decoupled** if

\[ z^{x_0,d_1}(t) = z^{x_0,d_2}(t), \quad \forall t \geq 0 \]

for all initial states \( x_0 \in \mathbb{R}^{n_x} \) and all locally integrable disturbances \( d_1 \) and \( d_2 \).
In order to find necessary and sufficient conditions for a bimodal system to be disturbance decoupled, we define the set

\[ R(x_0, T) := \{ x^{x_0,d}(T) \mid d \text{ is locally integrable} \} \]

for each initial state \( x_0 \in \mathbb{R}^{nx} \) and \( T \geq 0 \). It follows immediately that system (2.1) is disturbance decoupled if and only if for every \( x_0 \in \mathbb{R}^{nx} \) and \( T \geq 0 \) the difference between any two elements in \( R(x_0, T) \) is in the kernel of \( H \), or equivalently,

\[ \bigcup_{T \geq 0} \bigcup_{x_0 \in \mathbb{R}^{nx}} (R(x_0, T) + (-R(x_0, T))) \subseteq \ker H. \quad (2.3) \]

Neither the set \( R(x_0, T) \) nor \( R(x_0, T) + (-R(x_0, T)) \) is necessarily convex in general. As such, condition (2.3) is rather hard to check. However, by making use of controllable subspaces, as defined in equation (1.2), we can provide an equivalent geometric condition which is easier to verify.

**Theorem 2.3** The system (2.1) is disturbance decoupled if and only if

\[ \langle A_1 \mid \text{im} \ E \rangle + \langle A_2 \mid \text{im} \ E \rangle \subseteq \ker H. \quad (2.4) \]

Before we give a proof of Theorem 2.3, we state and prove the following three auxiliary lemmas.

**Lemma 2.4** Let \( A_1 \) and \( A_2 \) be two square matrices such that \( A_1 - A_2 = hc^T \). Then the rational vector \( c^T(sI - A_1)^{-1}E \) is identically zero if and only if so is \( c^T(sI - A_2)^{-1}E \).

**Proof.** We use the well-known identity

\[ (sI - X)^{-1} - (sI - Y)^{-1} = (sI - X)^{-1}(X - Y)(sI - Y)^{-1}, \]

with \( X = A_1 \) and \( Y = A_2 \). By premultiplying both sides by \( c^T \), post-multiplying by \( E \), and using \( A_1 - A_2 = hc^T \) we get

\[ c^T(sI - A_1)^{-1}E - c^T(sI - A_2)^{-1}E = c^T(sI - A_1)^{-1}hc^T(sI - A_2)^{-1}E. \]

Hence, if \( c^T(sI - A_2)^{-1}E \) is identically zero, then so is \( c^T(sI - A_1)^{-1}E \). By symmetry, the converse also holds. \( \blacksquare \)
Lemma 2.5 Let $A_1$ and $A_2$ be two square matrices such that $A_1 - A_2 = hc^T$. Then the subspace $\langle A_1 \mid \text{im} E \rangle + \langle A_2 \mid \text{im} E \rangle$ is the smallest subspace containing $\text{im} E$ that is invariant under both $A_1$ and $A_2$. Furthermore, if $c^T (sI - A_1)^{-1} E$ is not identically zero, then

$$\langle A_1 \mid \text{im} E \rangle + \langle A_2 \mid \text{im} E \rangle = \langle A_1 \mid \text{im} [h \ E] \rangle. \quad (2.5)$$

Proof. Let $V = \langle A_1 \mid \text{im} E \rangle + \langle A_2 \mid \text{im} E \rangle$ and $U = \langle A_1 \mid \text{im} [h \ E] \rangle$. The subspace $U$ contains $\text{im} h$ and is invariant under $A_1$, hence it is also invariant under $A_2 = A_1 - hc^T$. Since $U$ contains $\text{im} E$ and $\langle A_i \mid \text{im} E \rangle$ is the smallest $A_i$-invariant subspace containing $\text{im} E$, we have $\langle A_i \mid \text{im} E \rangle \subseteq U$ for $i = 1, 2$. Hence, the inclusion $V \subseteq U$ follows.

Suppose that

$$c^T (sI - A_1)^{-1} E = \sum_{k=0}^{\infty} \frac{1}{s^{k+1}} c^T A_1^k E$$

is not identically zero, and let $p$ be the smallest non-negative integer such that $c^T A_1^p E \neq 0$. From equation (1.2) it follows that for any element $y \in V_{\perp}$ it holds that

$$y^T A_1^k E = y^T A_2^k E = 0, \quad \forall k \geq 0.$$ 

In particular, by choosing $k = p + 1$ we obtain

$$0 = y^T A_2^{p+1} E = y^T (A_1 - hc^T)^{p+1} E = -y^T hc^T A_1^p E,$$

where we use that $c^T A_1^k E = 0$ for $0 \leq k \leq p - 1$. Since $c^T A_1^p E$ is nonzero, this implies that $y^T h = 0$. Hence, we get $h \in (V_{\perp})_{\perp} = V$. Consequently, for all $v_1 \in \langle A_1 \mid \text{im} E \rangle$ and $v_2 \in \langle A_2 \mid \text{im} E \rangle$ we have $A_1 (v_1 + v_2) = A_1 v_1 + A_2 v_2 + hc^T v_2 \in V$. As such, $V$ is $A_1$-invariant. Furthermore, $V$ contains both $\text{im} h$ and $\text{im} E$. It follows that $U \subseteq V$, since $U$ is the smallest $A_1$-invariant subspace containing $\text{im} h$ and $\text{im} E$. Hence, (2.5) holds. Since $U$ is invariant under both $A_1$ and $A_2$, so is the subspace $V$.

In the case that $c^T (sI - A_1)^{-1} E$ is identically zero, we have $c^T A_1^k E = 0$ for all integers $k \geq 0$. We claim that $A_1^k E = A_2^k E$ for all $k \geq 0$. To prove this claim, we employ mathematical induction on $k$. It clearly holds for $k = 0$. Suppose that $A_1^k E = A_2^k E$ holds for $k = 0, 1, \ldots, \ell$, then

$$A_1^{\ell+1} E = A_1 A_1^\ell E = (A_2 + hc^T) A_1^\ell E = A_2 A_1^\ell E = A_2^{\ell+1} E.$$ 

Hence, we have $\langle A_1 \mid \text{im} E \rangle = \langle A_2 \mid \text{im} E \rangle = V$. Consequently, also in this case $V$ is invariant under both $A_1$ and $A_2$. 


Since any subspace that contains im $E$ and is invariant under $A_1$ and $A_2$ must contain both $\langle A_1 \mid \text{im } E \rangle$ and $\langle A_2 \mid \text{im } E \rangle$, we see that $\mathcal{V}$ is the smallest of such subspaces.

\textbf{Lemma 2.6} If $c^T(sI - A_1)^{-1}E$ is identically zero, then for all initial states $x_0 \in \mathbb{R}^n$ and integrable disturbances $d_1$ and $d_2$ we have $c^T x^{0,d_1}(t) = c^T x^{0,d_2}(t)$ for all $t \geq 0$.

\textbf{Proof.} Let $\mathcal{V} = \langle A_1 \mid \text{im } E \rangle + \langle A_2 \mid \text{im } E \rangle$. It follows from Lemma 2.4 that $c^T A_1^k E = c^T A_2^k E = 0$ for $k \geq 0$. Hence, we get $\mathcal{V} \subseteq \ker c^T$. By Lemma 2.5, $\mathcal{V}$ is invariant under both $A_1$ and $A_2$. Let $v_1, v_2, \ldots, v_\ell$ be a basis for $\mathcal{V}$, and extend this to a basis $v_1, v_2, \ldots, v_{n_x}$ for $\mathbb{R}^{n_x}$. Let $S = [v_1 \ v_2 \ \ldots \ v_{n_x}]$, then the basis transformation $\xi = S^{-1} x$ results in the system

$$
\dot{\xi}(t) = \begin{cases}
\tilde{A}_1 \xi(t) + \tilde{E} d(t) & \text{if } \tilde{c}^T \xi(t) \leq 0, \\
\tilde{A}_2 \xi(t) + \tilde{E} d(t) & \text{if } \tilde{c}^T \xi(t) \geq 0.
\end{cases}
$$

Decompose $\xi$ as $\xi = \text{col}(\xi_1, \xi_2)$, where $\xi_1$ contains the first $\ell$ entries of $\xi$. Since $\text{im } E \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \ker c^T$, we see that the matrices $\tilde{A}_1, \tilde{A}_2, \tilde{E}$ and $\tilde{c}^T$ are of the form

$$
\begin{align*}
\tilde{A}_1 &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, & \tilde{A}_2 &= \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \\
\tilde{E} &= \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, & \tilde{c}^T &= \begin{bmatrix} 0 & c_2^T \end{bmatrix}.
\end{align*}
$$

In particular, $\xi_2$ satisfies

$$
\dot{\xi}_2 = \begin{cases}
A_{22} \xi_2 & \text{if } c_2^T \xi_2 \leq 0, \\
A_{22} \xi_2 & \text{if } c_2^T \xi_2 \geq 0.
\end{cases}
$$

Note that $\xi_2$ does not depend on the disturbance $d$. Therefore, the value of $c^T x = \tilde{c}^T \xi = c_2^T \xi_2$ does not depend on the disturbance. Hence, we see that $c^T x^{0,d_1}(t) = c^T x^{0,d_2}(t)$ for all $t \geq 0$, initial conditions $x_0$ and integrable disturbances $d_1, d_2$. \[\blacksquare\]

Now we are in a position to give a proof of Theorem 2.3.

\textbf{Proof of Theorem 2.3.} \textbf{Necessity:} Suppose that the system (2.1) is disturbance decoupled. Let $x_0$ be such that $c^T x_0 < 0$ and take $d_1(t) = d \in \mathbb{R}^{n_d}$ a constant vector, and $d_2(t) = 0$. Let $x_i(t) = x^{0,d_i}(t)$ for $i = 1, 2$ denote the state trajectories of the
system \((2.1)\) corresponding to the initial state \(x_0\) and disturbances \(d_i\), and let \(z_i(t) = Hx_i(t)\) denote their outputs. Since \(x_1\) and \(x_2\) are continuous, there exists an \(\epsilon > 0\) such that \(c^T x_i(t) < 0\) for all \(t \in (0, \epsilon)\) and \(i = 1, 2\). This means that for \(t \in (0, \epsilon)\) we have
\[
\dot{x}_i(t) = A_1 x_i(t) + Ed_i(t), \quad i = 1, 2.
\]
Since the system \((2.1)\) is disturbance decoupled, the outputs satisfy \(z_1(t) = z_2(t)\) for \(t \geq 0\). Therefore, we have
\[
Hx_1(t) = Hx_2(t), \quad t \geq 0.
\]
Note that \(d_1(t)\) and \(d_2(t)\) are both taken to be constant, so we can differentiate both sides \(k\) times, resulting in
\[
HA_1^k x_1(t) + HA_1^{k-1} Ed = HA_1^k x_2(t), \quad t \in (0, \epsilon), \quad k \geq 1.
\]
Taking the limit as \(t \downarrow 0\) and using \(x_1(0) = x_2(0)\) gives us
\[
HA_1^k Ed = 0, \quad k \geq 0.
\]
Since this holds for every vector \(d \in \mathbb{R}^q\), we can conclude that \(HA_1^k E = 0\) for all \(k \geq 0\). By choosing \(x_0\) such that \(c^T x_0 > 0\) and employing similar arguments, we obtain \(HA_2^k E = 0\) for all \(k \geq 0\). Consequently, \((2.4)\) holds.

**Sufficiency:** Let \(V = \langle A_1 \mid \text{im} E \rangle + \langle A_2 \mid \text{im} E \rangle\). In view of \((2.3)\), it suffices to show that \(R(x_0, T) - R(x_0, T) \subseteq V\), or equivalently \(V^\perp \subseteq (R(x_0, T) - R(x_0, T))^\perp\) for all \(x_0\) and \(T \geq 0\).

Let \(x_0\) be an initial state and \(d_1, d_2\) two disturbances. Also, let \(x_i(t) = x^{x_0, d_i}(t)\) for \(i = 1, 2\) denote the two corresponding trajectories of the system \((2.1)\). Let \(y\) be an element of \(V^\perp = \langle A_1 \mid \text{im} E \rangle^\perp \cap \langle A_2 \mid \text{im} E \rangle^\perp\). Then \(y^T A_1^k E = 0\) and \(y^T A_2^k E = 0\) for all \(k \geq 0\). In the case that \(c^T(sI - A_1)^{-1} E\) is not identically zero, we know from Lemma 2.5 that \(\text{im} h \subseteq V\). As such, we have \(y^T h = 0\). Together with \(y^T E = 0\), this yields
\[
y^T \dot{x}_i(t) = \begin{cases}
y^T A_1 x_i(t) & \text{if } c^T x_i(t) \leq 0 \\
y^T A_2 x_i(t) & \text{if } c^T x_i(t) \geq 0
\end{cases} = y^T A_1 x_i(t),
\]
for \(t \geq 0\) and \(i = 1, 2\). In the case that \(c^T(sI - A_1)^{-1} E\) is identically zero, it follows from Lemma 2.6 that \(c^T x_1(t) =
\(c^T x_2(t)\) for all \(t \geq 0\). Hence, we have \(hc^T(x_1(t) - x_2(t)) = 0\), which implies that
\[
y^T (x_1(t) - x_2(t)) = \begin{cases} 
y^T A_1(x_1(t) - x_2(t)), & c^T x_1(t) \leq 0 
y^T A_2(x_1(t) - x_2(t)), & c^T x_1(t) \geq 0 \end{cases}
\]
for \(t \geq 0\).

In conclusion, in both cases we have
\[
y^T (x_1(t) - x_2(t)) = y^T A_1(x_1(t) - x_2(t)),
\]
for all \(y \in \mathcal{V}^\perp\) and for almost all \(t \geq 0\). To study equation (2.6), we first suppose that \(\lambda\) is an eigenvalue of \(A_1^T\) and \(y \in \mathcal{V}^\perp\) satisfies
\[
(A_1^T - \lambda I)^k y = 0
\]
for some integer \(k \geq 1\). The vector \(y\) generates a Jordan chain \(y_1, y_2, \ldots, y_k\) for the eigenvalue \(\lambda\) as follows:
\[
y_j = (A_1^T - \lambda I)^{k-j} y \quad \text{for} \quad 1 \leq j \leq k.
\]
Since \(y_k = y \in \mathcal{V}^\perp\) and \(\mathcal{V}^\perp\) is \(A_1^T\)-invariant, we see that \(y_j \in \mathcal{V}^\perp\) for all \(j = 1, 2, \ldots, k\). We will prove by mathematical induction on \(j\) that
\[
y_j^T (x_1(t) - x_2(t)) = 0
\]
for \(j = 1, 2, \ldots, k\) and all \(t \geq 0\). For \(j = 1\), we have \(A_1^T y_1 = \lambda y_1\). Hence, it follows from (2.6) that
\[
\frac{d}{dt} [y_1^T (x_1(t) - x_2(t))] = \lambda y_1^T (x_1(t) - x_2(t)),
\]
for almost all \(t \geq 0\). This results in
\[
y_1^T (x_1(t) - x_2(t)) = e^{\lambda t} y_1^T (x_1(0) - x_2(0)) = 0,
\]
since \(x_1(0) = x_2(0)\). Now, assume that (2.8) holds for \(j = 1, 2, \ldots, \ell\) for some integer \(\ell\) with \(1 \leq \ell < k\). By using (2.6) and \(A_1^T y_{\ell+1} = \lambda y_{\ell+1} + y_\ell\), we obtain
\[
\frac{d}{dt} [y_{\ell+1}^T (x_1(t) - x_2(t))] = y_{\ell+1}^T A_1 (x_1(t) - x_2(t))
\]
\[
= (\lambda y_{\ell+1} + y_\ell)^T (x_1(t) - x_2(t))
\]
\[
= \lambda y_{\ell+1}^T (x_1(t) - x_2(t))
\]
for almost all \(t \geq 0\).
for almost all \( t \geq 0 \). Therefore, we have
\[
y_{t+1}^\top(x_1(t) - x_2(t)) = e^{At}y_{t+1}^\top(x_1(0) - x_2(0)) = 0.
\]
This completes the proof of (2.8). Clearly, (2.8) implies that \( y_j \in (R(x_0, T) - R(x_0, T))^\perp \) for all \( j, x_0 \) and \( T \geq 0 \).

To generalize this result to all \( y \in V^\perp \), we define \( M \subseteq \mathbb{C}^n \) to be the subspace \( M = V^\perp \oplus iV^\perp \). Consider \( A_{1}^\top \) as a linear map from \( \mathbb{C}^n \) to \( \mathbb{C}^n \). Since \( V^\perp \) is \( A_{1}^\top \)-invariant, so is \( M \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_r \) be the distinct eigenvalues of \( A_{1}^\top \) and define the corresponding root subspaces \( R_{\lambda_i} \) for \( i = 1, 2, \ldots, r \) as
\[
R_{\lambda_i}(A_{1}^\top) := \ker(A_{1}^\top - \lambda_i I)^{p_i},
\]
where \( p_i \) is the geometric multiplicity of the eigenvalue \( \lambda_i \). By [Gohberg et al., 2006, Thm. 2.1.5], we can decompose \( M \) as follows:
\[
M = \bigoplus_{i=1}^{r} (M \cap R_{\lambda_i}(A_{1}^\top)).
\]
For fixed \( x_0 \) and \( T \geq 0 \), we can consider \( R(x_0, T) - R(x_0, T) \) as a subset of \( \mathbb{C}^n \). Since each root subspace \( R_{\lambda_i} \) is spanned by a Jordan chain, it follows from the preceding argument on Jordan chains that \( M \subseteq (R(x_0, T) - R(x_0, T))^\perp \). Hence, \( V^\perp \subseteq (R(x_0, T) - R(x_0, T))^\perp \) for all \( x_0 \) and \( T \geq 0 \), which completes the proof.

For later use in the next two sections, and to relate our result to similar results for switched linear systems, we state the following corollary, which follows from combining Theorem 2.3 with Lemma 2.5.

**Corollary 2.7** The system (2.1) is disturbance decoupled if and only if there exists a subspace \( V \subseteq \mathbb{R}^{nx} \) that is invariant under both \( A_1 \) and \( A_2 \) such that \( \text{im} E \subseteq V \subseteq \ker H \).

**Example 2.8** We revisit Example 2.1. For this system, notice that \( A_1E = E \) and \( A_2E = E \), which implies that \( \langle A_1 \mid \text{im} E \rangle = \langle A_2 \mid \text{im} E \rangle = \text{im} E \). Since \( HE = 0 \), we see that this bimodal system satisfies (2.4), and hence it is disturbance decoupled.

**Remark 2.9** In [Yurtseven et al., 2012], the disturbance decoupling problem for switched linear systems is studied. The results presented in [Yurtseven et al., 2012, Thm. 3.7 and 3.9]
provide sufficient conditions for the disturbance decoupling of piecewise linear systems. Applied to the bimodal system (2.1), these conditions boil down to the conditions in Corollary 2.7, but with the extra condition that the subspace \( V \) and the matrices \( A_1 \) and \( A_2 \) should satisfy \( \text{im}(A_1 - A_2) = \text{im} h c^T \subseteq V \). This last condition implies that \( h \in V \), which is not necessary in the case that \( c^T(sI - A_1)^{-1}E \) is identically zero.

2.3 DISTURBANCE DECOUPLING BY STATE FEEDBACK

The next question we address is under what conditions a bimodal system can be rendered disturbance decoupled by means of static state feedback. To do so, we consider the bimodal system

\[
\dot{x}(t) = \begin{cases} 
A_1 x(t) + Bu(t) + Ed(t) & \text{if } c^T x(t) \leq 0 \\
A_2 x(t) + Bu(t) + Ed(t) & \text{if } c^T x(t) \geq 0
\end{cases} 
\]

\[ z(t) = Hx(t) \]

where \( u \in \mathbb{R}^{n_u} \) is the input, \( B \) is an \( n_x \times n_u \) input matrix, and \( x, z, d, A_1, A_2, E, H \) and \( c \) are as before. We assume that \( B \) has full column rank and that \( A_1 \) and \( A_2 \) satisfy the continuity condition (2.2).

In this section we provide necessary and sufficient conditions for the existence of a static state feedback law that renders the closed-loop system disturbance decoupled. We consider two forms of static feedback: mode-dependent and mode-independent.

2.3.1 Mode-dependent state feedback

Consider a mode-dependent static feedback law of the form

\[ u(t) = \begin{cases} 
F_1 x(t) & \text{if } c^T x(t) \leq 0 \\
F_2 x(t) & \text{if } c^T x(t) \geq 0
\end{cases} \]

for two matrices \( F_1, F_2 \in \mathbb{R}^{n_u \times n_x} \) with the property that \( c^T x = 0 \) implies \( F_1 x = F_2 x \), or equivalently, \( \ker c^T \subseteq \ker(F_1 - F_2) \). This implies that there exists a vector \( f \in \mathbb{R}^{n_u} \) such that

\[ F_1 - F_2 = fc^T. \]

20
In other words, we consider mode-dependent and continuous (in $x$) state feedback. Clearly, such a feedback results in the (continuous) closed-loop system

$$
\dot{x}(t) = \begin{cases} (A_1 + BF_1)x(t) + Ed(t) & \text{if } c^T x(t) \leq 0 \\ (A_2 + BF_2)x(t) + Ed(t) & \text{if } c^T x(t) \geq 0 \end{cases} \quad (2.12a)
$$

$$
z(t) = Hx(t). \quad (2.12b)
$$

In view of Corollary 2.7, we see that the closed-loop system (2.12) is disturbance decoupled if and only if there exist a subspace $\mathcal{V}$ and matrices $F_1$ and $F_2$ such that $\mathcal{V}$ is invariant under both $A_1 + BF_1$ and $A_2 + BF_2$, $\text{im} E \subseteq \mathcal{V} \subseteq \ker H$, and $\ker c^T \subseteq \ker (F_1 - F_2)$.

In order to check whether such a subspace exists or not, we need to introduce some nomenclature. Define the set of subspaces

$$
\mathcal{V}_{\text{md}}(H, \{A_1, A_2\}, B) := \{ \mathcal{V} \subseteq \ker H \mid \exists F_1, F_2 \text{ s.t. } (A_j + BF_j)\mathcal{V} \subseteq \mathcal{V}, j = 1, 2 \},
$$

where the subscript ‘md’ stands for mode-dependent. Let $\mathcal{V}$ and $\mathcal{W}$ be two subspaces in $\mathcal{V}_{\text{md}}(H, \{A_1, A_2\}, B)$. Then, since $\mathcal{V}$ and $\mathcal{W}$ are both $(A_1, B)$-invariant, the subspace $\mathcal{V} + \mathcal{W}$ is $(A_1, B)$-invariant as well. Similarly, we see that $\mathcal{V} + \mathcal{W}$ is $(A_2, B)$-invariant too. Therefore, $\mathcal{V}_{\text{md}}(H, \{A_1, A_2\}, B)$ is closed under subspace addition. Let $\mathcal{V}_{\text{md}}^*(H, \{A_1, A_2\}, B)$ be the largest of the subspaces in $\mathcal{V}_{\text{md}}(H, \{A_1, A_2\}, B)$. If the context is clear, we will denote it by $\mathcal{V}_{\text{md}}^*$.

Note that in the definition of $\mathcal{V}_{\text{md}}(H, \{A_1, A_2\}, B)$ in (2.13) we do not consider the continuity condition (2.11). However, for any subspace $\mathcal{V}$ in $\mathcal{V}_{\text{md}}(\ker H, \{A_1, A_2\}, B)$, there exist matrices $F_1$ and $F_2$ such that the feedback (2.10) is continuous in $x$ and $(A_i + BF_i)\mathcal{V} \subseteq \mathcal{V}$ for $i = 1, 2$, as shown in the following lemma.

**Lemma 2.10** If a subspace $\mathcal{V}$ is $(A_1, B)$-invariant and $(A_2, B)$-invariant, and $A_1 - A_2 = hc^T$, then there exist matrices $F_1, F_2 \in \mathbb{R}^{n_x \times n_x}$ such that $F_1 - F_2 = fc^T$ for some $f \in \mathbb{R}^{n_u}$ and $(A_i + BF_i)\mathcal{V} \subseteq \mathcal{V}$ for $i = 1, 2$.

**Proof.** $\mathcal{V}$ is $(A_1, B)$-invariant, so there exists a matrix $F_1$ such that $(A_1 + BF_1)\mathcal{V} \subseteq \mathcal{V}$. Since $\mathcal{V}$ is $(A_2, B)$-invariant as well, $\mathcal{V}$ is also $(hc^T, B)$-invariant, so $hc^T \mathcal{V} \subseteq \mathcal{V} + \text{im } B$. This implies that we have $h \in \mathcal{V} + \text{im } B$ or $\mathcal{V} \subseteq \ker c^T$. In the former case,
there exists an \( f \in \mathbb{R}^{n_u} \) such that \( h + Bf \in \mathcal{V} \). In the latter case, let \( f \) be any vector in \( \mathbb{R}^{n_u} \). Hence, in both cases we have \((h + Bf)c^T \mathcal{V} \subseteq \mathcal{V}\). Let \( F_2 = F_1 - fc^T \), then \( A_2 + BF_2 = A_1 + BF_1 - (h + Bf)c^T \), which implies that \((A_2 + BF_2)\mathcal{V} \subseteq \mathcal{V} \). □

The following theorem shows how we can use the subspace \( \mathcal{V}_{md}^* (H, \{A_1, A_2\}, B) \) to determine whether there exists a mode-dependent static state feedback controller that renders the system (2.9) disturbance decoupled.

**Theorem 2.11** There exists a mode-dependent static state feedback of the form (2.10) that renders the closed-loop system (2.12) disturbance decoupled if and only if

\[
\text{im } E \subseteq \mathcal{V}_{md}^* (H, \{A_1, A_2\}, B). 
\]

**Proof.** Sufficiency: Since \( \mathcal{V}_{md}^* \) is \((A_1, B)\)-invariant and \((A_2, B)\)-invariant, by Lemma 2.10 there exist matrices \( F_1 \) and \( F_2 \) such that \( F_1 - F_2 = fc^T \) for some \( f \in \mathbb{R}^{n_u} \) and \((A_i + BF_i)\mathcal{V}_{md}^* \subseteq \mathcal{V}_{md}^* \) for \( i = 1, 2 \). From the hypothesis, we have \( \text{im } E \subseteq \mathcal{V}_{md}^* \subseteq \ker H \). Therefore, \( E \in \mathcal{V}_{md}^* \) by Lemma 2.7 that mode-dependent static feedback given by (2.10) renders the closed-loop system (2.12) disturbance decoupled.

Necessity: Suppose that \( F_1 \) and \( F_2 \) are such that the input (2.10) renders the closed-loop system (2.12) disturbance decoupled. It follows from Corollary 2.7 that there exists a subspace \( \mathcal{V} \) that is invariant under both \( A_1 + BF_1 \) and \( A_2 + BF_2 \), and such that \( \text{im } E \subseteq \mathcal{V} \subseteq \ker H \). Therefore, \( \mathcal{V} \in \mathcal{V}_{md} (H, \{A_1, A_2\}, B) \). Hence, \( \text{im } E \subseteq \mathcal{V} \subseteq \mathcal{V}_{md}^* \).

In Section 2.5 we will provide an algorithm to compute the subspace \( \mathcal{V}_{md}^* (H, \{A_1, A_2\}, B) \). Once the condition \( \text{im } E \subseteq \mathcal{V}_{md}^* (H, \{A_1, A_2\}, B) \) is satisfied, one can construct the feedback matrices \( F_1 \) and \( F_2 \) by following the steps in the proof of Lemma 2.10.

### 2.3.2 Mode-independent state feedback

Consider the static state feedback law \( u = Fx \) for a matrix \( F \in \mathbb{R}^{n_u \times n_x} \). This can be seen as a special case of the mode-dependent state feedback, with \( F_1 = F_2 \). Such a feedback law results in the closed-loop system

\[
\dot{x}(t) = \begin{cases} 
(A_1 + BF)x(t) + Ed(t) & \text{if } c^T x(t) \leq 0 \\
(A_2 + BF)x(t) + Ed(t) & \text{if } c^T x(t) \geq 0.
\end{cases}
\] (2.14)
By Corollary 2.7, we see that the closed-loop system is disturbance decoupled if and only if there exist a subspace $V$ and a feedback matrix $F$ such that $V$ is invariant under both $A_1 + BF$ and $A_2 + BF$, and $\text{im } E \subseteq V \subseteq \ker H$. Similar to the mode-dependent case, we define the set of subspaces

$$V_{mi}(H, \{A_1, A_2\}, B) := \{ V \subseteq \ker H | \exists F \text{ s.t. } (A_j + BF)V \subseteq V \text{ for } j = 1, 2 \},$$

where the subscript ‘mi’ stands for mode-independent. The set $V_{mi}(H, \{A_1, A_2\}, B)$ is closed under subspace addition, and hence it has a largest element, denoted by $V^*_{mi}(H, \{A_1, A_2\}, B)$, or simply by $V^*_{mi}$ if the context is clear. In Section 2.5 we provide an algorithm to compute $V^*_{mi}$.

The following theorem can be proven by using similar arguments as employed in the proof of Theorem 2.11.

**Theorem 2.12** There exists a matrix $F \in \mathbb{R}^{n_u \times n_x}$ such that the state feedback $u(t) = Fx(t)$ renders the closed-loop system (2.9) disturbance decoupled if and only if

$$\text{im } E \subseteq V^*_{mi}(H, \{A_1, A_2\}, B).$$

### 2.4 DISTURBANCE DECOUPLING BY DYNAMIC FEEDBACK

In this section, we address the disturbance decoupling problem by dynamic feedback. Consider the bimodal system (2.9) together with the output

$$y(t) = Cx(t),$$

where $y \in \mathbb{R}^{n_y}$. The main goal of this section is to investigate under which conditions there exists a dynamic controller from $y$ to $u$ rendering the closed-loop system disturbance decoupled. Similar to the state feedback problem, we distinguish two cases: mode-dependent and mode-independent controllers.
2. DISTURBANCE DECOUPLING FOR BIMODAL SYSTEMS

2.4.1 Mode-dependent dynamic feedback

We start with the mode-dependent dynamic feedback controller given by

\[ \dot{w}(t) = \begin{cases} Kw(t) + L_1 y(t) & \text{if } c^T x \leq 0 \\ Kw(t) + L_2 y(t) & \text{if } c^T x > 0 \end{cases} \]  
(2.17a)

\[ u(t) = \begin{cases} Mw(t) + N_1 y(t) & \text{if } c^T x \leq 0 \\ Mw(t) + N_2 y(t) & \text{if } c^T x > 0 \end{cases} \]  
(2.17b)

where \( w \in \mathbb{R}^{n_w} \) is the state of the controller, \( u \in \mathbb{R}^{n_u} \) and \( y \in \mathbb{R}^{n_y} \) are as before, and the matrices \( K, L_1, L_2, M, N_1 \) and \( N_2 \) are of suitable sizes. Interconnecting this controller with the system given by (2.9) and (2.16) results in the closed-loop system

\[ \begin{bmatrix} \dot{x}(t) \\ \dot{w}(t) \end{bmatrix} = \begin{cases} A_{e,1} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + E_e d(t) & \text{if } c_e^T \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \leq 0 \\ A_{e,2} \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} + E_e d(t) & \text{if } c_e^T \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} \geq 0 \end{cases} \]  
(2.18a)

\[ z(t) = H_e \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}, \]  
(2.18b)

where

\[ A_{e,i} = \begin{bmatrix} A_i + BN_i C \\ L_i C \\ BM \\ K \end{bmatrix}, \quad i = 1, 2, \]  
(2.18c)

\[ E_e = \begin{bmatrix} E \\ 0 \end{bmatrix}, \quad H_e = \begin{bmatrix} H \\ 0 \end{bmatrix}, \quad c_e^T = \begin{bmatrix} c^T \\ 0 \end{bmatrix}. \]  
(2.18d)

We only consider mode-dependent feedback controllers that render the closed-loop system continuous (both in \( x \) and \( w \)). This amounts to imposing the following conditions on the matrices \( L_1, L_2, N_1 \) and \( N_2 \):

\[ \ker c^T \subseteq \ker (L_1 - L_2) C, \quad \ker c^T \subseteq \ker (N_1 - N_2) C. \]  
(2.19)

Equivalently, we assume that there are vectors \( \ell \in \mathbb{R}^{n_w} \) and \( n \in \mathbb{R}^{n_u} \) such that

\[ (L_1 - L_2) C = \ell c^T, \quad (N_1 - N_2) C = n c^T. \]  
(2.20)

As a result, we have \( \ker c_e^T \subseteq \ker (A_{e,1} - A_{e,2}) \).
The objective of this section is to find such a mode-dependent dynamic controller that renders the closed-loop system disturbance decoupled. By employing \((C, A_1, B)\)-pairs (see Section 1.5.1), the following theorem provides necessary and sufficient conditions for the existence of such a controller.

**Theorem 2.13** There exists a mode-dependent dynamic controller of the form (2.17) satisfying the continuity condition (2.19) such that the closed-loop system (2.18) is disturbance decoupled if and only if there exist subspaces \(\mathcal{T}\) and \(\mathcal{V}\) such that \((\mathcal{T}, \mathcal{V})\) is a \((C, A_1, B)\)-pair satisfying \(\im E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H\) and \(hc^\mathcal{T} \mathcal{V} \subseteq \mathcal{V} + \im B\).

**Proof.** Necessity: Assume that there exists such a controller given by \(K, L_1, L_2, M, N_1\) and \(N_2\). Let \(\mathbb{R}^{n_w}\) denote the state space of the controller. The (extended) state space of the interconnected system is then given by \(\mathbb{R}^{n_x} \times \mathbb{R}^{n_w}\). By Corollary 2.7, there exists a subspace \(\mathcal{V}_e \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_w}\) that is invariant under both \(A_{e,1}\) and \(A_{e,2}\), satisfying \(\im E \subseteq \mathcal{V}_e \subseteq \ker H\). For this subspace \(\mathcal{V}_e\), we define the following two subspaces of \(\mathbb{R}^{n_x}\):

\[
p(\mathcal{V}_e) := \{ x \in \mathbb{R}^{n_x} \mid \exists w \in \mathbb{R}^{n_w} \text{ such that } \begin{bmatrix} x \\ w \end{bmatrix} \in \mathcal{V}_e \},
\]

\[
i(\mathcal{V}_e) := \{ x \in \mathbb{R}^{n_x} \mid \begin{bmatrix} x \\ 0 \end{bmatrix} \in \mathcal{V}_e \},
\]

which can be seen as the projection of \(\mathcal{V}_e\) on \(\mathbb{R}^{n_x}\) and the intersection of \(\mathcal{V}_e\) and \(\mathbb{R}^{n_x} \times \{0\}\) respectively. Let \(\mathcal{T} = i(\mathcal{V}_e)\) and \(\mathcal{V} = p(\mathcal{V}_e)\). Since \(\mathcal{V}_e\) is \(A_{e,1}\)-invariant, \((\mathcal{T}, \mathcal{V})\) is a \((C, A_1, B)\)-pair (see e.g. [Trentelman et al., 2001, Theorem 6.2]). Next, we will show that this \((C, A_1, B)\)-pair \((\mathcal{T}, \mathcal{V})\) satisfies \(\im E \subseteq \mathcal{T}, \mathcal{V} \subseteq \ker H\) and \(hc^\mathcal{T} \mathcal{V} \subseteq \mathcal{V} + \im B\).

For any \(x \in \im E\), we have that \(\col(x, 0) \in \im E \subseteq \mathcal{V}_e\). Therefore, we get \(x \in i(\mathcal{V}_e) = \mathcal{T}\), hence we have \(E \subseteq \mathcal{T}\). For \(x \in \mathcal{V} = p(\mathcal{V}_e)\), there exists a \(w \in \mathbb{R}^{n_w}\) such that \(\col(x, w) \in \mathcal{V}_e \subseteq \ker H\). Then, we get \(Hx = H_e \col(x, w) = 0\) and hence \(\mathcal{V} \subseteq \ker H\).

Since \(L_1, L_2, N_1\) and \(N_2\) satisfy (2.19), there are vectors \(\ell\) and \(n\) such that (2.20) holds. Consequently, we have

\[
A_{e,1} - A_{e,2} = \begin{bmatrix} (h + Bn)c^\mathcal{T} \\ \ell c^\mathcal{T} \end{bmatrix}.
\]

(2.21)

Let \(x \in \mathcal{V}\). Then, a \(w \in \mathbb{R}^{n_w}\) such that \((x^\mathcal{T}, w^\mathcal{T})^\mathcal{T} \in \mathcal{V}_e\). Since \(\mathcal{V}_e\) is invariant under both \(A_{e,1}\) and \(A_{e,2}\), we have

\[
(A_{e,1} - A_{e,2}) \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} (h + Bn)c^\mathcal{T}x \\ \ell c^\mathcal{T}x \end{bmatrix} \in \mathcal{V}_e.
\]
Consequently, we obtain \((h + Bn)c^Tx \in \mathcal{V}\) and hence \(hc^T \mathcal{V} \subseteq \mathcal{V} + \text{im} \, B\).

**Sufficiency:** Let \((\mathcal{T}, \mathcal{V})\) be a such a \((C, A_1, B)\)-pair. Then there exist \(F\) and \(G\) such that

\[
(A_1 + BF) \mathcal{V} \subseteq \mathcal{V}, \quad (A_1 + GC) \mathcal{T} \subseteq \mathcal{T}.
\]

Furthermore, there exists a linear mapping \(N_1\) such that (see e.g. [Trentelman et al., 2001, Lemma 6.3])

\[
(A_1 + BN_1 C) \mathcal{T} \subseteq \mathcal{V}.
\]

Since \(hc^T \mathcal{V} \subseteq \mathcal{V} + \text{im} \, B\), we have \(\mathcal{V} \subseteq \ker c^T\) or \(h \in \mathcal{V} + \text{im} \, B\).

If the latter holds, then there exists an \(n \in \mathbb{R}^{n_u}\) such that \(h + Bn \in \mathcal{V}\). Then choose \(\ell \in \mathcal{V}\) such that \(h + Bn - \ell \in \mathcal{T}\).

If we have \(\mathcal{V} \subseteq \ker c^T\), then we can choose \(n \in \mathbb{R}^{n_u}\) and \(\ell \in \mathcal{V}\) arbitrarily. In both cases, we can find \(n\) and \(\ell\) such that \((h + Bn - \ell)c^T \mathcal{V} \subseteq \mathcal{T}\).

Let \(L_1 = BN_1 - G\) and define

\[
K = A_1 + BF + GC - BN_1 C, \quad L_2 = L_1 - \ell c^T, \\
M = F - N_1 C, \quad N_2 = N_1 - nc^T.
\]

and let \(K, L_1, L_2, M, N_1\) and \(N_2\) define a controller of the form (2.17), with \(n_w = n_x\). Note that \((L_1 - L_2)C = \ell c^T\) and \((N_1 - N_2)C = nc^T\), so \(L_1, L_2, N_1\) and \(N_2\) satisfy the continuity condition (2.19). The system matrices of the corresponding closed-loop system (2.18) are then given by

\[
A_{e,i} = \begin{bmatrix}
A_i + BN_i C & B(F - N_i C) \\
L_i C & A_1 + BF + GC - BN_i C
\end{bmatrix},
\]

for \(i = 1, 2\).

Let \(\mathcal{V}_e\) be the subspace of \(\mathbb{R}^{n_x} \times \mathbb{R}^{n_w}\) given by

\[
\mathcal{V}_e = \{\begin{bmatrix}s \\ 0\end{bmatrix} + \begin{bmatrix}v \\ 0\end{bmatrix} \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_w} \mid s \in \mathcal{T}, v \in \mathcal{V}\}.
\]

First we show that \(\mathcal{V}_e\) is invariant under both \(A_{e,1}\) and \(A_{e,2}\). For any \(s \in \mathcal{T}\) and \(v \in \mathcal{V}\) we have that

\[
A_{e,1} \begin{bmatrix}s \\ 0\end{bmatrix} = \begin{bmatrix}(A_1 + GC)s \\ 0\end{bmatrix} + \begin{bmatrix}(A_1 + BNC)s \\ (A_1 + BNC)s\end{bmatrix} - \begin{bmatrix}(A_1 + GC)s \\ (A_1 + GC)s\end{bmatrix},
\]

\[
A_{e,1} \begin{bmatrix}v \\ 0\end{bmatrix} = \begin{bmatrix}(A_1 + BF)v \\ (A_1 + BF)v\end{bmatrix}.
\]
are both elements of $\mathcal{V}_e$, so $\mathcal{V}_e$ is invariant under $A_{e,1}$. Using equation (2.21), $\ell \in \mathcal{V}$, and $(h + Bn - \ell) c^T \mathcal{V} \subseteq \mathcal{T}$, we see that for all $s \in \mathcal{T}$ and $v \in \mathcal{V}$ we have that

\[
(A_{e,1} - A_{e,2}) \begin{bmatrix} s \\ 0 \end{bmatrix} = \begin{bmatrix} (h + Bn) c^T s \\ \ell c^T s \end{bmatrix} = \begin{bmatrix} \ell c^T s \\ \ell c^T s \end{bmatrix} + \begin{bmatrix} (h + Bn - \ell) c^T s \\ 0 \end{bmatrix},
\]

\[
(A_{e,1} - A_{e,2}) \begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} (h + Bn) c^T v \\ \ell c^T v \end{bmatrix} = \begin{bmatrix} \ell c^T v \\ \ell c^T v \end{bmatrix} + \begin{bmatrix} (h + Bn - \ell) c^T v \\ 0 \end{bmatrix},
\]

are also both elements of $\mathcal{V}_e$. Consequently, $A_{e,2} \text{col}(s, 0)$ and $A_{e,2} \text{col}(v, v)$ are contained in $\mathcal{V}_e$ as well. Therefore, $\mathcal{V}_e$ is invariant under both $A_{e,1}$ and $A_{e,2}$.

Next, we show that $\mathcal{V}_e$ satisfies $\text{im} \ E_e \subseteq \mathcal{V}_e \subseteq \ker H_e$. For any point $\text{col}(x, w) \in \text{im} \ E_e$, we have $x \in \text{im} \ E \subseteq \mathcal{T}$ and $w = 0$. Consequently, $\text{col}(x, w) = \text{col}(x, 0) \in \mathcal{V}_e$ and hence $\text{im} \ E_e \subseteq \mathcal{V}_e$. Further, we have $x \in \mathcal{V} \subseteq \ker H$ for any $\text{col}(x, w) \in \mathcal{V}_e$. This implies that $H_e \text{col}(x, w) = Hx = 0$, i.e. $\text{col}(x, w) \in \ker H_e$. Then, we can conclude that $\mathcal{V}_e \subseteq \ker H_e$. Now we can use Corollary 2.7 to prove that the closed-loop system (2.18) is disturbance decoupled.

The conditions presented in Theorem 2.13 are existential in nature. Next, we articulate these conditions and provide easily verifiable conditions based on subspace algorithms. Recall that $\mathcal{T}^*(E, A_1, C)$ is the smallest $(C, A_1)$-invariant subspace containing $\text{im} \ E$.

**Theorem 2.14** There exists a mode-dependent dynamic controller of the form (2.17) satisfying the continuity condition (2.19) that renders the closed-loop system (2.18) disturbance decoupled if and only if

\[
\mathcal{T}^*(E, A_1, C) \subseteq \mathcal{V}^*_m(H, \{A_1, A_2\}, B).
\]

**Proof. Necessity:** If there exists such a controller, then by Theorem 2.13 there are subspaces $\mathcal{T}$ and $\mathcal{V}$ such that $(\mathcal{T}, \mathcal{V})$ is a $(C, A_1, B)$-pair, $\text{im} \ E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H$ and $hc^T \mathcal{V} \subseteq \mathcal{V} + \text{im} \ B$. We clearly have $\mathcal{T}^*(E, A_1, C) \subseteq \mathcal{T}$. The subspace $\mathcal{V}$ is $(A_1, B)$-invariant. Since $hc^T \mathcal{V} \subseteq \mathcal{V} + \text{im} \ B$, the subspace $\mathcal{V}$ is also
(A_2, B)-invariant. Therefore, we have \( \mathcal{V} \subseteq \mathcal{V}_{md}^*(H, \{A_1, A_2\}, B) \). Hence, we can conclude that

\[
T^*(E, A_1, C) \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \mathcal{V}_{md}^*(H, \{A_1, A_2\}, B).
\]

**Sufficiency:** Let \((\mathcal{T}, \mathcal{V})\) be the \((C, A_1, B)\)-pair given by \( \mathcal{T} = T^*(E, A_1, C) \) and \( \mathcal{V} = \mathcal{V}_{md}^*(H, \{A_1, A_2\}, B) \). Then we have \( \text{im} \, E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H \). Since \( \mathcal{V} \) is both \((A_1, B)\)-invariant and \((A_2, B)\)-invariant, we have \( A_i \mathcal{V} \subseteq \mathcal{V} + \text{im} \, B \) for \( i = 1, 2 \). As such, we obtain \( \text{hc} \, T \mathcal{V} = (A_1 - A_2) \mathcal{V} \subseteq \mathcal{V} + \text{im} \, B \). It follows from Theorem 2.13 that the closed-loop system (2.18) is disturbance decoupled.

### 2.4.2 Mode-independent dynamic feedback

As a special case, we consider in this section the linear time-invariant mode-independent feedback controller

\[
\begin{align*}
\dot{w}(t) &= Kw(t) + Ly(t) \\
u(t) &= Mw(t) + Ny(t)
\end{align*}
\]

where \( w \in \mathbb{R}^{n_w}, u \in \mathbb{R}^{n_u}, y \in \mathbb{R}^{n_y} \), and \( K, L, M \) and \( N \) are of suitable sizes. By interconnecting this controller with system given by (2.9) and (2.16), we obtain the closed-loop system (2.18) with the system matrices \( A_{e,1} \) and \( A_{e,2} \) now given by

\[
A_{e,i} = \begin{bmatrix} A_i + BNC & BM \\ LC & K \end{bmatrix}
\]

for \( i = 1, 2 \).

We can adapt Theorem 2.13 for mode-dependent dynamic controllers to obtain a similar, but more restrictive, result for mode-independent dynamic controllers.

**Theorem 2.15** There exists a mode-independent dynamic controller of the form (2.22) that renders the system given by (2.9) and (2.16) disturbance decoupled if and only if there exist subspaces \( \mathcal{T} \) and \( \mathcal{V} \) such that \((\mathcal{T}, \mathcal{V})\) is a \((C, A_1, B)\)-pair, \( \text{im} \, E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H \) and \( \text{hc}^T \mathcal{V} \subseteq \mathcal{T} \).

**Proof.** A proof of the statement follows from the proof of Theorem 2.13 by taking \( L_1 = L_2, N_1 = N_2, n = 0 \) and \( \ell = 0 \).

Note that the condition \( \text{hc}^T \mathcal{V} \subseteq \mathcal{T} \) in Theorem 2.15 is more restrictive than the condition \( \text{hc}^T \mathcal{V} \subseteq \mathcal{V} + \text{im} \, B \) that appears in Theorem 2.13.
Just as for the mode-dependent case, we would like to define some minimal $T^*$ and maximal $V^*$ such that $(T^*, V^*)$ is a $(C, A_1, B)$-pair that satisfies the conditions of Theorem 2.15 exactly when the system can be rendered disturbance decoupled by means of a mode-dependent dynamic feedback controller. For this reason, we define the set of subspaces

$$T_{mi}(E, \{A_1, A_2\}, C) := \{T \subseteq \mathbb{R}^n_x \mid \text{im } E \subseteq T, \quad (2.24) \}$$

and $\exists G$ s.t. $(A_j + GC)T \subseteq T$ for $j = 1, 2$.

Similar to the fact that the set $V_{mi}$ (defined in (2.15)) has a maximal element with respect to subspace addition, the set $T_{mi}$ has a minimal element. Let $T^*_{mi}(E, \{A_1, A_2\}, C)$ denote the smallest subspace in $T_{mi}$. In Section 2.5 we present an algorithm to compute $T^*_{mi}$.

The existence of a controller of the form (2.22) that renders the closed-loop system disturbance decoupled does not imply that $(T^*_{mi}, V^*_{mi})$ is a $(C, A_1, B)$-pair satisfying the conditions of Theorem 2.15, since $hc^T V^*_{mi} \subseteq T^*_{mi}$ is not necessarily satisfied. However, the following assertion holds.

**Theorem 2.16** There exists a controller of the form (2.22) that renders the system given by (2.9) and (2.16) disturbance decoupled if and only if at least one of the following two conditions holds

1. $T^*_{mi}([E \ h], \{A_1, A_2\}, C) \subseteq V^*_{mi}(H, \{A_1, A_2\}, B),$

2. $T^*_{mi}(E, \{A_1, A_2\}, C) \subseteq V^*_{mi}([H^T \ c]^T, \{A_1, A_2\}, B)$.

**Proof.** Sufficiency: If the first condition holds, then let

$$T = T^*_{mi}([E \ h], \{A_1, A_2\}, C), \quad V = V^*_{mi}(H, \{A_1, A_2\}, B).$$

Then, we have $h \in T$ which implies that $hc^T V \subseteq T$.

If the second condition holds, let

$$T = T^*_{mi}(E, \{A_1, A_2\}, C), \quad V = V^*_{mi}([H^T \ c]^T, \{A_1, A_2\}, B).$$

Then, we have $V \subseteq \ker c^T$ which implies that $hc^T V \subseteq T$.

In both cases we have that $(T, V)$ is a $(C, A_1, B)$-pair satisfying $\text{im } E \subseteq T \subseteq V \subseteq \ker H$ and $hc^T V \subseteq T$. Therefore, it follows from Theorem 2.15 that there exists a controller of the form (2.22) such that the closed-loop system is disturbance decoupled.

Necessity: Suppose there exists such a controller. By Theorem 2.15, there exist subspaces $T$ and $V$ such that $(T, V)$ is a
(\(C, A_1, B\))-pair, \(\text{im} E \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq \ker H\), and \(hc^T \mathcal{V} \subseteq \mathcal{T}\). The last condition implies that \(hc^T \mathcal{V} \subseteq \mathcal{V}\) and \(hc^T \mathcal{T} \subseteq \mathcal{T}\), and hence \((\mathcal{T}, \mathcal{V})\) is also a \((C, A_2, B)\)-pair. Therefore, we have \(\mathcal{T} \in T_{mi}(E, \{A_1, A_2\}, \ker H)\) and \(\mathcal{V} \in V_{mi}(H, \{A_1, A_2\}, B)\). Furthermore, \(hc^T \mathcal{V} \subseteq \mathcal{T}\) also implies that we have \(h \in \mathcal{T}\) or \(\mathcal{V} \subseteq \ker c^T\). If \(h \in \mathcal{T}\), then \(\mathcal{T}\) is an element of \(T_{mi}(E, \{A_1, A_2\}, C)\), which means that

\[
T_{mi}^s([E \ h], \{A_1, A_2\}, C) \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq V_{mi}^s(H, \{A_1, A_2\}, B).
\]

If \(\mathcal{V} \subseteq \ker c^T\), then \(\mathcal{V} \in V_{mi}([H^T \ c]^T, \{A_1, A_2\}, B)\). Hence, we get

\[
T_{mi}(E, \{A_1, A_2\}, C) \subseteq \mathcal{T} \subseteq \mathcal{V} \subseteq V_{mi}([H^T \ c]^T, \{A_1, A_2\}, B).
\]

In conclusion, at least one of the two conditions in the statement holds.

\[\blacksquare\]

2.5 Subspace Algorithms

In this section we first propose subspace algorithms for computing \(V_{md}^s(H, \{A_1, A_2\}, B)\) and \(V_{mi}^s(H, \{A_1, A_2\}, B)\). Both algorithms are similar to the invariant subspace algorithm for computing \(V^s(H, A_1, B)\) for linear systems (see e.g. [Trentelman et al., 2001]), and to the subspace algorithms proposed in [Yurtseven et al., 2012] for switched linear systems. Afterwards, we will provide an algorithm for computing \(T_{mi}^s(E, \{A_1, A_2\}, C)\).

2.5.1 Algorithm for \(V_{md}^s(H, \{A_1, A_2\}, B)\)

For computing \(V_{md}^s(H, \{A_1, A_2\}, B)\), we propose the following algorithm. We first define

\[
V_0 = \ker H.
\]

Then, for \(i \geq 0\), we define

\[
V_{i+1} = V_i \cap A_1^{-1}(V_i + \text{im} B),
\]

if \(h \in V_i + \text{im} B\), and otherwise

\[
V_{i+1} = V_i \cap A_1^{-1}(V_i + \text{im} B) \cap \ker c^T.
\]

It is clear that we have \(V_{i+1} \subseteq V_i\) for all \(i \geq 0\) and hence there is a \(k \leq n_x\) such that \(V_k = V_{k+1}\). Moreover, it follows from the definition of \(V_i\) that we then have \(V_{k+2} = V_{k+1}\). Therefore, we get \(V_i = V_k\) for all \(i \geq k\).
Theorem 2.17 Let $\mathcal{V}_i$ be defined as in algorithm (2.25). Then for $q = \min \{ k \in \mathbb{N} \mid \mathcal{V}_k = \mathcal{V}_{k+1} \} \leq n_x$ we have

$$\mathcal{V}_q = \mathcal{V}^*_\text{md}(H, \{A_1, A_2\}, B).$$

Proof. As the subspaces $\mathcal{V}_i$ are nested, we have $\mathcal{V}_q \subseteq \mathcal{V}_0 = \ker H$. Since $\mathcal{V}_q$ satisfies $\mathcal{V}_q = \mathcal{V}_{q+1}$, it follows that $\mathcal{V}_q = \mathcal{V}_q \cap A_1^{-1}(\mathcal{V}_q + \operatorname{im} B)$ if $h \in \mathcal{V}_q + \operatorname{im} B$, and $\mathcal{V}_q = \mathcal{V}_q \cap A_1^{-1}(\mathcal{V}_q + \operatorname{im} B) \cap \ker c^T$ otherwise. In both cases we have $A_1 \mathcal{V}_q \subseteq \mathcal{V}_q + \operatorname{im} B$, so $\mathcal{V}_q$ is $(A_1, B)$-invariant. Furthermore, we have $h \in \mathcal{V}_q + \operatorname{im} B$ or $\mathcal{V}_q \subseteq \ker c^T$, which implies that $hc^T \mathcal{V}_q \subseteq \mathcal{V}_q + \operatorname{im} B$. Hence, $A_2 \mathcal{V}_q \subseteq A_1 \mathcal{V}_q + hc^T \mathcal{V}_q \subseteq \mathcal{V}_q + \operatorname{im} B$, so $\mathcal{V}_q$ is $(A_2, B)$-invariant as well. Therefore, we see that $\mathcal{V}_q$ is an element of $\mathcal{V}^*_\text{md}(H, \{A_1, A_2\}, B)$, and hence $\mathcal{V}_q \subseteq \mathcal{V}^*_\text{md}$.

To prove that we have $\mathcal{V}^*_\text{md} \subseteq \mathcal{V}_q$ as well, we use mathematical induction on $i$. Firstly, we have that $\mathcal{V}^*_\text{md} \subseteq \mathcal{V}_0 = \ker H$. Secondly, assume that $\mathcal{V}^*_\text{md} \subseteq \mathcal{V}_i$ for some $i \geq 0$. Since $\mathcal{V}^*_\text{md}$ is both $(A_1, B)$-invariant and $(A_2, B)$-invariant, it is $(hc^T, B)$-invariant as well. Therefore, we have

$$hc^T \mathcal{V}^*_\text{md} \subseteq \mathcal{V}^*_\text{md} + \operatorname{im} B \subseteq \mathcal{V}_i + \operatorname{im} B.$$

Hence, we get $h \in \mathcal{V}_i + \operatorname{im} B$ or $\mathcal{V}^*_\text{md} \subseteq \ker c^T$. In both cases, it holds that $\mathcal{V}^*_\text{md} \subseteq \mathcal{V}_{i+1}$. Therefore, we see that $\mathcal{V}^*_\text{md} \subseteq \mathcal{V}_k$ for all $k \geq 0$. In particular, we have $\mathcal{V}^*_\text{md} \subseteq \mathcal{V}_q$. ■

2.5.2 Algorithm for $\mathcal{V}^*_\text{mi}(H, \{A_1, A_2\}, B)$

To compute $\mathcal{V}^*_\text{mi}(H, \{A_1, A_2\}, B)$, we refer to Algorithm 5.3 in [Yurtseven et al., 2012], which in our case simplifies to

$$\mathcal{V}_0 = \ker H \quad (2.26a)$$

and

$$\mathcal{V}_{i+1} = \mathcal{V}_i \cap A_1^{-1}(\mathcal{V}_i + \operatorname{im} B) \cap (A_1 - A_2)^{-1}(\mathcal{V}_i) \quad (2.26b)$$

for $i \geq 0$.

2.5.3 Algorithm for $\mathcal{T}^*_\text{mi}(E, \{A_1, A_2\}, C)$

By making use of the well-known duality between controlled invariance and conditioned invariance (see e.g. [Trentelman...]}
et al., 2001, Theorem 5.6], we adapt the algorithm (2.26) for computing $v^*_{mi}$ to obtain the following algorithm. We define

$$T_0 = \text{im } E,$$  \hfill (2.27a)

and

$$T_{i+1} = \text{im } E + A_1(T_i \cap \ker C) + hc^T T_i$$ \hfill (2.27b)

for $i \geq 0$. It is easy to see that $T_i \subseteq T_{i+1}$ for $i \geq 0$. Since $n_x$ is finite and $T_i \in \mathbb{R}^{n_x}$ for all $i \geq 0$, it follows that there is a $k$ such that $T_k = T_{k+1}$. Furthermore, it follows from the definition of $T_i$ that we have $T_i = T_k$ for all $i \geq k$. The next theorem shows that this algorithm indeed gives us $T^*_{mi}(E, \{A_1, A_2\}, C)$. We omit the proof, since it follows from similar arguments as employed in the proof of Theorem 2.17.

**Theorem 2.18** Let $T_i$ be defined as in algorithm (2.27). Then for $q = \min\{k \in \mathbb{N} \mid T_k = T_{k+1}\} \leq n_x$ we have

$$T_q = T^*_{mi}(E, \{A_1, A_2\}, C).$$

### 2.6 Conclusions

In this chapter, we studied the disturbance decoupling problem for continuous piecewise linear bimodal systems. The main contributions of this chapter include necessary and sufficient conditions for such systems to be disturbance decoupled as well as a complete characterization of the solvability of the disturbance decoupling problem with mode-independent and mode-dependent feedback controllers. Furthermore, we provided subspace algorithms in order to compute the minimal and maximal subspaces that are used in the presented conditions for disturbance decoupling by both state feedback and dynamic feedback.

Future research possibilities include the extension of the presented results to general piecewise affine dynamical systems, which will be the subject of the next chapter.