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Newton-Cartan gravity revisited

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**rijksuniversiteit
groningen**

Newton-Cartan gravity revisited

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Chapter 1

Introduction

1.1 Crawling out of the cave

For thousands of years people have tried to understand the world which they live in on a fundamental level. For this purpose they developed, among others, scientific methods which started more or less from everyday experience. However, this everyday experience, which still shapes our modern human intuition, has its limitations. As became more and more clear at the end of the 19th century, our everyday experience forms just the tip of the iceberg.

The first paradigm shift came with the dawning of Quantum Mechanics in 1900, which showed that the classical laws of physics cannot blindly be applied to the subatomic world. Whereas for us the world seems to be continuous, on atomic scales nature turns out to be quantized, and the clear difference between pointlike matter and waves disappears. The second paradigm shift came in 1905, when Einstein presented his theory of Special Relativity [1]. This theory rejects the absolute nature of time, which was and still is for a lot of people the obvious thing to believe, and unites space and time into one entity called spacetime. Ten years later, in 1915, Einstein replaced Newton's theory of gravity by his theory of General Relativity [2], stating that gravity is a manifestation of the spacetime curvature described by the so-called Einstein equations. The theory of General Relativity completely reshaped the universally held notion of spacetime. While in Newtonian physics and Special Relativity spacetime was just a static and "God-given" arena on which all the physics takes place, in General Relativity spacetime is a dynamical background which has its own dynamics determined by its content. As such, spacetime and everything in it become intimately related. General Relativity also opened the doors to modern theories of cosmology. Observations by Hubble implied that the universe is expanding, a possibility which was also suggested by the application of General Relativity to the universe as a whole, but which troubled Einstein and others because of the firm believe in a static universe. This conviction of a static universe was also based on everyday experience: a human

life is simply too short compared to cosmic timescales to see the stars on our night sky change their patterns. In a similar way a human body is too big to experience quantum effects and is moving too slowly to notice that space and time are really intertwined. If we lived on a planet orbiting a massive black hole bending spacetime significantly, we would be comfortable with the idea that the angles of a triangle don't add up to 180 degrees and that some objects distort spacetime in such a way that even light cannot escape. Riemannian geometry would then probably have been found before Euclidean geometry instead of the other way around. If we lived at lengthscales of Angströms instead of meters, the strange world of Quantum Mechanics where matter shows interference patterns wouldn't be that strange anymore. In this sense our everyday experience is like Plato's cave [3], with the world of the physical extremes lying outside. Theoretical physics with its mathematical formulation allows us then to peek outside this cave.¹

However, old theories like in Newtonian physics are not considered to be “wrong”. They just happen to have a smaller region of validity than the new theory. This motivates the so-called *correspondence principle*, which was first formulated by Niels Bohr in the context of Quantum Mechanics [4], and states that in certain limits the new theory should reproduce the old theory.

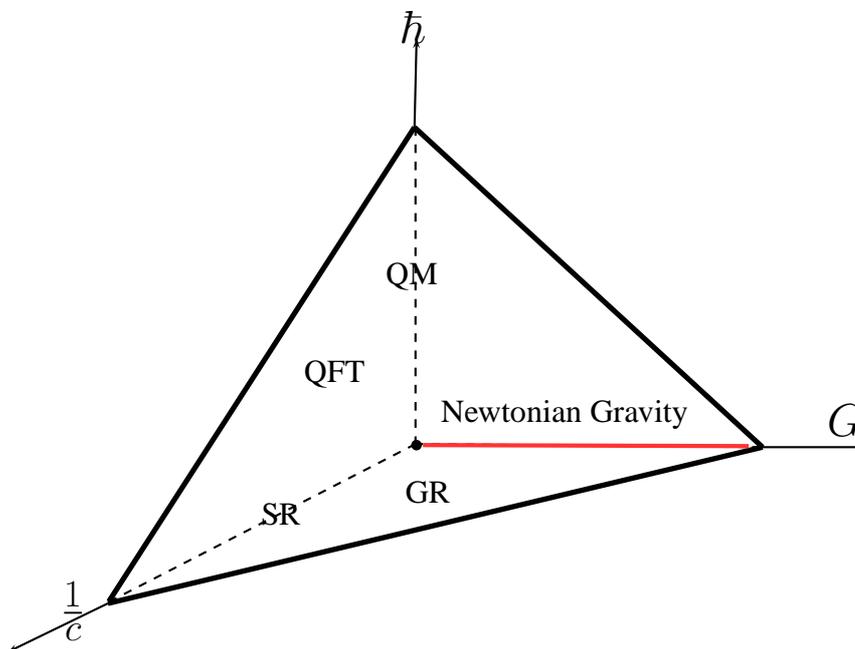


Figure 1.1: The “dimensional pyramid”, taken from [5]. The planes with $\hbar > 0$ indicate the quantum regime, the planes with $G > 0$ indicate the gravitational regime, and the planes with $\frac{1}{c} > 0$ indicate the relativistic regime. The red line $\{\hbar = 0, G > 0, \frac{1}{c} = 0\}$ indicates the Newtonian

¹An example outside physics is Darwin's theory of evolution, in which species gradually change but in general too slowly to be noticed *directly*, resembling the movement of the stars. An example in geophysics is given by Wegener's theory of continental drift.

regime, which will be the one of interest in this thesis.

This means that in the Newtonian limit, which according to fig.(1.1) can be characterized by sending the speed of light to infinity, General Relativity should reproduce Newton's theory of gravity in order to be consistent, which it does. But also General Relativity has its limitations. Namely, it is only applicable beyond a characteristic length scale known as the Planck scale, because at smaller length scales the notion of spacetime becomes ill-defined due to quantum fluctuations. The obvious solution to this problem seemed to be to make the theory consistent with the rules of Quantum Mechanics, but this turned out to be extremely difficult: quantizing General Relativity as an ordinary field theory results in a theory which predicts infinities as outcome for e.g. graviton scattering. In the language of Quantum Field Theory, the theory which unifies Special Relativity and Quantum Mechanics, it is said that General Relativity is an effective field theory. This means that beyond a certain energy/length scale new physics appears. General Relativity is ignorant of this new physics and can only be trusted below this energy scale.

The holy grail of high energy physics is to obtain a well-defined theory of quantum gravity, which lies on the tilded edge $\{\hbar > 0, G > 0, \frac{1}{c} > 0\}$ of fig.(1.1). One such attempt is Loop Quantum Gravity [6, 7]. Quantum Mechanics turned the infinite answers of classical physics applied to black body radiation into finite answers by quantizing the energy of the radiation. Similarly, Loop Quantum Gravity tries to get rid of the infinities which plague the canonical quantization of General Relativity by quantizing spacetime. However, it is not yet clear if Loop Quantum Gravity reproduces General Relativity in the classical limit in which Planck's constant goes to zero [8]. Another attempt of a theory of quantum gravity which tries to go beyond General Relativity is String Theory [9–11]. This theory postulates that all the different particles, which according to Quantum Mechanics describe matter and the interactions experienced by it, are actually different vibrational modes of tiny strings. This string-like character of matter and fundamental interactions only manifests itself clearly at the Planck scale. String Theory does one specific prediction which again contradicts our human intuition: the spacetime inside Plato's cave may seem to have four dimensions, but outside the cave (Super)string Theory demands as a consistency condition that spacetime has six extra spatial dimensions. To account for observations these extra dimensions have to be very small, such that until now they haven't been noticed yet by particle accelerators. One exciting and highly non-trivial fact of String Theory is that it seems to be able to reproduce Einstein's theory of General Relativity: one vibrational mode of the fundamental string turns out to be a massless spin-2 mode, and as such has exactly the same properties as one would expect from the particle mediating gravity, which can be identified as the graviton. Einstein's equations of General Relativity plus stringy corrections appear as a condition for quantum consistency

of the theory. Besides providing a consistent theory of quantum gravity, String Theory also has the potential to unify gravity with the other fundamental interactions encountered in nature. The correspondence principle tells us that at low energies String Theory reduces to so-called “Supergravities” [12] in four spacetime dimensions, which depend on the specific compactification of the six extra dimensions. Superstring Theory needs a symmetry called supersymmetry² in spacetime in order to interpret the spectrum of vibrations. This symmetry relates particles of different spin. The Supergravities mentioned earlier are then supersymmetric extensions of General Relativity. Because supersymmetry, as mathematically elegant as it is, is still not found experimentally, it is possible that supersymmetry-wise we are still locked in Plato’s cave.

1.2 General covariance and gauge symmetries

In the development of modern physics the role of symmetries cannot be emphasized enough. Whereas the laws of physics arrange the events we want to describe in spacetime, the symmetries arrange the laws of physics themselves by restricting their possible forms! In short, a symmetry constitutes a change of the physical system without changing the physical outcome. The following two symmetry principles are very important.

The first symmetry principle is that of *covariance*, stating that the coordinates one uses to describe events in spacetime are just labels. As such, the laws of physics should not depend on the choice of coordinates. This principle already holds in Newtonian physics for inertial observers, which are all connected by Galilei transformations, and with Newtonian gravity one can add accelerations in the form of time-dependent spatial translations to these transformations. The equations of motion for Newtonian gravity then take their simplest form when written in these coordinates. In General Relativity however one deals with the principle of *general covariance*, stating that the laws of physics are invariant under *general* coordinate transformations. As such the form of the field equations of General Relativity are the same for *all* observers. For Einstein this was an important step in developing his theory of General Relativity, because via the equivalence principle it implied the description of gravity in terms of differential geometry.

The second symmetry principle is that of *gauge invariance*, which first showed up in Maxwell’s theory of electromagnetism. There it was found that the electromagnetic field can be reformulated in terms of a spacetime vector potential. This vector potential how-

²Historically, supersymmetry was first introduced on the world-sheet to add fermionic degrees of freedom. That this world-sheet supersymmetry can be turned into supersymmetry in spacetime is highly non-trivial. Another formulation of Superstring Theory called the Green-Schwarz formulation starts from a manifestly spacetime-supersymmetric theory.

ever is not uniquely defined; one can add the spacetime gradient of a general function to it without changing the resulting electromagnetic field. In such a way infinitely many different vector potentials, all connected via so-called gauge transformations, result in the very same electromagnetic field. For the electromagnetic field such a gauge transformation can be regarded as the element of the symmetry group of the circle in an abstract, internal space. The theory can then be formulated on a spacetime, where at each point a circle is attached which is not part of spacetime itself.

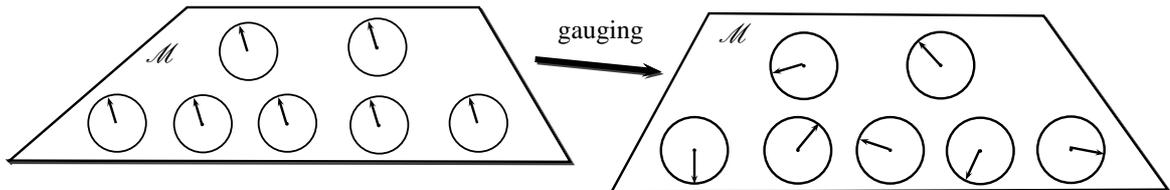


Figure 1.2: A pictorial representation of a gauging. The circles on the spacetime manifold \mathcal{M} represent an abstract space in which the fields transform. A global symmetry is one in which the field is rotated in the same way in every spacetime point. Gauging this symmetry makes it local, meaning that now the field is allowed to be rotated differently at every spacetime point.

Until now it seems that one cannot get around the vector potential if Quantum Mechanics and electromagnetism are unified; in nature it is found that matter is coupled to the vector potential, and not to the electric or magnetic field separately. The appearance of gauge symmetries in Maxwell's theory of electromagnetism was extended by Yang and Mills [13]. In Yang-Mills theories one promotes a global symmetry to a local one, i.e. symmetry transformations depending on the spacetime coordinates. While in electromagnetism the local symmetry was that of a circle, Yang and Mills enlarged the symmetry groups to those describing higher dimensional objects. The gauge principle then provides a clear and simple procedure how to couple matter to the different forces they experience. It seems that the fundamental subatomic interactions of nature can be very accurately described by these gauge theories: the Standard Model, which describes the different interactions between fundamental subatomic particles except for gravity, is formulated in terms of a Yang-Mills theory.

So, whereas the principle of *covariance* deals with spacetime symmetries, the principle of *gauge invariance* deals with symmetries in some abstract space attached at each point in spacetime. However, the so-called hole argument made clear to Einstein that the coordinate transformations of General Relativity must be regarded as what we now understand to be gauge transformations. Shortly after the development of Yang-Mills theories it was found that in close analogy, although not completely similarly, General Relativity can also be reformulated as a gauge theory [14, 15]. In this procedure the abstract, internal space at each point in the gauge theory must be related to the tangent space at that

point. Taking symmetries as a guideline in constructing theories, in the sixties some people wondered how much symmetry one could invoke in interacting quantum field theories without making the theory trivial. It turned out that the internal symmetries could not be mixed up with the symmetries of Special Relativity, a theorem which is now known as the O’Raifeartaigh-Coleman-Mandula theorem. An important assumption in this theorem is that the symmetries are generated by Lie algebras. A way to circumvent this no-go theorem is to go to so-called super Lie algebras, in which the symmetry parameters are Grassmann variables, making them fermionic instead of bosonic. The resulting symmetry is the earlier-mentioned supersymmetry. This symmetry relates bosons, which mediate the forces between matter, and fermions, which constitute the matter. One could thus say that supersymmetry removes the old dichotomy of matter and forces which permeates theories from Newtonian physics until the Standard Model! Applying a gauging procedure to the supersymmetry transformations results in an elegant way to obtain the simplest theory of Supergravity [16, 98].

Having seen the enormous role of gauge theories in modern physics, one could wonder to what extent gauge symmetries determine a set of field equations. To clarify the role of gauge symmetries, the photon in classical electrodynamics is given as an example. A photon has two polarizations. However, to describe a photon in a Lorentz-covariant way, the smallest representation giving room to these two polarization states is the vector representation. This representation has four components, giving two redundant degrees of freedom. Gauge symmetry allows one to get rid of these two degrees of freedom, and as such to describe the photon in a Lorentz-covariant way as a vector. Using gauge symmetries one is thus able to make spacetime symmetries, in this case those of Special Relativity, manifest. Something similar is true for Einstein’s field equations of General Relativity. The precise meaning of the principle of general covariance was not well-understood by Einstein when he launched his theory of General Relativity.³ Einstein originally gave too much credit to the notion of general covariance, as was pointed out by others soon after the publication of his field equations. Kretschmann observed that practically *any* field equation could be made invariant under general coordinate transformations [20], and as such also a theory of Newtonian gravity. Perhaps motivated by this remark Elie Cartan showed explicitly how to geometrize Newtonian gravity within the language of differential geometry only a few years later in 1923, and made the equations of motion for Newtonian gravity general-covariant instead of Galilei-covariant. Such an extension of spacetime covariance is more general: one can use a so-called Stückelberg trick to make a theory invariant under arbitrary gauge symmetries [21]. Such a trick consists of adding new fields to a theory to make the field equations invariant under the gauge transformations one wishes, which is

³See e.g. [18] or [19].

also the case for Newton-Cartan theory.⁴ The geometrization which Cartan used can thus be regarded as a Stückelberg trick, albeit a nontrivial one. This theory is now known as the theory of Newton-Cartan [22], and its framework forms the basis of this thesis.

1.3 Motivation: Why crawling back into the cave?

So why do we study Newtonian gravity and Newton-Cartan theory if Einstein came up with a theory which is more widely applicable? The first reason is that, although the theory of General Relativity conceptually and mathematically is very elegant, it is much more complicated than Newton's theory at the computational level. Since our own world is Newtonian, Newton's theory of gravity suffices in a lot of everyday applications.

A second reason is that one can gain insight into certain problems in General Relativity, which often become simpler because degrees of freedom decouple in the Newtonian limit and as such one can focus on a specific subsector of the relativistic theory. One example of this is cosmology [23], where for structure formation in the early universe one can turn to Newtonian approximations. Another example is the so-called cosmic no-hair theorem, which states that solutions of the Einstein equations with positive cosmological constant converge to the deSitter solution. This theorem is easier to analyze in the framework of Newton-Cartan cosmology [24].

A third reason for studying Newtonian gravity is the so-called AdS/CFT correspondence [25, 26], which allows one to reformulate a strongly coupled gravitational theory on Anti-de Sitter spacetimes as a weakly coupled field theory without gravity on the boundary of the spacetime, and vice versa. Some years ago this correspondence revived the interest in non-relativistic physics, because one can describe certain commonly encountered condensed matter systems via the AdS/CFT correspondence with solutions of gravitational theories exhibiting non-relativistic isometries [27, 28]. In most applications, the non-relativistic limit is taken on the field theory side, whereas it can be interesting to take the limit also on the gravity side in a covariant way, resulting in Newton-Cartan theory [90]. An explicit proposal for the resulting Newton-Cartan geometry in such a limit is the so-called Quantum Hall Effect [119], in which the AdS-space is replaced by flat space.⁵ However, in this thesis the emphasis will be on the construction of such theories,

⁴A simple example of the Stückelberg trick is given by the theory of a massless vector field exhibiting a $U(1)$ gauge symmetry. Adding a mass term to the Lagrangian explicitly breaks this gauge symmetry. One can then simply add a scalar field to the theory and restore the $U(1)$ gauge symmetry by assigning also a gauge transformation to the scalar field. In this way the $U(1)$ gauge symmetry has been restored by adding a scalar degree of freedom. From that perspective it would be more correct to speak of “gauge redundancies” instead of gauge symmetries!

⁵Such a flat space is considered to be the limit in which the AdS radius becomes infinite, giving a Minkowski background. This limit is important for realistic applications (our universe doesn't appear to

and not on applications in the AdS/CFT correspondence.

A fourth reason is that one can go one order beyond the Newtonian approximation called the “post-Newtonian approximation”. This approximation, which will not be discussed in this thesis, turns out to be remarkably effective even in regimes where the gravitational fields are strong and bodies are moving fast [32]. The reason for this effectiveness is not clear yet.

The fifth and last motivation can be summarized by a quote of Feynman [33]: “Psychologically we must keep all the theories in our heads, and every theoretical physicist who is any good knows six or seven different theoretical representations for exactly the same physics.” Newton-Cartan theory forces one to reconsider notions like general covariance, spacetime and gauge symmetries in general, and as such can deepen one’s understanding of General Relativity and gravity in general. Together, all these considerations motivate a better understanding of Newtonian gravity and Newton-Cartan theory.

On the other hand, one should be careful in using Newton-Cartan theory to draw lessons for General Relativity. One particular important problem at which one should be careful is quantum gravity. If one performs a Hamiltonian analysis of General Relativity, the Hamiltonian consists only of constraints and thus vanishes. This implies a “frozen” universe in which nothing changes in time. The reason for this vanishing Hamiltonian is the absence of absolute structures in General Relativity, and this problem is known as the “problem of time”. In Newton-Cartan theory one does not have this problem, because there is a preferred foliation of spacetime by the absolute time which characterizes Newtonian physics. Another problem one is facing in quantum gravity is that it is difficult to define observables. An intuitive reason for this is that in probing very small length scales, one needs a certain amount of energy, creating a black hole with an event horizon which is bigger than the probed spacetime (see e.g. [34]). But in Newton-Cartan one does not have black hole solutions.⁶ For this reason and others people have tried to quantize Newton-Cartan theory [5, 35], but it is unclear what such a theory of Newtonian quantum gravity means because the Newtonian limit involves by definition low energy scales, whereas quantum effects only play a role at high energy scales. Also, these Newtonian theories of gravity don’t have gravitational waves as solutions, which constitute the propagating degrees of freedom for relativistic gravitational theories. Of course, one could study the non-relativistic Schrödinger equation with a Newtonian gravitational potential. However, just as the analysis of the Hydrogen atom in ordinary Quantum Mechanics doesn’t teach one anything about Quantum Electrodynamics because ordinary

be AdS) but has its own subtleties, see e.g. [31].

⁶The notion of an event horizon is a relativistic aspect. One does have singularities due to e.g. point masses, just like in classical mechanics and field theories.

Quantum Mechanics is inherently non-relativistic and the electromagnetic field is treated as a classical background field instead of being quantized, an analysis of Quantum Mechanics coupled to Newtonian gravity is not shedding any light on a relativistic theory of quantum gravity. Combined with the earlier remarks that Newton-Cartan theory seems to lack the structures which makes quantizing gravity hard in the first place, one should not have too much hope to learn anything new about relativistic quantum gravity by quantizing Newton-Cartan theory.

1.4 Outline

Before we turn to the new insights into Newton-Cartan theory, which form the topic of this thesis, we will first review some topics to give the reader a solid background. In the second chapter some preliminaries are given about Galilean, Special and General Relativity, and the Newtonian limit of General Relativity is reviewed, as well as some Supersymmetry and Supergravity. In the third chapter both relativistic and non-relativistic particles, strings and branes are treated from the point of view of sigma models, and their symmetries are investigated. This third chapter ends the review of the necessary concepts. In the fourth chapter, which is based on [37], it is shown how Newtonian gravity can be obtained by gauging the so-called Bargmann algebra. This Bargmann algebra is a centrally-extended Galilei algebra, and this central extension plays a very important role in the gauging procedure. This procedure reproduces Newtonian gravity in the guise of the earlier mentioned Newton-Cartan theory. Some constraints which are rather ad-hoc in the traditional Newton-Cartan procedure are shown to follow from curvature constraints in the gauge theory. The gauging procedure, outlined in chapter 4, can be extended to theories of gravitating strings and branes, with or without a cosmological constant. This is done in two ways. The first is a bottom-up approach, in which one gauges the spatial translations to arrive directly at the class of so-called Galilean observers. The second one is a top-down approach, in which one gauges the extended stringy Galilei algebra and imposes constraints to arrive at the Galilei observers. In this procedure the central extension of the point particle algebra is replaced by a general extension, which again plays an important role in the resulting gravity theory. This is done in chapter 5, which is based on [38, 39], and gives “stringy” extensions of Newton-Cartan gravity. A supersymmetric extension of Newton-Cartan gravity in three dimensions will be addressed in chapter 6, which is based on [40]. Because the defence of this thesis has had some delay, developments which succeeded the research done in this thesis are also briefly mentioned. These developments, together with conclusions and an outlook, will be given in chapter 7.

Chapter 2

Relativity, gravity and symmetries

In this chapter some basic notions necessary for the following chapters are introduced. First, some Newtonian physics will be treated, along with the Galilean Relativity Principle. After that, Einstein's theory of Special and General Relativity will be touched, including the Newtonian limit. We will end this chapter with some preliminaries about supersymmetry. For a detailed treatment on General Relativity, see e.g. [41–44]. For details about differential geometry one can consult [45]. Supersymmetry is introduced in e.g. [46–49].

2.1 Galilean Relativity

In Newtonian mechanics space and time are decoupled. Newtonian space is a flat manifold \mathbb{R}^{D-1} , and time $x^0 = t$ is absolute. This absolute time means that once different observers have synchronized their clocks they will stay synchronized, regardless of their relative motion in space. This allows one to define a notion of D -dimensional Newtonian spacetime, which is foliated by the absolute time function t and where each foliation is just flat space \mathbb{R}^{D-1} . The laws of physics are then stated to be the same for the class of observers on which no forces act, the so-called inertial observers. This is the Galilean Relativity principle. The spatial coordinates $\{x^i\}$ and time coordinate $\{t\}$ of these inertial observers are connected via the Galilei group,¹

$$\begin{aligned}t' &= t + \zeta^0, \\x'^i &= A^i_j x^j + v^i t + \zeta^i.\end{aligned}\tag{2.1}$$

Here $\{\zeta^0, \zeta^i\}$ are constant temporal and spational translations, v^i is the Galilean boost parameter and $A^i_j \in SO(D-1)$ is a spatial rotation with an inverse denoted by A_j^i ,

$$A^i_j A_j^k = \delta_j^k.\tag{2.2}$$

¹For a detailed group-theoretical exposure of the Galilei group and the corresponding algebra, see [50].

In particular, note that the Galilean boost only involves the transformation of $\{x^i\}$. These Galilei transformations constitute the spacetime symmetries of Newtonian physics. In an inertial frame in Cartesian coordinates the motion of a free particle with trajectory $\{x^i(t)\}$ is then given by

$$\ddot{x}^i = 0, \quad (2.3)$$

where a dot denotes derivation with respect to t . The solution is a straight path in Newtonian space(time),

$$x^i(t) = w^i t + d^i, \quad \{w^i, d^i\} \in \mathbb{R}^{D-1}. \quad (2.4)$$

We can then regard (2.1) as the group of transformations connecting all these straight paths.

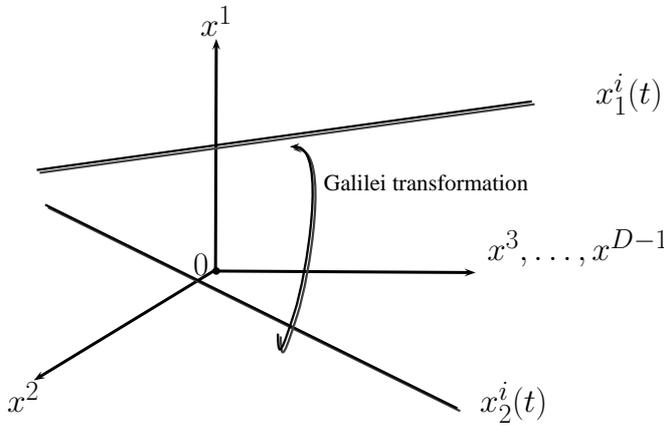


Figure 2.1: Galilei transformations as connections between straight lines $x_1^i(t)$ and $x_2^i(t)$ in \mathbb{R}^{D-1} .

So-called inertial or "fictitious" forces appear if one considers Newton's laws for accelerating observers. For example, if we consider a time-dependent rotation $A_j^i(t)$,

$$x'^i = A_j^i(t)x^j, \quad (2.5)$$

the equations of motion (2.3) in this accelerated frame become²

$$\ddot{x}^i + A_k^i \ddot{A}^k_j x^j + 2A_k^i \dot{A}^k_j \dot{x}^j = 0, \quad (2.6)$$

where the prime is dropped. Inertial forces are called as such because one can put them to zero by going to an inertial frame of reference, in this case a non-rotating one. The second term in eqn.(2.6) contains the centrifugal force, while the third term is the so-called Coriolis force.

²This can be compared to the usual expression in three spatial dimensions for an acceleration described in a rotating frame, $\mathbf{a}_{rot} = \mathbf{a}_{rest} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - \dot{\boldsymbol{\omega}} \times \mathbf{r} - 2\boldsymbol{\omega} \times \mathbf{v}$, where \mathbf{a} is the acceleration, \mathbf{r} is the position vector, \mathbf{v} is the velocity vector and $\boldsymbol{\omega}$ is the angular velocity vector.

2.2 Newtonian gravity

Newtonian gravity is an instantaneous force between gravitating masses. The Newtonian gravitational force $F(r)$ between two particles separated a distance r with gravitational masses M and m in $(D - 1)$ spatial dimensions is given by

$$F(r) = \frac{GMm}{r^{D-2}}. \quad (2.7)$$

Here G is Newton's constant, which we consider to be independent of the spacetime dimension D . Notice in particular the absence of any time-dependent factor, indicating that gravity is propagating with an infinite speed. If we write the path of a particle with inertial mass m in spherical coordinates with radial coordinate $r(t)$, the radial acceleration $\ddot{r}(t)$ due to a gravitational field caused by a mass M is given by

$$\begin{aligned} \ddot{r}(t) &= \frac{-GM}{r^{D-2}} \\ &\equiv -\frac{\partial\Phi}{\partial r}. \end{aligned} \quad (2.8)$$

Here $\Phi(r)$ is defined to be the Newtonian potential. Note that we divided out the mass m , which is possible because the inertial and gravitational mass of a particle are experimentally determined to be equal up to a very high accuracy. Now, the volume $V_d(R)$ of a d -dimensional sphere with radius R is given by³

$$\begin{aligned} V_d(R) &= \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})} R^d, \\ &\equiv S_d R^d, \quad S_d \equiv V_d(R=1), \end{aligned} \quad (2.9)$$

where we defined the volume of the d -dimensional unit sphere S_d . In particular, $V_2(R) = 4\pi R^2$ and $V_3(R) = \frac{4}{3}\pi R^3$. We can use this expression to integrate the relation (2.8) over a $(D - 2)$ -dimensional sphere ∂V with radius r , giving

$$\begin{aligned} \oint_{\partial V} \frac{\partial\Phi}{\partial r} d^{D-2}x &= \frac{GM}{r^{D-2}} V_{D-2}(r), \\ &= S_{D-2} GM, \end{aligned} \quad (2.10)$$

where on the right hand side the factors of r cancel. However, we can write the mass M as an integral over space,

$$M = \int_V \rho(x) d^{D-1}x, \quad (2.11)$$

where $\rho(x)$ is the mass density. Using Gauss' theorem

$$\oint_{\partial V} \frac{\partial\Phi}{\partial r} d^{D-2}x = \int_V r^{-2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial\Phi}{\partial r} \right) d^{D-1}x, \quad (2.12)$$

³The Gamma function $\Gamma(x)$ has the properties that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(x+1) = x\Gamma(x)$.

we see that, upon equating the integrands (2.11) and (2.12), eqn.(2.10) gives the Poisson equation in spherical coordinates,

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \Phi}{\partial r} \right) = S_{D-2} G \rho(r). \quad (2.13)$$

In Cartesian coordinates this equation reads

$$\Delta \Phi(x) = S_{D-2} G \rho(x), \quad (2.14)$$

where $\Delta \equiv \delta^{ij} \partial_i \partial_j$ is the spatial Laplacian. It is important to emphasize that the potential $\Phi(x)$ is a scalar under the Galilei group (2.1), but not under (time-dependent) accelerations. The equations of motion for a particle moving in this potential is then eqn.(2.8), which in Cartesian coordinates reads

$$\ddot{x}^i + \partial^i \Phi(x) = 0. \quad (2.15)$$

This equation and eqn.(2.14) are invariant under the Galilei group (2.1), but also under the additional transformations

$$x'^i = x^i + \xi^i(t), \quad \Phi'(x') = \Phi(x) - \delta_{ij} \ddot{\xi}^i(t) x^j. \quad (2.16)$$

Eqn.(2.16) means locally one can always use an acceleration $\ddot{\xi}^i$ to put $\Phi'(x') = 0$ and erase every appearance of gravity. Or the other way around: one can always redefine $\Phi(x)$ via eqn.(2.16) and change the acceleration of an observer in the gravitational field without changing the physics. Note that this is possible because inertial and gravitational mass are found to be equal, and that this mass is always positive. If mass could be both positive *and* negative, as for electric charges, we could simply flip the sign of the mass to see if we are dealing with a uniform gravitational field or an accelerating frame of reference. If inertial and gravitational masses weren't equal, gravity wouldn't couple to all masses in the same way. The transformations (2.16) hint to the idea that gravity can be regarded as an inertial force, just as the Coriolis force or centrifugal force in eqn.(2.6). This idea goes under the name of “the equivalence principle”, and was used by Einstein to its full extent in the theory of General Relativity. It is this property, the ability to make it disappear locally in spacetime, which makes gravity fundamentally different from the other forces in nature.

2.3 Special Relativity

In the theory of Special Relativity one focusses on the class of inertial observers. As in Galilean relativity these observers are postulated to be equivalent, which means that their experimental outcomes should agree with each other. On top of that it is postulated they will all measure the same speed of light c , a postulate which is motivated by the experiments of Michelson and Morley and the Maxwell equations. This postulate will

change the Galilei symmetries significantly when velocities approach c . The group of spacetime transformations connecting these inertial observers is called the Poincaré group,

$$x'^A = \Lambda^A_B x^B + \zeta^A, \quad (2.17)$$

where $\Lambda^A_B \in SO(D-1, 1)$ are Lorentz transformations and ζ^A are spacetime translations. These are global transformations, as the parameters do not depend on the spacetime coordinates, and form the symmetry group of the theory. The transformations (2.17) are derived by the demand that they keep the spacetime interval

$$\begin{aligned} ds^2 &= \eta_{AB} dx^A dx^B \\ &= -c^2 dt^2 + \delta_{ij} dx^i dx^j \end{aligned} \quad (2.18)$$

invariant. The Minkowski metric $\eta_{AB} = \text{diag}(-c^2, +1, \dots, +1)$ is then a non-degenerate metric on Minkowski spacetime obeying

$$\Lambda^C_A \Lambda^D_B \eta_{CD} = \eta_{AB}. \quad (2.19)$$

From the transformations (2.17) it can be seen that time is not absolute, as x'^0 is not equal to x^0 necessarily. Because gravity is a long range force which is always attractive, the simplest guess to incorporate it in Einstein's relativistic framework would be to introduce a massless Lorentz-scalar field, being the relativistic counterpart of the Newton potential $\Phi(x)$. However, such a theory has some observational problems; the deflection of light cannot be described, and the prediction of the precession of Mercury is also wrong.⁴ Gravity turns out to be more subtle.

2.4 General Relativity

The conceptual basis of the theory of General Relativity is the so-called equivalence principle, which was already mentioned in the context of Newtonian gravity, see eqn.(2.16). This principle can be stated as

Locally in spacetime, the laws of physics for freely-falling particles in a gravitational field are the same as those in a uniformly accelerating frame.

"Freely-falling" means there are no forces acting on the particle. This implies that, locally in spacetime, *every* observer can accelerate such that he/she doesn't experience gravity, and hence can use the theory of Special Relativity. This motivated Einstein to use the language of differential geometry to describe gravity. Namely, if spacetime is represented by a manifold, its curvature manifests itself only globally, like gravity. This makes the identification of gravity as spacetime-curvature plausible. In General Relativity the Minkowski

⁴See [43] for a detailed treatment of relativistic scalar-gravity theories.

metric η_{AB} is then replaced by a (non-degenerate) metric $g_{\mu\nu}(x)$ with Lorentzian signature, which has its own dynamics. The equivalence principle then always guarantees locally the existence of general coordinate transformations which transform $g_{\mu\nu}(x)$ into η_{AB} .⁵ With this the group of Poincaré transformations (2.17) is extended to the group of general coordinate transformations. Let's make this a bit more explicit.

The metric $g_{\mu\nu}(x)$ transforms under a general coordinate transformation $x^\rho \rightarrow x'^\rho(x^\mu)$ as

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\lambda}{\partial x'^\nu} g_{\rho\lambda}(x), \quad (2.20)$$

such that the line element $ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu$ is a scalar. This can be seen as merely a field redefinition of the tensor in new coordinates, and we say that the tensor transforms covariantly under general coordinate transformations.⁶ The metric is also invertible, and the inverse is written as $g^{\mu\nu}$:

$$g^{\mu\nu} g_{\nu\rho} = \delta_\rho^\mu. \quad (2.21)$$

Under the Poincaré transformations (2.17) the partial derivative ∂_μ transforms covariantly, but under general coordinate transformations it does not. As in gauge theories, see appendix B, this motivates the introduction of a covariant derivative ∇_μ which per construction transforms in a covariant way:

$$\begin{aligned} \nabla_\mu T_{\nu\rho\dots}^{\lambda\sigma\dots} &= \partial_\mu T_{\nu\rho\dots}^{\lambda\sigma\dots} + \Gamma_{\mu\theta}^\lambda T_{\nu\rho\dots}^{\theta\sigma\dots} + \Gamma_{\mu\theta}^\sigma T_{\nu\rho\dots}^{\lambda\theta\dots} + \dots \\ &\quad - \Gamma_{\mu\nu}^\theta T_{\theta\rho\dots}^{\lambda\sigma\dots} - \Gamma_{\mu\rho}^\theta T_{\nu\theta\dots}^{\lambda\sigma\dots} - \dots \end{aligned} \quad (2.22)$$

The connection components $\Gamma_{\mu\nu}^\rho$ then transform inhomogeneously,

$$\Gamma'_{\mu\nu}{}^\rho(x') = \frac{\partial x'^\rho}{\partial x^\lambda} \frac{\partial x^\sigma}{\partial x'^\mu} \frac{\partial x^\theta}{\partial x'^\nu} \Gamma_{\sigma\theta}^\lambda(x) + \frac{\partial x'^\rho}{\partial x^\lambda} \frac{\partial^2 x^\lambda}{\partial x'^\mu \partial x'^\nu}, \quad (2.23)$$

just as the gauge field of a Yang-Mills theory transforms inhomogeneously under the gauge transformations. Note that for the Poincaré transformations (2.17) the inhomogeneous term drops out of the transformation (2.23). The covariant derivative is per construction a linear derivative operator obeying the Leibnitz rule, and becomes a partial derivative on scalar fields. In General Relativity the connection $\Gamma_{\mu\nu}^\rho$ is usually *uniquely* determined by the following two constraints:

- Metric compatibility, $\nabla_\rho g_{\mu\nu} = 0$,
- Zero torsion, $\Gamma_{[\mu\nu]}^\rho = 0$.

⁵And also that the first derivative of the metric vanishes, whereas the second derivative does not.

⁶From the passive point of view, one went from one chart representing an observer with coordinates $\{x^\mu\}$, to another chart representing an observer with coordinates $\{x'^\rho(x^\mu)\}$. Both observers describe the metric at the same point on the spacetime manifold \mathcal{M} . From the active point of view we stay in the same coordinate chart representing an observer, and move the point (event) in spacetime. These two pictures are dual to each other, see e.g. appendix C.1 of [42].

The first constraint consists of $\frac{D^2(D+1)}{2}$ equations, whereas the second equation leaves $\frac{D^2(D+1)}{2}$ independent components for the connection. As such these two constraints imply that the connection is uniquely determined by the metric and we can say that the metric carries all the geometric information of the spacetime manifold. Because the metric is invertible, metric compatibility also implies $\nabla_\rho g^{\mu\nu} = 0$. The previously mentioned counting shows that the two constraints can be solved uniquely for $\Gamma_{\mu\nu}^\rho$ by writing down $\nabla_\rho g_{\mu\nu} - \nabla_\mu g_{\nu\rho} - \nabla_\nu g_{\rho\mu} = 0$, giving the so-called Levi-Civita connection

$$\Gamma_{\mu\nu}^\rho = \frac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}), \quad (2.24)$$

often denoted by the Christoffel symbols $\{\Gamma_{\mu\nu}^\rho\}$. Now, whereas partial derivatives commute, for covariant derivatives one can check that

$$[\nabla_\rho, \nabla_\sigma]V^\mu = R_{\nu\rho\sigma}^\mu(\Gamma)V^\nu \quad (2.25)$$

for any vector V^ν , where we have defined the Riemann tensor

$$R_{\nu\rho\sigma}^\mu(\Gamma) = \partial_\rho\Gamma_{\nu\sigma}^\mu - \partial_\sigma\Gamma_{\nu\rho}^\mu + \Gamma_{\nu\sigma}^\lambda\Gamma_{\lambda\rho}^\mu - \Gamma_{\nu\rho}^\lambda\Gamma_{\lambda\sigma}^\mu. \quad (2.26)$$

This tensor describes the curvature of a manifold with zero torsion and metric compatibility, and is completely determined by the metric. It obeys the following identities:

$$\begin{aligned} R_{\mu\nu\rho\sigma}(\Gamma) &= -R_{\mu\nu\sigma\rho}(\Gamma), & R_{\mu\nu\rho\sigma}(\Gamma) &= R_{\rho\sigma\mu\nu}(\Gamma), \\ R_{\mu[\nu\rho\sigma]}(\Gamma) &= 0, & \nabla_{[\lambda}R_{\mu\nu]\rho\sigma}(\Gamma) &= 0. \end{aligned} \quad (2.27)$$

The last identities are known as the Bianchi identities. Taking traces of the Riemann tensor gives the corresponding Ricci tensor $R_{\mu\nu}(\Gamma)$ and Ricci scalar $R(\Gamma)$,

$$R_{\mu\nu}(\Gamma) = R_{\mu\rho\nu}^\rho(\Gamma), \quad R(\Gamma) = g^{\mu\nu}R_{\mu\nu}(\Gamma). \quad (2.28)$$

Having defined these curvatures we can make the statement that gravity is a manifestation of spacetime curvature more precise. Originally Einstein derived the equations governing spacetime dynamics by ‘‘covariantizing’’ the Poisson equation (2.14). More specifically, he looked for a geometric reformulation of eqn.(2.14) which is invariant under general coordinate transformations by the equivalence principle. The tensorial extension of the mass density ρ in eqn.(2.14) is the energy-momentum 2-tensor $T_{\mu\nu}$, which is symmetric in its indices and obeys the covariant conservation of energy and momentum, $\nabla_\mu T^{\mu\nu} = 0$. The left hand side of eqn.(2.14), which is a second order differential equation in the Newton potential $\Phi(x)$, should then generalize to a symmetric 2-tensor $G_{\mu\nu}$ constructed out of the Riemann tensor,⁷ obeying $\nabla_\mu G^{\mu\nu} = 0$. This $G_{\mu\nu}$ can be found by using the Bianchi identities of (2.27), and the result $G_{\mu\nu} = R_{\mu\nu}(\Gamma) - \frac{1}{2}R(\Gamma)g_{\mu\nu}$ is known as the Einstein

⁷The Riemann tensor contains up to second order derivatives of the metric.

tensor. The most general expression which one can then find obeying these demands are the Einstein equations,

$$R_{\mu\nu}(\Gamma) - \frac{1}{2}R(\Gamma)g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu}. \quad (2.29)$$

The coupling constant κ^2 , depending on the spacetime dimension D , is determined via the correspondence principle mentioned in the Introduction by the Newtonian limit in section 2.7, and Λ is the notorious cosmological constant allowed due to metric compatibility. The Newtonian limit of eqn.(2.29) indeed gives the Poisson equation as will be checked later, supplemented by the cosmological constant Λ . In a more formal approach the vacuum equations of motion (2.29) with $T_{\mu\nu} = \Lambda = 0$ are derived from the action consisting of the simplest scalar density one can write down involving second order derivatives of the metric, namely

$$\mathcal{L}_{EH} = \sqrt{-g}R(\Gamma). \quad (2.30)$$

The corresponding action is called the Einstein-Hilbert action, which can be supplemented by a cosmological constant Λ . If matter is added via the matter Lagrangian \mathcal{L}_{matter} , one obtains finally

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{\kappa^2} R(\Gamma) - 2\Lambda + \mathcal{L}_{matter} \right). \quad (2.31)$$

The equations of motion are given by varying eqn.(2.31) with respect to the metric, giving indeed eqn(2.29). Now the energy-momentum tensor $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}, \quad (2.32)$$

and is covariantly conserved due to the invariance of the action (2.31) under general coordinate transformations, as can be easily shown. The Einstein equations (2.29) determine the dynamics of the spacetime, in which we can consider fields, particles, strings, branes etc. The path of a particle is determined by the postulate that particles move along geodesics, neglecting the back-reaction such a particle can have on the spacetime geometry. Such a geodesic is described by the equation

$$\ddot{x}^\rho + \Gamma_{\mu\nu}^\rho \dot{x}^\mu \dot{x}^\nu = 0, \quad (2.33)$$

where a dot denotes derivation with respect to the affine parameter τ and the connection $\Gamma_{\mu\nu}^\rho$ is the Levi-Civita connection (2.24).

This review of General Relativity is for general spacetime dimension D , but we mention two special cases. For $D = 2$ the Einstein-Hilbert action is a topological term, and the resulting Einstein equations $R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu}$ are trivially satisfied. This can be checked via the expression for the two-dimensional Riemann tensor:

$$D = 2 : R_{\mu\nu\rho\sigma} = Rg_{[\mu[\rho}g_{\sigma]\nu]}. \quad (2.34)$$

For $D = 3$ one can check that the number of independent components of the Riemann tensor, $\frac{1}{12}D^2(D^2 - 1)$, equals the number of independent components of the Ricci tensor, $\frac{1}{2}D(D + 1)$. As such the Riemann tensor is completely determined by the Ricci tensor,

$$D = 3 : \quad R_{\mu\nu\rho\sigma} = 4g_{[\mu[\rho}R_{\sigma]\nu]} - Rg_{[\mu[\rho}g_{\sigma]\nu]}, \quad (2.35)$$

and the vacuum equations then leave no room for gravitational waves. As we will see in section 2.7, this means that as a consequence the Newtonian limit in this case gives a world without gravitational interaction between point particles. This doesn't imply that General Relativity in three dimensions is completely trivial, see e.g. [51].

2.5 The hole argument

Because this thesis is all about describing gravity by gauge theories, here we briefly discuss the so-called hole argument. It was invented by a puzzled Einstein in order to show that general covariance is incompatible with determinism and to justify his temporary rejecting of general-covariant field equations. The puzzle was solved by interpreting the general coordinate transformations as gauge transformations [44, 107]. The argument makes the implications of general covariance clear, and shows that events in a spacetime do not have any physical meaning without the metric.

We will focus on the vacuum Einstein equations of General Relativity without cosmological constant, which determine the time-evolution of the metric in the absence of matter and energy:

$$G_{\mu\nu}[g_{\rho\lambda}(x)] = 0. \quad (2.36)$$

The metric is a tensor under general coordinate transformations $x^\mu \rightarrow x'^\mu(x^\nu)$, which is expressed by eqn.(2.20). This transformation can be regarded in the active sense: the coordinates $\{x\}$ and $\{x'(x)\}$ in eqn.(2.20) are defined in the same chart and as such refer to *different* points [42]. Under the general coordinate transformation (2.20) the Einstein equations (2.36) are covariant:

$$G'_{\mu\nu}[g'_{\rho\lambda}(x')] = 0. \quad (2.37)$$

Now imagine one has found a solution $g_{\mu\nu}(x)$ of (2.36). By covariance the transformed metric $g'_{\mu\nu}(x')$ can be constructed via the coordinate transformation (2.20), which solves eqn.(2.37). However, we can reset $\{x'\}$ in $g'_{\mu\nu}(x')$ to its old value $\{x\}$ giving $g'_{\mu\nu}(x)$, which also solves (2.37):

$$G'_{\mu\nu}[g'_{\rho\lambda}(x)] = 0. \quad (2.38)$$

The following question now arises: as $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x)$ seem to be two different metrics in the same coordinate system, what is the relation between them? If $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x)$ are physically different, general covariance allows one to construct an infinite amount of

physically new solutions $g'_{\mu\nu}(x)$ from $g_{\mu\nu}(x)$, but with the same initial data.

For Einstein it was tempting to think that $g'_{\mu\nu}(x)$ and $g_{\mu\nu}(x)$ are physically different, because they *look* different. So for the moment let's give in with this temptation and consider a spacetime manifold \mathcal{M} with a region $H \subset \mathcal{M}$ which is non-empty: $H \neq \emptyset$. The points of \mathcal{M} are interpreted as events. Now consider a general coordinate transformation, such that

- outside H one has $x^\mu = x'^\mu$,
- inside H one has $x^\mu \neq x'^\mu$,
- on the boundary of H these two transformations are smoothly connected.

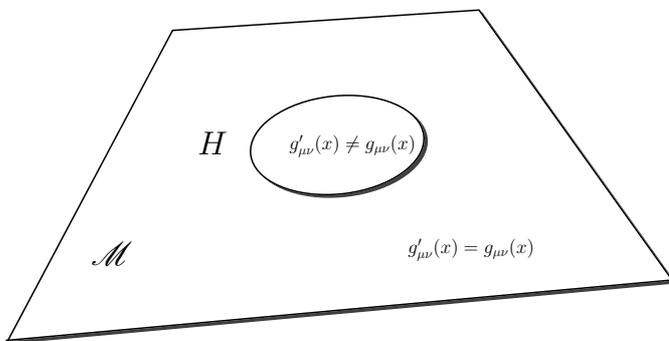


Figure D.1: The manifold \mathcal{M} with the hole H . The coordinate transformation shifts only the points inside the hole H . As such, $g'_{\mu\nu}(x) \neq g_{\mu\nu}(x)$ inside the hole, and $g'_{\mu\nu}(x) = g_{\mu\nu}(x)$ outside the hole.

As such the region H is called a “hole”. Note that this argument can only be made because the transformations involved are local.

The following subtlety then arised for Einstein: his equations describe the evolution of the metric, and a set of initial data should suffice to determine the metric $g_{\mu\nu}(x)$ uniquely through spacetime. Everything is fine outside the hole. But once the hole is entered, one can suddenly use covariance to obtain from the metric $g_{\mu\nu}(x)$ the mathematically different metric $g'_{\mu\nu}(x)$, as is shown in fig.(2.5). If these two metrics are also different physically, then covariance implies that the Einstein equations are not deterministic. Namely, the same initial data results in different solutions inside the hole.

The solution to us is clear: $g'_{\mu\nu}(x)$ and $g_{\mu\nu}(x)$ must be physically the same. One must conclude that *mathematically, points on a manifold can be distinguished without a metric, but physically they cannot.* Points (events) and their coordinates can only be physically interpreted *after* one introduces a metric, and as such a spacetime always consists of a

manifold \mathcal{M} equipped with a metric structure. But in the hole argument one tacitly assumes that the points, labeled by $\{x^\mu\}$ and $\{x'^\mu\}$, have a meaning *before* the metric is considered. This is deceiving and simply wrong. In this sense General Relativity must be regarded as a gauge theory. If we write the general coordinate transformation infinitesimally as $\delta x^\mu = \xi^\mu(x)$, one has the induced gauge transformation

$$\delta_\xi g_{\mu\nu}(x) \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x) = 2\nabla_{(\mu}\xi_{\nu)}. \quad (2.39)$$

Under this gauge transformation the vacuum Einstein equation $G_{\mu\nu} = 0$ is invariant.

Let's consider the Schwarzschild metric, being a solution to the vacuum Einstein equations (2.36), as an example [106]. In spherical coordinates $\{t, r, \Omega\} = \{t, r, \theta, \phi\}$ the space-time interval is written as⁸

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1}dr^2 + r^2d\Omega^2. \quad (2.40)$$

Then the following transformation is chosen:

$$\begin{aligned} t &\rightarrow t' = t, \\ r &\rightarrow r' = f^{-1}(r), \\ \Omega &\rightarrow \Omega' = \Omega, \end{aligned} \quad (2.41)$$

where the inverse is for notational convenience. The function $f^{-1}(r)$ has the following properties:

- $f^{-1}(r) = r$ outside H ,
- $f^{-1}(r) \neq r$ inside H ,
- on the boundary of H these two transformations are smoothly connected.

As such the hole H is defined only by spatial coordinate transformations. Under the transformation (2.41) the spacetime interval (2.40) becomes

$$ds'^2 = -\left(1 - \frac{2M}{f(r')}\right)dt^2 + \left(1 - \frac{2M}{f(r')}\right)^{-1}\left(\frac{\partial f}{\partial r'}\right)^2dr'^2 + f^2(r')d\Omega^2, \quad (2.42)$$

which by covariance equals ds^2 .⁹

⁸We take $G = c = 1$ for convenience.

⁹This is just a choice to keep the argument as simple as possible; one could of course also involve the time coordinate in the transformations.

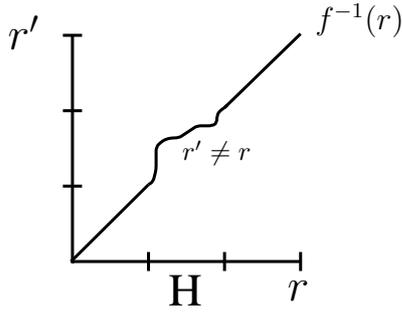


Figure D.2: The coordinate transformation which defines the hole H in spacetime [106].

Now choose $r' = r$ in eqn.(2.42) to get

$$d\tilde{s}^2 = -\left(1 - \frac{2M}{f(r)}\right)dt^2 + \left(1 - \frac{2M}{f(r)}\right)^{-1} \left(\frac{\partial f}{\partial r}\right)^2 dr^2 + f^2(r)d\Omega^2. \quad (2.43)$$

The spacetime interval (2.40) corresponds to $g_{\mu\nu}(x)$, whereas the spacetime interval (2.43) corresponds to $g'_{\mu\nu}(x)$. Comparison shows that they are mathematically different inside the hole,

$$ds^2 \neq d\tilde{s}^2 \quad r \in H. \quad (2.44)$$

If we now consider an event with coordinate r inside the hole H , we could naively think that for the metric with interval (2.40) the event is on a sphere with area $4\pi r^2$, while for (2.43) the event is on a sphere with area $4\pi f(r)^2$. Also, for (2.40) the horizon seems to be located at $r = 2M$, while for (2.43) the horizon is at $f(r) = 2M$. So the two metrics seem to give physically different predictions. However, as we saw, this reasoning is wrong. Only *after* writing the metric (2.43) we can interpret the coordinate $r' = f^{-1}(r)$ and the corresponding points on the manifold. The two metrics (2.40) and (2.43) must be associated to two diffeomorphic spacetime manifolds, describing the same physics. So, the moral of the story is:

*“Thou shalt not speculate about an event
before the metric is on hand.”*

Historically, we can conclude that Einstein was troubled because he didn't recognize the metric to be a gauge field under general coordinate transformations.

2.6 The vielbein formalism

Matter as we know it is described by fermions. These fermions are described by fields transforming under spinorial representations of the Lorentz group. These fields are not representations of the group of general coordinate transformations. To deal with these spinor fields one needs to introduce the so-called vielbein formulation. We will see that this vielbein gives at every spacetime point a map from spacetime to the tangent space,

in which one can define arbitrary representations of the Lorentz group. The vielbein formulation of General Relativity will also be crucial when we regard the theory as a gauge theory of the Poincaré group.

Up to now we have regarded tensors in a so-called coordinate basis $\{e_{(\mu)} = \partial_{\mu}\}$. However, we can also introduce a new set of basis vectors $\{e_{(A)}\}$ which is orthonormal. The relation between both bases is given by the so-called vielbein e_{μ}^A via $e_{(\mu)} = e_{\mu}^A e_{(A)}$. Because the set $\{e_{(A)}\}$ is chosen to be orthonormal, we have the relation

$$g^{\mu\nu} e_{\mu}^A e_{\nu}^B = \eta^{AB}. \quad (2.45)$$

Likewise, we can define the inverse of e_{μ}^A , denoted by e^{μ}_A . The vielbein and its inverse allow us to write

$$g_{\mu\nu} = e_{\mu}^A e_{\nu}^B \eta_{AB}, \quad e^{\mu}_A e^{\nu}_B g_{\mu\nu} = \eta_{AB}. \quad (2.46)$$

For this reason the vielbein is also called the 'square root of the metric'. Because e^{μ}_A is defined to be the inverse of e_{μ}^A , we also have the relations

$$e_{\mu}^A e^{\mu}_B = \delta_B^A, \quad e_{\mu}^A e^{\nu}_A = \delta_{\mu}^{\nu}. \quad (2.47)$$

Notice that, using the vielbein, we have defined a coordinate frame in which the metric locally looks flat. Physically, this frame corresponds to a freely-falling observer which does not experience the effects of gravity; that such a choice of frame is possible is guaranteed by the equivalence principle.

Now we are in a position to describe spinor fields in curved spacetime. Namely, the vielbeins are maps, defined at every point, from the spacetime manifold to the tangent space and vice versa. And it is in this tangent space that one can define the spinorial representations of the Lorentz group. If spacetime is flat, we can choose $e_{\mu}^A = \delta_{\mu}^A$ and $e^{\mu}_A = \delta^{\mu}_A$, such that the distinction between curved indices $\{\mu\}$ and flat indices $\{A\}$ vanishes. In general, curved indices on tensors can be converted into flat indices and vice versa via the vielbein. For example, the components of a vector can be rewritten via

$$e_{\mu}^A V^{\mu} = V^A, \quad e^{\mu}_B V^B = V^{\mu}. \quad (2.48)$$

Given a set of vielbeins $\{e_{\mu}^A\}$, one has the freedom of performing a local Lorentz transformation $\Lambda^A_B(x)$ without changing the metric $g_{\mu\nu}$, as is clear from (2.46) and (2.19):

$$e_{\mu}^{\prime A} = \Lambda^A_B(x) e_{\mu}^B. \quad (2.49)$$

These local Lorentz transformations $\Lambda^A_B(x)$, being elements of $SO(D-1, 1)$, have $\frac{1}{2}D(D-1)$ independent components, whereas the vielbein has D^2 components, leaving $\frac{1}{2}D(D+1)$ independent components for the metric $g_{\mu\nu}(x)$. This is the right number for a

symmetric two-tensor.

If we consider a tensor $T_{CD\dots}^{AB\dots}$ in the tangent space at a point, the partial derivative $\partial_\mu T_{CD\dots}^{AB\dots}$ does not transform homogeneously under the group of local Lorentz transformations. We again introduce a connection $\omega_\mu^A{}_B$, but this time in the tangent space, which is called the spin connection:

$$\begin{aligned} \nabla_\mu T_{CD\dots}^{AB\dots} = & \partial_\mu T_{CD\dots}^{AB\dots} - \omega_\mu^A{}_F T_{CD\dots}^{FB\dots} - \omega_\mu^B{}_F T_{CD\dots}^{AF\dots} - \dots \\ & + \omega_\mu^F{}_C T_{FD\dots}^{AB\dots} + \omega_\mu^F{}_D T_{CF\dots}^{AB\dots} + \dots \end{aligned} \quad (2.50)$$

We can also consider the covariant derivative of tensors with both flat and curved indices. The rule is that for every curved index one gets a Levi-Civita connection $\Gamma_{\mu\nu}^\rho$, whereas for every flat index one gets a spin connection $\omega_\mu^A{}_B$. But a tensor should not depend on our choice of basis. If we write e.g. the covariant derivative of a vector in both an orthonormal basis and a coordinate basis and demand that they are equal, we obtain the so-called vielbein postulate¹⁰

$$\nabla_\mu e_\nu^A = \partial_\mu e_\nu^A - \Gamma_{\mu\nu}^\rho e_\rho^A - \omega_\mu^A{}_B e_\nu^B = 0. \quad (2.51)$$

Metric compatibility $\nabla_\rho g_{\mu\nu} = 0$ and the vielbein postulate together imply that the spin connection is antisymmetric in $\{AB\}$. Note that from now on we won't care anymore about the position of the flat indices $\{A, B, C, \dots\}$, and simply write ω_μ^{AB} . The vielbein postulate (2.51) can be uniquely solved for $\Gamma_{(\mu\nu)}^\rho$ in terms of the vielbein and spin connection:

$$\Gamma_{(\mu\nu)}^\rho = e^\rho{}_A \left(\partial_{(\mu} e_{\nu)}^A - \omega_{(\mu}{}^{AB} e_{\nu)}^B \right). \quad (2.52)$$

The zero-torsion condition $\Gamma_{[\mu\nu]}^\rho = 0$ then gives the additional constraint

$$\partial_{[\mu} e_{\nu]}^A - \omega_{[\mu}{}^{AB} e_{\nu]}^B \equiv R_{\mu\nu}^A = 0. \quad (2.53)$$

Here we defined $R_{\mu\nu}^A = e_\rho^A \Gamma_{[\mu\nu]}^\rho$ for future convenience. The spin connection has as many independent components as $R_{\mu\nu}^A$, namely $\frac{1}{2}D^2(D-1)$, and the spin connection only appears algebraically in $R_{\mu\nu}^A$ multiplied by vielbeins. This means we can solve (2.53) uniquely for the spin connection. Writing

$$R_{\mu\nu}^A e_\rho^A + R_{\rho\mu}^A e_\nu^A - R_{\nu\rho}^A e_\mu^A = 0, \quad (2.54)$$

we obtain the solution

$$\omega_\mu^{AB}(e, \partial e) = 2e^{\lambda[A} \partial_{[\lambda} e_{\mu]}^{B]} + e_\mu^C e^{\lambda A} e^{\rho B} \partial_{[\lambda} e_{\rho]}^C. \quad (2.55)$$

If we write an infinitesimal local Lorentz transformation as $\Lambda^A{}_B = \delta_B^A + \lambda^A{}_B$, where $\lambda_{AB} = \lambda_{[AB]}$ because of (2.19), the spin connection transforms accordingly to

$$\delta\omega_\mu^{AB} = \partial_\mu \lambda^{AB} + 2\lambda^{C[A} \omega_\mu^{B]C}. \quad (2.56)$$

¹⁰For an explicit derivation of this result, see e.g. [41].

This follows from the explicit solution (2.55) via $\delta e_\mu^A = \lambda^A{}_{B\mu} e_\mu^B$.

We conclude this section by coming back to the claim that the vielbein formalism allows us to define spinorial fields in curved spacetime. If the $\frac{1}{2}D(D-1)$ matrices¹¹ $\{\frac{1}{2}\Gamma_{AB}\}$ form the spin- $\frac{1}{2}$ representation of the Lorentz algebra, a spinor ψ transforms as

$$\delta\psi = \frac{1}{4}\lambda^{AB}\Gamma_{AB}\psi. \quad (2.57)$$

The covariant derivative on a spinor field $\psi(x)$ is then defined via

$$\nabla_\mu\psi = \partial_\mu\psi - \frac{1}{4}\omega_\mu{}^{AB}\Gamma_{AB}\psi, \quad (2.58)$$

which we will denote by $D_\mu\psi$, such that $D_\mu\psi$ is covariant with respect to local Lorentz transformations. Of course, this can also be done for other representations of the Lorentz algebra. For instance, the covariant derivative on the vector-spinor $\psi_\mu(x)$ is defined as

$$\begin{aligned} \nabla_\mu\psi_\nu &= \partial_\mu\psi_\nu - \frac{1}{4}\omega_\mu{}^{AB}\Gamma_{AB}\psi_\nu - \Gamma_{\mu\nu}^\lambda\psi_\lambda \\ &= D_\mu\psi_\nu - \Gamma_{\mu\nu}^\lambda\psi_\lambda. \end{aligned} \quad (2.59)$$

One can check that these covariant derivatives indeed transform tensorially under both general coordinate transformations and the local Lorentz transformations.

2.7 The Newtonian limit of General Relativity

As was noted in the Introduction, the correspondence principle dictates that under certain conditions the theory of General Relativity should reproduce Newton's theory of gravity. These certain conditions are known as the Newtonian limit, and for the point particle with embedding coordinates $\{x^\mu(\tau)\} = \{x^0(\tau) = ct, x^i(\tau)\}$ in a Minkowski background this limit is defined by three requirements:

- (1) $\dot{x}^0 \gg \dot{x}^i$,
- (2) $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon f_{\mu\nu}$ with $\epsilon \ll 1$,
- (3) $g_{0j} = 0$ and $g_{\mu\nu} = g_{\mu\nu}(x^i)$.

The first requirement means that the longitudinal “velocity” is much larger than the transverse velocity, and captures the non-relativistic limit: the speed $|v|$ of the particle is small compared to the speed of light c , which we keep explicitly. Effectively this means that one typically has $\mathcal{O}(\epsilon) = \mathcal{O}(\frac{v^2}{c^2})$. The second requirement means that gravity is weak, such that we can expand the metric around the Minkowski vacuum $\eta_{\mu\nu}$ and only work at first order in the perturbation $f_{\mu\nu}$, or $\mathcal{O}(\epsilon)$. The third requirement means that the line

¹¹Spinor indices have been suppressed. The form of the matrices $\{\frac{1}{2}\Gamma_{AB}\}$ is given in appendix A.

element is invariant under the transformation $x^0 \rightarrow -x^0$, which corresponds to a static¹² gravitational field.

First we consider the Newtonian limit for the geometry. The requirement of weak curvature reads

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon f_{\mu\nu}, \quad g^{\mu\nu} = \eta^{\mu\nu} - \epsilon f^{\mu\nu}, \quad \epsilon \ll 1, \quad (2.60)$$

such that $g_{\mu\nu}g^{\nu\rho} = \delta_{\mu}^{\rho} + \mathcal{O}(\epsilon^2)$. From now on we understand that all the objects we are writing down are of order $\mathcal{O}(\epsilon)$ and omit the expansion parameter ϵ . The expansion (2.60) gives us for the connection and Ricci tensor

$$\begin{aligned} \Gamma_{\mu\nu}^{\rho} &= \frac{1}{2}\eta^{\rho\sigma} \left(\partial_{\mu}f_{\nu\sigma} + \partial_{\nu}f_{\mu\sigma} - \partial_{\sigma}f_{\mu\nu} \right), \\ R_{\mu\nu} &= \partial_{\sigma}\Gamma_{\mu\nu}^{\sigma} - \partial_{\mu}\Gamma_{\sigma\nu}^{\sigma}. \end{aligned} \quad (2.61)$$

This implies that the Einstein tensor becomes¹³

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}R\eta_{\mu\nu} \\ &= \partial^{\sigma}\partial_{(\mu}f_{\nu)\sigma} - \frac{1}{2}\partial^2 f_{\mu\nu} - \frac{1}{2}\partial_{\mu}\partial_{\nu}f - \frac{1}{2}\eta_{\mu\nu}(\partial^{\sigma}\partial^{\rho}f_{\sigma\rho} - \partial^2 f). \end{aligned} \quad (2.62)$$

Now, to further simplify the linearized Einstein equations (2.62) we first define

$$\bar{f}_{\mu\nu} \equiv f_{\mu\nu} - \frac{1}{2}f\eta_{\mu\nu}. \quad (2.63)$$

With this definition the linearized Einstein equations become

$$-\frac{1}{2}\partial^2\bar{f}_{\mu\nu} + \partial^{\sigma}\partial_{(\mu}\bar{f}_{\nu)\sigma} - \frac{1}{2}\eta_{\mu\nu}\partial^{\sigma}\partial^{\rho}\bar{f}_{\sigma\rho} = \kappa^2 T_{\mu\nu}. \quad (2.64)$$

These equations can even be more simplified by using an infinitesimal coordinate transformation $x^{\mu} \rightarrow x'^{\mu} = x^{\mu} - \xi^{\mu}(x)$, which transforms the metric into

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x) = g_{\mu\nu}(x) + 2\partial_{(\mu}\xi_{\nu)}(x), \quad (2.65)$$

such that¹⁴

$$\begin{aligned} f &\rightarrow f' = f + 2\partial^{\lambda}\xi_{\lambda}, \\ \bar{f}_{\mu\nu} &\rightarrow \bar{f}'_{\mu\nu} = \bar{f}_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)} - \eta_{\mu\nu}\partial^{\lambda}\xi_{\lambda}, \\ \partial^{\mu}\bar{f}_{\mu\nu} &\rightarrow \partial'^{\mu}\bar{f}'_{\mu\nu} = \partial^{\mu}\bar{f}_{\mu\nu} + \partial^2\xi_{\nu}. \end{aligned} \quad (2.66)$$

¹²A *stationary* spacetime admits a timelike Killing vector field, which means that one can always find a coordinate system in which the metric is time-independent. A *static* spacetime has the same properties, plus the additional requirement that $g_{j0}(x) = 0$.

¹³Note that the trace $f = \eta^{\mu\nu}f_{\mu\nu} + \mathcal{O}(\epsilon^2)$ is taken with respect to the Minkowski metric. We denote the D -dimensional spacetime Laplacian $\partial^2 = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}$, whereas the spatial Laplacian is denoted by $\Delta = \delta^{ij}\partial_i\partial_j$.

¹⁴The partial derivative is transformed as $\partial'_{\mu} = \partial_{\mu} + \partial_{\mu}\xi^{\lambda}\partial_{\lambda}$, but because ξ^{μ} is infinitesimal and $f_{\mu\nu}$ is considered to be a perturbation, they can both be considered to be of $\mathcal{O}(\epsilon)$. This gives that at order $\mathcal{O}(\epsilon)$ one has $\partial'^{\mu}\bar{f}'_{\mu\nu} = \partial^{\mu}\bar{f}_{\mu\nu}$. This also explains why the covariant derivative becomes a partial derivative in the transformation (2.65).

According to the last transformation one can obtain $\partial^\mu \bar{f}'_{\mu\nu} = 0$ in the new coordinate system by solving the equation

$$\partial^\mu \bar{f}_{\mu\nu} = -\partial^2 \xi_\nu \quad (2.67)$$

for the parameter ξ_ν . As is common in gauge theories, this doesn't completely fix the parameter ξ_ν ; one can still perform gauge transformations which obey the harmonic condition $\partial^2 \xi_\nu = 0$. Dropping the primes, the linearized Einstein equations (2.64) then become quite simple in this particular coordinate system,

$$\partial^2 \bar{f}_{\mu\nu} = -2\kappa^2 T_{\mu\nu}. \quad (2.68)$$

Eqn.(2.68) is the linear approximation of the Einstein equations in the gauge-choice (2.67), and expresses the weak curvature approximation on the geometry.

Now we consider the Newtonian limit for the matter and energy distribution. For that we take as energy momentum tensor one of non-interacting matter particles following worldlines $\dot{x}^\mu(\tau)$, or so-called dust. Such a dust is characterized by a proper matter density $\rho(x^\mu)$ and velocity $\dot{x}^\mu(\tau)$. The energy momentum tensor reads

$$T^{\mu\nu} = \rho(x^\mu) \dot{x}^\mu \dot{x}^\nu. \quad (2.69)$$

In the non-relativistic limit $v \ll c$ one has $T_{00} = \rho(x^\mu)c^2$ showing therefore its interpretation as an energy density, whereas the other components vanish, $T_{i0} = T_{ij} = 0$. The static approximation states that effectively the matter density $\rho = \rho(x^i)$ does not depend on time, and so $\partial_0 f_{\mu\nu} = 0$. Plugging these two approximations into (2.68) gives

$$\Delta \bar{f}_{00} = -2\kappa^2 c^2 \rho, \quad (2.70)$$

$$\Delta \bar{f}_{kl} = 0. \quad (2.71)$$

If we demand that \bar{f}_{kl} vanishes at spatial infinity, then the solution of the second equation is

$$\bar{f}_{kl} = 0. \quad (2.72)$$

This gives for the trace \bar{f}

$$\bar{f} = \eta^{\mu\nu} \bar{f}_{\mu\nu} = -c^{-2} \bar{f}_{00}. \quad (2.73)$$

Comparing (2.70) with the Poisson equation (2.14) relates \bar{f}_{00} and the Newton potential $\phi(x)$:

$$\bar{f}_{00} = \frac{-2\kappa^2 c^2}{G S_{D-2}} \phi(x). \quad (2.74)$$

Using that eqn.(2.63) implies that $\bar{f} = \left(\frac{2-D}{2}\right)f$, one can write

$$f_{\mu\nu} = \bar{f}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu} f = \bar{f}_{\mu\nu} + \frac{1}{(2-D)}\bar{f}\eta_{\mu\nu}. \quad (2.75)$$

From this expression it is clear that $D \neq 2$, which is what we will assume in what follows. From eqn.2.75 we obtain the components

$$f_{00} = \frac{3-D}{2-D} \bar{f}_{00} = -\frac{2\kappa^2 c^2 \phi}{G S_{D-2}} \frac{3-D}{2-D}, \quad (2.76)$$

$$f_{ij} = -\frac{\bar{f}_{00}}{2-D} \delta_{ij} = \frac{2\kappa^2 c^2 \phi}{G(2-D) S_{D-2}} \delta_{ij}, \quad (2.77)$$

Note that in the static and weak field approximation space is still curved, $f_{ij} \neq 0$.¹⁵ Now we turn to the geodesic equation. We will see that the assumption of non-relativistic velocities imposes that only the term with Γ_{00}^μ enters the geodesic equation. So even though space is curved, due to the non-relativistic velocities of the test particles these particles will not experience spatial curvature; all the terms containing \dot{x}^i -terms are beyond order $\mathcal{O}(\epsilon)$. Also, the assumption of static gravitational fields, $\partial_0 f_{\mu\nu} = 0$, kicks out the time derivative of the metric in Γ_{00}^μ and puts $\Gamma_{00}^0 = 0$. To be more concrete, in the non-relativistic limit $\dot{x}^\mu(\tau) \approx (c, \bar{0})$, the geodesic equation becomes

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{00}^\mu \frac{dx^0}{d\tau} \frac{dx^0}{d\tau} = 0. \quad (2.78)$$

The Christoffel symbol Γ_{00}^μ reads

$$\Gamma_{00}^\mu = -\frac{1}{2} \eta^{\mu\rho} \partial_\rho f_{00}. \quad (2.79)$$

Because $\Gamma_{00}^0 = 0$ as the gravitational field is static, the equations of motion for $\{x^0\}$ read $\ddot{x}^0 = 0$. This allows for the (global) choice of static gauge $t = \tau$. Using eqn.(2.76), one can write

$$\Gamma_{00}^i = \frac{2\kappa^2 c^2}{G S_{D-2}} \frac{D-3}{D-2} \delta^{ij} \partial_j \phi. \quad (2.80)$$

With this the geodesic equation reduces to

$$\frac{d^2 x^i}{dt^2} + \frac{2\kappa^2 c^4}{G S_{D-2}} \frac{D-3}{D-2} \delta^{ij} \partial_j \phi = 0. \quad (2.81)$$

Now, according to the correspondence principle eqn.(2.81) should reduce to $\ddot{x}^i + \partial^i \phi = 0$. Using (2.9), the coupling constant κ^2 for $D > 3$ must then be given by

$$\kappa^2 = \frac{D-2}{D-3} \frac{2\pi^{\frac{D-3}{2}} G}{\Gamma(\frac{D-1}{2}) c^4}. \quad (2.82)$$

E.g, $\kappa_4^2 = \frac{8\pi G}{c^4}$ and $\kappa_5^2 = \frac{3\pi^2 G}{2c^4}$. From eqn.(2.81) it becomes clear that for $D = 3$ particles are not influenced by gravity, which expresses the fact mentioned in section 2.4 that General Relativity for $D = 3$ does not have propagating degrees of freedom. So we can write

¹⁵This can be checked by taking the weak field limit of the Schwarzschild line element. Defining the Newtonian potential as $\phi(r) = -\frac{GM}{r}$ sourced by a mass M , the weak field limit of the Schwarzschild line element becomes $ds^2 = -(c^2 + 2\phi)dt^2 + \left(1 - \frac{2\phi}{c^2}\right)dr^2 + r^2 d\Omega^2$. Notice that the potential in the g_{rr} -component is suppressed by a factor of c^2 compared to the g_{tt} -component.

down Newtonian gravity for $D = 3$, but it cannot be considered to be the Newtonian limit of General Relativity in three spacetime dimensions. The case $D = 2$ is not considered because the Einstein equations are trivially satisfied; the Einstein-Hilbert action is for $D = 2$ a topological term, see eqn.(2.34).

Having discussed the Newtonian limit of General Relativity, some comments should be made. First, from the expression $\Gamma_{00}^i = \partial^i \phi$ and the transformation (2.23) one can deduce how ϕ transforms under an infinitesimal coordinate transformation $\delta x^0 = 0, \delta x^i = \xi^i(t)$ in the static gauge, namely

$$\delta\phi(x) = \xi^i \partial_i \phi - x_i \xi^{ii}, \quad (2.83)$$

which is indeed the infinitesimal form of the transformation (2.16). Second, we consider a massive particle with mass $m \ll M$ moving in a gravitational potential due to a mass M , where the distance between m and M is r . In the Newtonian regime one then typically has

$$mv^2 \sim \frac{GMm}{r}, \quad (2.84)$$

stating that the kinetic energy of the particle is of the same order as the gravitational potential. If we denote the Schwarzschild radius by $r_s = \frac{2GM}{c^2}$, eqn.(2.84) implies that

$$\frac{v^2}{c^2} \sim \frac{r_s}{r}. \quad (2.85)$$

So we see that, keeping G fixed, we could interpret the expansion parameter ϵ in $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon f_{\mu\nu}$ as $\epsilon = \kappa^2$, and regard the Newtonian limit as an expansion in c^{-2} .¹⁶ Third, the Newtonian limit depends heavily on the fact that one is considering point particles. We will see in chapter 5 how to define a non-relativistic theory in which the Newtonian potential ϕ is being replaced by a tensor potential depending on what kind of objects one is considering in the gravitational field (particles, strings, branes, etc.). This will not be done by taking a non-relativistic limit at the level of equations of motion. Instead, we will take a so-called Inönü-Wigner contraction [54] on the symmetry algebra of the relativistic theory without gravity, and apply a gauging procedure in order to obtain a non-relativistic theory of gravity. Fourth, because one takes the Newtonian limit on the level of the field equations a natural question to ask is what happens to the spacetime geometry. After being exposed to General Relativity, it seems unnatural to leave the language of differential geometry, because it is so succesful in describing gravity in the framework of Special Relativity. There is no reason to believe that differential geometry cannot be used for Galilean Relativity. In the next section we will discuss the metric structure of classical mechanics, which will be a first step in reformulating Newtonian gravity into a metric theory, called Newton-Cartan. This structure is obtained in the gauging procedure just mentioned. This Newton-Cartan theory can also be obtained by performing the Newtonian limit on General Relativity in which c^{-2} is treated as the expansion parameter [52].

¹⁶Note that all the time derivatives ∂_0 are of order $\mathcal{O}(c^{-1})$.

2.8 Galilean metrical structure

Looking at the theory of Special Relativity one can wonder which spacetime interval is invariant under the Galilei group (2.1), giving the Newtonian spacetime a metrical structure. To answer this question we rewrite the Galilean transformations (2.1) as

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + \zeta^\mu, \quad (2.86)$$

where now $\Lambda^\mu{}_\nu$ is *not* an element of the Lorentz group, but

$$\Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu} = \begin{pmatrix} \frac{\partial x'^0}{\partial x^0} & \frac{\partial x'^0}{\partial x^i} \\ \frac{\partial x'^i}{\partial x^0} & \frac{\partial x'^i}{\partial x^j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & A \end{pmatrix}.$$

A Galilei-invariant contravariant metric $g^{\mu\nu}$ should obey

$$\Lambda^\mu{}_\rho \Lambda^\nu{}_\lambda g^{\rho\lambda} = g^{\mu\nu}. \quad (2.87)$$

This gives the restrictions

$$\begin{aligned} g^{ij} &= v^i v^j g^{00} + 2v^{(i} A^{j)}{}_k g^{0k} + A^i{}_k A^j{}_l g^{kl}, \\ g^{0j} &= v^j g^{00} + A^j{}_k g^{0k}, \end{aligned} \quad (2.88)$$

which for general rotations and boosts are solved by

$$g^{\mu 0} = 0, \quad g^{ij} = \delta^{ij}. \quad (2.89)$$

To investigate the Galilei-invariance of a covariant metric $\tilde{g}_{\mu\nu}$, where the tilde stresses the fact that we cannot regard this metric as the inverse of $g^{\mu\nu}$ since this inverse is not uniquely defined, we identify the inverse¹⁷ $\Lambda_\nu{}^\mu$ of $\Lambda^\mu{}_\nu$ as

$$\Lambda_\nu{}^\mu = \frac{\partial x^\mu}{\partial x'^\nu} = \begin{pmatrix} \frac{\partial x^0}{\partial x'^0} & \frac{\partial x^0}{\partial x'^i} \\ \frac{\partial x^i}{\partial x'^0} & \frac{\partial x^i}{\partial x'^j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -A^{-1}v & A^{-1} \end{pmatrix}.$$

Galilei-invariance of $\tilde{g}_{\mu\nu}$ is now expressed as

$$\Lambda_\rho{}^\mu \Lambda_\lambda{}^\nu \tilde{g}_{\mu\nu} = \tilde{g}_{\rho\lambda}. \quad (2.90)$$

This equation gives the constraints

$$\begin{aligned} \tilde{g}_{00} &= \tilde{g}_{00} - A_k{}^i v^k A_l{}^j v^l \tilde{g}_{ij} - A_k{}^i v^k \tilde{g}_{i0}, \\ \tilde{g}_{i0} &= A_i{}^k \tilde{g}_{k0} - A_i{}^k A_m{}^l v^m \tilde{g}_{kl}, \\ \tilde{g}_{ij} &= A_i{}^k A_j{}^l \tilde{g}_{kl}. \end{aligned} \quad (2.91)$$

These constraints are solved by

$$\tilde{g}_{i0} = \tilde{g}_{ij} = 0. \quad (2.92)$$

¹⁷Notice that we use relativistic notation, $\Lambda_\nu{}^\mu \equiv [\Lambda^\mu{}_\nu]^{-1}$.

and leave only \tilde{g}_{00} as constant, nonzero entry. We now rename these metrics as

$$\tilde{g}_{\mu\nu} \equiv \tau_{\mu\nu}, \quad g^{\mu\nu} \equiv h^{\mu\nu}. \quad (2.93)$$

Scaling $\tau_{00} = 1$, the two metrics which are Galilei-invariant then for $D = 4$ read explicitly

$$\tau_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad h^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As such we can only assign Galilei-invariant lengths to spatial separations via $h^{\mu\nu}$ or to temporal separations via $\tau_{\mu\nu}$. It is now clear that the Galilei group keeps invariant two separate metrics $h^{\mu\nu}$ and $\tau_{\mu\nu}$ which are degenerate:¹⁸

$$h^{\mu\nu}\tau_{\nu\rho} = 0. \quad (2.94)$$

Having a metric structure, one can wonder whether it would be possible to describe Newtonian gravity as a metric theory, as in General Relativity. This degenerate metric structure will indeed be the starting point of the theory of Newton-Cartan, which will be considered in chapter 4.

2.9 Supersymmetry and Supergravity

As was mentioned in the Introduction, supersymmetry, which we will abbreviate as SUSY, is a symmetry which relates bosons and fermions. In non-supersymmetric quantum field theories both the “internal” symmetries, or gauge symmetries, and the spacetime symmetries are generated by elements of a Lie algebra which are bosonic. The O’Raifeartaigh-Coleman-Mandula theorem states that for interacting theories in more than two spacetime dimensions these two classes of symmetries don’t talk to each other, i.e. they commute. The intuitive reason for this is that enlarging the symmetries by making internal and spacetime symmetries non-commuting would constrain a field theory in such a way that the individual momenta of particles would be conserved instead of the total momentum, and hence there is no scattering. As such the *total* symmetry algebra of an ordinary quantum field theory can be written as a direct sum of the Poincaré algebra and the gauge algebra. The so-called Haag-Lopuszanski-Sohnius theorem [53] shows that the symmetries of four dimensional quantum field theories can be extended with fermionic generators, giving the SUSY algebra. This algebra is a so-called \mathbb{Z}_2 -graded algebra, meaning that for bosonic

¹⁸In chapter 3 we will see that we can take a particular combination of the Newton-Cartan metrics and a vector field m_μ , namely $h_{\mu\nu} - 2m_{(\mu}\tau_{\nu)}$, which is also invariant under boosts. The vector field is the gauge field of the central extension explained in section 3.4.

generators B and fermionic generators F one has schematically

$$\begin{aligned} [B, B] &= \{F, F\} = B, \\ [B, F] &= F. \end{aligned} \tag{2.95}$$

Here $\{, \}$ denotes an anticommutator. Such an algebra is also called a super-Lie algebra, and circumvents the O’Raifeartaigh-Coleman-Mandula theorem. In practice one can define interacting supersymmetric field theories for any spacetime dimension $D = 1, \dots, 11$; higher dimensions forces one to consider multiplets with spin higher than two, which we discuss at the end of this section. Here we will focus on the case of $D = 4$. The algebra defining so-called minimal or $\mathcal{N} = 1$ SUSY is an extension of the algebra which generates the Poincaré transformations (2.17), called the super-Poincaré algebra. Besides Lorentz transformations and spacetime translations, the algebra also includes supertransformations. It reads¹⁹

$$\begin{aligned} [P_A, P_B] &= 0, & [P_A, Q] &= 0, \\ [M_{BC}, P_A] &= -2\eta_{A[B}P_{C]}, & \{Q, Q\} &= -\frac{1}{2}\Gamma^A C^{-1}P_A, \\ [M_{CD}, M_{EF}] &= 4\eta_{[C[E}M_{F]D]}, & [M_{AB}, Q] &= -\frac{1}{2}\Gamma_{AB}Q. \end{aligned} \tag{2.96}$$

When we realize this algebra on the corresponding gauge fields, we use parameters λ^A_B (Lorentz transformations), ζ^A (spacetime translations) and ϵ (supertransformations). Because the supercharge Q can be chosen to be a Majorana spinor, and a Dirac spinor in 4 dimensions has $2^{D/2} = 4$ complex components, the Q ’s (and their corresponding parameters ϵ) have 4 real components. The commutator $[P_A, Q] = 0$ tells us that the SUSY is rigid, i.e. the parameters are constant, and the commutator $[M_{AB}, Q]$ tells us that the supercharge Q transforms as a spinor. As such these two commutators are fixed. The interesting commutator however is $\{Q, Q\} \sim P$, which often is described by saying that “the square of a supertransformation is a spacetime translation”, connecting the supertransformations between fermions and bosons with their spacetime transformations. It can be motivated in several ways, but the most straightforward way is to check the graded Jacobi identities

$$(-1)^{ac}[A, [B, C]] + (-1)^{ab}[B, [C, A]] + (-1)^{bc}[C, [A, B]] = 0, \tag{2.97}$$

where $a, b, c \in \mathbb{Z}_2$ are the gradings of the generators A, B, C respectively, and where an anticommutator for two fermionic generators is implicitly understood. Writing

$$\{Q, Q\} = c_1\Gamma^A C^{-1}P_A + c_2\Gamma^{AB}C^{-1}M_{AB} \tag{2.98}$$

¹⁹For our spinor conventions, see appendix A.2. We don’t write spinor indices explicitly here. Notice that the C^{-1} -matrix in the commutator $\{Q, Q\}$ is used to pull down a spinor index on Γ^A .

for two coefficients c_1 and c_2 , the Jacobi identity (2.97) then shows that $c_1 = -\frac{1}{2}$ and $c_2 = 0$. For any supersymmetric field theory, all the fields then fall into irreducible representations of the algebra (2.96) called supermultiplets. The commutator $[M_{AB}, Q]$ implies that the states in such a multiplet differ in helicity/spin λ by steps of $1/2$. The multiplets are usually analyzed by writing down the $\{Q, Q\}$ -commutator in the rest frame for which we need the equations of motion, which makes the analysis on-shell. First one can analyze the massive case. The fact that $P_A P^A$ is a Casimir means that all the states in a multiplet have the same mass m . Choosing $C = \Gamma_0$, in the rest frame $P_A = (m, 0, 0, 0)$ one then obtains

$$\{Q, Q\} = \frac{1}{2}m\mathbb{1}. \quad (2.99)$$

By rescaling Q with a factor $\sqrt{2/m}$, the anticommutator (2.99) describes a Clifford algebra in 4 Euclidean dimensions, having $2^{4/2} = 4$ states. The analog of Γ_5 in this Clifford algebra is $Q_1 Q_2 Q_3 Q_4$, which is an operator acting on states of which the eigenvalue tells you whether this state is fermionic or bosonic. Being traceless and Hermitian, it has two eigenvalues equal to $+1$, and two eigenvalues equal to -1 , which is the statement that on-shell a supermultiplet contains an equal number of bosonic and fermionic degrees of freedom. For massless multiplets one can choose the frame $P_A = (E, 0, 0, E)$, which gives

$$\{Q, Q\} = \frac{1}{2}E(\mathbb{1} + \Gamma_{03}). \quad (2.100)$$

Because Γ_{03} is traceless and squares to one, the matrix $(\mathbb{1} + \Gamma_{03})$ has two eigenvalues equal to $+2$, and two eigenvalues equal to 0 . This splits the algebra into two parts, of which the non-degenerate part gives again a Clifford algebra as in the massive case, but now in *two* dimensions. Half of these states are fermionic, and half of them are bosonic.

One can also write down $\mathcal{N} > 1$ versions of the algebra (2.96) by considering \mathcal{N} supercharges $Q^{(i)}$, which changes the $\{Q, Q\}$ commutator in the algebra (2.96) into

$$\{Q^{(i)}, Q^{(j)}\} = -\frac{1}{2}\delta_j^i \Gamma^A C^{-1} P_A. \quad (2.101)$$

In practice this means that a supermultiplet is a combination of several $\mathcal{N} = 1$ multiplets. For these extended SUSY algebras one can also introduce so-called central extensions, a concept we will address in section 3.4 and which we will use for the $\mathcal{N} = 2$ algebra in chapter 6.²⁰

Having seen the successes of gauge theories in the Standard Model, a natural thing to try is to gauge the SUSY algebra (2.96), which amounts to making all the corresponding parameters spacetime-dependent. The commutator $\{Q, Q\} \sim P$ then implies that we

²⁰In that chapter we will denote the two parameters of the supertransformations by ϵ_{\pm} , for reasons that will become clear.

also obtain local spacetime translations as a symmetry. These local translations, which are described by parameters $\zeta^A(x)$, smell like general coordinate transformations, which are the symmetries of General Relativity. Of course, local translations are not *exactly* equivalent to general coordinate transformations, but it turns out that upon using curvature constraints a local translation can be rewritten as a sum of a general coordinate transformation and other symmetry transformations; see the remarks after eqn.(B.10). As such the local translations are effectively removed from the algebra, leaving us with general coordinate transformations, local Lorentz transformations and local supertransformations. So the important lesson to learn is that gauging SUSY introduces gravity in our theory! This statement can indeed be made more precise, and the resulting theory is minimal ($\mathcal{N} = 1$) Supergravity.²¹ The corresponding gauge field of the generator Q is the vector-spinor ψ_μ called the gravitino, and the gravitational supermultiplet of $\mathcal{N} = 1$ supergravity then consists of the metric and one gravitino. Unlike $\mathcal{N} = 1$ supergravity, the transformation rules of $\mathcal{N} = 2$ supergravity cannot be obtained by a direct gauging of the corresponding SUSY algebra. The theory contains two gravitino states $|\lambda = 3/2\rangle$ and one vector state $|\lambda = 1\rangle$. An easy way to see this is to denote the graviton state by $|\lambda = 2\rangle$ (of which there is only one), and note that schematically

$$\begin{aligned} Q^{(1,2)}|\lambda = 2\rangle &= |\lambda = 3/2\rangle, \\ Q^{(1)}Q^{(2)}|\lambda = 2\rangle &= -Q^{(2)}Q^{(1)}|\lambda = 2\rangle = |\lambda = 1\rangle. \end{aligned} \quad (2.102)$$

A supermultiplet always contains an equal amount of fermionic and bosonic degrees of freedom; this is necessary to realize SUSY (see e.g. [46]). Often these amounts will only be equal by using the equations of motion for the fields. Such a closure of the symmetries is called 'on-shell closure'. An on-shell counting for the graviton, i.e. by using the gravitational field equations, shows that the graviton has two degrees of freedom. The two gravitini together have $2 \times 2 = 4$ on-shell degrees of freedom [12]. Comparison of these two numbers shows that one needs $4 - 2 = 2$ extra bosonic degrees of freedom to realize on-shell closure of SUSY on the fields. These two bosonic degrees of freedom are provided by one vector field. Similarly, \mathcal{N} -extended supergravity will contain, among others, \mathcal{N} gravitini $\psi_\mu^{(i)}$, and hence extra bosonic fields.

Finally, we discuss the bound on the spacetime dimension D and amount of SUSY \mathcal{N} for Supergravity theories. Until now there are no consistent Supergravity theories with massless fields having spin larger than two due to the fact it is not known how to describe consistent couplings of these higher spin fields to gravity [56], see also e.g. [57, 58]. This leads to the bound $N \leq 8$ and $D \leq 11$.²² To show this, let's consider $\mathcal{N} = 1$ supergravity in $D = 12$. In $D = 12$ one can choose the supercharge Q to be a Majorana or Weyl spinor

²¹For a very nice introduction to Supergravity, see [55].

²²This is under the assumption of one timelike direction.

(but not both). This means that the number of *real* components of Q is $2^{12/2} = 64$. In the massless case this gives us, following the remarks after eqn.(2.100), $64/2 = 32$ raising and lowering operators, of which half of them will lower the helicity/spin. This means these lowering operators take us from $|\lambda = +4\rangle$ to $|\lambda = -4\rangle$, giving fields with helicity/spin larger than two. Increasing D or \mathcal{N} only makes the problem worse. For this reason the only consistent Supergravity theories are known to exist for $D \leq 11$, where the $D = 11$ case has an amount of supersymmetry $\mathcal{N} = 1$.²³ A similar argument explains the bound on \mathcal{N} .

²³We will see in chapter 6 that a non-trivial, non-relativistic notion of supersymmetry requires that $\mathcal{N} > 1$. With non-trivial we mean that two SUSY-transformations give a local (space or time) translation. The reason is that for $\mathcal{N} = 1$ the SUSY-transformations decouple from the local translations, giving only a central charge transformation. This suggests that in order for non-relativistic supergravity theories as limits of their relativistic counterparts to be non-trivial, we have the bound $D \leq 10$.

Chapter 3

Particles and strings

In this chapter the dynamics of particles, strings and branes will be discussed. To obtain a proper understanding of the notion of symmetries of these objects, sigma models will be treated. After that particles and strings will be described in terms of these sigma models, both relativistic and non-relativistic. We will see that the difference between relativistic and non-relativistic particles and strings lies in their target space symmetries, whereas their world-volume symmetries are the same.

3.1 Symmetries and sigma models

First some attention is being paid to non-linear sigma models and their symmetries [12, 59, 60]. These models have a wide range of applications in physics, from the strong interaction and condensed matter systems to String Theory. This more general discussion will turn out to be useful if we consider particles and strings and their gravitational interactions.

A non-linear sigma model is defined in a $(p + 1)$ dimensional world-volume Σ with metric $\gamma_{\bar{\alpha}\bar{\beta}}(\sigma)$, with a collection of D scalar fields $\{\phi^\mu(\sigma)\}$ on a target space \mathcal{N} with metric $g_{\mu\nu}(\phi)$. The dynamics of such a model is described by the following action:

$$S = T \int_{\Sigma} d^{p+1}\sigma \sqrt{-\gamma} g_{\mu\nu}(\phi) \partial_{\bar{\alpha}}\phi^\mu \partial_{\bar{\beta}}\phi^\nu \gamma^{\bar{\alpha}\bar{\beta}}(\sigma). \quad (3.1)$$

Here $\gamma = \det(\gamma_{\bar{\alpha}\bar{\beta}})$ and T is a coupling constant. The world-volume symmetries of this action are

$$\begin{aligned} \delta\phi^\mu &= \xi^{\bar{\alpha}} \partial_{\bar{\alpha}}\phi^\mu, \\ \delta\gamma_{\bar{\alpha}\bar{\beta}} &= \xi^{\bar{\epsilon}} \partial_{\bar{\epsilon}}\gamma_{\bar{\alpha}\bar{\beta}} + \partial_{\bar{\alpha}}\xi^{\bar{\epsilon}} \gamma_{\bar{\epsilon}\bar{\beta}} + \partial_{\bar{\beta}}\xi^{\bar{\epsilon}} \gamma_{\bar{\alpha}\bar{\epsilon}}, \\ \delta\sqrt{-\gamma} &= \partial_{\bar{\alpha}}\left(\sqrt{-\gamma}\xi^{\bar{\alpha}}\right), \end{aligned} \quad (3.2)$$

with appropriate boundary conditions on the world-volume vector $\xi^{\bar{\alpha}}$. These world-volume symmetries do not depend on the target space background $g_{\mu\nu}(\phi)$.

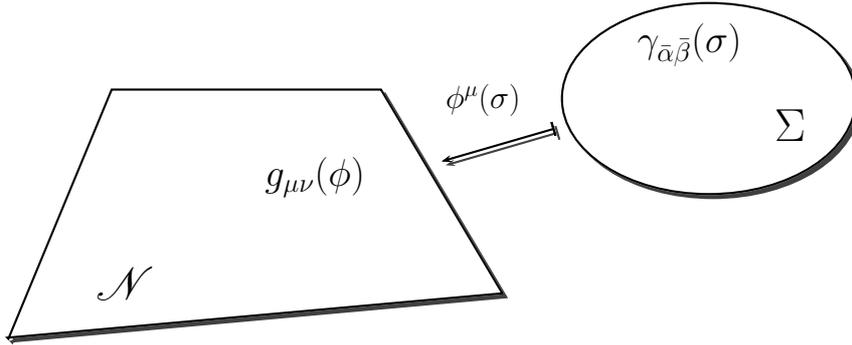


Figure 3.1: The sigma model. The D scalar fields $\phi^\mu(\sigma) : \Sigma \rightarrow \mathcal{N}$ provide maps from a world-volume Σ to a target space \mathcal{N} .

We will now focus on the target space symmetries. There is an interesting interplay between the world-volume and target space [60]. Although on the target space the metric $g_{\mu\nu}(\phi)$ is a background, from the world-volume point of view this background can be regarded as an (infinite) set of coupling constants which couples the fields $\{\phi^\mu(\sigma)\}$ and their derivatives $\{\partial_{\bar{\alpha}}\phi^\mu(\sigma)\}$, as can be seen by performing a Taylor expansion of $g_{\mu\nu}(\phi)$ around $\phi^\mu = 0$. Under a general coordinate transformation on the target space we have

$$\phi^\mu \rightarrow \phi'^\mu(\phi'), \quad g'_{\mu\nu}(\phi') = \frac{\partial\phi^\rho}{\partial\phi'^\mu} \frac{\partial\phi^\lambda}{\partial\phi'^\nu} g_{\rho\lambda}(\phi), \quad (3.3)$$

which is just a reformulation of (2.20). We interpret this transformation as a field redefinition of $\phi^\mu(\sigma)$, accompanied by a field redefinition of the background $g_{\mu\nu}(\phi)$. The action (3.1) is invariant under these two redefinitions, and thus eqn.(3.3) constitutes a symmetry of the theory. However, because we changed both the fundamental fields $\phi^\mu(\sigma)$ and the background $g_{\mu\nu}(\phi)$, the transformation (3.3) is called a *pseudo-symmetry* of the target space. These pseudo-symmetries don't have Noether charges associated to them.¹

Proper symmetries of the target space on the other hand are defined by those transformations which act only on the fundamental fields $\phi^\mu(\sigma)$, and do have associated Noether charges. We will refer to these symmetries as the isometries of the target space. Infinitesimally we write

$$\delta\phi^\mu = -k^\mu(\phi). \quad (3.4)$$

The change in the metric $g_{\mu\nu}(\phi)$ due to the transformation (3.4) is given by

$$\begin{aligned} \delta g_{\mu\nu}(\phi) &= g_{\mu\nu}(\phi - k) - g_{\mu\nu}(\phi) \\ &= -k^\rho \partial_\rho g_{\mu\nu}(\phi). \end{aligned} \quad (3.5)$$

¹Note that general covariance (or “invariance under diffeomorphisms”) of the Einstein equations implies that the energy-momentum tensor is covariantly conserved, as was noted after eqn.(2.32). As such the general coordinate transformations are not pseudo-symmetries anymore, because in the Einstein-Hilbert action we consider the metric to be the fundamental field.

The transformation (3.4) also implies

$$\delta\left(\partial_{\bar{\alpha}}\phi^{\mu}\right) = \partial_{\bar{\alpha}}\left(\delta\phi^{\mu}\right) = -\partial_{\bar{\alpha}}k^{\mu}(\phi) = -\partial_{\bar{\alpha}}\phi^{\nu}\partial_{\nu}k^{\mu}, \quad (3.6)$$

where we used the fact that δ and $\partial_{\bar{\alpha}}$ commute as we keep the world-volume coordinates $\{\sigma^{\bar{\alpha}}\}$ fixed, and the chain rule. Varying the action (3.1) and using the transformations (3.5) and (3.6), it follows that the transformation (3.4) constitutes a proper symmetry only if the Lie derivative of $g_{\mu\nu}(\phi)$ with respect to k^{ρ} vanishes:

$$\mathcal{L}_k g_{\mu\nu}(\phi) = 2\nabla_{(\mu}k_{\nu)} = 0. \quad (3.7)$$

With the isometry transformation (3.4) we can associate a conserved world-volume current

$$j^{\bar{\alpha}} = \frac{\partial L}{\partial(\partial_{\bar{\alpha}}\phi^{\mu})}k^{\mu}. \quad (3.8)$$

Using the conservation equation $\nabla_{\bar{\alpha}}j^{\bar{\alpha}} = 0$, the Noether charge $Q[k]$ associated to the Killing vector k can be obtained by integrating over a spatial world-sheet section with coordinates $\{\sigma^{\bar{1}}, \dots, \sigma^{\bar{p}}\}$:

$$Q[k] = \int d^p\sigma j^{\bar{0}}. \quad (3.9)$$

This language of fundamental fields, backgrounds, world-volumes, target spaces, pseudo symmetries and proper symmetries will now be used in analyzing particles and strings, both relativistically and non-relativistically.

3.2 Relativistic point particles

We saw that the spacetime metric $g_{\mu\nu}(x)$ is determined by the Einstein equations (2.29), whereas the particle's trajectory is determined by the geodesic equation (2.33). We can also regard such a particle in terms of a $p = 0$ sigma model. The world-volume then collapses to a worldline, and the target space becomes the spacetime \mathcal{M} of General Relativity. We therefore relabel the fields in (3.1) as follows:

$$\begin{aligned} \sigma^{\bar{\alpha}} &\rightarrow \tau, \\ \phi^{\rho}(\sigma) &\rightarrow x^{\rho}(\tau). \end{aligned} \quad (3.10)$$

The D fields $x^{\rho}(\tau)$, describing the path of the particle, are regarded as fundamental worldline fields. However, to describe massive particles the sigma model must be modified because the equation of motion for the world-volume metric implies that the particle follows a light-like geodesic. To achieve this, the sigma model action (3.1) can be modified such that it gives the same equations of motion as the usual point particle action and has a sensible massless limit. To show this, a mass term is added to the $p = 0$ sigma model action (3.1):²

$$S = \frac{1}{2} \int_{\tau_i}^{\tau_f} \left(e^{-1} g_{\mu\nu}(x) \dot{x}^{\mu} \dot{x}^{\nu} - em^2 c^2 \right) d\tau, \quad (3.11)$$

²Here $e(\tau) = \sqrt{-\gamma_{\bar{0}\bar{0}}(\tau)}$.

where c is the speed of light. The einbein $e(\tau)$ is a world-line scalar density, as is clear from the world-volume transformations (3.2). Using the (algebraic) equation of motion for the einbein,

$$e(\tau) = (mc)^{-1} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \quad (3.12)$$

and plugging this solution back into the action (3.11), one recovers the familiar massive point particle action

$$S = -mc \int_{\tau_i}^{\tau_f} \sqrt{-g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\tau. \quad (3.13)$$

The action (3.11) (and (3.13)) is world-line reparametrization invariant, as we expect from a sigma model.³

We now turn to flat backgrounds, i.e. we take $g_{\mu\nu} = \eta_{\mu\nu}$ in (3.13):

$$S = -mc \int_{\tau_i}^{\tau_f} \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} d\tau. \quad (3.14)$$

The corresponding Lagrangian L is invariant under the target space transformations

$$\delta x^\mu = \lambda^\mu{}_\nu x^\nu + \zeta^\mu, \quad (3.15)$$

which are just the Poincaré transformations. The action is invariant under infinitesimal world-line reparametrizations $\delta\tau = \xi(\tau)$,

$$\delta S = -mc \left(\xi \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} \right) \Big|_{\tau_i}^{\tau_f} = 0, \quad (3.16)$$

because of the boundary conditions $\xi(\tau_i) = \xi(\tau_f) = 0$. The canonical momenta p_μ read

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{mc \dot{x}_\mu}{\sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}, \quad (3.17)$$

where L is the Lagrangian. Varying the action (3.14) then gives the equations of motion

$$\frac{d}{d\tau} \left(\frac{mc \dot{x}_\mu}{\sqrt{-\dot{x}_\mu \dot{x}^\mu}} \right) = \dot{p}_\mu = 0, \quad \text{or} \quad \ddot{x}^\mu = \frac{\dot{L}}{L} \dot{x}^\mu. \quad (3.18)$$

Now, the components of the canonical momentum (3.17) are not independent. One can see from their expression that the constraint

$$V_0 \equiv p_\mu p^\mu + m^2 c^2 = 0 \quad (3.19)$$

holds. This is called a primary constraint, because it is satisfied due to the definition of p_μ , without using the equations of motion. The origin of this primary constraint is that the relation $p_\mu(\dot{x}^\nu)$ is not invertible; the Jacobian matrix

$$\frac{\partial p_\mu}{\partial \dot{x}^\nu} = \frac{\partial^2 L}{\partial \dot{x}^\mu \partial \dot{x}^\nu} \quad (3.20)$$

³Note that upon writing the metric as $g_{\mu\nu} = e_\mu{}^A e_\nu{}^B \eta_{AB}$, the action (3.13) is also invariant under infinitesimal local Lorentz transformations in the tangent space applied on the vielbein, $\delta e_\mu{}^A = \lambda^A{}_{B\mu} e_\mu{}^B$.

has one eigenvector with zero eigenvalue, namely \dot{x}^μ , stating that the determinant of the Jacobian matrix (3.20) vanishes. The canonical Hamiltonian corresponding to the Lagrangian (3.14) also vanishes:

$$H_{can} = p_\mu \dot{x}^\mu - L = 0, \quad (3.21)$$

which means that all the dynamics is captured by (3.19). This is to be expected; the canonical Hamiltonian describes the τ -evolution of the system, but the action is invariant under τ -reparametrizations. As such the parameter τ is not a dynamical degree of freedom, but a gauge degree of freedom. The total Hamiltonian is then defined to be

$$H = H_{can} + \lambda_0(\tau)V_0 = \lambda_0(\tau)[p_\mu p^\mu + m^2 c^2], \quad (3.22)$$

where $\lambda_0(\tau)$ is a Lagrange multiplier.

In the first order formalism the coordinates $\{x^\mu\}$ and momenta $\{p^\mu\}$ are a priori independent. We rewrite the Lagrangian as

$$L = p_\mu \dot{x}^\mu - H = p_\mu \dot{x}^\mu - \lambda_0(\tau)[p_\mu p^\mu + m^2 c^2]. \quad (3.23)$$

Hamilton's equations for $\{x^\mu\}$, $\{p^\mu\}$ and $\lambda_0(\tau)$ respectively then read

$$\begin{aligned} \dot{p}_\mu &= 0, \\ \dot{x}^\mu &= 2\lambda_0(\tau)p^\mu, \\ p_\mu p^\mu + m^2 c^2 &= 0. \end{aligned} \quad (3.24)$$

Unlike in the second order formalism, here we can express \dot{x}^μ in terms of p^μ due to the Lagrange multiplier $\lambda_0(\tau)$. Comparison with (3.18) and (3.19) shows the equivalence. The choice $\lambda_0(\tau) = (2m)^{-1}$ in this first order formalism corresponds in the second order formalism to

$$\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} \equiv c \rightarrow \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2. \quad (3.25)$$

This means that for this choice the evolution parameter τ becomes an affine parameter of the particle and that the particle moves on a timelike geodesic. In terms of the world-line theory this choice corresponds to taking the induced world-line metric $\gamma_{\bar{0}\bar{0}} = \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ to be constant. The Hamilton equations of motion (3.24) can then be written as

$$\ddot{x}^\mu = 0, \quad p_\mu p^\mu + m^2 c^2 = 0, \quad (3.26)$$

which are the geodesic equation in flat spacetime plus the relativistic energy-momentum relation.

3.3 Non-relativistic particle without gravity

We next consider the action of a non-relativistic free particle,⁴

$$S = \frac{m}{2} \int_{\tau_i}^{\tau_f} \frac{\dot{x}^i \dot{x}_i}{\dot{x}^0} d\tau. \quad (3.27)$$

This action defines a one-dimensional field theory, where the fundamental fields are given by $\{x^0(\tau), x^i(\tau)\}$. The Lagrangian is invariant under world-line reparametrizations $\tau \rightarrow \tau'(\tau)$. These transformations constitute the world-line symmetries, and it explains the appearance of $\{\dot{x}^0\}$ in the denominator of the Lagrangian of (3.27). The target-space symmetries are given by the Galilei transformations (2.1), which infinitesimally read

$$\begin{aligned} \delta_H x^0 &= \zeta^0, \\ \delta_J x^i &= \lambda^i_j x^j, \quad \delta_G x^i = \lambda^i x^0, \quad \delta_P x^i = \zeta^i. \end{aligned} \quad (3.28)$$

These transformations constitute the proper symmetries of the theory. An important observation is that the Lagrangian transforms as a total derivative under boosts, a fact which we will treat in the next section. One can define the canonical momenta p_0 and p_i :

$$p_0 = \frac{\partial L}{\partial \dot{x}^0} = -\frac{m}{2} \frac{\dot{x}^i \dot{x}_i}{(\dot{x}^0)^2}, \quad p_i = \frac{\partial L}{\partial \dot{x}^i} = m \frac{\dot{x}^i}{\dot{x}^0}. \quad (3.29)$$

These momenta obey the primary constraint

$$V_0 = p^i p_i + 2m p_0 = 0, \quad (3.30)$$

which should be recognized as the non-relativistic dispersion relation. Varying the action (3.27) with respect to $\{x^0\}$ and $\{x^i\}$ respectively gives the equations of motion

$$\dot{p}_0 = 0, \quad \dot{p}_i = 0 \rightarrow \ddot{x}^i = \frac{\ddot{x}^0}{\dot{x}^0} \dot{x}^i. \quad (3.31)$$

The equations of motion for $\{x^0\}$ and $\{x^i\}$ are not independent; from the primary constraint (3.30) one can already see that $\dot{x}^i p_i = 0$ implies $\dot{p}_0 = 0$. Due to the world-line reparametrization invariance of the action (3.27) the canonical Hamiltonian $H_{can} = p_\mu \dot{x}^\mu - L$ vanishes, and so we can write

$$H = H_{can} + \lambda_0(\tau) V_0 = \lambda_0(\tau) (p^i p_i + 2m p_0), \quad (3.32)$$

with $\lambda_0(\tau)$ a Lagrange multiplier. In the first order formalism the Lagrangian of (3.27) becomes

$$\begin{aligned} L &= p_0 \dot{x}^0 + p_i \dot{x}^i - H \\ &= p_0 \dot{x}^0 + p_i \dot{x}^i - \lambda_0(\tau) (p^i p_i + 2m p_0). \end{aligned} \quad (3.33)$$

⁴Note that we use curved indices $\{0, i\}$ for notational convenience.

To show the equivalence between the Lagrangians (3.33) and (3.27) we calculate Hamilton's equations of motion for $\{x^0\}$, $\{x^i\}$, $\{p_0\}$, $\{p_i\}$ and $\lambda_0(\tau)$ respectively:

$$\begin{aligned} \dot{p}_0 &= 0, & \dot{p}_i &= 0, \\ \dot{x}^0 &= 2m\lambda_0, & \dot{x}^i &= 2\lambda_0 p^i, \\ p^i p_i + 2mp_0 &= 0. \end{aligned} \quad (3.34)$$

Due to its linear appearance the conjugate momentum $\{p_0\}$ acts as a Lagrange multiplier. From the third and fourth equation we can eliminate λ_0 to get

$$p_i = m \frac{\dot{x}^i}{\dot{x}^0}, \quad (3.35)$$

which equals the conjugate momentum $\{p_i\}$ in eqn.(3.29). Choosing the Lagrange multiplier as $\lambda_0 = (2m)^{-1}$ corresponds to

$$\dot{x}^0 = 1, \quad (3.36)$$

which implies the static gauge $x^0 = \tau$ up to a constant.

A natural question to pose is if one could add a term to the Lagrangian of (3.27) such that the Lagrangian is invariant under the Galilei transformations, instead of transforming into a total derivative. This feature, in which the Lagrangian transforms into a total derivative, is known as *quasi-invariance* of the Lagrangian. In an attempt to make the Lagrangian invariant, a total τ -derivative $\dot{f}(x, \dot{x})$ can be added to the Lagrangian,

$$L \rightarrow L + \dot{f}(x, \dot{x}), \quad (3.37)$$

without changing the equations of motion. To make the Lagrangian invariant under the Galilei transformations, the function $f(x, \dot{x})$ must then obey the constraints

$$\begin{aligned} \delta_H \dot{f}(x, \dot{x}) &= \delta_P \dot{f}(x, \dot{x}) = \delta_J \dot{f}(x, \dot{x}) = 0, \\ \delta_G \dot{f}(x, \dot{x}) &= -m \dot{x}^i \lambda_i. \end{aligned} \quad (3.38)$$

Using the infinitesimal Galilei transformations (3.28), these constraints become

$$\begin{aligned} \delta_H f &= \zeta^0 \frac{\partial f}{\partial x^0} = \text{cst.}, \\ \delta_P f &= \zeta^i \frac{\partial f}{\partial x^i} = \text{cst.}, \\ \delta_J f &= \lambda^i_j x^j \frac{\partial f}{\partial x^i} + \lambda^i_j \dot{x}^j \frac{\partial f}{\partial \dot{x}^i} = \text{cst.}, \\ \delta_G f &= \lambda^i x^0 \frac{\partial f}{\partial x^i} + \lambda^i \dot{x}^0 \frac{\partial f}{\partial \dot{x}^i} = -m \lambda_i \dot{x}^i. \end{aligned} \quad (3.39)$$

This set of differential equations does not have a solution for $f(x, \dot{x})$, as can be easily checked. However, there is another way to make the Lagrangian invariant: we can extend

Newtonian spacetime by an extra coordinate $s \in \mathbb{R}$ and consider⁵

$$L_{ext} = \frac{m}{2} \left(\frac{\dot{x}^i \dot{x}_i}{\dot{x}^0} + 2\dot{s} \right). \quad (3.40)$$

Per construction the extra coordinate $\{s\}$ obeys

$$\delta_G s = -\lambda_i x^i, \quad \delta_H s = \text{cst.}, \quad \delta_P s = \text{cst.}, \quad \delta_J s = 0, \quad (3.41)$$

such that the Lagrangian (3.40) is invariant under the Galilei group. Note that the mass m plays the role of the conjugate momentum to $\{s\}$.

3.4 Central extensions

Central extensions are perhaps best known for their appearances when quantizing a classical theory. For example, classical String Theory exhibits a conformal symmetry on the world-sheet, generated by the so-called Witt or Virasoro algebra. If one quantizes the string, normal-ordering ambiguities in the world-sheet fields dictate a central extension for this algebra. However, (central) extensions also show up in classical physics whenever the Lagrangian transforms as a total derivative under certain symmetry transformations [85]. For a point particle this means that $\delta L = \dot{\Theta}$ for some function $\Theta(x^\mu)$. This is precisely the case for the action (3.27). This action is invariant under the Galilei transformations (2.1), which infinitesimally are given by eqn.(3.28). However, the Lagrangian L transforms as a total derivative under an infinitesimal Galilean boost:

$$\delta L = \frac{d}{d\tau} (m x_i \lambda^i) = \dot{\Theta}, \quad \Theta = m x_i \lambda^i. \quad (3.42)$$

Due to this the Noether charge belonging to the Galilean boosts becomes $Q_G = p^i \delta x^i - \Theta = m \dot{x}^i \lambda^i x^0 - m x_i \lambda^i$. The Noether charges belonging to the Galilei transformations (3.28) are then given by

$$Q_H = p_0 \zeta^0, \quad Q_J = p_i \lambda^i_j x^j, \quad Q_P = p_i \zeta^i, \quad Q_G = p_i \lambda^i x^0 - m x^i \lambda_i. \quad (3.43)$$

Using the Poisson brackets

$$\{F, G\}_{PB} = \frac{\partial F}{\partial x^\mu} \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial G}{\partial x^\mu} \quad (3.44)$$

such that $\{x^\mu, p_\nu\}_{PB} = \delta^\mu_\nu$, the symmetry transformations (3.28) are generated by the Noether charges via the Poisson brackets:

$$\delta_X x^\mu = -\{Q_X, x^\mu\}_{PB}. \quad (3.45)$$

⁵The construction of a Lagrangian which is invariant under a group of symmetries can be done more rigorously by using the formalism of Maurer-Cartan forms.

For example, $\delta_P x^i = -\{Q_P, x^i\}_{PB} = \zeta^i$. The Poisson brackets of the Noether charges (3.43) also obey the Lie algebra with structure constants f_{XY}^Z generating the symmetries of the action (3.27),

$$\{Q_X, Q_Y\}_{PB} = f_{XY}^Z Q_Z, \quad (3.46)$$

and the Jacobi identities. This hints to the centrally extended Galilei algebra, which is known as the Bargmann algebra. Namely, one may verify that the Poisson bracket of the Noether charge Q_G corresponding to infinitesimal Galilei boosts $\delta x^i = \lambda^i$ with the Noether charge Q_P corresponding to infinitesimal spatial translations $\delta x^i = \zeta^i$ indicates the existence of the central generator Z :

$$\{Q_G, Q_P\}_{PB} = -m\zeta_i \lambda^i. \quad (3.47)$$

With this it is clear that the Z generator is needed to obtain massive representations of the Galilei algebra.

Another place where central extensions are found is in extended supersymmetry algebras, which was already mentioned in section 2.9. One can use the graded Jacobi identities (2.97) to show that the $\mathcal{N} = 2$ algebra in $D = 4$ allows for the two central extensions \mathcal{V} and \mathcal{Z} in the following way:

$$\{Q^{(i)}, Q^{(j)}\} = -\frac{1}{2}\delta^{ij}\Gamma^A C^{-1} P_A + \frac{1}{2}\varepsilon^{ij}(\Gamma_5 \mathcal{Z} + i\mathcal{V})C^{-1}, \quad i, j = 1, 2. \quad (3.48)$$

Here ε^{ij} is the two-dimensional epsilon symbol. Being central extensions means that \mathcal{V} and \mathcal{Z} commute with all the other generators of the supersymmetry algebra, making them Lorentz scalars. From a field theory point of view these central extensions allow one to introduce massive multiplets without completely breaking the supersymmetry, see e.g. [48]. We will use the central extension \mathcal{Z} in chapter 6 to introduce the notion of non-relativistic supersymmetry.

3.5 Non-relativistic particle with gravity

One can introduce Newtonian gravity for the point particle by coupling a potential $\phi(x)$ to the field $\{\dot{x}^0\}$:

$$L = \frac{m}{2} \left(\frac{\delta_{ij} \dot{x}^i \dot{x}^j}{\dot{x}^0} - 2\dot{x}^0 \phi(x) \right). \quad (3.49)$$

The transformations we now consider are (2.16), which infinitesimally read

$$\begin{aligned} \delta_H x^0 &= \zeta^0, \\ \delta_F x^i &= \xi^i(x^0), \quad \delta_J x^i = \lambda^i_j x^j. \end{aligned} \quad (3.50)$$

Here $\xi^i(x^0)$ is an arbitrary differentiable function describing an arbitrary time-dependent acceleration. Under spatial rotations J and temporal translations H the Lagrangian is

invariant. Under an acceleration F the Lagrangian transforms as

$$\begin{aligned}\delta_F L &= m \left(\frac{\dot{x}^i \dot{\xi}_i}{\dot{x}^0} - \dot{x}^0 \delta_F \phi(x) \right) \\ &= m \left(\frac{d}{d\tau} \left(\frac{x^i \dot{\xi}_i}{\dot{x}^0} \right) - \frac{d}{d\tau} \left(\frac{\dot{\xi}_i}{\dot{x}^0} \right) x^i - \dot{x}^0 \delta_F \phi(x) \right).\end{aligned}\quad (3.51)$$

So if we choose⁶

$$\delta_F \phi(x) = -\frac{1}{\dot{x}^0} \frac{d}{d\tau} \left(\frac{\dot{\xi}_i}{\dot{x}^0} \right) x^i, \quad (3.52)$$

the Lagrangian is quasi-invariant under (3.50). However, these symmetries are pseudo-symmetries of the theory. The reason is that the transformation (3.52) of the potential $\phi(x)$, which can be considered as a background field, is not induced by the fundamental fields $\{x^i(\tau)\}$,

$$\delta_F \phi(x) \neq \frac{\partial \phi}{\partial x^i} \delta_F x^i. \quad (3.53)$$

As such the transformation (3.52) expresses covariance of the theory under arbitrary accelerations, and therefore these transformations don't have corresponding Noether charges. Calculating the momenta $\{p_0\}$ and $\{p_i\}$, the primary constraint (3.30) is replaced by

$$V_0 = p^i p_i + 2mp_0 - 2m^2 \phi(x) = 0, \quad (3.54)$$

from which it is clear that the equations of motion for $\{x^0\}$ and $\{x^i\}$ are not independent. To calculate the Lie algebra which generates (3.50) we write $\{x^0 = t\}$ for convenience and expand the translation parameter $\xi^i(t)$ as follows:

$$\xi^i(t) = \sum_{n=0}^N \frac{1}{n!} a_{(n)}^i t^n \equiv \sum_{n=0}^N \xi_{(n)}^i, \quad N \rightarrow \infty. \quad (3.55)$$

An infinitesimal translation on the transverse coordinate $\{x^i\}$ is then written as

$$\delta_{F^{(n)}} x^i = \frac{1}{n!} a_{(n)}^i t^n. \quad (3.56)$$

The non-zero commutators of the corresponding (infinite dimensional!) Lie algebra are then easily calculated to be

$$\begin{aligned}[H, F_i^{(n)}] &= F_i^{(n-1)}, \quad F_i^{(-1)} = 0, \\ [J_{ij}, F_k^{(n)}] &= -2\delta_{k[i} F_{j]}^{(n)}, \\ [J_{ij}, J_{kl}] &= 4\delta_{[i[k} J_{l]j]}.\end{aligned}\quad (3.57)$$

⁶Note that here and in the following chapter we leave out the push-forward term in the infinitesimal transformation of the Newton potential, i.e. we define the variation δ from now on as $\delta\Phi(x) \equiv \Phi'(x') - \Phi(x)$ because we are interested in the covariance of the action (3.49) under the field redefinition of the embedding coordinates. In [43] this variation is denoted as $\tilde{\delta}$. Note also that in the transformation (2.83) we used the conventional definition of the variation. The difference between these two definitions is merely the earlier-mentioned push-forward term.

Notice that we don't require N to be finite in the expansion (3.55) in order for the algebra to close. It is also clear that the translations commute and do not allow for a central extension. This can be understood from the Jacobi identities of this algebra as follows. Imagine that we keep N in eqn.(3.55) finite. For $N = 1$ we get the Galilei algebra, for $N = 2$ a constant-acceleration extended Galilei algebra [61], etc. Then the Jacobi identities imply that the only possible central extension C in the translations is between

$$[F^{(N)}, F^{(N-1)}] \sim C^{N,N-1}. \quad (3.58)$$

This can be seen by looking at the following sequence:

$$\begin{aligned} [H, [F^{(0)}, F^{(1)}]] + cycl &= 0 \\ [H, [F^{(0)}, F^{(2)}]] + cycl &= 0 \rightarrow C^{0,1} = 0 \\ [H, [F^{(1)}, F^{(2)}]] + cycl &= 0 \rightarrow C^{2,0} = 0 \\ [H, [F^{(1)}, F^{(3)}]] + cycl &= 0 \rightarrow C^{1,2} = 0 \\ &\vdots \end{aligned} \quad (3.59)$$

Because we send $N \rightarrow \infty$ in eqn.(3.55) as the accelerations are arbitrary differentiable functions of the time $\{t\}$, the central extension drops out, and hence

$$[F_i^{(n)}, F_j^{(m)}] = 0. \quad (3.60)$$

From this analysis and checking all the Jacobi identities it is clear that indeed the Galilei algebra (for which $N = 1$) does allow for a central extension giving the Bargmann algebra, whereas the algebra (3.57) does not allow for such a central extension.

3.6 Relativistic strings

We now go from particles, which were considered to be one-dimensional sigma models, to strings, which are two-dimensional sigma models. The relativistic string in an arbitrary target space background $g_{\mu\nu}(x)$ is described by the Nambu-Goto action

$$S = -T \int_{\Sigma} d^2\sigma \sqrt{-\gamma}, \quad \gamma_{\bar{\alpha}\bar{\beta}} = \partial_{\bar{\alpha}}x^{\mu} \partial_{\bar{\beta}}x^{\nu} g_{\mu\nu}(x), \quad (3.61)$$

where $\gamma = \det(\gamma_{\bar{\alpha}\bar{\beta}})$ is the determinant of the induced world-sheet metric $\gamma_{\bar{\alpha}\bar{\beta}}$. The geometric interpretation of this Nambu-Goto action is that it is proportional to the area which the world-sheet Σ traverses in the target space. This action can be obtained from the sigma model action (3.1) by using the equations of motion for the world-volume metric in terms of the induced metric $\gamma_{\bar{\alpha}\bar{\beta}}$ and plugging this expression back into the action. This nontrivial feature makes quantization of the string in a flat background possible, because

as such one can get rid of the square root in the Nambu-Goto action. The equations of motion are⁷

$$\gamma^{\bar{\alpha}\bar{\beta}} \left(\nabla_{\bar{\alpha}} \partial_{\bar{\beta}} x^{\rho} + \partial_{\bar{\alpha}} x^{\mu} \partial_{\bar{\beta}} x^{\nu} \Gamma_{\mu\nu}^{\rho} \right) = 0, \quad (3.62)$$

where $\Gamma_{\mu\nu}^{\rho} = \{\mu\nu\}$. More generally, we can define any p -brane via the Nambu-Goto action (3.61) such that the geodesic equation (3.62) describes the dynamics of the world-volume.

The canonical momenta corresponding to the Lagrangian of (3.61) are

$$p_{\mu} = \frac{\partial L}{\partial \dot{x}^{\mu}} = -T \sqrt{-\gamma} \gamma^{\bar{\alpha}\bar{0}} \partial_{\bar{\alpha}} x_{\mu}, \quad (3.63)$$

where a dot denotes derivation with respect to $\tau = \sigma^{\bar{0}}$, whereas a prime⁸ will denote derivation with respect to $\sigma^{\bar{1}}$. One then has the identities

$$p_{\mu} x'^{\mu} = -T \sqrt{-\gamma} \gamma^{\bar{\alpha}\bar{0}} \gamma_{\bar{\alpha}\bar{1}} \sim \delta_{\bar{1}}^{\bar{0}} = 0, \quad (3.64)$$

and

$$p_{\mu} p^{\mu} + T^2 x'_{\mu} x'^{\mu} = -T^2 \gamma \gamma^{\bar{0}\bar{0}} + T^2 \gamma_{\bar{1}\bar{1}} = 0, \quad (3.65)$$

where for the last identity we used the inverse relation

$$\gamma^{\bar{\alpha}\bar{\beta}} = \frac{1}{\gamma} \begin{pmatrix} \gamma_{\bar{1}\bar{1}} & -\gamma_{\bar{0}\bar{1}} \\ -\gamma_{\bar{0}\bar{1}} & \gamma_{\bar{0}\bar{0}} \end{pmatrix}.$$

So the primary constraints $\{V_0, V_1\}$ are given by

$$\begin{aligned} V_0 &= p_{\mu} p^{\mu} + T^2 x'_{\mu} x'^{\mu} = 0, \\ V_1 &= p_{\mu} x'^{\mu} = 0. \end{aligned} \quad (3.66)$$

Because the canonical Hamiltonian satisfies

$$H_{can} = \int d\sigma \left(p_{\mu} \dot{x}^{\mu} - L \right) = 0, \quad (3.67)$$

all the dynamics of the string is captured by the primary constraints (3.66), as for the point particle. The Hamiltonian is then written as the sum of the primary constraints,

$$H = \int d\sigma \left(\lambda_0 V_0 + \lambda_1 V_1 \right), \quad (3.68)$$

for the two Lagrange multipliers $\{\lambda_0(\sigma), \lambda_1(\sigma)\}$.

⁷Using that $\gamma^{\mu\nu} \Gamma_{\mu\nu}^{\rho} = -\frac{1}{\sqrt{-\gamma}} \partial_{\lambda} \left(\sqrt{-\gamma} \gamma^{\lambda\rho} \right)$ for any Levi-Civita connection $\Gamma_{\mu\nu}^{\rho}$.

⁸This prime shouldn't be confused with the prime used for a coordinate transformation or the prime used for a flat longitudinal direction $\{a'\}$.

3.7 Non-relativistic strings

The action describing non-relativistic strings in a flat background is given by

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-\bar{\gamma}} \left(\bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} x^i \partial_{\bar{\beta}} x^j \delta_{ij} \right), \quad (3.69)$$

where $\bar{\gamma}_{\bar{\alpha}\bar{\beta}}$ is the pull-back of the longitudinal metric $\eta_{\alpha\beta}$, i.e.

$$\bar{\gamma}_{\bar{\alpha}\bar{\beta}} = \partial_{\bar{\alpha}} x^\alpha \partial_{\bar{\beta}} x^\beta \eta_{\alpha\beta}. \quad (3.70)$$

One can check that in the ‘‘point particle limit’’, where $\bar{\gamma}_{\bar{\alpha}\bar{\beta}} \rightarrow \bar{\gamma}_{00} = -(x^0)^2$ and the string tension T becomes the particle mass m , the action (3.69) reduces to eqn.(3.27). The action (3.69) is invariant under world-sheet reparametrizations and the following ‘‘stringy’’ Galilei symmetries:

$$\delta x^\alpha = \lambda^\alpha{}_\beta x^\beta + \zeta^\alpha, \quad \delta x^i = \lambda^i{}_j x^j + \lambda^i{}_\beta x^\beta + \zeta^i, \quad (3.71)$$

where $(\zeta^\alpha, \zeta^i, \lambda^i{}_j, \lambda^i{}_\alpha, \lambda^\alpha{}_\beta)$ parametrize a (constant) longitudinal translation, transverse translation, transverse rotation, ‘‘stringy’’ boost transformation and longitudinal rotation, respectively. Notice how the longitudinal space remains relativistic, while the transverse space exhibits Galilean symmetries. We will see in chapter 5 that this is a consequence of how one obtains the action (3.69) via a limit procedure applied on the Nambu-Goto action (3.61), or equivalently, how the contraction procedure on the relativistic algebra is applied. The equations of motion for $\{x^i\}$ corresponding to the action (3.69) are given by

$$\partial_{\bar{\alpha}} \left(\sqrt{-\bar{\gamma}} \bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\beta}} x^i \right) = 0. \quad (3.72)$$

The non-relativistic Lagrangian defined by (3.69) is invariant under a stringy boost transformation only up to a total world-sheet divergence. The implications for this fact, namely an extension of the underlying symmetry algebra, will be treated in chapter 5. The canonical momenta read

$$\begin{aligned} p_\alpha &= -T \sqrt{-\bar{\gamma}} \partial_{\bar{\alpha}} x^i \partial_{\bar{\beta}} x_i \left(\frac{1}{2} \bar{\gamma}^{\bar{\alpha}\bar{0}} \bar{\gamma}^{\bar{0}\bar{\beta}} - \bar{\gamma}^{\bar{\alpha}\bar{\epsilon}} \bar{\gamma}^{\bar{0}\bar{\beta}} \right) \partial_{\bar{\epsilon}} x_\alpha, \\ p_i &= -T \sqrt{-\bar{\gamma}} \bar{\gamma}^{\bar{\alpha}\bar{0}} \partial_{\bar{\alpha}} x_i. \end{aligned} \quad (3.73)$$

These momenta obey the two primary constraints⁹

$$\begin{aligned} V_0 &= p_\alpha \varepsilon^{\alpha\beta} x'_\beta + \frac{1}{2} \left(T^{-1} p^i p_i + T x'^i x'_i \right) = 0, \\ V_1 &= p_\alpha x'^\alpha + p_i x'^i = 0, \end{aligned} \quad (3.74)$$

and the Hamiltonian can again be written in the form (3.68). This closes our discussion of strings, particles and branes. In the next chapter we will discuss how to obtain Newtonian gravity in the guise of Newton-Cartan theory by applying a gauging procedure to the Bargmann algebra.

⁹We define here the two-dimensional epsilon symbol $\varepsilon^{10} = -\varepsilon^{01} = 1$ of the longitudinal space.

Chapter 4

Newtonian Gravity and the Bargmann Algebra

4.1 Introduction

As was mentioned in the Introduction, Einstein's formulation of gravity can be obtained by performing a formal gauging procedure of the Poincaré algebra [14, 15]. In this procedure one associates to each generator of the Poincaré algebra a gauge field. Next, one imposes constraints on the curvature tensors of these gauge fields such that the translational symmetries of the algebra get converted into general coordinate transformations. At the same time the gauge field of the Lorentz transformations gets expressed into (derivatives of) the Vierbein gauge field which is the only independent gauge field. One thus obtains an off-shell formulation of Einstein gravity. On-shell Einstein gravity is obtained by imposing the usual Einstein equations of motion.

One may consider the non-relativistic version of the Poincaré algebra and Einstein gravity independently. It turns out that the relevant non-relativistic version of the Poincaré algebra is a particular contraction of the Poincaré algebra trivially extended with a 1-dimensional algebra that commutes with all the generators. This contraction yields the so-called Bargmann algebra, which is the centrally extended Galilean algebra. On the other hand, taking the non-relativistic limit of General Relativity leads to the well-known non-relativistic Newtonian gravity in flat space. The Newton-Cartan theory is a geometric reformulation of this Newtonian theory, mimicking as much as possible the geometric formulation of General Relativity [22, 62]. A notable difference with the relativistic case is the occurrence of a degenerate metric.

The question we pose in this chapter is: can we derive the Newton-Cartan formulation of Newtonian gravity directly from gauging the Bargmann algebra in the same way that Einstein gravity may be derived from gauging the relativistic Poincaré algebra as described

above?¹ The answer will be yes, but there are some subtleties involved. This is partly due to the fact that the standard procedure leads to spin-connection fields that not only depend on the temporal and spatial vielbeins but also on the gauge field corresponding to the central charge generator. These connections have to be fixed appropriately via further curvature constraints, in order to obtain the Poisson equation.

The outline of this chapter is as follows. In section 2 we first review how Einstein gravity may be obtained by gauging the Poincaré algebra. To keep the discussion in this section as general as possible we leave the dimension D of spacetime arbitrary. Next, we briefly review in section 3 the Newton-Cartan formulation of Newtonian gravity, since this is the theory we wish to end up with in the non-relativistic case. We next proceed, in section 4, with gauging the Bargmann algebra. In a first step we introduce a set of curvature constraints that convert the spatial (time) translational symmetries of the algebra into spatial (time) general coordinate transformations. We next impose a vielbein postulate for the vielbeins in the temporal and spatial directions. In a final step we impose further curvature constraints on the theory in order to recover the non-relativistic Poisson equation in terms of the boost curvature, plus a similar equation of motion for the rotational curvature. In the last section we give conclusions.

4.2 Einstein Gravity and Gauging the Poincaré Algebra

In this section we review how the basic ingredients of Einstein gravity may be obtained by applying a formal gauging procedure to the Poincaré algebra. We leave the dimension D of spacetime in this section arbitrary.

Our starting point is the D -dimensional Poincaré algebra $\mathfrak{iso}(D-1, 1)$ with generators $\{P_A, M_{AB}\}$, which is the bosonic part of the algebra (2.96):

$$\begin{aligned} [P_A, P_B] &= 0, \\ [M_{BC}, P_A] &= -2\eta_{A[B}P_{C]}, \\ [M_{CD}, M_{EF}] &= 4\eta_{[C[E}M_{F]D]}. \end{aligned} \tag{4.1}$$

Here the indices ($A = 0, 1, \dots, D-1$) are regarded as abstract indices of some internal space, as is usually done in gauge theories; only later these indices will be identified with tangent-space indices. Associating a gauge field e_μ^A to the local P -transformations with spacetime dependent parameters $\zeta^A(x)$, and a gauge field ω_μ^{AB} to the local Lorentz transformations with spacetime dependent parameters $\lambda^{AB}(x)$, we obtain from appendix

¹The gauging of the Bargmann algebra, from a somewhat different point of view, has been considered before in [63, 64].

B the transformation rules

$$\delta e_\mu^A = \partial_\mu \zeta^A - \omega_\mu^{AB} \zeta^B + \lambda^{AB} e_\mu^B, \quad (4.2)$$

$$\delta \omega_\mu^{AB} = \partial_\mu \lambda^{AB} + 2\lambda^{C[A} \omega_\mu^{B]C}. \quad (4.3)$$

and the curvatures

$$R_{\mu\nu}^A(P) = 2\left(\partial_{[\mu} e_{\nu]}^A - \omega_{[\mu}^{AB} e_{\nu]}^B\right), \quad (4.4)$$

$$R_{\mu\nu}^{AB}(M) = 2\left(\partial_{[\mu} \omega_{\nu]}^{AB} - \omega_{[\mu}^{CA} \omega_{\nu]}^{BC}\right). \quad (4.5)$$

In order to make contact with gravity we wish to replace the local P -transformations of all gauge fields by general coordinate transformations and to interpret e_μ^A as the vielbein, with the inverse vielbein field e_A^μ defined by eqn.(2.47),

$$e_\mu^A e^\mu_B = \delta_B^A, \quad e_\mu^A e^\nu_A = \delta_\mu^\nu. \quad (4.6)$$

To show how this replacement can be achieved by imposing curvature constraints we first consider the general identity (B.10) for a gauge algebra and corresponding gauge fields $\{B_\mu^A\}$:

$$0 = \delta_{gct}(\xi^\lambda) B_\mu^A + \xi^\lambda R_{\mu\lambda}^A - \sum_{\{C\}} \delta(\xi^\lambda B_\lambda^C) B_\mu^A. \quad (4.7)$$

If we now relate the parameters ξ^λ and ζ^A via

$$\xi^\lambda = e_A^\lambda \zeta^A, \quad (4.8)$$

we can bring the P -transformation of e_μ^A in the sum in eqn.(4.7) to the left-hand side of the equation to obtain

$$\delta_P(\zeta^B) e_\mu^A = \delta_{gct}(\xi^\lambda) e_\mu^A + \xi^\lambda R_{\mu\lambda}^A(P) - \delta_M(\xi^\lambda \omega_\lambda^{AB}) e_\mu^A. \quad (4.9)$$

We see that the difference between a P -transformation and a general coordinate transformation is a curvature term and a Lorentz transformation. More generally, we deduce from the identity (4.7) that, whenever a gauge field transforms under a P -transformation, the P -transformations of this gauge field can be replaced by a general coordinate transformation plus other symmetries of the algebra by putting the curvature of the gauge field to zero. Since the vielbein is the only field that transforms under the P -transformations, see (4.2), we are led to impose the following so-called conventional constraint:

$$R_{\mu\nu}^A(P) = 0. \quad (4.10)$$

Such a conventional constraint also allows one to solve for the Lorentz gauge field ω_μ^{AB} in terms of (derivatives of) the vielbein and its inverse, and this gives the solution (2.55):

$$\omega_\mu^{AB}(e, \partial e) = 2e^{\lambda[A} \partial_{[\lambda} e_{\mu]}^{B]} + e_\mu^C e^{\lambda A} e^{\rho B} \partial_{[\lambda} e_{\rho]}^C. \quad (4.11)$$

What remains is a theory with the vielbein e_μ^A as the only independent field transforming under local Lorentz transformations and general coordinate transformations and with ω_μ^{AB} as the dependent spin connection field.²

A Γ -connection may be introduced by imposing the vielbein postulate (2.51)

$$\nabla_\mu e_\nu^A = D_\mu e_\nu^A - \Gamma_{\mu\nu}^\rho e_\rho^A = 0, \quad (4.12)$$

where D_μ is the Lorentz-covariant derivative. The antisymmetric part of this equation, together with the curvature constraint (4.10), shows that the antisymmetric part of the Γ -connection is zero, i.e. there is no torsion. From the vielbein postulate (4.12) one may solve the Γ -connection in terms of the vielbein and its inverse as follows:

$$\Gamma_{\mu\nu}^\rho = e^\rho_A D_\mu e_\nu^A. \quad (4.13)$$

Finally, a non-degenerate metric and its inverse can be defined as

$$g_{\mu\nu} = e_\mu^A e_\nu^B \eta_{AB}, \quad g^{\mu\nu} = e_A^\mu e_B^\nu \eta^{AB}. \quad (4.14)$$

This concludes our description of the basic ingredients of off-shell Einstein gravity and the Poincaré algebra. These basic ingredients are an independent non-degenerate metric $g_{\mu\nu}$ and a dependent Γ -connection $\Gamma_{\mu\nu}^\rho$ or, in the presence of flat indices, an independent vielbein field e_μ^A and a dependent spin-connection field ω_μ^{AB} . The theory can be put on-shell by imposing the Einstein equations of motion (2.29), in which the Riemann tensor is expressed via eqn.(2.25) and the vielbein postulate as³

$$R_{\nu\rho\sigma}^\mu(\Gamma) = -e^\mu_A e_{\nu B} R_{\rho\sigma}^{AB}(M), \quad (4.15)$$

where $R_{\rho\sigma}^{AB}(M)$ is the curvature associated to Lorentz transformations, eqn.(4.5).

Besides the gravitational dynamics, the geodesic equation for a point particle can also be obtained by applying a gauging procedure to the action of a particle in flat spacetime (3.23). This procedure is outlined in [72]. It clarifies why the gauge field e_μ^A can be identified with the vielbein and that this identification is only possible if the spacetime translations are removed.

4.3 Newton-Cartan Gravity

From now on we restrict the discussion to $D = 4$, i.e. one time and three space directions. We wish to review Newton-Cartan gravity as a geometric rewriting of Newtonian gravity

²From the variation of eqn.(4.10) one can also solve for the variation $\delta\omega_\mu^{AB}$. With this one can check that a Lorentz transformation on the dependent spin connection is still given by eqn.(4.3).

³Note that via the Bianchi identities and the vielbein postulate this relation give us the familiar conditions $R_{[\nu\rho\sigma]}^\mu(\Gamma) = 0$ and $\nabla_{[\lambda} R_{\mu\nu]\rho\sigma}(\Gamma) = 0$, see eqns.(2.27).

[22, 62], which was considered in section 2.2. This geometric re-formulation is motivated by the following observation. From the Newtonian point of view the equations of motion (2.15) describe a *curved* trajectory in a *flat* three-dimensional space. We now wish to re-interpret the same equations as a geodesic in a *curved* four-dimensional spacetime. Indeed, one may rewrite the equations (2.15) as the geodesic equations of motion

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0, \quad (4.16)$$

provided that one chooses the coordinate $\{x^0\}$ equal to the evolution parameter τ and takes the following expression for the non-zero connection fields:

$$\Gamma_{00}^i = \delta^{ij} \partial_j \phi, \quad (4.17)$$

where we have used the Euclidean three-metric. At this point $\Gamma_{\nu\rho}^\mu$ is a symmetric connection independent of the metric. The coordinate choice $x^0 = \tau$ corresponds to choosing the static gauge. The corresponding D -dimensional spacetime is called the Newton-Cartan spacetime \mathcal{M} . The only non-zero component of the Riemann tensor corresponding to the connection (4.17) is

$$R^i{}_{0j0} = \delta^{ik} \partial_k \partial_j \phi. \quad (4.18)$$

If one now imposes the equations of motion $R_{00} = 4\pi G\rho$ one obtains the Poisson equation (5.42). To write the Poisson equation in a covariant way we first must introduce a metric.

As it stands, the Γ -connection defined by (4.17) cannot follow from a non-degenerate four-dimensional metric. One way to see this is to consider the Riemann tensor that is defined by this Γ -connection. The Riemann tensor, defined in terms of a metric connection based upon a non-degenerate metric, satisfies the symmetry properties (2.27). One may easily verify that these properties are not satisfied by the Riemann tensor (4.18). Another way to see that a degenerate metric is unavoidable is to consider the relativistic Minkowski metric and its inverse

$$\eta_{\mu\nu}/c^2 = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_3/c^2 \end{pmatrix}, \quad \eta^{\mu\nu} = \begin{pmatrix} -1/c^2 & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix}. \quad (4.19)$$

Taking the limit $c \rightarrow \infty$ naturally leads to a degenerate covariant temporal metric $\tau_{\mu\nu}$ with three zero eigenvalues and a degenerate contra-variant spatial metric $h^{\mu\nu}$ with one zero eigenvalue. We conclude that the Galilei group keeps invariant *two* metrics $\tau_{\mu\nu}$ and $h^{\mu\nu}$ which are degenerate, i.e. $h^{\mu\nu} \tau_{\nu\rho} = 0$. This is precisely the Galilean metric structure derived in section 2.8. Since $\tau_{\mu\nu}$ is effectively a 1×1 matrix we will below use its vielbein version which is defined by a covariant vector τ_μ defined by $\tau_{\mu\nu} = \tau_\mu \tau_\nu$.

Looking at section 2.8, a degenerate spatial metric $h^{\mu\nu}$ of rank 3 and a degenerate temporal vielbein τ_μ of rank 1, together with a symmetric connection $\Gamma_{\mu\nu}^\rho$ on \mathcal{M} , that

depends on these two degenerate metrics, can be introduced as follows [65]. First of all, the degeneracy implies that

$$h^{\mu\nu}\tau_\nu = 0. \quad (4.20)$$

We next impose metric compatibility:⁴

$$\nabla_\rho h^{\mu\nu} = 0, \quad \nabla_\rho \tau_\mu = 0. \quad (4.21)$$

The covariant derivative ∇ is with respect to a connection $\Gamma_{\mu\nu}^\rho$. The second of these conditions indicates that

$$\tau_\mu = \partial_\mu f(x) \quad (4.22)$$

for a scalar function $f(x)$. In General Relativity metric compatibility and the absence of torsion allows one to write down the connection in terms of the metric and its derivatives in a unique way, see eqn. (4.13). In the present analysis, the connection $\Gamma_{\mu\nu}^\rho$ is not uniquely determined by the metric compatibility conditions (4.21). This can be seen from the fact that these conditions are preserved by the shift

$$\Gamma_{\mu\nu}^\rho \rightarrow \Gamma_{\mu\nu}^\rho + h^{\rho\lambda} K_{\lambda(\mu}\tau_{\nu)} \quad (4.23)$$

for an arbitrary two-form $K_{\mu\nu}$ [52]. Using this arbitrary two-form it is possible to write down the most general connection which is compatible with (4.21). In order to do this, one needs to introduce new tensors, the spatial inverse metric $h_{\mu\nu}$ and the temporal inverse vielbein τ^μ which are defined by the following properties:

$$\begin{aligned} h^{\mu\nu}h_{\nu\rho} &= \delta_\rho^\mu - \tau^\mu\tau_\rho, & \tau^\mu\tau_\mu &= 1, \\ h^{\mu\nu}\tau_\nu &= 0, & h_{\mu\nu}\tau^\nu &= 0. \end{aligned} \quad (4.24)$$

Geometrically the tensor $h^{\mu\nu}h_{\nu\rho}$ is a projection operator from the spacetime to the spatial sections, whereas $\tau^\mu\tau_\rho$ is a projection operator from spacetime to the temporal direction. Note that from the conditions (4.24) it follows that

$$\nabla_\rho h_{\mu\nu} = -2\tau_{(\mu}h_{\nu)\sigma}\nabla_\rho\tau^\sigma \quad (4.25)$$

which is not zero in general. The most general connection compatible with (4.21) is then [52]

$$\Gamma_{\mu\nu}^\sigma = \tau^\sigma\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\sigma\rho}\left(\partial_\nu h_{\rho\mu} + \partial_\mu h_{\rho\nu} - \partial_\rho h_{\mu\nu}\right) + h^{\sigma\lambda}K_{\lambda(\mu}\tau_{\nu)}. \quad (4.26)$$

We note that the original independent connection (4.17) is quite different from the metric connection defined in (4.26). Nevertheless, given extra conditions discussed below, the Newton-Cartan theory with the metric connection (4.26) reproduces Newtonian gravity.

⁴Note that we do not impose metric compatibility on $h_{\mu\nu}$ and τ^μ !

To see how this goes, it is convenient to use adapted coordinates $f(x) = x^0$ in the condition (4.22). The metric relations (4.24) then imply

$$\begin{aligned}\tau_\mu &= \delta_\mu^0, & \tau^\mu &= (1, \tau^i), \\ h^{\mu 0} &= 0, & h_{\mu 0} &= -h_{\mu i} \tau^i,\end{aligned}\tag{4.27}$$

or in explicit matrix form

$$\begin{aligned}h^{\mu\nu} &= \begin{pmatrix} 0 & 0 \\ 0 & h^{ij} \end{pmatrix}, & \tau_{\mu\nu} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\ h_{\mu\nu} &= \begin{pmatrix} h_{ij} \tau^i \tau^j & -h_{ij} \tau^j \\ -h_{ij} \tau^j & h_{ij} \end{pmatrix}, & \tau^{\mu\nu} &= \begin{pmatrix} 1 & \tau^i \\ \tau^i & \tau^i \tau^j \end{pmatrix}.\end{aligned}\tag{4.28}$$

The choice of adapted coordinates is preserved by the coordinate transformations

$$\begin{aligned}x^0 &\rightarrow x^0 + \zeta^0, \\ x^i &\rightarrow x^i + \xi^i(x),\end{aligned}\tag{4.29}$$

where ζ^0 is a constant and $\xi^i(x)$ depends on both space and time. The finite spatial transformation generated by $\xi^i(x)$ is invertible. In adapted coordinates $f(x) = x^0$ the connection coefficients (4.26) are given by [52]

$$\begin{aligned}\Gamma_{00}^i &= h^{ij}(\partial_0 h_{j0} - \frac{1}{2} \partial_j h_{00} + K_{j0}) \equiv h^{ij} \Phi_j, \\ \Gamma_{0j}^i &= h^{ik}(\frac{1}{2} \partial_0 h_{jk} + \partial_{[j} h_{k]0} - \frac{1}{2} K_{jk}) \equiv h^{ik}(\frac{1}{2} \partial_0 h_{jk} + \Omega_{jk}), \\ \Gamma_{jk}^i &= \{^i_{jk}\}, & \Gamma_{\mu\nu}^0 &= 0,\end{aligned}\tag{4.30}$$

where $\{^i_{jk}\}$ are the usual Christoffel symbols (2.24) with respect to the metric h_{ij} with inverse h^{ij} .

We now replace the original equations of motion $R_{00} = 4\pi G\rho$ by the covariant Ansatz

$$R_{\mu\nu} = 4\pi G\rho \tau_\mu \tau_\nu\tag{4.31}$$

and verify that this leads to Newtonian gravity. In adapted coordinates these equations imply that

$$R_{ij} = R_{i0} = 0.\tag{4.32}$$

The condition $R_{ij} = 0$ implies that the spatial hypersurfaces are flat,⁵ i.e. one can choose a coordinate frame with $\Gamma_{jk}^i = 0$ such that the spatial metric is given by

$$h_{ij} = \delta_{ij}, \quad h^{ij} = \delta^{ij}.\tag{4.33}$$

⁵Note that *only* in three dimensions a vanishing Ricci tensor implies a vanishing Riemann tensor, due to the three-dimensional identity (2.35), which for spatial indices reads $R_{ijkl} = 4g_{[i[k}R_{l]j]} - Rg_{[i[k}g_{l]j]}$.

This implies

$$\begin{aligned}\Gamma_{0j}^i &= h^{ik}\Omega_{jk} \leftrightarrow \Omega_{ij} = h_{k[j}\Gamma_{i]0}^k, \\ \Gamma_{00}^i &= h^{ij}\Phi_j \leftrightarrow \Phi_i = h_{ij}\Gamma_{00}^j.\end{aligned}\quad (4.34)$$

The choice of a flat metric further reduces the allowed coordinate transformations (4.29) to⁶

$$t \rightarrow t + \zeta^0, \quad x^i \rightarrow A^i_j(t)x^j + \xi^i(t), \quad (4.35)$$

where $A^i_j(t)$ is an element of $\text{SO}(3)$.

To derive the Poisson equation from the Ansatz (4.31) two additional conditions must be invoked. The first is the Trautman condition [66]:

$$h^{\sigma[\lambda}R^{\mu]}_{(\nu\rho)\sigma}(\Gamma) = 0. \quad (4.36)$$

In General Relativity this condition is automatically satisfied for the metric $g_{\mu\nu}$, as can be checked via the identities (2.27). In adapted coordinates the Trautman condition (4.36) implies

$$\partial_0\Omega_{mi} - \partial_{[m}\Phi_{i]} = 0, \quad \partial_{[k}\Omega_{mi]} = 0. \quad (4.37)$$

Although Φ_i and Ω_{ij} are not tensors under general coordinate transformations⁷ as they are part of Christoffel symbols, both equations of (4.37) are separately covariant under (4.35) which can be checked explicitly. Using the definitions (4.34) of Φ_i and Ω_{ij} one may verify that the conditions (4.37) are equivalent to the manifestly tensorial equation

$$\partial_{[\rho}K_{\mu\nu]} = 0 \rightarrow K_{\mu\nu} = 2\partial_{[\mu}m_{\nu]}, \quad (4.38)$$

where m_μ is a vector field determined up to the derivative of some scalar field.

The second condition we need is that Ω_{ij} , see (4.30), depends only on time, not on space coordinates [52, 65]. In [65] three possible conditions on the Riemann tensor are discussed that lead to the desired restriction on Ω_{ij} :

$$\begin{aligned}h^{\rho\lambda}R^\mu_{\nu\rho\sigma}(\Gamma)R^\nu_{\mu\lambda\varsigma}(\Gamma) &= 0, \quad \text{or} \\ \tau_{[\lambda}R^\mu_{\nu]\rho\sigma}(\Gamma) &= 0, \quad \text{or} \\ h^{\sigma[\lambda}R^\mu]_{\nu\rho\sigma}(\Gamma) &= 0.\end{aligned}\quad (4.39)$$

These three conditions are the so-called Ehlers conditions. Each condition separately leads to the condition $\partial_k\Omega_{ij} = 0$ in adapted coordinates and thus $\Omega_{ij} = \Omega_{ij}(t)$. One can next set

⁶We write $\{x^0 = t\}$.

⁷See also [73] for a detailed discussion on accelerations in Newton-Cartan theory.

$\Omega'_{ij} \equiv 0$, or equivalently $\Gamma'_{0j}{}^i \equiv 0$, see (4.34), by a time-dependent rotation $x'^i = A^i{}_j(t)x^j$ [52]. The conditions (4.37) imply that in the new coordinate system $\partial'_{[i}\Phi'_{j]} = 0$ and hence that $\Phi'_i = \partial'_i\Phi$ for some scalar field Φ . This implies that

$$\Gamma'_{00}{}^i = \delta^{ij}\partial'_j\Phi \quad (4.40)$$

in this coordinate system. The equations (4.31) thus lead to the Poisson equation:

$$R_{00} = \partial_i\Gamma_{00}{}^i = \delta^{ij}\partial_i\partial_j\phi = 4\pi G\rho. \quad (4.41)$$

Finally, we should also recover the geodesic equation (4.16). Using adapted coordinates and performing the above time-dependent rotation indeed gives the desired equations:⁸

$$\ddot{x}'^0(\tau) = 0, \quad \ddot{x}'^i(\tau) + \partial'^i\Phi = 0. \quad (4.42)$$

This completes the proof that Newton-Cartan gravity, formulated in terms of two degenerate metrics (see eqn. (4.20)), and supplied with the Trautman condition (4.36) and the Ehlers conditions (4.39), precisely leads to the equations of Newtonian gravity. The differences between Newton-Cartan gravity and General Relativity are summarized in the following table:

	Newton-Cartan theory	General Relativity
Metric:	Degenerate due to absolute time	Non-degenerate
Connection:	Determined up to arbitrary two-form	Uniquely determined
Curvature:	Only in space-time direction	Spacetime

Table 4.1: The differences between Newton-Cartan theory and General Relativity.

In the next section we will show how the same Newton-Cartan theory, including the Trautman and Ehlers conditions, follows from gauging the Bargmann algebra.

4.4 The Bargmann algebra

The Bargmann algebra is the Galilean algebra augmented with a central generator⁹ Z and can be obtained as follows. We first extend the Poincaré algebra $\mathfrak{iso}(D-1, 1)$ to the direct sum of the Poincaré algebra and a commutative subalgebra \mathfrak{g}_M spanned by Z :

$$\mathfrak{iso}(D-1, 1) \rightarrow \mathfrak{iso}(D-1, 1) \oplus \mathfrak{g}_M. \quad (4.43)$$

⁸Notice the difference with the Newtonian limit on General Relativity as discussed in section 2.7, where space is curved but the $(\dot{x}^i$ -terms drop out of the geodesic equation because these are considered to be negligible.

⁹In $D = 3$ dimensions three such central generators can be introduced [67, 68].

We next perform a Inönü-Wigner contraction [54] on this algebra. Such a contraction is a singular transformation on the Lie algebra, which is a vector space. This contraction reads

$$P_0 \rightarrow \frac{1}{\omega^2}Z + H, \quad P_a \rightarrow \frac{1}{\omega}P_a, \quad J_{a0} \rightarrow \frac{1}{\omega}G_a, \quad \omega \rightarrow 0, \quad (4.44)$$

which is then interpreted as the non-relativistic limit. Notice that in the contraction of the commutator $[J_{a0}, P_0]$ one obtains the potentially dangerous term $\omega^{-3}[G_a, Z]$, which is however zero because Z is assumed to be a central element and as such commutes with all the other Galilei generators. The contraction of P_0 is motivated by considering the non-relativistic approximation of P_0 for a massive free particle,¹⁰

$$P_0 = +\sqrt{c^2 P_a P^a + M^2 c^4} \approx M c^2 + \frac{P_a P^a}{2M}, \quad (4.45)$$

where $\omega = c^{-1}$. The contracted algebra is the so-called Bargmann algebra $\mathfrak{b}(D-1, 1)$ which has the following non-zero commutation relations:

$$\begin{aligned} [J_{ab}, J_{cd}] &= 4\delta_{[a[c}J_{d]b]}, & [J_{ab}, P_c] &= -2\delta_{c[a}P_{b]}, \\ [J_{ab}, G_c] &= -2\delta_{c[a}G_{b]}, & [G_a, H] &= -P_a, \\ [G_a, P_b] &= -\delta_{ab}Z. \end{aligned} \quad (4.46)$$

For $Z = 0$ this is the Galilean algebra. Note that the last commutator is in line with the Poisson brackets (3.47).

4.5 Gauging the Bargmann algebra

We now gauge the Bargmann algebra (4.46) following the same procedure we applied to the Poincaré algebra (4.1) in Section 4.2.

Compared to the Poincaré case the gauge fields and parameters corresponding to the Bargmann algebra split up into a spatial and temporal part:

$$\begin{aligned} e_\mu^A &\rightarrow \{e_\mu^{\underline{0}}, e_\mu^a\}, & \omega_\mu^{AB} &\rightarrow \{\omega_\mu^{ab}, \omega_\mu^a_{\underline{0}} \equiv \omega_\mu^a\} \\ \zeta^A &\rightarrow \{\zeta^{\underline{0}}, \zeta^a\}, & \lambda^{AB} &\rightarrow \{\lambda^{ab}, \lambda^a_{\underline{0}} \equiv \lambda^a\}. \end{aligned} \quad (4.47)$$

The gauge field corresponding to the generator Z will be called m_μ and its gauge parameter will be called σ . We label $e_\mu^{\underline{0}} = \tau_\mu$ and $\zeta^{\underline{0}} = \tau$, where the parameter τ shouldn't be confused with the evolution parameter τ we use in the point particle actions. The

¹⁰Note that there are more contractions which lead to the same algebra.

variations of the gauge fields corresponding to the different generators are given by

$$\begin{aligned}
H : \quad & \delta\tau_\mu = \partial_\mu\tau, \\
P : \quad & \delta e_\mu^a = \partial_\mu\zeta^a - \omega_\mu^{ab}\zeta^b + \lambda^{ab}e_\mu^b + \lambda^a\tau_\mu - \tau\omega_\mu^a, \\
G : \quad & \delta\omega_\mu^a = \partial_\mu\lambda^a - \lambda^b\omega_\mu^{ab} + \lambda^{ab}\omega_\mu^b, \\
J : \quad & \delta\omega_\mu^{ab} = \partial_\mu\lambda^{ab} + 2\lambda^{c[a}\omega_\mu^{b]c}, \\
Z : \quad & \delta m_\mu = \partial_\mu\sigma - \zeta^a\omega_\mu^a + \lambda^a e_\mu^a.
\end{aligned} \tag{4.48}$$

One can notice the non-relativistic nature of these variations by the boost transformations G . Namely, under a boost the spatial vielbein e_μ^a transforms to a temporal vielbein τ_μ but not vice versa. It is now the vector field m_μ of the central extension which transforms under a boost to the spatial vielbein.

The curvatures of the gauge fields read

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]}, \tag{4.49}$$

$$\begin{aligned}
R_{\mu\nu}^a(P) &= 2(D_{[\mu}e_{\nu]}^a - \omega_{[\mu}^a\tau_{\nu]}) \\
&= 2(\partial_{[\mu}e_{\nu]}^a - \omega_{[\mu}^{ab}e_{\nu]}^b - \omega_{[\mu}^a\tau_{\nu]}),
\end{aligned} \tag{4.50}$$

$$R_{\mu\nu}^{ab}(J) = 2(\partial_{[\mu}\omega_{\nu]}^{ab} - \omega_{[\mu}^{ca}\omega_{\nu]}^{bc}) \tag{4.51}$$

$$\begin{aligned}
R_{\mu\nu}^a(G) &= 2D_{[\mu}\omega_{\nu]}^a, \\
&= 2(\partial_{[\mu}\omega_{\nu]}^a - \omega_{[\mu}^{ab}\omega_{\nu]}^b),
\end{aligned} \tag{4.52}$$

$$R_{\mu\nu}(Z) = 2(\partial_{[\mu}m_{\nu]} - \omega_{[\mu}^a e_{\nu]}^a). \tag{4.53}$$

The derivative D_μ is covariant with respect to the J -transformations and as such only contains the ω_μ^{ab} gauge field. Using the general formula (4.7) we convert the P and H transformations into general coordinate transformations in space and time. We write the parameter of the general coordinate transformations ξ^λ in (4.7) as

$$\xi^\lambda = e^\lambda_a \zeta^a + \tau^\lambda \tau. \tag{4.54}$$

Here we have used the inverse spatial vielbein e^λ_a and the inverse temporal vielbein τ^λ defined by

$$e_\mu^a e^\mu_b = \delta_b^a, \quad \tau^\mu \tau_\mu = 1, \tag{4.55}$$

$$\tau^\mu e_\mu^a = 0, \quad \tau_\mu e^\mu_a = 0, \tag{4.56}$$

$$e_\mu^a e^\nu_a = \delta_\mu^\nu - \tau_\mu \tau^\nu. \tag{4.57}$$

These conditions are the vielbein version of the conditions (4.24), and imply the variations¹¹

$$\delta e^\mu_a = -e^\mu_b e^\nu_a \delta e_\nu^b - \tau^\mu e^\nu_a \delta \tau_\nu, \tag{4.58}$$

$$\delta \tau^\mu = -\tau^\mu \tau^\nu \delta \tau_\nu - e^\mu_b \tau^\nu \delta e_\nu^b. \tag{4.59}$$

¹¹Using the explicit forms (4.28) we can now do a check of counting, as we did in the relativistic case

Now we observe that only the gauge fields e_μ^a , τ_μ and m_μ transform under the P and H transformations. These are the fields that should remain independent. Namely, the fields e_μ^a , τ_μ are going to be interpreted as vielbeins. The field m_μ cannot be solved for; it is associated to a central extension, and as such does not occur with vielbeins in any of the gauge curvatures. The spin connections are expected to become dependent fields. These demands motivate the following constraints:

$$R_{\mu\nu}(H) = R_{\mu\nu}{}^a(P) = R_{\mu\nu}(Z) = 0, \quad (4.60)$$

of which the last two are called conventional. The Bianchi identities, see appendix C, then lead to additional relations between curvatures:

$$R_{[\lambda\mu}{}^{ab}(J)e_{\nu]}{}^b = -R_{[\lambda\mu}{}^a(G)\tau_{\nu]}, \quad e_{[\lambda}{}^a R_{\mu\nu]}{}^a(G) = 0. \quad (4.61)$$

The constraint $R_{\mu\nu}(H) = 0$ gives the condition $\partial_{[\mu}\tau_{\nu]} = 0$ and hence we may take τ_μ as in the condition (4.22). The other two conventional constraints, $R_{\mu\nu}{}^a(P) = R_{\mu\nu}(Z) = 0$, enable us to solve for the spin connection gauge fields $\{\omega_\mu^{ab}, \omega_\mu^a\}$ in terms of the other gauge fields, so that indeed only e_μ^a , τ_μ and m_μ remain as independent fields.

To solve for ω_μ^{ab} , we write

$$R_{\mu\nu}{}^a(P)e_\rho^a + R_{\rho\mu}{}^a(P)e_\nu^a - R_{\nu\rho}{}^a(P)e_\mu^a = 0. \quad (4.62)$$

From this it follows that

$$\omega_\mu^{ab} = 2e^{\nu[a}\partial_{[\nu}e_{\mu]}{}^{b]} + e_\mu{}^c e^{\nu a} e^{\rho b} \partial_{[\nu}e_{\rho]}{}^c - \tau_\mu e^{\rho[a}\omega_{\rho}{}^{b]}. \quad (4.63)$$

Next we solve for ω_μ^a . We substitute (4.63) into $R_{\mu\nu}{}^a(P) = 0$ and contract this with $e^{\mu b}$ and τ^ν . This gives the condition

$$e^{\mu(a}\omega_{\mu}{}^{b)} = 2e^{\mu(a}\partial_{[\mu}e_{\nu]}{}^{b)}\tau^\nu. \quad (4.64)$$

Furthermore, $R_{\mu\nu}(Z) = 0$ can be contracted with $e^\mu{}_a$ and τ^μ to give the following conditions:

$$e^{\mu[a}\omega_{\mu}{}^{b]} = e^{\mu a}e^{\nu b}\partial_{[\mu}m_{\nu]}, \quad \tau^\mu\omega_{\mu}{}^a = 2\tau^\mu e^{\nu a}\partial_{[\mu}m_{\nu]}. \quad (4.65)$$

Using the constraints (4.64) and (4.65) one arrives at the following solution for ω_μ^a :

$$\omega_\mu^a = e^{\nu a}\partial_{[\mu}m_{\nu]} + e^{\nu a}\tau^\rho e_{\mu}{}^b\partial_{[\nu}e_{\rho]}{}^b + \tau_\mu\tau^\nu e^{\rho a}\partial_{[\nu}m_{\rho]} + \tau^\nu\partial_{[\mu}e_{\nu]}{}^a. \quad (4.66)$$

At this point we are left with the independent fields e_μ^a , τ_μ and m_μ . Furthermore, the theory is still off-shell; no equations of motion have been imposed.

after eqn.(2.49). The number of independent components of $h_{\mu\nu}$ is given by those of h_{ij} and τ^i , which is $\frac{1}{2}(D+2)(D-1)$ in total. This equals the number of independent components of e_μ^a minus the number of components of $\lambda^a{}_b$, namely $D(D-1) - \frac{1}{2}(D-1)(D-2)$. We don't subtract the number of components of the boost parameter, because $h_{\mu\nu}$ is not invariant under $\delta e_\mu^a = \lambda^a{}_\tau \tau_\mu$. The metric $h^{\mu\nu}$, which is effectively a symmetric $(D-1) \times (D-1)$ matrix, is invariant under both boost and rotation transformations of the inverse vielbein $e^\mu{}_a$, and one gets the equality $\frac{1}{2}D(D-1) = D(D-1) - \frac{1}{2}(D-1)(D-2) - (D-1)$.

4.6 Newton-Cartan Gravity

To make contact with the formulation of Newton-Cartan gravity presented in Section 4.3 we need to introduce a Γ -connection. First a vielbein postulate for the spatial vielbein is imposed,

$$\partial_\mu e_\nu^a - \omega_\mu^{ab} e_\nu^b - \omega_\mu^a \tau_\nu - \Gamma_{\mu\nu}^\rho e_\rho^a = 0. \quad (4.67)$$

Notice that this postulate implies that $\nabla_\mu e_\nu^a = \omega_\mu^a \tau_\nu$, or

$$\nabla_\rho h_{\mu\nu} = 2\omega_\rho^a e_{(\mu}^a \tau_{\nu)}, \quad (4.68)$$

which can be compared with eqn.(4.25). The second vielbein postulate is for the temporal vielbein,

$$\partial_\mu \tau_\nu - \Gamma_{\mu\nu}^\lambda \tau_\lambda = 0, \quad (4.69)$$

which is the second condition of (4.21). Note that invariance of the first vielbein postulate (4.67) under local Galilei boosts is guaranteed by the second vielbein postulate (4.69). These two vielbein postulates together imply

$$\Gamma_{\mu\nu}^\rho = \tau^\rho \partial_{(\mu} \tau_{\nu)} + e^{\rho a} \left(\partial_{(\mu} e_{\nu)}^a - \omega_{(\mu}^{ab} e_{\nu)}^b - \omega_{(\mu}^a \tau_{\nu)} \right). \quad (4.70)$$

This connection is symmetric due to the curvature constraints $R_{\mu\nu}^a(P) = R_{\mu\nu}(H) = 0$, and satisfies the metric conditions (4.21).

An important difference between the metric compatibility conditions given in (4.21) and in (4.67, 4.69) is that the latter define the connection Γ uniquely. From (4.26) and (4.70) we find that

$$K_{\mu\nu} = 2\omega_{[\mu}^a e_{\nu]}^a, \quad (4.71)$$

with ω_μ^a given by (4.66). This implies via the $R(M) = 0$ constraint that

$$K_{\mu\nu} = 2\partial_{[\mu} m_{\nu]}, \quad (4.72)$$

which solves the condition (4.38). The Riemann tensor corresponding to (4.70) can now be expressed in terms of the curvature tensors of the gauge algebra:

$$\begin{aligned} R_{\nu\rho\sigma}^\mu(\Gamma) &= \partial_\rho \Gamma_{\nu\sigma}^\mu - \partial_\sigma \Gamma_{\nu\rho}^\mu + \Gamma_{\nu\sigma}^\lambda \Gamma_{\lambda\rho}^\mu - \Gamma_{\nu\rho}^\lambda \Gamma_{\lambda\sigma}^\mu \\ &= -e^\mu_a \left(R_{\rho\sigma}^a(G) \tau_\nu + R_{\rho\sigma}^{ab}(J) e_{\nu b} \right). \end{aligned} \quad (4.73)$$

Here we have used (4.60). The Trautman condition (4.36), applied to (4.73), is equivalent to the first constraint of (4.61).

We know from the analysis in section 3 that, in order to make contact with the Newton-Cartan formulation, we must impose the Ehlers conditions (4.39). One can show that each of the three Ehlers conditions (4.39) is equivalent to the single curvature constraint

$$R_{\mu\nu}^{ab}(J) = 0. \quad (4.74)$$

Substituting this result into (4.61) leads to the following constraints on $R_{\mu\nu}{}^a(G)$:

$$R_{[\lambda\mu}{}^a(G)\tau_{\nu]} = 0, \quad e_{[\lambda}{}^a R_{\mu\nu]}{}^a(G) = 0. \quad (4.75)$$

The contraction of (4.75) with $e^\mu{}_a$ and τ^μ gives

$$e^\mu{}_a e^\nu{}_b R_{\mu\nu}{}^c(G) = 0, \quad \tau^\mu e^\nu{}^a R_{\mu\nu}{}^b(G) = 0. \quad (4.76)$$

This implies that the only non-zero component of $R_{\mu\nu}{}^a(G)$ is

$$\tau^\mu e^\nu{}^a R_{\mu\nu}{}^b(G) = \delta^{c(a} R^b)_{\underline{0c\underline{0}}}(\Gamma) \quad (4.77)$$

which in flat coordinates is precisely the only non-zero component (4.18) of the Riemann tensor that occurs in the Newton-Cartan formulation. Under a local boost this equation transforms, upon using the first Bianchi identity of (4.61), to an equation of motion for the rotational curvature.¹² The full set of equations of motion are

$$\tau^\mu e^\nu{}^a R_{\mu\nu}{}^a(G) = 4\pi G\rho, \quad \tau^\mu e^\nu{}^a R_{\mu\nu}{}^{ab}(J) = 0. \quad (4.78)$$

Similarly to the relativistic case, one can also obtain the geodesic equation from a gauging procedure similar to the analysis in [72]. For that it is important that one starts with the point particle action (3.40) which is *invariant* under the Galilei group instead of *quasi-invariant*.

At this point we have made contact with the Newton-Cartan gravity theory presented in Section 4.3. We have the same Γ -connection and (degenerate) metrics. It can be shown that these lead to the desired Poisson equation following the same steps as in Section 4.3. The explicit form of the Newton potential in terms of the gauge fields will be considered in the next chapter.

4.7 Conclusions

In this chapter we have shown how, just like Einstein gravity, the Newton-Cartan formulation of Newtonian gravity can be obtained by a gauging procedure. The Lie algebra underlying this procedure is the Bargmann algebra given in (4.46). To obtain the correct Newton-Cartan formulation we need to impose constraints on the curvatures. In a first step we impose the curvature constraints (4.60). They enable us to convert the spatial (time) translational symmetries of the Bargmann algebra into spatial (time) general coordinate transformations. At the same time they enable us to solve for the spin-connection

¹²We already have imposed the constraint (4.74), but we should make a distinction between constraints and equations of motion. Note that the inverse vielbein τ^μ transforms under a local boost, $\delta\tau^\mu = -\lambda^a e^\mu{}_a$, while $e^\mu{}_a$ is boost-invariant.

gauge fields $\omega_\mu{}^a$ and $\omega_\mu{}^{ab}$ in terms of the remaining gauge fields $e_\mu{}^a, \tau_\mu$ and m_μ , see eqs. (4.63) and (4.66). For this to work it is essential that we work with a non-zero central element Z in the algebra. So far, we work off-shell without comparing equations of motion.

In a second step we impose the vielbein postulates (4.67) and (4.69). These enable us to solve for the Γ connection thereby solving the Trautman condition (4.36) automatically. In order to obtain the correct Poisson equation we impose in a third step the additional curvature constraints (4.74) which are equivalent to each of the three Ehlers conditions (4.39). The Poisson equation is obtained from the relation (4.77) between the curvature of the dependent field $\omega_\mu{}^a$ and the Newton-Cartan Riemann tensor in the form (4.18) and gives also a similar equation of motion of the rotational curvature.. The independent gauge fields $e_\mu{}^a$ and τ_μ describe the degenerate metrics of Newton-Cartan gravity.

One of the original motivations of this analysis was the possible role of Newton-Cartan gravity in non-relativistic applications of the AdS-CFT correspondence. In most applications the relativistic symmetries of the AdS bulk theory are broken by the vacuum solution one considers.¹³ This is the case if one considers the Schrodinger or Lifshitz algebras. The situation changes if one considers the Galilean Conformal Algebra instead. It has been argued that in that case the bulk gravity theory is given by an extension of the Newton-Cartan theory where the spacetime metric is degenerate with *two* zero eigenvalues corresponding to the time and the radial directions [92]. This leads to a foliation where the time direction is replaced by a two-dimensional AdS₂ space. This requires a contraction of the Poincaré algebra in which the Bargmann algebra is replaced by a deformed string Galilean algebra or, if one includes the cosmological constant, by a stringy Newton-Hooke algebra [82, 86].¹⁴ This construction will be considered in the next chapter.

¹³For other aspects of Newton-Cartan gravity, see, e.g., [76, 90]

¹⁴For other applications of the Newton-Hooke algebra see, e.g., [77, 78].

Chapter 5

“Stringy” Newton-Cartan gravity

5.1 Introduction

To apply General Relativity in practical situations it is often convenient to consider the Newtonian limit which is defined as the limit of small velocities $v \ll c$ with respect to the speed of light c , and a slowly varying and weak gravitational field. This limit was discussed in section 2.7, but is not the unique non-relativistic limit of General Relativity. It is a specific limit which is based upon the assumption that particles are the basic entities and it further makes the additional assumption of a slowly varying and weak gravitational field. In this chapter we will encounter different limits which are based upon strings or, more general, branes, as the basic objects, and which do not necessarily assume a slowly varying and weak gravitational field.

For practical purposes, it is convenient in the Newtonian limit to consider not only free-falling frames but to include all frames corresponding to a so-called “Galilean observer” [5, 64]. These are all frames that are accelerated, with arbitrary (time-dependent) acceleration, with respect to a free-falling frame. An example of a frame describing a Galilean observer with constant acceleration [61] is the one attached to the Earth’s surface, thereby ignoring the rotation of the Earth. Newton showed that in the constant-acceleration frames the gravitational force is described by a time-independent scalar potential $\Phi(x^i)$ ($i = 1, \dots, D - 1$). In frames with time-dependent acceleration the potential becomes an arbitrary function $\Phi(x)$ of the spacetime coordinates. A noteworthy difference between General Relativity and Newtonian gravity is that, while in General Relativity *any* observer can locally in spacetime use a general coordinate transformation to make the metric flat, in Newtonian gravity only the Galilean observers can use an acceleration to make the Newton potential disappear.

The equations of motion corresponding to a Galilean observer are invariant under the so-called “acceleration-extended” Galilei symmetries. This corresponds to an extension of

the Galilei symmetries in which the (constant) space translations and boost transformations have been gauged resulting into a theory which is invariant under spatial translations having an arbitrary time-dependency.¹ The gravitational potential can be viewed as the “background gauge field” necessary to realize these time-dependent translations. Starting from a free particle in a Newtonian spacetime, there are now two ways to derive the equations of motion for a Galilean observer from a gauging principle. If one is only interested in the physics observed by a Galilean observer it is sufficient to gauge the constant space translations by promoting the corresponding (constant) parameters to arbitrary functions of time. This automatically includes the gauging of the boost transformations. The equation of motion of a particle is then obtained by deforming the free equation of motion with the background gravitational potential $\Phi(x)$ such that the resulting equation is invariant under the acceleration-extended Galilei symmetries. The Poisson equation of $\Phi(x)$ can be obtained by realizing that it is the only equation, of second order in the spatial derivatives, that is invariant under the acceleration-extended Galilei symmetries.

It is natural to extend the above ideas and the gauging procedure of chapter 4 to strings. This will give us information about the gravitational forces as experienced by a non-relativistic string instead of a particle. Although the symmetries involved are different, the ideas are the same as in the particle case discussed above. The starting point in this case is a string moving in a flat Minkowski background. Taking the non-relativistic limit leads to the action for a non-relativistic string [80, 81, 83] that is invariant under a “stringy” version of the Galilei symmetries. The transformations involved, which will be specified later, are similar to the particle case except that now not only time but also the spatial direction along the string plays a special role. This leads to an $M_{1,1}$ -foliation of spacetime. Again, the Lagrangian is only invariant up to a total derivative (in the world-sheet coordinates) and hence we obtain an extension of the “stringy” Galilei algebra which involves two additional generators Z_a and $Z_{a'b'} = -Z_{b'a'}$ ($a' = 0, 1$). Due to the extra index structure these generators provide general extensions rather than central extensions of the stringy Galilei algebra [86].

Any two free-falling frames are now connected by a stringy Galilei transformation. A “stringy” Galilean observer is then defined as an observer with respect to any frame that is accelerated, with arbitrary (time and longitudinal coordinate dependent) acceleration, with respect to a free-falling frame. The corresponding acceleration-extended “stringy” Galilei symmetries are obtained by gauging the translations in the spatial directions transverse to the string by promoting the corresponding parameters to arbitrary functions of the world-sheet coordinates. These transformations involve the constant transverse trans-

¹The group of acceleration-extended Galilei symmetries is also called the Milne group [96]. It is generated by the algebra (3.57).

lations and the stringy boost transformations, which are linear in the world-sheet coordinates.

Again, there are two ways to obtain the equations of motion for a stringy Galilean observer. Either we start from the string in a Minkowski background and gauge the transverse translations. In the string case this requires the introduction of a background gravitational potential $\Phi_{\alpha\beta}(x) = \Phi_{\beta\alpha}(x)$ ($\alpha = 0, 1$), as was also pointed out in [92]. This is a striking difference with General Relativity where, independent of whether particles or strings are the basic objects, one always ends up with the same metric function $g_{\mu\nu}(x)$. This is related to the fact that in the non-relativistic case spacetime is a foliation and that the dimension of the foliation space depends on the nature of the basic object (particles, strings or branes).

The equation of motion for $\Phi_{\alpha\beta}(x)$ can be obtained by requiring that it is of second order in the transverse spatial derivatives and invariant under the acceleration-extended stringy Galilei transformations. Alternatively, one gauges the full deformed stringy Galilei algebra and imposes a set of kinematical constraints, like in the particle case. In the string case one requires that both the curvature of spatial rotations transverse to the string as well as the curvature of rotations among the foliation directions vanishes. This leads to a flat foliation corresponding to an $M_{1,1}$ -foliation of spacetime as well as to flat transverse directions. One next introduces the equations of motion making use of the (non-invertable) temporal and spatial metric and Christoffel symbols corresponding to the stringy Newton-Cartan spacetime. To make contact with a stringy Galilean observer one imposes gauge-fixing conditions which reduce the symmetries to the acceleration-extended stringy Galilei ones. As expected, the two approaches lead to precisely the same expression for the equation of motion of a fundamental string as well as of the gravitational potential $\Phi_{\alpha\beta}(x)$ itself.

This chapter is organized as follows. In section 2 we review, as a warming-up exercise, the particle case for zero cosmological constant. In section 3 we derive the relevant expressions for the stringy extension. In section 4 we apply the gauging procedure of the last chapter to the full (deformed) stringy Galilei symmetries. To study applications of the AdS/CFT correspondence based on the symmetry algebra corresponding to a non-relativistic string it is necessary to include a (negative) cosmological constant Λ . We will address this issue both for particles and strings in the last section.

5.2 The Particle Case

Our starting point is the action (3.14). Following [81, 82] we take the non-relativistic limit by rescaling the longitudinal coordinate $x^0 \equiv t$ and the mass m with a parameter ω and

taking $\omega \gg 1$:

$$x^0 \rightarrow \omega x^0, \quad m \rightarrow \omega m, \quad \omega \gg 1. \quad (5.1)$$

This rescaling is such that the kinetic term remains finite. This results into the following action:

$$S \approx -m\omega^2 \int \dot{x}^0 \left(1 - \frac{\dot{x}^i \dot{x}_i}{2\omega^2 (\dot{x}^0)^2} \right) d\tau, \quad i = 1, \dots, D-1. \quad (5.2)$$

The first term on the right-hand-side, which is a total derivative, can be cancelled by coupling the particle to a constant background gauge field A_μ by adding a term

$$S_I = m \int A_\mu \dot{x}^\mu d\tau, \quad (5.3)$$

and choosing $A_0 = \omega^2$ and $A_i = 0$ [83]. Because this A_μ can be written as a total derivative the associated field-strength vanishes, such that no dynamics for the background gauge field is introduced. However, this gauge field shifts the energy spectrum of all the particles which couple to it; the energy p_0 of such a particle is shifted with an amount of $m\omega^2$, such that it cancels the divergent rest energy. The limit $\omega \rightarrow \infty$ then yields the non-relativistic action (3.27):

$$S = \frac{m}{2} \int \frac{\dot{x}^i \dot{x}^j \delta_{ij}}{\dot{x}^0} d\tau. \quad (5.4)$$

This action is invariant under worldline reparametrizations and the Galilei symmetries (3.27). The equations of motion corresponding to the action (5.4) are given by eqn. (3.31):²

$$\ddot{x}^i = \frac{\ddot{x}^0}{\dot{x}^0} \dot{x}^i. \quad (5.5)$$

The non-relativistic Lagrangian (5.4) is invariant under boosts only up to a total τ -derivative, i.e.,

$$\delta L = \frac{d}{d\tau} (m x^i \lambda^j \delta_{ij}). \quad (5.6)$$

This leads to a modified Noether charge giving rise to a centrally extended Galilei algebra containing an extra so-called central charge generator Z , as we saw in the last chapter.

The above results apply to free-falling frames without any gravitational interactions. Such frames are connected to each other via the Galilei symmetries. We now wish to extend these results to include frames that apply to a Galilean observer, i.e. that are accelerated with respect to the free-falling frames, with arbitrary (time-dependent) acceleration. As explained in the introduction we can do this via two distinct gauging procedures. The first procedure is convenient if one is only interested in the physics experienced by a Galilean observer. In that case it is sufficient to gauge the transverse translations by replacing the constant parameters ζ^i by arbitrary time-dependent functions $\zeta^i \rightarrow \xi^i(x^0)$. From

²As was mentioned after eqn.(3.31), the equation of motion for $\{x^0\}$ and $\{x^i\}$ corresponding to the action (5.4) are not independent. When we will include gravity in (5.4) via the worldline-reparametrization invariant coupling $\dot{x}^0 \Phi(x)$, see (5.8), this will again be the case, as was mentioned after eqn.(3.54).

the worldline point of view, usually such a gauging would consist in introducing $(D - 1)$ worldline gauge fields $A^i(\tau)$, assigning them the gauge transformations $\delta A^i = \dot{\xi}^i$ which makes them Stückelberg fields, and defining covariant derivatives $D_\tau x^i = \dot{x}^i(\tau) - A^i(\tau)$, such that the action (5.4) becomes

$$S = \frac{m}{2} \int \frac{D_\tau x^i D_\tau x^j \delta_{ij}}{\dot{x}^0} d\tau. \quad (5.7)$$

However, we don't want to obtain $(D - 1)$ fundamental worldline fields $A^i(\tau)$, but a background field $\Phi(x)$. As such the gauging we will consider results in a pseudo-symmetry, as discussed in section 3.1. Introducing the background field $\Phi(x)$ to the action (5.4) leads to the following “gauged” action containing the gravitational potential $\Phi(x)$:³

$$S = \frac{m}{2} \int d\tau \left(\frac{\dot{x}^i \dot{x}^j \delta_{ij}}{\dot{x}^0} - 2\dot{x}^0 \Phi(x) \right). \quad (5.8)$$

The action (5.8) is invariant under worldline reparametrizations and the acceleration-extended symmetries (we write x^0 as t from now on)

$$\delta t = \zeta^0, \quad \delta x^i = \lambda^i_j x^j + \xi^i(t), \quad (5.9)$$

provided that the “background gauge field” $\Phi(x)$ transforms as follows:

$$\delta \Phi(x) = -\frac{1}{\dot{t}} \frac{d}{d\tau} \left(\frac{\dot{\xi}_i}{\dot{t}} \right) x^i + \partial_0 g(t). \quad (5.10)$$

The second term with the arbitrary function $g(t)$ represents a standard ambiguity in any potential describing a force and gives a boundary term in the action (5.8). This action leads to the following modified equation of motion describing a particle moving in a gravitational potential:

$$\ddot{x}^i + (\dot{t})^2 \delta^{ij} \partial_j \Phi(x) = \frac{\ddot{t}}{\dot{t}} \dot{x}^i. \quad (5.11)$$

Notice how (5.10) and (5.11) simplify if one takes the static gauge

$$t = \tau, \quad (5.12)$$

for which $\dot{t} = 1$ and $\ddot{t} = 0$. Using this static gauge we see that for *constant* accelerations $\ddot{\xi}^i = \text{constant}$, it is sufficient to introduce a time-independent potential $\Phi(x^i)$ but that for time-dependent accelerations we need a potential $\Phi(x)$ that depends on both the time and the transverse spatial directions.

The equation of motion of $\Phi(x)$ itself is easiest obtained by requiring that it is second order in spatial derivatives and invariant under the acceleration-extended Galilei symmetries

³Note that $\Phi(x)$ is a background field representing a set of coupling constants from the world-line point of view. Since these coupling constants also transform we are dealing not with a “proper” symmetry but with a “pseudo” or “sigma-model” symmetry, see section 3.1.

(5.9) and (5.10). Since the variation of $\Phi(x)$, see eqn. (5.10), contains an arbitrary function of time and is linear in the transverse coordinate, it is clear that the unique second-order differential operator satisfying this requirement is the Laplacian $\Delta \equiv \delta^{ij} \partial_i \partial_j$. Requiring that the source term is provided by the mass density function $\rho(x)$, which transforms as a scalar with respect to (5.9), this leads to the following Poisson equation

$$\Delta \Phi(x) = S_{D-2} G \rho(x), \quad (5.13)$$

where we have introduced Newton's constant G for dimensional reasons, and S_{D-2} is the volume of a $(D-2)$ -dimensional sphere.

The second gauging procedure is relevant if one is interested in describing the physics in more frames than the set of accelerated ones. In that case one needs to gauge *all* the symmetries of the Bargmann algebra, as was explained in the last chapter. After eqn.(4.73) the independent gauge fields are given by $\{\tau_\mu, e_\mu^a, m_\mu\}$. The dynamics of the Newton-Cartan point particle is now described by the following action [5]:

$$L = \frac{m}{2} \left(\frac{h_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}{\tau_\rho \dot{x}^\rho} - 2m_\mu \dot{x}^\mu \right). \quad (5.14)$$

Alternatively, this action can be written as

$$L = \frac{m}{2} N^{-1} \dot{x}^\mu \dot{x}^\nu \left(h_{\mu\nu} - 2m_\mu \tau_\nu \right) \quad (5.15)$$

with $N \equiv \tau_\mu \dot{x}^\mu$.

The first term in this Lagrangian can be seen as the covariantization of the Lagrangian of (5.4) with the Newton-Cartan metrics $h_{\mu\nu}$ and τ_μ . The presence of the central charge gauge field m_μ represents the ambiguity when trying to solve the Γ -connection in terms of the (singular) metrics of Newton-Cartan spacetime. The Lagrangian (5.14) is quasi-invariant under the gauged Bargmann algebra; under Z -transformations $\delta m_\mu = \partial_\mu \sigma$ the Lagrangian (5.14) transforms as a total derivative, while for the other transformations the Lagrangian is invariant. In fact, the $m_\mu \dot{x}^\mu$ term in (5.14) is needed in order to render the action invariant under boost transformations which transform both the spatial metric $h_{\mu\nu}$ and the central charge gauge field m_μ as follows:

$$\delta h_{\mu\nu} = 2\lambda^a e_{(\mu}^a \tau_{\nu)}, \quad \delta m_\mu = \lambda^a e_\mu^a. \quad (5.16)$$

Varying the Lagrangian (5.14) gives, after a lengthy calculation,⁴ the geodesic equation

$$\ddot{x}^\mu + \Gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho = \frac{\dot{N}}{N} \dot{x}^\mu. \quad (5.17)$$

Here $N \equiv \tau_\mu \dot{x}^\mu = \dot{f}$, which in adapted coordinates becomes $N = \dot{t}$. In these adapted coordinate one obtains the geodesic equation (4.16), as was given in the last chapter. The

⁴Some details are given in appendix D.1.

Γ -connection is given by (4.70). The geodesic equation (5.17) can be regarded as the covariantization of (5.11).

Unlike the particle dynamics, the gravitational dynamics cannot be obtained from an action in a straightforward way, see e.g. [93, 112]. The equation describing the dynamics of Newton-Cartan spacetime may be written in terms of the Ricci-tensor of the Γ -connection as follows:

$$R_{\mu\nu}(\Gamma) = S_{D-2} G_{\rho\tau} \tau_{\mu\nu}. \quad (5.18)$$

In terms of the curvatures we can write down the equations of motion as eqn.(4.78). To make contact with the equations for a Galilean observer, derived in the first gauging procedure, one must impose the kinematical constraint (4.74), i.e. that the curvature corresponding to the $(D-1)$ -dimensional spatial rotations vanishes. It should be stressed that one is not forced to impose this curvature constraint, and one could stay more general and try to solve the resulting theory of gravity for a curved transverse space. We will see that the constraint (4.74) can be considered as an Ansatz for the transverse Newton-Cartan metric $h^{\mu\nu}$ to be flat. It is also convenient to choose adapted coordinates $f(x) = t$ in eqn.(4.22). This reduces the general coordinate transformations to constant time translations and spatial translations with an arbitrary space-time dependent parameter.

The kinematical constraint (4.74) enables us to do two things. First, we can now choose a flat Cartesian coordinate system in the $(D-1)$ spatial dimensions, because the transverse space is flat as can be seen from eqn.(4.73):

$$R^i{}_{jkl}(\Gamma) = 0. \quad (5.19)$$

The solution (4.63) implies that the spatial components $\omega_i{}^{ab}$ of the gauge field of spatial rotations is zero in such a coordinate system, which expresses the fact that the transverse Christoffel symbols vanish:

$$\Gamma^i{}_{jk} \sim \delta_a^i \delta_b^j \omega_k{}^{ab} = 0. \quad (5.20)$$

This choice of coordinates restricts the spatial rotations to those that have a time-dependent parameter only. Second, due to the same kinematical constraint (4.74) the time component $\omega_0{}^{ab}$ of the same gauge field is a pure gauge; $R_{\mu\nu}{}^{ab}(J)$ is the field-strength of an $SO(D-1)$ gauge theory and contains only $\omega_\mu{}^{ab}$, as can be seen from (5.49). As such the constraint (4.74) allows one to gauge-fix $\omega_\mu{}^{ab}$ to zero,⁵ and this restricts the spatial rotations to having constant parameters only. Via (4.70) one can show that this implies

$$\Gamma^i{}_{0j} \sim \delta_a^i \delta_b^j \omega_0{}^{ab} = 0. \quad (5.21)$$

⁵Explicitly, one can write $R_{\mu\nu}{}^{ab}(J) = 2D_{[\mu}\omega_{\nu]}{}^{ab}$ and $\delta\omega_\mu{}^{ab} = D_\mu\lambda^{ab}$, where D_μ is the gauge covariant derivative. Putting $R_{\mu\nu}{}^{ab}(J) = 0$ imposes the constraint $\omega_\mu{}^{ab} = D_\mu f^{ab}$ on the gauge field for some function f^{ab} . Performing then a gauge transformation on $\omega_\mu{}^{ab}$ and choosing the gauge parameter to be $\lambda^{ab} = -f^{ab}$, the result follows.

The same choice of a Cartesian coordinate system also restricts the spatial translations to having only time-dependent parameters. This reduces the symmetries acting on the spacetime coordinates to the acceleration-extended Galilei symmetries given in eqn. (5.9). The central charge transformations now only depend on time and do not act on the spacetime coordinates. The vielbein postulate tells us that the only remaining connection component Γ_{00}^i can be written as $\Gamma_{00}^i = \partial^i \Phi(x)$, where

$$\Phi(x) = m_0(x) - \frac{1}{2} \delta_{ij} \tau^i(x) \tau^j(x) + \partial_0 m(x). \quad (5.22)$$

Here m_0 and $\partial_i m$ are the time component and spatial gradient components of the extension gauge field m_μ , and τ^i are the space components of the inverse temporal vielbein τ^μ . Using the transformation properties of Γ_{00}^i one can show that $\Phi(x)$, defined by eqn. (5.22), indeed transforms like in eqn. (5.10) under the acceleration-extended Galilei symmetries.⁶

One can show that after gauge-fixing the Newton-Cartan symmetries to the acceleration-extended Galilei symmetries, as described above, the Lagrangian (5.14) reduces to

$$L = \frac{m}{2} \left(\frac{\delta_{ij} \dot{x}^i \dot{x}^j}{\dot{x}^0} + \dot{x}^0 (\delta_{ij} \tau^i \tau^j - 2m_0 - 2\partial_0 m) \right), \quad (5.23)$$

where a boundary term has been discarded.⁷ Upon comparison with the action (5.8) this again identifies the potential as in (5.22). Note that the $\tau^i \dot{x}^i$ terms cancel, reflecting the choice of gauge (5.21) and indicating that this particular reference frame is non-rotating. Similarly, eqn.(5.18) reduces in this gauge to the Poisson equation (5.13).

As expected, having the same symmetries, the equations of motion (5.17) and (5.18) reduce to precisely the equations of motion (5.11) and (5.13) we obtained in the first gauging procedure.

5.3 From Particles to Strings

We now consider instead of particles of mass m strings with tension T moving in a D -dimensional Minkowski spacetime, with metric $\eta_{\mu\nu}$ ($\mu = 0, 1, \dots, D-1$). The action describing the dynamics of such a string is given by the Nambu-Goto action (3.61) (we take $c = 1$):

$$S = -T \int d^2\sigma \sqrt{-\gamma}, \quad (5.24)$$

⁶The fact that Φ transforms with the double time derivative of ξ^i shows that it indeed transforms as a component of the Γ -connection; see also eqn.(2.83).

⁷We have made use of the fact that, because $x^\mu = x^\mu(\tau)$, the τ -derivative of a general function $f(x)$ can be written as $\dot{f}(x) = \dot{x}^0 \partial_0 f(x) + \dot{x}^i \partial_i f(x)$, which in the static gauge becomes $\dot{f}(x) = \partial_0 f(x) + \dot{x}^i \partial_i f(x)$.

where $\sigma^{\bar{\alpha}}$ ($\bar{\alpha} = 0, 1$) are the world-sheet coordinates and γ is the determinant of the induced world-sheet metric $\gamma_{\bar{\alpha}\bar{\beta}}$:

$$\gamma_{\bar{\alpha}\bar{\beta}} = \partial_{\bar{\alpha}}x^{\mu}\partial_{\bar{\beta}}x^{\nu}\eta_{\mu\nu}. \quad (5.25)$$

The action (5.24) is invariant under world-sheet reparametrizations. Like in the particle case, the Lagrangian corresponding to this action is invariant under Poincaré transformations in the target spacetime.

Following [81, 82] we take the non-relativistic limit by rescaling the longitudinal coordinate $x^{\alpha} = (x^0 \equiv t, x^1)$ with a parameter ω and taking $\omega \gg 1$:⁸

$$x^{\alpha} \rightarrow \omega x^{\alpha}, \quad \omega \gg 1. \quad (5.26)$$

This results into the following action ($i = 2, \dots, D-1$):

$$S \approx -T\omega^2 \int d^2\sigma \sqrt{-\bar{\gamma}} \left(1 + \frac{1}{2\omega^2} \bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}}x^i \partial_{\bar{\beta}}x^j \delta_{ij} \right), \quad (5.27)$$

where $\bar{\gamma}_{\bar{\alpha}\bar{\beta}}$ is the pull-back of the longitudinal metric $\eta_{\alpha\beta}$, i.e.

$$\bar{\gamma}_{\bar{\alpha}\bar{\beta}} = \partial_{\bar{\alpha}}x^{\alpha} \partial_{\bar{\beta}}x^{\beta} \eta_{\alpha\beta}. \quad (5.28)$$

Unlike the world-sheet metric (5.25), the pull-back used in (5.28) is given by a 2×2 -matrix, and as such is invertible. This means that the inverse metric $\bar{\gamma}^{\bar{\alpha}\bar{\beta}}$ can be explicitly given: it is the pull-back of the longitudinal inverse metric $\eta^{\alpha\beta}$,

$$\bar{\gamma}^{\bar{\alpha}\bar{\beta}} = \partial_{\alpha}\sigma^{\bar{\alpha}} \partial_{\beta}\sigma^{\bar{\beta}} \eta^{\alpha\beta}, \quad (5.29)$$

such that $\bar{\gamma}^{\bar{\alpha}\bar{\beta}} \bar{\gamma}_{\bar{\beta}\bar{\epsilon}} = \delta_{\bar{\epsilon}}^{\bar{\alpha}}$.

The divergent term on the right hand side of eqn. (5.27) is a total world-sheet derivative [81]. This can be seen by using the identity $\eta_{[\beta[\alpha}\eta_{\gamma]\delta]} = -\frac{1}{2}\varepsilon_{\beta\delta}\varepsilon_{\alpha\gamma}$, which holds in two dimensions and in which $\varepsilon_{\alpha\gamma}$ is the two-dimensional epsilon symbol. This allows one to write

$$\begin{aligned} \sqrt{-\bar{\gamma}} &= \frac{1}{2}\varepsilon^{\bar{\alpha}\bar{\beta}}\varepsilon_{\alpha\beta}\partial_{\bar{\alpha}}x^{\alpha}\partial_{\bar{\beta}}x^{\beta} \\ &= \partial_{\bar{\alpha}}\left(\frac{1}{2}\varepsilon^{\bar{\alpha}\bar{\beta}}\varepsilon_{\alpha\beta}x^{\alpha}\partial_{\bar{\beta}}x^{\beta}\right). \end{aligned} \quad (5.30)$$

The divergent term can be canceled by coupling the string to a constant background 2-form potential $B_{\mu\nu}$ via the following Wess-Zumino term:

$$S_I = T \int d^2\sigma \varepsilon^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}}x^{\mu} \partial_{\bar{\beta}}x^{\nu} B_{\mu\nu}, \quad (5.31)$$

⁸Note that, unlike the particle mass, the tension T does not get rescaled.

and choosing the constant field components $B_{\mu\nu}$ such that

$$B_{\alpha\beta} = \frac{1}{2}\omega^2\varepsilon_{\alpha\beta}, \quad B_{i\alpha} = B_{ij} = 0. \quad (5.32)$$

The resulting field-strength of $B_{\mu\nu}$ is zero, similar to the particle case. The limit $\omega \rightarrow \infty$ of the sum of (5.27) and (5.31) then leads to the following non-relativistic action:

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-\bar{\gamma}} \left(\bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} x^i \partial_{\bar{\beta}} x^j \delta_{ij} \right). \quad (5.33)$$

This action is invariant under world-sheet reparametrizations and the following “stringy” Galilei symmetries:

$$\delta x^\alpha = \lambda^\alpha{}_\beta x^\beta + \zeta^\alpha, \quad \delta x^i = \lambda^i{}_j x^j + \lambda^i{}_\beta x^\beta + \zeta^i, \quad (5.34)$$

where $(\zeta^\alpha, \zeta^i, \lambda^i{}_j, \lambda^i{}_\alpha, \lambda^\alpha{}_\beta)$ parametrize a (constant) longitudinal translation, transverse translation, transverse rotation, “stringy” boost transformation and longitudinal rotation, respectively. As for the point particle, the equations of motion for the longitudinal and transverse components are not independent. The equations of motion for $\{x^i\}$ corresponding to the action (5.33) are given by

$$\partial_{\bar{\alpha}} \left(\sqrt{-\bar{\gamma}} \bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\beta}} x^i \right) = 0. \quad (5.35)$$

The non-relativistic Lagrangian defined by (5.33) is invariant under a stringy boost transformation only up to a total world-sheet divergence:

$$\delta L = \partial_{\bar{\alpha}} \left(-T \sqrt{-\bar{\gamma}} \frac{\partial \sigma^{\bar{\alpha}}}{\partial x^\alpha} \lambda_i{}^\alpha x^i \right), \quad (5.36)$$

where (5.29) has been used. This leads to a modified Noether charge giving rise to an extension of the stringy Galilei algebra containing two extra generators: $Z_{a'}$ and $Z_{a'b'}$ ($a' = (0, 1)$) [86]. The corresponding extended stringy Galilei algebra will be given later.

We now wish to connect to the physics as experienced by a “stringy” Galilean observer by gauging the translations in the spatial directions transverse to the string. In this procedure we replace the constant parameters ζ^i by functions $\xi^i(x^\alpha)$ depending only on the foliation coordinates. Applying this gauging to the non-relativistic action (5.33) leads to the following gauged action containing a gravitational potential $\Phi_{\alpha\beta}$:

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-\bar{\gamma}} \left(\bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} x^i \partial_{\bar{\beta}} x^j \delta_{ij} - 2\eta^{\alpha\beta} \Phi_{\alpha\beta} \right). \quad (5.37)$$

This action can be compared with the point particle action (5.8).⁹ The string action (5.37) is invariant under world-sheet reparametrizations and the acceleration-extended stringy Galilei symmetries [86]

$$\delta x^\alpha = \lambda^\alpha{}_\beta x^\beta + \zeta^\alpha, \quad \delta x^i = \lambda^i{}_j x^j + \xi^i(x^\alpha). \quad (5.38)$$

⁹Note that $\bar{\gamma}_{\bar{\alpha}\bar{\beta}}$ corresponds to a factor $-(\dot{x}^0)^2$ in the particle action.

The local transverse translations are only realized provided that the background potentials $\Phi_{\alpha\beta}$ transform as follows:

$$\delta\Phi_{\alpha\beta} = -\frac{1}{2\sqrt{-\bar{\gamma}}}\eta_{\alpha\beta}\partial_{\bar{\alpha}}\left(\sqrt{-\bar{\gamma}}\bar{\gamma}^{\bar{\alpha}\bar{\beta}}\partial_{\bar{\beta}}\xi_i\right)x^i + \nabla_{(\alpha}g_{\beta)}(x^\epsilon), \quad (5.39)$$

for arbitrary $g_\beta(x^\epsilon)$. Eqn. (5.39) is the string analog of eqn. (5.10). The action (5.37) leads to the following modified equations of motion for the transverse coordinates $\{x^i\}$:

$$\partial_{\bar{\alpha}}\left(\sqrt{-\bar{\gamma}}\bar{\gamma}^{\bar{\alpha}\bar{\beta}}\partial_{\bar{\beta}}x^i\right) + \sqrt{-\bar{\gamma}}\eta^{\alpha\beta}\partial^i\Phi_{\alpha\beta} = 0. \quad (5.40)$$

These equations of motion simplify if we choose the static gauge

$$x^\alpha = \sigma^{\bar{\alpha}}. \quad (5.41)$$

In this gauge we have that $\bar{\gamma}_{\bar{\alpha}\bar{\beta}} = \eta_{\alpha\beta}$.

The equation of motion of $\Phi_{\alpha\beta}(x)$ itself is easiest obtained by requiring that it is second order in spatial derivatives and invariant under the acceleration-extended stringy Galilei symmetries (5.38) and (5.39). Since the variation of $\Phi_{\alpha\beta}(x)$, see eqn. (5.39), contains an arbitrary function of the longitudinal coordinates and is linear in the transverse coordinates, it follows that the unique second-order differential operator satisfying the above requirement is the Laplacian $\Delta \equiv \delta^{ij}\partial_i\partial_j$. Requiring that the source term is provided by the mass density function $\rho(x)$, which transforms as a scalar with respect to (5.38), this leads to the following Poisson equation:

$$\Delta\Phi_{\alpha\beta}(x) = S_{D-2}G\rho(x)\eta_{\alpha\beta}. \quad (5.42)$$

This finishes our first approach where we only gauge the transverse translations. In this approach we have presented both the equations of motion for the transverse coordinates $\{x^i\}$ of a string, see eqn. (5.40), as well as the bulk equations of motion for the gravitational potential $\Phi_{\alpha\beta}$, see eqn. (5.42).

5.4 Gauging the stringy Galilei algebra

We now proceed with the second gauging procedure in which we gauge the full deformed stringy Galilei algebra. This algebra consists of longitudinal translations, transverse translations, longitudinal Lorentz transformations, ‘‘boost’’ transformations, transverse rotations and two distinct extension transformations. As a first step one associates a gauge

field to each of these symmetries:

$$\begin{aligned}
\tau_\mu^{a'} & : && \text{longitudinal translations} \\
e_\mu^a & : && \text{transverse translations} \\
\omega_\mu^{a'b'} & : && \text{longitudinal Lorentz transformations} \\
\omega_\mu^{aa'} & : && \text{“boost” transformation} \\
\omega_\mu^{ab} & : && \text{transverse rotations} \\
m_\mu^{a'}, m_\mu^{a'b'} & : && \text{extension transformations.}
\end{aligned} \tag{5.43}$$

At the same time the constant parameters describing the transformations are promoted to arbitrary functions of the spacetime coordinates $\{x^\mu\}$:

$$\begin{aligned}
\tau^{a'}(x^\mu) & : && \text{longitudinal translations} \\
\zeta^a(x^\mu) & : && \text{transverse translations} \\
\lambda^{a'b'}(x^\mu) & : && \text{longitudinal Lorentz transformations} \\
\lambda^{aa'}(x^\mu) & : && \text{“boost” transformations} \\
\lambda^{ab}(x^\mu) & : && \text{transverse rotations} \\
\sigma^{a'}(x^\mu), \sigma^{a'b'}(x^\mu) & : && \text{extension transformations.}
\end{aligned} \tag{5.44}$$

The nonzero commutators of the undeformed stringy Galilei algebra read

$$\begin{aligned}
[G_{bc'}, H_{a'}] &= \eta_{a'c'} P_b, & [J_{bc}, P_a] &= -2\eta_{a[b} P_{c]}, \\
[G_{cd'}, M_{e'f'}] &= 2\eta_{d'[e'} G_{|c|f'}], & [J_{cd}, G_{e'f'}] &= -2\eta_{e[c} G_{d]f'}, \\
[J_{cd}, J_{ef}] &= 4\eta_{[c[e} J_{f]d]} & [M_{b'c'}, H_{a'}] &= -2\eta_{a'[b'} H_{c'}],
\end{aligned} \tag{5.45}$$

where $a' = \underline{0}, \underline{1}$ are the two longitudinal foliating directions and $a = 2, \dots, D-1$ are the $D-2$ transverse directions. Note that the Lorentz algebra $\mathfrak{so}(1, 1)$ of the two-dimensional foliation space is Abelian while for general p-branes, where the symmetries of the foliation space are generated by the algebra $\mathfrak{so}(1, p)$, this would not be the case. The extensions suggested by the Poisson brackets corresponding to the non-relativistic string action (5.33) are given by [87]

$$\begin{aligned}
[P_a, G_{bb'}] &= \eta_{ab} Z_{b'}, & [G_{aa'}, G_{bb'}] &= -\eta_{ab} Z_{a'b'}, \\
[H_{a'}, Z_{b'c'}] &= 2\eta_{a'[b'} Z_{c']}, & [Z_{a'b'}, M_{c'd'}] &= 4\eta_{[a'[c'} Z_{d']b'}], \\
[Z_{a'}, M_{b'c'}] &= 2\eta_{a'[b'} Z_{c']}.
\end{aligned} \tag{5.46}$$

The gauge transformations of the gauge fields (5.43) corresponding to the generators of the deformed stringy Galilei algebra are given by

$$\begin{aligned}
\delta\tau_\mu^{a'} &= \partial_\mu\tau^{a'} - \tau^{b'}\omega_\mu^{a'b'} + \lambda^{a'b'}\tau_\mu^{b'} , \\
\delta e_\mu^a &= \partial_\mu\zeta^a - \zeta^b\omega_\mu^{ab} + \lambda^{ab}e_\mu^b + \lambda^{aa'}\tau_\mu^{a'} - \tau^{a'}\omega_\mu^{aa'} , \\
\delta\omega_\mu^{a'b'} &= \partial_\mu\lambda^{a'b'} , \\
\delta\omega_\mu^{aa'} &= \partial_\mu\lambda^{aa'} - \lambda^{ab'}\omega_\mu^{a'b'} + \lambda^{a'b'}\omega_\mu^{ab'} + \lambda^{ab}\omega_\mu^{ba'} - \lambda^{ba'}\omega_\mu^{ab} , \\
\delta\omega_\mu^{ab} &= \partial_\mu\lambda^{ab} + 2\lambda^{c[a}\omega_\mu^{b]c} , \\
\delta m_\mu^{a'} &= \partial_\mu\sigma^{a'} + \lambda^{aa'}e_\mu^a - \zeta^a\omega_\mu^{aa'} + \lambda^{a'b'}m_\mu^{b'} - \sigma^{b'}\omega_\mu^{a'b'} + \tau^{b'}m_\mu^{a'b'} - \sigma^{a'b'}\tau_\mu^{b'} , \\
\delta m_\mu^{a'b'} &= \partial_\mu\sigma^{a'b'} - \lambda^{aa'}\omega_\mu^{ab'} + \lambda^{ab'}\omega_\mu^{aa'} + \sigma^{c'[a'}\omega_\mu^{b']c'} + \lambda^{c'[a'}m_\mu^{b']c'} ,
\end{aligned} \tag{5.47}$$

where we have used the gauge parameters (5.44). The corresponding curvatures are given by¹⁰

$$\begin{aligned}
R_{\mu\nu}^{a'}(H) &= 2D_{[\mu}\tau_{\nu]}^{a'} , \\
R_{\mu\nu}^a(P) &= 2\left(D_{[\mu}e_{\nu]}^a - \omega_{[\mu}^{aa'}\tau_{\nu]}^{a'}\right) , \\
R_{\mu\nu}^{a'b'}(M) &= 2\partial_{[\mu}\omega_{\nu]}^{a'b'} , \\
R_{\mu\nu}^{aa'}(G) &= 2D_{[\mu}\omega_{\nu]}^{aa'} , \\
R_{\mu\nu}^{ab}(J) &= 2\left(\partial_{[\mu}\omega_{\nu]}^{ab} - \omega_{[\mu}^{ca}\omega_{\nu]}^{bc}\right) , \\
R_{\mu\nu}^{a'}(Z) &= 2\left(D_{[\mu}m_{\nu]}^{a'} + e_{[\mu}^a\omega_{\nu]}^{aa'} - \tau_{[\mu}^{b'}m_{\nu]}^{a'b'}\right) , \\
R_{\mu\nu}^{a'b'}(Z) &= 2\left(D_{[\mu}m_{\nu]}^{a'b'} + \omega_{[\mu}^{aa'}\omega_{\nu]}^{ab'}\right) ,
\end{aligned} \tag{5.48}$$

where M , G and J indicate the generators corresponding to longitudinal Lorentz transformations, “boost” transformations and transverse rotations, respectively. The derivative D_μ is covariant with respect to these three transformations. Besides the gauge transformations all gauge fields transform under general coordinate transformations with parameters $\xi^\mu(x^\mu) = (\xi^\alpha(x^\mu), \xi^i(x^\mu))$.

Like in the particle case we would like to express the Γ -connection in terms of the previous gauge fields. In order to do that we first impose a set of so-called conventional constraints on the curvatures of the gauge fields:

$$R_{\mu\nu}^{a'}(H) = R_{\mu\nu}^a(P) = R_{\mu\nu}^{a'}(Z) = 0 . \tag{5.49}$$

¹⁰For general p-branes we would have $\delta\omega_\mu^{a'b'} = \partial_\mu\lambda^{a'b'} + 2\lambda^{c'[a'}\omega_\mu^{b']c'}$ and $R_{\mu\nu}^{a'b'}(M) = 2\left(\partial_{[\mu}\omega_{\nu]}^{a'b'} - \omega_{[\mu}^{c'a'}\omega_{\nu]}^{b'c'}\right)$. In two spacetime dimensions one can write the single Lorentz boost as $\lambda^{a'b'} = \lambda\varepsilon^{a'b'}$ indicating that the Lorentz algebra is trivially Abelian.

These constraints are required to convert the local H and P transformations into general coordinate transformations via the identity (B.10). Besides this, the constraints (5.49) also imply that the gauge fields ω_μ^{ab} , $\omega_\mu^{aa'}$ and $\omega_\mu^{a'b'}$ become dependent:

$$\omega_\mu^{ab} = \partial_{[\mu} e_{\nu]}^a e^{\nu b} - \partial_{[\mu} e_{\nu]}^b e^{\nu a} + e_\mu^c \partial_{[\nu} e_{\rho]}^c e^{\nu a} e^{\rho b} - \tau_\mu^{a'} e^{\rho [a} \omega_\rho^{b]a'}, \quad (5.50)$$

$$\begin{aligned} \omega_\mu^{aa'} &= 2\tau_\mu^{b'} \left(\tau^{\nu b'} e^{\rho a} [\partial_{[\nu} m_{\rho]}^{a'} - \omega_{[\nu}^{a'c'} m_{\rho]}^{c'}] - e^{\nu a} m_\nu^{a'b'} \right) \\ &\quad + 2e_\mu^b \tau^{\rho a'} e^{\nu (b} \partial_{[\nu} e_{\rho]}^a) + e_\mu^b e^{\nu b} e^{\rho a} [\partial_{[\nu} m_{\rho]}^{a'} - \omega_{[\nu}^{a'b'} m_{\rho]}^{b'}], \end{aligned} \quad (5.51)$$

$$\omega_\mu^{a'b'} = \partial_{[\mu} \tau_{\nu]}^{a'} \tau^{\nu b'} - \partial_{[\mu} \tau_{\nu]}^{b'} \tau^{\nu a'} + \tau^{\nu a'} \tau^{\rho b'} \tau_\mu^{c'} \partial_{[\nu} \tau_{\rho]}^{c'}. \quad (5.52)$$

The solution for $\omega_\mu^{a'b'}$ is familiar from the Poincaré theory, see (2.55), and reflects the fact that the foliation space is given by a two-dimensional Minkowski spacetime. The same constraints have a third effect, namely that they lead to constraints on the curl of the gauge field $\tau_\mu^{a'}$. More precisely, the conventional constraint $R_{\mu\nu}^{a'}(H) = 0$ can not only be used to solve for the spin connection $\omega_\mu^{a'b'}$, see eqn. (5.52). Substituting this solution back into the constraint also implies that the following projections of $\partial_{[\mu} \tau_{\nu]}^{a'}$ vanish:

$$e^{\mu a} \tau^{\nu (a'} \partial_{[\mu} \tau_{\nu]}^{b')} = 0, \quad e^\mu_a e^\nu_b \partial_{[\mu} \tau_{\nu]}^{a'} = 0. \quad (5.53)$$

It is instructive to verify how the other two spin connections are solved for. First, the conventional constraints $R_{\mu\nu}^a(P) = 0$ can not only be used to solve for the spin connection ω_μ^{ab} , see eqn. (5.50), but also for the following projections of the spin connection field $\omega_\mu^{aa'}$:

$$e^{\mu (a} \omega_\mu^{b)b'} = 2\tau^{\nu b'} e^{\mu (a} \partial_{[\mu} e_{\nu]}^b), \quad \omega_\rho^{a[a' \tau^{b']\rho} = -\tau^{\mu a'} \tau^{\nu b'} \partial_{[\mu} e_{\nu]}^a. \quad (5.54)$$

Making different contractions of the third conventional constraint $R_{\mu\nu}^{a'}(Z) = 0$ one can solve for two more projections of the same spin connection field:

$$\tau^{\mu b'} \omega_\mu^{aa'} = 2\tau^{\mu b'} e^{\nu a} \left(\partial_{[\mu} m_{\nu]}^{a'} - \omega_{[\mu}^{a'c'} m_{\nu]}^{c'} \right) - 2e^{\mu a} m_\mu^{a'b'}, \quad (5.55)$$

$$e^{\mu [a} \omega_\mu^{b]a'} = e^{\mu a} e^{\nu b} \left(\partial_{[\mu} m_{\nu]}^{a'} - \omega_{[\mu}^{a'b'} m_{\nu]}^{b'} \right). \quad (5.56)$$

Combining the solutions (5.54), (5.55) and (5.56) for the different projections and using the decomposition

$$\omega_\mu^{aa'} = \tau_\mu^{b'} \tau^{\nu b'} \omega_\nu^{aa'} + e_\mu^b e^{\nu (b} \omega_\nu^{a)a'} + e_\mu^b e^{\nu [b} \omega_\nu^{a]a'}, \quad (5.57)$$

one can solve for the spin connection field $\omega_\mu^{aa'}$, see (5.51). Finally, it turns out that beyond the contractions already considered there is one more contraction of the conventional constraint $R_{\mu\nu}^{a'}(Z) = 0$. It leads to the following constraint on the gauge field $m_\mu^{a'b'}$:

$$\tau^{\mu [c'} m_\mu^{d]a'} = \tau^{\mu c'} \tau^{\nu d'} \left(\partial_{[\mu} m_{\nu]}^{a'} - \omega_{[\mu}^{a'b'} m_{\nu]}^{b'} \right). \quad (5.58)$$

This constraint relates the longitudinal projection of $D_{[\mu}m_{\nu]}^{a'}$ to a certain projection of the gauge field $m_{\mu}^{a'b'}$, but does not allow one to solve $m_{\mu}^{a'b'}$ completely; the other projections remain unspecified! We will return to the meaning of the constraint (5.58) after eqn.(5.74).

At this point the symmetries of the theory are the general coordinate transformations, the longitudinal Lorentz transformations, “boost” transformations, transverse rotations and extension transformations, all with parameters that are arbitrary functions of the spacetime coordinates. The gauge fields $\tau_{\mu}^{a'}$ of longitudinal translations and e_{μ}^a of transverse translations are identified as the (singular) longitudinal and transverse vielbeins. One may also introduce their inverses (with respect to the longitudinal and transverse subspaces) $\tau^{\mu}_{a'}$ and e^{μ}_a :

$$\begin{aligned} e_{\mu}^a e^{\mu}_b &= \delta_b^a, & e_{\mu}^a e^{\nu}_a &= \delta_{\mu}^{\nu} - \tau_{\mu}^{a'} \tau^{\nu}_{a'}, & \tau^{\mu}_{a'} \tau_{\mu}^{b'} &= \delta_{a'}^{b'}, \\ \tau^{\mu}_{a'} e_{\mu}^a &= 0, & \tau_{\mu}^{a'} e^{\mu}_a &= 0. \end{aligned} \quad (5.59)$$

The spatial and temporal vielbeins are related to the spatial metric $h^{\mu\nu}$ with “inverse” $h_{\mu\nu}$, and the temporal metric $\tau_{\mu\nu}$ with “inverse” $\tau^{\mu\nu}$, as follows:

$$\begin{aligned} \tau_{\mu\nu} &= \tau_{\mu}^{a'} \tau_{\nu}^{b'} \eta_{a'b'}, & \tau^{\mu\nu} &= \tau^{\mu}_{a'} \tau^{\nu}_{b'} \eta^{a'b'}, \\ h_{\mu\nu} &= e_{\mu}^a e_{\nu}^b \delta_{ab}, & h^{\mu\nu} &= e^{\mu}_a e^{\nu}_b \delta^{ab}. \end{aligned} \quad (5.60)$$

These tensors satisfy the Newton-Cartan metric conditions

$$\begin{aligned} h^{\mu\nu} h_{\nu\rho} + \tau^{\mu\nu} \tau_{\nu\rho} &= \delta_{\rho}^{\mu}, & \tau^{\mu\nu} \tau_{\mu\nu} &= 2, \\ h^{\mu\nu} \tau_{\nu\rho} &= h_{\mu\nu} \tau^{\nu\rho} = 0. \end{aligned} \quad (5.61)$$

We note that for the point particle one would have $\tau^{\mu\nu} \tau_{\mu\nu} = 1$ instead of $\tau^{\mu\nu} \tau_{\mu\nu} = 2$.

A Γ -connection can be introduced by imposing the following vielbein postulates:

$$\begin{aligned} \partial_{\mu} e_{\nu}^a - \omega_{\mu}^{ab} e_{\nu}^b - \omega_{\mu}^{aa'} \tau_{\nu}^{a'} - \Gamma_{\nu\mu}^{\lambda} e_{\lambda}^a &= 0, \\ \partial_{\mu} \tau_{\nu}^{a'} - \omega_{\mu}^{a'b'} \tau_{\nu}^{b'} - \Gamma_{\nu\mu}^{\rho} \tau_{\rho}^{a'} &= 0. \end{aligned} \quad (5.62)$$

These vielbein postulates allow one to solve for Γ uniquely. The torsion $\Gamma_{[\nu\mu]}^{\rho}$, given by

$$\Gamma_{[\nu\mu]}^{\rho} = \tau^{\rho}_{a'} R_{\mu\nu}^{a'}(H) + e^{\rho}_a R_{\mu\nu}^a(P), \quad (5.63)$$

vanishes because of the constraints $R(P) = R(H) = 0$, and with this the vielbein postulates give the solution

$$\Gamma_{\nu\mu}^{\rho} = \tau^{\rho}_{a'} \left(\partial_{(\mu} \tau_{\nu)}^{a'} - \omega_{(\mu}^{a'b'} \tau_{\nu)}^{b'} \right) + e^{\rho}_a \left(\partial_{(\mu} e_{\nu)}^a - \omega_{(\mu}^{ab} e_{\nu)}^b - \omega_{(\mu}^{aa'} \tau_{\nu)}^{a'} \right) \quad (5.64)$$

in terms of the dependent spin connections $\omega_\mu^{ab}, \omega_\mu^{aa'}$ and $\omega_\mu^{a'b'}$. If one plugs in the explicit solutions of these spin connections, one obtains

$$\begin{aligned} \Gamma_{\mu\nu}^\rho &= \frac{1}{2}\tau^{\rho\sigma}\left(\partial_\nu\tau_{\sigma\mu} + \partial_\mu\tau_{\sigma\nu} - \partial_\sigma\tau_{\mu\nu}\right) + \frac{1}{2}h^{\rho\sigma}\left(\partial_\nu h_{\sigma\mu} + \partial_\mu h_{\sigma\nu} - \partial_\sigma h_{\mu\nu}\right) \\ &\quad + h^{\rho\sigma}K_{\sigma(\mu}{}^{a'}\tau_{\nu)}{}^{a'}, \end{aligned} \quad (5.65)$$

where $K_{\mu\nu}{}^{a'} = -K_{\nu\mu}{}^{a'}$ is given by the covariant curl of $m_\mu{}^{a'}$:

$$K_{\mu\nu}{}^{a'} = 2D_{[\mu}m_{\nu]}{}^{a'}. \quad (5.66)$$

An important observation is that $m_\mu{}^{a'b'}$ does not appear in (5.65). The origin of this absence is the fact that the expression (5.64) is invariant under the shift transformations

$$\omega_\mu{}^{aa'} \rightarrow \omega_\mu{}^{aa'} + \tau_\mu{}^{b'}X_{a'b'}^a, \quad (5.67)$$

where $X_{a'b'}^a = X_{[a'b]}^a$ is an arbitrary shift parameter. The field $m_\mu{}^{a'b'}$ appears in the form $X_{a'b'}^a = e^\lambda{}_a m_\lambda{}^{a'b'}$ in the solution of $\omega_\mu{}^{aa'}$, and as such $m_\mu{}^{a'b'}$ will drop out of the connection (5.64), and thus out of (5.65).

The Riemann tensor can be obtained, using the vielbein postulates, from the curvatures of the spin connection fields:

$$R^\mu{}_{\nu\rho\sigma}(\Gamma) = -\tau^\mu{}_{a'}R_{\rho\sigma}{}^{a'b'}(M)\tau_\nu{}^{b'} - e^\mu{}_a R_{\rho\sigma}{}^{ab}(J)e_{\nu b} - e^\mu{}_a R_{\rho\sigma}{}^{aa'}(G)\tau_{\nu a'}. \quad (5.68)$$

Note that this Riemann tensor has no dependence on the gauge field $m_\mu{}^{a'b'}$, as was argued.

At this stage the independent fields are given by $\{\tau_\mu{}^{a'}, e_\mu{}^a, m_\mu{}^{a'}\}$, whereas we saw that $m_\mu{}^{a'b'}$ was partially solved for via eqn. (5.58) and does not enter the dynamics.¹¹ The dynamics of a Newton-Cartan string is now described by the following Lagrangian:

$$L = -\frac{T}{2}\sqrt{-\det(\tau)}\tau^{\bar{\alpha}\bar{\beta}}\partial_{\bar{\alpha}}x^\mu\partial_{\bar{\beta}}x^\nu\left(h_{\mu\nu} - 2m_\mu{}^{a'}\tau_\nu{}^{a'}\right), \quad (5.69)$$

where the induced world-sheet metric $\tau_{\bar{\alpha}\bar{\beta}}$ is given by

$$\tau_{\bar{\alpha}\bar{\beta}} \equiv \partial_{\bar{\alpha}}x^\mu\partial_{\bar{\beta}}x^\nu\tau_{\mu\nu}. \quad (5.70)$$

eqn. (5.69) is the stringy generalization of the particle action (5.15). The first term in eqn. (5.69) can be seen as the covariantization of the Lagrangian of (5.33) with the Newton-Cartan metrics $h_{\mu\nu}$ and $\tau_{\mu\nu}$, where the induced world-sheet metric (5.70) is the covariantization of (5.28) with $\tau_{\mu\nu}$. Analogously to the point particle, the Lagrangian (5.69) is quasi-invariant under the gauged deformed stringy Galilei algebra. Under $Z_{a'}$ -transformations $\delta m_\mu{}^{a'} = \partial_\mu\sigma^{a'}$ the Lagrangian (5.69) transforms as a total derivative,

¹¹An analogous results holds for the dynamics of the non-relativistic string, see eqn. (32) of [87].

while the other transformations leave the Lagrangian invariant. In particular, this applies to the $Z_{a'b'}$ -transformations which are given by

$$\delta m_\mu^{a'} = -\sigma^{a'b'} \tau_\mu^{b'} \quad \text{or} \quad \tau^{\mu[a'} \delta m_\mu^{b']} = \sigma^{a'b'}. \quad (5.71)$$

The latter way of writing shows that the projection $\tau^{\mu[a'} m_\mu^{b']}$ of the gauge field $m_\mu^{a'}$ can be gauged away. The $m_{(\mu}^{a'} \tau_{\nu)}^{a'}$ term in the Lagrangian (5.69) is needed in order to render the action invariant under boost transformations which transform both the spatial metric $h_{\mu\nu}$ and the extension gauge field $m_\mu^{a'}$ as follows:

$$\delta h_{\mu\nu} = 2\lambda^{aa'} e_{(\mu}^a \tau_{\nu)}^{a'}, \quad \delta m_\mu^{a'} = \lambda^{aa'} e_\mu^a. \quad (5.72)$$

Like in the particle case, the presence of the extension gauge field $m_\mu^{a'}$ represents an ambiguity when trying to solve the Γ -connection in terms of the (singular) metrics (5.60) of Newton-Cartan spacetime. Namely, the metric compatibility conditions on $h^{\mu\nu}$ and $\tau_{\mu\nu}$,

$$\nabla_\rho h^{\mu\nu} = \nabla_\rho \tau_{\mu\nu} = 0, \quad (5.73)$$

give the solution (5.65), but $K_{\mu\nu}^{a'} = -K_{\nu\mu}^{a'}$ is an ambiguity which is not fixed by the metric compatibility conditions. It is the specific solution (5.64) of the vielbein postulates which fixes this ambiguity to be (5.66). A new feature of the string case is that the ambiguity $K_{\mu\nu}^{a'}$ has its own ambiguity. In other words: there is an ambiguity in the ambiguity! To show how this works we first note that from eqn. (5.65) it follows that the longitudinal projection of (5.66) does not contribute to the connection because it is multiplied by $h^{\rho\sigma}$. This is equivalent to saying that the expression (5.65) is invariant under the shift transformations

$$K_{\mu\nu}^{a'} \rightarrow K_{\mu\nu}^{a'} + \tau_{[\mu}^{c'} \tau_{\nu]}^{b'} Y_{b'c'}^{a'} \quad (5.74)$$

for arbitrary parameters $Y_{b'c'}^{a'}$. We will now argue that this ambiguity in $K_{\mu\nu}^{a'}$ is related to the second extension gauge field, m_μ^{ab} , which in contrast to $m_\mu^{a'}$ does *not* enter the Lagrangian (5.69). We have seen that the absence of $m_\mu^{a'b'}$ in the dynamics follows from the shift symmetry (5.67), which prevents the field $m_\mu^{a'b'}$ to enter the Γ -connection. We now come back to the role of the constraint (5.58). Using eqn. (5.66) we see that this constraint relates a certain projection of $m_\mu^{a'b'}$ to the longitudinal projection of the ambiguity $K_{\mu\nu}^{a'}$. This longitudinal projection of the ambiguity is precisely the part that drops out of the expression for Γ corresponding to the shift invariance of (5.65) under (5.74). Therefore, the constraint (5.58) implies that a certain projection of the extension gauge field $m_\mu^{a'b'}$ can be regarded as an ‘‘ambiguity in the ambiguity’’.

Summarizing, we conclude that the extension gauge field $m_\mu^{a'}$, like in the particle case, corresponds to an ambiguity in the Γ -connection. This gauge field occurs in the string action (5.69). A new feature, absent in the particle case, is that the extension of the algebra contains also a generator $Z_{a'b'}$, which is needed in order to close the extended

stringy Galilei algebra.¹² As a result there is a second extension gauge field $m_\mu^{a'b'}$, which corresponds to an ambiguity in the ambiguity. This extension gauge field does not occur in the string action (5.69).

Having clarified the role of the extension gauge fields we now vary the Lagrangian (5.69) which gives, after a calculation¹³ similar to the one leading to (5.17),

$$\tau^{\bar{\alpha}\bar{\beta}} \left(\nabla_{\bar{\alpha}} \partial_{\bar{\beta}} x^\rho + \partial_{\bar{\alpha}} x^\mu \partial_{\bar{\beta}} x^\nu \Gamma_{\mu\nu}^\rho \right) = 0, \quad (5.75)$$

where the Γ -connection is given by (5.64). This geodesic equation can be seen as the covariantization of (5.40), and in the particle case reduces to (5.17) as one would expect. The equations describing the dynamics of stringy Newton-Cartan spacetime are given by

$$R_{\mu\nu}(\Gamma) = S_{D-2} G \rho \tau_{\mu\nu}, \quad (5.76)$$

just as for the point particle. The Ricci tensor however now is given in terms of the Γ -connection (5.64). As for the particle case (4.78) we can write down the bulk dynamics eqn.(5.76) also in term of the gauge curvatures by contracting eqn.(5.76) with vielbeine and using eqn.(5.68):

$$\begin{aligned} R_{a'c'}^{a'd'}(M) + R_{ac'}^{ad'}(G) &= -S_{D-2} G \rho \eta_{c'd'}, \\ R_{ac}^{ad}(J) &= 0, \\ R_{a'c}^{a'c'}(M) + R_{ac'}^{ac}(J) + R_{ac}^{ac'}(G) &= 0. \end{aligned} \quad (5.77)$$

To make contact with a Galilean observer we now impose the additional kinematical constraints

$$R_{\mu\nu}^{a'b'}(M) = R_{\mu\nu}^{ab}(J) = 0. \quad (5.78)$$

Here J refers to the generators of spatial rotations, whereas M refers to the generator of a longitudinal boost which was absent for the particle. It should be stressed that one is not forced to impose these curvature constraints, and one could stay more general and try to solve the resulting theory of gravity for a curved longitudinal and transverse space. In particular, in adding a cosmological constant in the next section, we will impose a different constraint for the longitudinal space. The first constraint of (5.78) allows one to gauge-fix $\omega_\mu^{a'b'} = 0$, expressing the flatness of the longitudinal space. This solves the constraints (5.53) and allows one to go to the so-called adapted coordinates, in which $\tau_\mu^{a'}$ is given by

$$\tau_\mu^{a'} = \delta_\mu^{a'}. \quad (5.79)$$

¹²I.e. one cannot extend the algebra with *only* $Z_{a'}$; the algebra extension is a package deal, giving also $Z_{a'b'}$.

¹³Some details are given in appendix D.1.

In terms of these adapted coordinates the longitudinal and transverse vielbeins and their inverses are given by

$$\begin{aligned}\tau_\mu^{a'} &= (\delta_\alpha^{a'}, 0), & e_\mu^a &= (-e_k^a \tau_{a'}^k, e_i^a), \\ \tau^{\mu a'} &= (\delta_{a'}^\alpha, \tau^i_{a'}), & e^\mu_a &= (0, e^i_a),\end{aligned}\quad (5.80)$$

in terms of the independent components $\tau^i_{a'}$ and the transverse vielbeins e_i^a together with their inverse e^i_a . Note that in adapted coordinates the transverse vielbein is non-singular in the transverse space, i.e.

$$e_i^a e^j_a = \delta_i^j, \quad e_i^a e^i_b = \delta_b^a. \quad (5.81)$$

The second kinematical constraint of (5.78) expresses the choice of flat transverse directions. It implies, using eqn. (5.68), that $R^i_{jkl}(\Gamma) = 0$ and allows us to choose a flat Cartesian coordinate system in the transverse space such that

$$e_i^a = \delta_i^a, \quad e^i_a = \delta^i_a. \quad (5.82)$$

As such the constraints (5.78) can be regarded as metric Ansätze in which one is looking for solutions of the metrics describing both a flat transverse space and a flat foliation space. All metric components can now be expressed in terms of the only nontrivial components $\tau^i_{a'}$:

$$\begin{aligned}\tau_\mu^{a'} &= (\delta_\alpha^{a'}, 0), & e_\mu^a &= (-\tau^{a'}_{a'}, \delta_i^a), \\ \tau^{\mu a'} &= (\delta_{a'}^\alpha, \tau^i_{a'}), & e^\mu_a &= (0, \delta^i_a),\end{aligned}\quad (5.83)$$

where we do not distinguish anymore between (longitudinal, transverse) curved indices (α, i) and (longitudinal, transverse) flat indices (a', a) .

Plugging the conventional constraints (5.49) and the kinematical constraints (5.78) into the Bianchi identities (C.6) we find that

$$R_{\alpha\beta}(\Gamma) = -\delta_{(\alpha}^{a'} \delta_{\beta)}^{b'} e^\rho_a \tau^\sigma_{b'} R_{\rho\sigma}{}^{aa'}(G) \quad (5.84)$$

are the only nonzero components of the Ricci tensor. Furthermore, the remaining nonzero curvatures $R(J)$ and $R(Z)$ are constrained by the following algebraic identities:

$$R_{[\lambda\mu}{}^{aa'}(G)\tau_{\nu]}^{a'} = R_{[\lambda\mu}{}^{aa'}(J)e_{\nu]}^a - R_{[\lambda\mu}{}^{a'b'}(Z)\tau_{\nu]}^{b'} = 0. \quad (5.85)$$

The kinematical constraint $R_{\mu\nu}{}^{ab}(M) = 0$ also allows one to gauge-fix $\omega_\mu{}^{ab} = 0$. We will now show that in this gauge

$$\Gamma^i_{\alpha j} = 0, \quad \Gamma^i_{\alpha\beta} = \partial^i \Phi_{\alpha\beta}, \quad (5.86)$$

where the latter equation defines the gravitational potential $\Phi_{\alpha\beta}$.

We first show that $\Gamma_{\alpha j}^i = 0$. Using the expressions (5.83), eqn. (5.64) and the fact that $\omega_j^{a'b'} = \omega_\mu^{ab} = 0$ we find that $\Gamma_{\alpha j}^i$ is given by

$$\Gamma_{a'j}^i = \frac{1}{2}(-\partial_j \tau^{i a'} - \omega_j^{i a'}). \quad (5.87)$$

Next, using expressions (5.50)-(5.52), we find that

$$\omega_j^{ia} = -\partial_{[i} m_{j]a} - \partial^{(i} \tau^{j)a}, \quad (5.88)$$

where we have used that $\omega_i^{a'b'} = 0$. Furthermore, the gauge-fixing condition $\omega_k^{ij} = 0$ is already satisfied but the gauge-fixing condition $\omega_\alpha^{ab} = 0$ leads to the constraint

$$\omega_{a'}^{ij} = -\partial_{[i} m_{j]a'} - \partial^{[i} \tau^{j]a'} = 0. \quad (5.89)$$

This constraint equation implies that $m_{ia'}$ can be written as

$$m_{ia'} = -\tau^i_{a'} - \partial_i m_{a'}, \quad (5.90)$$

where $m_{a'}$ are the transverse spatial gradient components of $m_{ia'}$. Substituting the expression for $\omega_j^{ia'}$ into that of $\Gamma_{a'j}^i$ the result becomes proportional to the righthand-side of the constraint equation (5.89) and hence we find $\Gamma_{a'j}^i = 0$.

We next show that $\Gamma_{\alpha\beta}^i$ can be written as $\partial^i \Phi_{\alpha\beta}$ defining a gravitational potential $\Phi_{\alpha\beta}$. Using (5.64) we derive the following expression:¹⁴

$$\Gamma_{a'b'}^i = -\partial_{(a'} \tau^{i b')} - \omega_{(a'}^{i b')}, \quad (5.91)$$

where we have used that $\omega_\alpha^{a'b'} = \omega_\alpha^{ij} = 0$. Following eqs. (5.50)-(5.52) we find that $\omega_{a'}^{ib'}$ is given by

$$\omega_{a'}^{ib'} = \partial_{a'} m_{ib'} - \partial_i m_{a'b'} + \tau^k_{a'} \partial_{[k} m_{i]b'} + \frac{1}{2} \tau^k_{a'} (\partial_i \tau^k_{b'}) + \frac{1}{2} \tau^k_{a'} \partial_k \tau^i_{b'} + 2m_i^{a'b'}. \quad (5.92)$$

Substituting this expression for $\omega_{a'}^{ib'}$ back into that of $\Gamma_{a'b'}^i$ and using (5.90) we indeed find that $\Gamma_{a'b'}^i = \partial^i \Phi_{a'b'}$ with

$$\Phi_{\alpha\beta}(x) = m_{(\alpha\beta)}(x) - \frac{1}{2} \delta_{ij} \tau^i_\alpha(x) \tau^j_\beta(x) + \partial_{(\alpha} m_{\beta)}(x), \quad (5.93)$$

where $m_{(\alpha\beta)} = m_{(\alpha}{}^{a'} \delta_{\beta)}^{a'}$. This is the stringy generalisation of eqn. (5.22).

Using the expressions for the components of the Γ -connection calculated above we may now verify that the Newton-Cartan geodesic equation (5.75) and the Poisson equation

¹⁴Remember that we do not distinguish anymore between flat indices a' and curved indices α .

(5.76) corresponding to the second gauging procedure reduce to the equations (5.40) and (5.42) derived in the first gauging procedure. After gauge-fixing the Newton-Cartan symmetries to the acceleration-extended Galilei symmetries as described above, the Lagrangian (5.69) reduces to the Lagrangian associated to the action (5.37), with the potential $\Phi_{\alpha\beta}$ given by (5.93) and $\bar{\gamma}_{\bar{\alpha}\bar{\beta}} = \tau_{\bar{\alpha}\bar{\beta}}$.¹⁵

$$L = -\frac{T}{2} \sqrt{-\det(\tau)} \tau^{\bar{\alpha}\bar{\beta}} \left(\partial_{\bar{\alpha}} x^i \partial_{\bar{\beta}} x^j \delta_{ij} + \partial_{\bar{\alpha}} x^\alpha \partial_{\bar{\beta}} x^\beta [\tau^i{}_{\alpha} \tau^j{}_{\beta} \delta_{ij} - 2m_{(\alpha\beta)} - 2\partial_{(\alpha} m_{\beta)}] \right). \quad (5.94)$$

The longitudinal components $R_{\alpha\beta}(\Gamma)$ of the Ricci tensor become

$$R_{\alpha\beta}(\Gamma) = -\delta_{(\alpha}^{a'} \delta_{\beta)}^{b'} e^{\rho}{}_{a'} \tau^{\sigma}{}_{b'} R_{\rho\sigma}{}^{aa'}(G) = \delta^{ij} \partial_i \partial_j \Phi_{\alpha\beta}, \quad (5.95)$$

such that indeed (5.76) gives the stringy Poisson equation (5.42). This finishes our discussion of the string moving in a flat Minkowski spacetime. In the next section we will consider the addition of a cosmological constant.

5.5 Adding a Cosmological Constant

To discuss Anti-de Sitter (AdS) backgrounds we first take a look at the particle case. In the relativistic case the addition of a negative cosmological constant means that the Poincaré algebra is replaced by an AdS algebra. In the non-relativistic case the Bargmann algebra is replaced by the so-called Newton-Hooke algebra. However, instead of gauging this algebra we will take another approach.¹⁶ It turns out that, when taking the non-relativistic limit of a particle moving in an AdS background, which is a Λ -deformation of the Minkowski background, one ends up with a non-relativistic particle action which is a particular case of the non-relativistic particle action for a Galilean observer with zero cosmological constant but with the following non-zero-value of the gravitational potential:

$$\Phi(x^i) = -\frac{1}{2} \Lambda x^i x^j \delta_{ij}, \quad (5.96)$$

where $\{x^i\}$ are the transverse coordinates. The action is invariant under the so-called Newton-Hooke symmetries which are a Λ -deformation of the Galilei symmetries. The Newton-Hooke algebra can be obtained by performing a Inönü-Wigner contraction on the algebra, which schematically looks like the following:

¹⁵After the gauge-fixing one has $\tau_{\bar{\alpha}\bar{\beta}} = \partial_{\bar{\alpha}} x^\alpha \partial_{\bar{\beta}} x^\beta \eta_{\alpha\beta}$.

¹⁶For the explicit gauging of the Newton-Hooke algebra, see [115]. Note that in the relativistic case the conventional constraint also removes the local translations from the spin connection. Non-relativistically, something similar happens for the spin connection of the boosts.

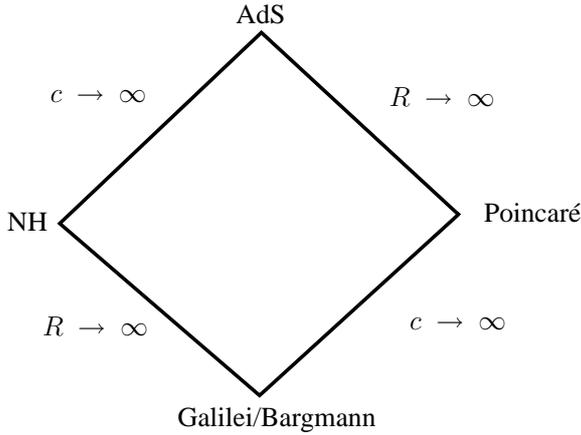


Figure 5.1: The different contractions one can take on the AdS algebra in D spacetime dimensions, $\mathfrak{so}(2, D - 2)$. The parameter R is the radius of curvature, whereas c is the speed of light.

All Newton-Hooke symmetries can be viewed as particular time-dependent transverse translations. Therefore, when gauging the transverse translations, it does not matter whether one gauges the Galilei or Newton-Hooke symmetries, in both cases one ends up with the same theory but with a different interpretation of the potential. When gauging the Galilei symmetries one interprets the potential $\Phi(x)$ as a purely gravitational potential $\phi(x)$, i.e. $\Phi(x) = \phi(x)$. On the other hand, when gauging the Newton-Hooke symmetries one writes $\Phi(x)$ as the sum of a purely gravitational potential $\phi(x)$ and a Λ -dependent part, i.e.

$$\Phi(x) = \phi(x) - \frac{1}{2}\Lambda x^i x^j \delta_{ij}. \quad (5.97)$$

In both cases, turning off gravity amounts to setting $\phi(x) = 0$. For $\Lambda = 0$ this implies $\Phi(x) = 0$ but for $\Lambda \neq 0$ this implies $\Phi(x^i) = \frac{1}{2}\Lambda x^i x^j \delta_{ij}$. These different conditions lead to different surviving symmetries: (centrally extended) Galilei symmetries for $\Lambda = 0$ versus (centrally extended) Newton-Hooke symmetries [88, 89] for $\Lambda \neq 0$.

It is now a relatively straightforward task to generalize the above discussion to a string moving in an AdS background. Taking the non-relativistic limit of a string moving in such a background leads to a non-relativistic action that is invariant under a stringy version of the Newton-Hooke symmetries [82, 87]. Note that this action is Λ -deformed in two ways: (i) there is a Λ -dependent potential term in the action like in the particle case and (ii) the foliation metric is deformed from $M_{1,1}$ ($\Lambda = 0$) to AdS_2 ($\Lambda \neq 0$). The latter deformation, which leads to an AdS_2 -foliation of spacetime, is trivial in the particle case. All stringy Newton-Hooke symmetries can be viewed as particular world-sheet dependent transverse translations. It is therefore sufficient to gauge the symmetries for the case $\Lambda = 0$ only, which amounts to gauging the stringy Galilei symmetries. In a second stage one obtains the $\Lambda \neq 0$ case by a different interpretation of the potential $\Phi_{\alpha\beta}(x)$ and by replacing the flat foliation space by an AdS_2 spacetime. To be concrete, in analogy to the particle case,

we gauge the stringy Galilei symmetries only and, next, write the background potential $\Phi_{\alpha\beta}(x)$, which is needed for this gauging, as

$$\Phi_{\alpha\beta}(x) = \phi_{\alpha\beta}(x) + \frac{1}{4}\Lambda x^i x^j \delta_{ij} \tau_{\alpha\beta}, \quad (5.98)$$

where $\phi_{\alpha\beta}(x)$ is the purely gravitational potential and $\tau_{\alpha\beta}$ is an AdS_2 -metric. At the same time we have replaced the flat foliation by an AdS_2 space leading to an AdS_2 -foliation of spacetime.¹⁷

In this way it is a relatively simple manner to obtain the geodesic equations of motion for a fundamental string in a cosmological background and to derive the equations of motion for the potential $\Phi_{\alpha\beta}(x)$ itself.

5.5.1 The Particle Case

In taking the non-relativistic limit of a particle moving in an AdS background (which is a Λ -deformation of the Minkowski background) one ends up with the action of a non-relativistic particle moving in a harmonic oscillator potential. This is a particular case of the non-relativistic particle action for a Galilean observer with zero cosmological constant but with a particular non-zero-value of the potential $\Phi(x)$. In view of this it is convenient to write the potential $\Phi(x)$ as the sum of a purely gravitational potential $\phi(x)$ and an effective background potential $\phi_\Lambda(x)$ describing the harmonic oscillator due to the cosmological constant:

$$\Phi(x) = \phi(x) + \phi_\Lambda(x). \quad (5.99)$$

Notice that eqn.(5.99) points out a conceptual difference between the relativistic and non-relativistic notion of a cosmological constant, which will also be true for the string. Namely, according to (5.99) one is always able to redefine the potential $\Phi(x)$ in order to absorb the cosmological constant into $\Phi(x)$. But in the relativistic case such a redefinition of Λ into the metric $g_{\mu\nu}(x)$ is not possible. The non-relativistic particle action in the presence of a cosmological constant is invariant under the Newton-Hooke symmetries which is a Λ -deformation of the Galilei symmetries we considered in section 2. A particularly useful feature of the Newton-Hooke symmetries is that the Λ -deformed symmetries can all be viewed as particular time-dependent transverse translations. This means that, when gauging the Galilei symmetries like we did in section 2, the Newton-Hooke symmetries are automatically included. The consequence of this is that, although we cannot perform the second gauging procedure of section 2, i.e. gauge the full Newton-Hooke algebra, it is straightforward to apply the first gauging procedure, i.e. gauge the transverse translation

¹⁷When gauging the full (deformed) stringy Galilei symmetries one of the kinematical constraints which have to be imposed in order to restrict to a stringy Galilean observer, for $\Lambda \neq 0$, is that the curvature corresponding to rotations amongst the longitudinal directions is proportional to Λ . This leads to a flat foliation for $\Lambda = 0$ but an AdS_2 -foliation for $\Lambda \neq 0$.

leading to arbitrary accelerations between different frames, as is appropriate for a Galilean observer. Independent of whether we are starting from the Galilei or Newton-Hooke symmetries, when we gauge the transverse translations we end up with precisely the same answer which we already derived in section 2, but with a different interpretation of the potential $\Phi(x)$. The difference is seen when we turn off gravity. Without a cosmological constant, turning off gravity means setting $\Phi(x) = \phi(x) = 0$ and there is no background potential, i.e. $\phi_\Lambda(x) = 0$. However, when $\Lambda \neq 0$, turning off gravity means a different thing since now we want to end up with a non-zero background potential $\phi_\Lambda(x) \neq 0$. According to eqn. (5.99) it means setting $\Phi(x) = \phi_\Lambda(x)$ or $\phi(x) = 0$. One can view this as a different gauge condition and that is the reason why, in the presence of a non-zero cosmological constant, the symmetries that relate inertial frames is given by the Newton-Hooke symmetries instead of the Galilei symmetries. For a Galilean observer, however, we end up with precisely the same geodesic equation and bulk equation of motion we derived in the absence of a cosmological constant in the previous section.

Before showing how the Newton-Hooke symmetries arise as the transformations that relate inertial frames, it is instructive to first re-derive the Galilei symmetries starting from a Galilean observer. Consider the acceleration-extended Galilei symmetries given in eqs. (5.9) and (5.10). Without a cosmological constant, turning off gravity means setting $\Phi(x) = 0$. Given the transformation rule (5.10) of the background potential $\Phi(x)$ this implies the following restriction on the transverse translations:

$$\frac{d}{d\tau} \left(\frac{\dot{\xi}^i}{\dot{t}} \right) = 0, \quad (5.100)$$

where we have ignored the standard ambiguity in the potential represented by the function $g(t)$ in eqn. (5.10). This restriction implies that $\dot{\xi}^i = \lambda^i \dot{t}$ or $\xi^i(t) = \lambda^i t + \zeta^i$. This brings us back to the Galilei transformations.

We now turn to the case of a non-zero cosmological constant Λ . It turns out that, when taking the non-relativistic limit as is described in section 5.2 of a particle moving in an (A)dS background,¹⁸ one ends up with a particle moving in an effective background potential $\phi_\Lambda = -\frac{1}{2}\Lambda x^i x^i$ describing a harmonic oscillator [89]:

$$S = \frac{m}{2} \int \left(\frac{\dot{x}^i \dot{x}^j \delta_{ij}}{\dot{t}} + \dot{t} \Lambda x^i x^j \delta_{ij} \right) d\tau. \quad (5.101)$$

We take the convention in which $\Lambda > 0$ describes a dS space, whereas $\Lambda < 0$ gives an AdS space. In the following we will consider the AdS case only. The action (5.101) is nothing

¹⁸For this the cosmological constant Λ must be rescaled with a factor of ω^{-2} . This is related to the fact that if one wants to obtain the Newton-Hooke algebra from the *AdS* algebra by contraction, the radius of curvature R needs to be rescaled with ω .

else than the action (5.8), with $\Phi(x)$ being the harmonic oscillator potential,

$$\Phi(x) = \phi_\Lambda(x) = -\frac{1}{2}\Lambda x^i x^i. \quad (5.102)$$

Viewed as a gauge condition, and using the transformation rule (5.10), this equation is invariant under transverse translations that satisfy the following constraint:

$$\frac{1}{t} \frac{d}{d\tau} \left(\frac{\xi^i}{t} \right) = \Lambda \xi^i. \quad (5.103)$$

Here we have again ignored the ambiguity in the potential represented by the function $g(t)$ in eqn. (5.10). For $\Lambda < 0$, i.e. AdS space, the restriction (5.103) on ξ^i is solved by¹⁹

$$\xi^i(t) = \lambda^i R \sin\left(\frac{t}{R}\right) + \zeta^i \cos\left(\frac{t}{R}\right), \quad (5.104)$$

where

$$R^2 \equiv -\frac{1}{\Lambda}. \quad (5.105)$$

Note that for $\Lambda \rightarrow 0$ or $R \rightarrow \infty$ this expression reduces to the Galilei result $\xi^i(t) = \lambda^i t + \zeta^i$.

The complete transformation rules are now obtained by combining the transformations (5.104) with the constant time translations and the spatial rotations:

$$\delta t = \zeta^0, \quad \delta x^i = \lambda^i_j x^j + \lambda^i R \sin\left(\frac{t}{R}\right) + \zeta^i \cos\left(\frac{t}{R}\right). \quad (5.106)$$

This defines the Newton-Hooke algebra whose non-zero commutators are given by [88, 89]:

$$\begin{aligned} [P_a, H] &= R^{-2} G_a, & [G_a, H] &= -P_a, \\ [M_{ab}, P_c] &= -2\eta_{c[a} P_{b]}, & [M_{ab}, G_c] &= -2\eta_{c[a} G_{b]}, \\ [M_{ab}, M_{cd}] &= 4\eta_{[a[c} M_{d]b]}. \end{aligned} \quad (5.107)$$

Here H, P_a, G_a and M_{ab} are the generators of time translations, spatial translations, boosts and spatial rotations, with parameters $\zeta^0, \zeta^a, \lambda^a$ and λ^{ab} , respectively. We note that the cosmological constant shows up in the $[P_a, H]$ commutator, but not in the $[P_a, P_b]$ commutator.²⁰ This is consistent with the fact that the transverse space is flat. We also observe that at this stage the Newton-Hooke algebra (5.107) does not contain a central extension like the Bargmann algebra, i.e. $[P_a, G_b] = 0$. Similar to the Galilei particle action (5.4) the Newton-Hooke particle action (5.101) suggests a central extension: the

¹⁹For $\Lambda > 0$, i.e. dS space, one obtains a similar expression but with the sine and cosine replaced by their hyperbolic counterparts.

²⁰Note that upon gauging the Newton-Hooke algebra the cosmological constant only appears in the boost-curvature and the transformation of the boost spin connection, see [115].

corresponding Lagrangian is quasi-invariant under both boosts *and* translations, described by the parameter (5.104):

$$\begin{aligned}\delta L &= \frac{d}{d\tau} \left(\frac{m \delta_{ij} x^i \dot{\xi}^j}{\dot{t}} \right) \\ &= \frac{d}{d\tau} \left(m x^i \lambda^j \delta_{ij} \cos\left(\frac{t}{R}\right) - m x^i \zeta^j \delta_{ij} \sin\left(\frac{t}{R}\right) \right).\end{aligned}\quad (5.108)$$

This is most easily seen by using the restriction (5.103) directly in the variation of the Lagrangian corresponding to the action (5.101). In the limit $R \rightarrow \infty$, i.e. $\Lambda \rightarrow 0$ the variation (5.108) reduces to the variation (5.6). Calculating the Noether charges Q_P and Q_G for the translations and the boosts respectively, the Poisson brackets suggest the same central extension Z as for the Galilei particle:

$$[P_a, G_b] = \delta_{ab} Z. \quad (5.109)$$

Given the transformation rules (5.106), it is straightforward to calculate the commutators between the different transformations and to verify that they are indeed given by the Newton-Hooke algebra (5.107). As explained above, when viewed as the symmetries of the Newton-Hooke particle described by the action (5.101), one obtains a centrally-extended Newton-Hooke algebra. The contraction $R \rightarrow \infty$ on this algebra reproduces the Bargmann algebra. This is the non-relativistic analog of the fact that the $R \rightarrow \infty$ contraction on the $(A)dS$ algebra yields the Poincaré algebra.

To obtain the cosmological constant in the gauging procedure of the Bargmann algebra we relate the expression for the potential (5.22) in terms of the gauge field components to the potential (5.99):

$$\begin{aligned}\Phi(x) &= m_0(x) - \frac{1}{2} \delta_{ij} \tau^i(x) \tau^j(x) + \partial_0 m(x) \\ &= \phi(x) - \frac{1}{2} \Lambda x^i x^j \delta_{ij}.\end{aligned}\quad (5.110)$$

The Poisson equation (5.13) can then be written as

$$\Delta \phi(x) = S_{D-2} G \rho(x) + (D-1) \Lambda, \quad (5.111)$$

where D is the dimension of spacetime.

5.5.2 The String Case

We now wish to discuss the string case following the same philosophy as we used for the particle case. Like in the particle case, we write the potential $\Phi_{\alpha\beta}(x)$ as the sum of a purely gravitational potential and a background potential that represents the extra gravitational force represented by the non-zero cosmological constant Λ :

$$\Phi_{\alpha\beta}(x) = \phi_{\alpha\beta}(x) + \phi_{\alpha\beta,\Lambda}(x). \quad (5.112)$$

We first consider the case of a zero cosmological constant and show how the stringy Galilei symmetries are recovered after turning off gravity. According to eqn. (5.39) the condition $\Phi_{\alpha\beta}(x) = 0$ leads to the following restriction on the transverse translations:

$$\partial_{\alpha} \left(\sqrt{-\bar{\gamma}} \bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\beta}} \xi^i \right) = 0, \quad (5.113)$$

where we have ignored the standard ambiguity in $\Phi_{\alpha\beta}(x)$ represented by the arbitrary functions $g_{\beta}(x^{\epsilon})$ in eqn. (5.39). This restriction is the stringy analogue of the restriction (5.100) we found in the particle case. It is precisely the same restriction one finds if one requires that the non-relativistic string action (5.33) is invariant under transverse translations. The solution of eqn. (5.113) is given by $\xi^i(x^{\alpha}) = \lambda^i_{\beta} x^{\beta} + \zeta^i$, which can be checked using expression (5.29) of $\bar{\gamma}^{\bar{\alpha}\bar{\beta}}$. This brings us back to the stringy Galilei symmetries given in eqn. (5.34).

We now consider a non-zero cosmological constant Λ . It turns out that when one considers the non-relativistic limit of a string moving in an AdS background one ends up with an effective background potential given by [81]

$$\phi_{\alpha\beta,\Lambda} = \frac{1}{4} \Lambda x^i x^j \delta_{ij} \tau_{\alpha\beta}, \quad (5.114)$$

where $\tau_{\alpha\beta}$ is an AdS₂-metric. At the same time one should replace the flat foliation of spacetime by an AdS₂-foliation. This means that both in the definition of $\bar{\gamma}_{\bar{\alpha}\bar{\beta}}$ given in eqn. (5.28) and the action (5.37) one should replace the flat metric $\eta_{\alpha\beta}$ by the AdS₂-metric $\tau_{\alpha\beta}$. Setting also $\Phi_{\alpha\beta}(x) = \frac{1}{4} \Lambda x^i x^j \delta_{ij} \tau_{\alpha\beta}$ in eqn.(5.37), one obtains the action [81]

$$S = -\frac{T}{2} \int d^2\sigma \sqrt{-\bar{\gamma}} \left(\bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} x^i \partial_{\bar{\beta}} x^j \delta_{ij} + \Lambda x^i x^j \delta_{ij} \right), \quad (5.115)$$

with $\bar{\gamma}_{\bar{\alpha}\bar{\beta}}$ given by

$$\bar{\gamma}_{\bar{\alpha}\bar{\beta}} = \partial_{\bar{\alpha}} x^{\alpha} \partial_{\bar{\beta}} x^{\beta} \tau_{\alpha\beta}. \quad (5.116)$$

The replacement of $\eta_{\alpha\beta}$ by $\tau_{\alpha\beta}$ also applies to the transformation rule (5.39). This leads to the following modified restriction on the transverse translations:

$$\frac{1}{\sqrt{-\bar{\gamma}}} \partial_{\bar{\alpha}} \left(\sqrt{-\bar{\gamma}} \bar{\gamma}^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\beta}} \xi^i \right) = -\Lambda \xi^i. \quad (5.117)$$

Note that we have again ignored the arbitrary functions $g_{\beta}(x^{\epsilon})$ in eqn. (5.39). For $\Lambda < 0$, i.e. AdS space, the restriction (5.117) is solved for by the following expression for $\xi^i(x^{\alpha})$:

$$\xi^i(x^{\alpha}) = \lambda^i_0 \sqrt{z^2 + R^2} \sin\left(\frac{t}{R}\right) + \lambda^i_1 z + \zeta^i \frac{\sqrt{z^2 + R^2}}{R} \cos\left(\frac{t}{R}\right), \quad (5.118)$$

where we have written $x^{\alpha} = \{t, z\}$ and used that $\Lambda = -R^{-2}$. Note that for $R \rightarrow \infty$ this expression reduces to the stringy Galilei one given by $\xi^i(x^{\alpha}) = \lambda^i_{\beta} x^{\beta} + \zeta^i$.

The complete transformation rules are obtained by combining the transformation rules (5.118) with the spatial transverse rotations and the isometries of the AdS₂-space that act on $x^\alpha = \{t, z\}$. The form of the latter transformations in an explicit coordinate frame is given in appendix E, see eqn. (G.14), where a few useful properties of the AdS₂ foliation space have been collected. All these transformations together define the stringy Newton-Hooke algebra:

$$\begin{aligned}
[H_{a'}, H_{b'}] &= R^{-2} M_{a'b'}, & [M_{b'c'}, H_{a'}] &= -2\eta_{a'[b'} H_{c']}, \\
[M_{c'd'}, M_{e'f'}] &= 4\eta_{[c'[e'} M_{f']d'}], \\
[P_a, H_{a'}] &= R^{-2} M_{aa'}, & [J_{cd}, J_{ef}] &= 4\eta_{[c[e} J_{f]d}], \\
[G_{bc'}, H_{a'}] &= \eta_{a'c'} P_b, & [J_{bc}, P_a] &= -2\eta_{a[b} P_{c]}, \\
[G_{cd'}, M_{ef}] &= 2\eta_{d'[e'} G_{|c|f'}], & [J_{cd}, G_{ef}] &= -2\eta_{e[c} G_{d]f'}.
\end{aligned} \tag{5.119}$$

Note that the generators $\{H_{a'}, M_{a'b'}\}$ span an $\mathfrak{so}(2, 1)$ algebra describing the isometries of the AdS₂-foliation. Using the transformation rules given above and in appendix E one may calculate the different commutators and verify that the algebra defined by (5.119) is satisfied. Notice how the cosmological constant ends up in the $[H_{a'}, H_{b'}]$ and $[P_a, H_{a'}]$ commutators, but not in the $[P_a, P_b]$ commutator. This is consistent with the fact that the transverse space is flat but that the two-dimensional longitudinal space is not flat. Like in the case of the point particle, the stringy Newton-Hooke algebra (5.119) allows for an extension [81]. This is motivated by the fact that the Lagrangian L corresponding to the string action (5.115) with the potential (5.114) transforms as a total derivative under the boosts and translations described by the parameters (5.118):

$$\delta L = \partial_{\bar{\alpha}} \left(-T \sqrt{-\bar{\gamma}} \bar{\gamma}^{\bar{\alpha}\bar{\beta}} x^i \partial_{\bar{\beta}} \xi_i \right). \tag{5.120}$$

This is most easily seen by using the restriction (5.117) directly in the variation of the Lagrangian corresponding to (5.115). For $R \rightarrow \infty$ the variation (5.120) reduces to the variation (5.36), and in the particle case it reduces to the variation (5.108). The resulting extension suggested by the Poisson brackets is given by eqn. (5.46).

We now fit the cosmological constant into the gauging procedure for the string. One important difference with the point particle case is that the foliation space for the string becomes AdS₂, whereas for the particle this foliation space is trivially flat. To accomplish this AdS₂-foliation we change the on-shell curvature constraint (5.78) for the foliation space, whereas for the transverse space we keep it unaltered:

$$R_{\mu\nu}{}^{a'b'}(M) = \Lambda \tau_{[\mu}{}^{a'} \tau_{\nu]}{}^{b'}, \quad R_{\mu\nu}{}^{ab}(J) = 0. \tag{5.121}$$

This gives an AdS₂ space in the longitudinal direction and a flat transverse space. We then choose coordinates such that

$$\begin{aligned}\tau_\mu^a &= (\tau_\alpha^{a'}, 0), & e_\mu^a &= (-\tau_\alpha^{a'} \tau_\alpha^{a'}, \delta_i^a), \\ \tau^\mu_{a'} &= (\tau^\alpha_{a'}, \tau^i_{a'}), & e^\mu_a &= (0, \delta_a^i),\end{aligned}\tag{5.122}$$

where now we are not able to choose $\tau_\alpha^{a'} = \delta_\alpha^{a'}$, as we did in (5.80). Using the coordinates chosen in appendix E one can choose

$$\tau_\alpha^{a'} = \left(\left(1 + \frac{z^2}{R^2}\right)^{1/2} \delta_0^{a'}, \left(1 + \frac{z^2}{R^2}\right)^{-1/2} \delta_1^{a'} \right),\tag{5.123}$$

$$\tau^\alpha_{a'} = \left(\left(1 + \frac{z^2}{R^2}\right)^{-1/2} \delta_{a'}^0, \left(1 + \frac{z^2}{R^2}\right)^{1/2} \delta_{a'}^1 \right).\tag{5.124}$$

In view of this we should carefully distinguish between the curved longitudinal coordinates $\{\alpha\}$ and the flat longitudinal coordinates $\{a'\}$. In contrast, from now on we will not distinguish between flat and curved transverse coordinates $\{a\}$ and $\{i\}$ because the transverse space is flat. With the coordinates (5.122) the constraints (5.121) allow for the gauge choice

$$\omega_\mu^{ab} = 0, \quad \omega_i^{a'b'} = 0.\tag{5.125}$$

The condition $\omega_i^{ab} = 0$ is trivially satisfied, but an explicit calculation reveals that

$$\omega_\alpha^{ij} = -\tau_\alpha^{a'} \left(\partial^{[i} \tau^{j]}_{a'} + \partial_{[i} m_{j]}^{a'} \right) = -\frac{1}{2} \Gamma_{\alpha j}^i = 0.\tag{5.126}$$

So the gauge condition $\omega_\alpha^{ij} = 0$ sets the connection component $\Gamma_{\alpha j}^i$ to zero, as in the Galilei string case. From (5.126) we again arrive at (5.90). One should now be careful in distinguishing between $\tau^i_{a'}$, which is nonzero in general, and $\tau_i^{a'}$, which is zero for the coordinate choice (5.122). With the spin connections (5.125) and (5.126) one can show that the expression for the connection, eqn.(5.64), implies that again $\Gamma_{\alpha\beta}^i = \partial^i \Phi_{\alpha\beta}$, i.e. the Γ -connection can also for the AdS₂-foliation be written as the transverse gradient of a potential. The potential $\Phi_{\alpha\beta}$ is now given by

$$\Phi_{\alpha\beta} = m_{a'} \omega_{(\alpha}^{a'b'} \tau_{\beta)}^{b'} + \tau_{(\alpha}^{a'} \partial_{\beta)} m_{a'} + \tau_{(\alpha}^{a'} m_{\beta)}^{a'} - \frac{1}{2} \tau_{(\alpha}^{a'} \tau_{\beta)}^{b'} \tau^j_{a'} \tau^j_{b'},\tag{5.127}$$

which should be compared to the potential for the flat foliation, eqn. (5.93). To describe the splitting described in the beginning of this section with the background given by (5.114), we put the potential (5.127) equal to (5.112). That the set of gauge fields appearing on the right hand side of (5.127) can give rise to an arbitrary symmetric $\Phi_{\alpha\beta}$ can be seen by taking, for example, the realization $m_{a'} = \tau^i_{a'} = 0$ (and thus, via (5.90), $m_i^{a'} = 0$) in the potential (5.127) and expressing the remaining longitudinal components m_α^a in terms of $\Phi_{\alpha\beta}$. The symmetric longitudinal projection of $m_\mu^{a'}$ is then given by

$$\tau^{\alpha(a'} m_\alpha^{b')} = \tau^{\alpha a'} \tau^{\beta b'} \Phi_{\alpha\beta},\tag{5.128}$$

whereas the antisymmetric longitudinal projection of $m_\mu^{a'}$, given by $\tau^{\alpha[a' m_\alpha^{b']}$, can be gauged away via a $Z_{a'b'}$ -transformation as is clear from eqn.(5.71). As such $m_\mu^{a'}$ can be expressed in terms of $\Phi_{\alpha\beta}$. With $\{\Gamma_{\alpha\beta}^i, \Gamma_{\alpha\beta}^\epsilon\}$ being the only nonzero connection coefficients, the longitudinal components of the Ricci tensor become

$$\begin{aligned} R_{\alpha\beta}(\Gamma) &= \Delta\Phi_{\alpha\beta} + R_{\alpha\beta}(AdS_2) \\ &= \Delta\phi_{\alpha\beta} + (D-1)\Lambda\tau_{\alpha\beta}, \end{aligned} \quad (5.129)$$

where we have used that $R_{\alpha\beta}(AdS_2) = \Lambda\tau_{\alpha\beta}$. Therefore, the nonzero components of the Poisson equation (5.76) read as follows [92]:

$$\Delta\phi_{\alpha\beta} = \left(S_{D-2}G\rho - (D-1)\Lambda \right) \tau_{\alpha\beta}, \quad (5.130)$$

where D is the dimension of spacetime. This concludes our discussion of the addition of the cosmological constant to the theory.

5.6 Conclusions and outlook

We have shown how the theory of Newton-Cartan can be extended from particles moving in a flat background to strings moving in a cosmological background. One way to obtain the desired equations corresponding to these extensions is to gauge the transverse translations. This necessitates the introduction of a new field, which is identified as the gravitational potential. The resulting equations of motion are the ones used by a Galilean observer. Alternatively, one can first gauge the full extended (stringy) Galilei algebra and, next, gauge-fix some of the symmetries in order to obtain the symmetries that are appropriate to a Galilean observer. The (central) extensions of the algebras involved play a crucial role in this procedure. To obtain the (stringy) Newton-Cartan theory, conventional constraints are imposed to convert the spacetime translations into general coordinate transformations and to make the spin connections dependent fields. Further on-shell constraints are imposed on the curvature of the transverse space and, in the string case, on the curvature of the foliation space. The transverse space is chosen to be flat, whereas for the string the on-shell constraint on the longitudinal boost curvature can be chosen such that one obtains either a flat foliation (corresponding to the stringy Galilei group) or an AdS_2 -foliation (corresponding to the stringy Newton-Hooke group). The first choice describes the non-relativistic limit of a string moving in a Minkowski background, whereas the second choice describes the non-relativistic limit of a string moving in an AdS_D background. The analysis can easily be extended to arbitrary branes, in which case one should use extended brane Galilei algebras [87].

It is interesting to compare our results with the literature on the application of Newton-Cartan theory in the non-relativistic limit of the AdS/CFT correspondence. This has been

discussed in, e.g., [90, 91] where some subtleties of this application are discussed. In [92] it was noted that the non-relativistic limit on the CFT-side of the correspondence should give (an infinite-dimensional extension of) the so-called Galilei conformal algebra. This Galilean conformal algebra can be obtained by contracting the relativistic conformal algebra $\mathfrak{so}(D, 2)$. It differs from the Schrödinger algebra in that first, the Galilei conformal algebra scales space and time in the same way and second, it does *not* allow for a central extension playing the role of mass. The Galilean conformal algebra is then the boundary realization of the stringy Newton-Hooke algebra in the bulk [94]. The dual gravity theory should correspondingly be a Newton-Cartan theory with an AdS_2 -foliation describing strings, instead of the usual \mathbb{R} -foliation which describes particle Newton-Cartan theory. The gauging procedure outlined in this chapter provides the framework of developing such a theory from a gauge perspective.

It is known that the Newton-Cartan theory can be obtained from a dimensional reduction of General Relativity along a null-Killing vector, see e.g. [93, 95].²¹ The fact that the Killing vector is null provides one with the degenerate metric structure which is characteristic for Newton-Cartan theory. The central charge gauge field m_μ is related to the Kaluza-Klein vector corresponding to this null direction. It would be interesting to investigate if the stringy version of the Newton-Cartan theory presented in this chapter can also be obtained by a null-reduction from higher dimensions such that the deformation potentials $m_\mu^{a'}$ and $m_\mu^{a'b'}$ obtain a similar Kaluza-Klein interpretation. This possibility should be related to the fact that the extended p-brane algebra in D dimensions is a subalgebra of the “multitemporal” conformal algebra $\mathfrak{so}(D + 1, p + 2)$ in one dimension higher [87]. One way to obtain null-directions is to start from a relativistic string and to T-dualize along its spatial world-sheet direction. The T-dual picture is a pp-wave which has a null-direction [82]. One could now use this null direction for a Kaluza-Klein reduction along the lines of [95] and see whether one obtains the stringy NC theory constructed in this chapter.

Finally, the results from this chapter, which of course are classical, can be compared to the non-relativistic limit of string theory. In [83] a particular non-relativistic limit of closed string theories is taken in which no graviton appears in the closed string spectrum. As such these theories are called “non-gravitational”, but still exhibit all the duality relations known from relativistic string theories. However, in calculating amplitudes between wound strings, an instantaneous gravitational force in the form of a scalar potential is found between these strings. It would be interesting to see how the trace of the tensor potential is related to the scalar potential of this particular non-relativistic string theory.

²¹In [95] also a proposal for an action describing the NC bulk dynamics has been made. For AdS/CFT applications this is a very desirable feature.

Chapter 6

Supersymmetric Newton-Cartan gravity

6.1 Introduction

By now we know that non-relativistic Newtonian gravity can be reformulated in a geometric way, invariant under general coordinate transformations, thus mimicking General Relativity. By (partially) gauge fixing general coordinate transformations, non-geometric formulations can be obtained. The extreme case is the one in which one gauge fixes such that one only retains the Galilei symmetries, corresponding to a description in free-falling frames, in which there is no gravitational force. A less extreme case is obtained by gauge fixing such that one not only considers free-falling frames, but also includes frames that are accelerated, with an arbitrary time-dependent acceleration, with respect to a free-falling frame. These observers are called ‘Galilean observers’ and the corresponding formulation of non-relativistic gravity is called ‘Galilean gravity’¹. In such a frame, the gravitational force is described by the Newton potential Φ . Such frames are related to each other by the so-called ‘acceleration extended’ Galilei symmetries, consisting of an extension of the Galilei symmetries in which constant spatial translations become time-dependent ones. In this chapter, we will construct a supersymmetric version of both Newton-Cartan gravity, as well as Galilean gravity, and show how they are related via a partial gauge fixing.

In chapter four we showed how four-dimensional Newton-Cartan gravity can be obtained by gauging the Bargmann algebra.² An important step in this gauging procedure is the imposition of a set of constraints on the curvatures corresponding to the algebra [98]. The purpose of these constraints is to convert the abstract time and space translations

¹The case in which constant accelerations are considered, instead of time-dependent ones, leads to ordinary Newtonian gravity, described by a time-independent Newton potential.

²The Bargmann algebra does not contain any conformal symmetries. Non-relativistic conformal (super)algebras, and their relation to Newton-Cartan space-time, were investigated in [76,97].

of the Bargmann algebra into general coordinate transformations. When gauging the Poincaré algebra, as we reviewed in section 4.2, one imposes that the curvature corresponding to the spacetime translations, vanishes:

$$R_{\mu\nu}{}^A(P) = 0, \quad \mu, A = 0, 1, 2, 3. \quad (6.1)$$

These constraints are conventional constraints. The same set of constraints serves another purpose: it can be used to solve for the spin-connection fields corresponding to the Lorentz transformations in terms of the other gauge fields. This is different from the non-relativistic case where setting the curvature corresponding to time translations equal to zero is a true constraint:

$$R_{\mu\nu}(H) = 2\partial_{[\mu}\tau_{\nu]} = 0. \quad (6.2)$$

This constraint cannot be used to solve for any spin connection. Instead, it allows us to write the temporal Vierbein τ_μ as

$$\tau_\mu(x^\nu) = \partial_\mu\tau(x^\nu) \quad (6.3)$$

for an arbitrary scalar function $\tau(x^\nu)$. One can use the time reparametrizations to choose this function equal to the absolute time which foliates the Newtonian space-time:

$$\tau(x^\nu) = x^0 \equiv t, \quad \tau_\mu(x^\nu) = \delta_\mu^0. \quad (6.4)$$

This can be viewed as a gauge condition that fixes the time reparametrizations with local parameters $\xi^0(x^\mu)$ to constant time translations:³

$$\xi^0(x^\nu) = \xi^0. \quad (6.5)$$

One also imposes the conventional constraint that the curvature of the spatial translations equals zero:

$$R_{\mu\nu}{}^a(P) = 0. \quad (6.6)$$

However, this constraint by itself is not sufficient to solve for both the spin connection fields corresponding to the spatial translations as well as the spin connection fields corresponding to the boost transformations. In order to achieve that one needs to extend the Galilei algebra to the Bargmann algebra and impose that the curvature corresponding to the central extension vanishes as well. Together with (6.6) this conventional constraint can be used to solve for all spin-connection fields. The invariance of the non-relativistic theory under central charge transformations corresponds to particle number conservation which is indeed a non-relativistic property.

³With the exception of sections 2.1 and 4, we will assume that any parameter, without any spacetime dependence indicated, is constant. This should be contrasted to fields where we do not always indicate the explicit spacetime dependence.

It is the purpose of this chapter to extend the construction of chapter four to the supersymmetric case by gauging a supersymmetric extension of the Bargmann algebra. An $\mathcal{N} = 1$ supersymmetric extension of the Bargmann algebra was considered in [99]. According to this algebra, the anti-commutator of two supercharges leads to a central charge transformation. We are however primarily interested in a non-trivial supersymmetric extension in which the anti-commutator of the fermionic generators contains the generators corresponding to time and space translations. It turns out that this can only be achieved provided we consider an $\mathcal{N} = 2$ supersymmetric extension of the Bargmann algebra [81]. The analysis of [81] also leads to a realization of this algebra, as global symmetries, on the embedding coordinates of a non-relativistic superparticle propagating in a flat Newtonian space-time.

For technical reasons explained below, we consider from now on only the case of three spacetime dimensions, i.e. $D = 3$. Three-dimensional gravity is interesting by itself, both relativistically as well as non-relativistically. We saw in sections 2.4 and 2.7 that the relativistic theory does not have any local degrees of freedom and there is no interaction between static sources. However, moving particles can still exhibit non-trivial scattering [100]. In contrast, in the non-relativistic Newtonian theory, there is an attractive gravitational Newton force that goes as the inverse of the distance between point masses. This theory can thus not be viewed as a non-relativistic limit of General Relativity. Indeed, in the latter, there is no attractive force between static sources, while Newton gravity does exhibit such a gravitational attraction. Coming back to the supersymmetric extensions of non-relativistic gravity, we note that supersymmetric extensions of the three-dimensional Bargmann algebra were considered in [101].

When gauging the $\mathcal{N} = 2$ super-Bargmann algebra, one must at some point impose that the super-covariant extension of the bosonic curvature $R_{\mu\nu}(H)$ equals zero:

$$\hat{R}_{\mu\nu}(H) = 0. \quad (6.7)$$

This is the supersymmetric generalization of the constraint (6.2). We find that under supersymmetry this constraint leads to another constraint that sets the super-covariant curvature corresponding to one of the two gravitini, $\psi_{\mu+}$, equal to zero:

$$\hat{\psi}_{\mu\nu+} = 0. \quad (6.8)$$

In the same way that the time reparametrizations, up to constant time translations, can be used to fix the temporal dreibein according to (6.4), one may now use one of the two local supersymmetries, with arbitrary fermionic parameters $\epsilon_+(x^\mu)$, to set the $\psi_{\mu+}$ gravitini equal to zero:

$$\psi_{\mu+} = 0. \quad (6.9)$$

This gauge choice fixes the local ϵ_+ -supersymmetry to constant ones:

$$\epsilon_+(x^\mu) = \epsilon_+ . \quad (6.10)$$

The remaining supersymmetry, with parameters $\epsilon_-(x^\nu)$ can be non-trivially gauged. Only the commutator of a constant and a gauged supersymmetry leads to a (local) spatial translation. We find that the commutator of two constant supersymmetries leads to a (constant) time translation while the commutator of two gauged supersymmetries leads to a (local) central charge transformation. It turns out that one can gauge-fix the global (with parameter ϵ_+) supersymmetry, but not the local supersymmetry (with parameter $\epsilon_-(x^\nu)$). This explains why we need at least two supersymmetries to obtain a non-trivial (i.e. where the commutator of two supersymmetries gives a space or time translation) supersymmetry algebra .

The above paragraph refers to a so-called ‘full gauging’, in which all symmetries are gauged. This leads to a geometric description of Newtonian supergravity, that uses a temporal and spatial dreibein and is invariant under arbitrary general coordinate transformations. This theory can appropriately be called ‘Newton-Cartan supergravity’. The case in which we consider a description that is only invariant under the acceleration extended Galilei symmetries, is obtained by a ‘medium gauging’ and the corresponding supergravity theory can be called ‘Galilean supergravity’. In this chapter, we will obtain the medium gauging from the fully gauged Newton-Cartan supergravity by a partial gauge fixing. The Galilean supergravity we thus obtain, contains a field, corresponding to the Newton potential, as well as a fermionic superpartner. The Newton potential of Galilean supergravity replaces the temporal and spatial dreibeins of Newton-Cartan supergravity. We find that, in order to write down the supersymmetry transformation rules, we also have to introduce a ‘dual Newton potential’. The Newton potential and its dual can be seen as real and imaginary parts of a meromorphic function, whose singularities indicate the positions of added point-like sources.

All the above arguments are equally valid when gauging the four-dimensional $\mathcal{N} = 2$ super-Bargmann algebra. However, in the four-dimensional case we are dealing with the additional complication that in the relativistic case the algebra can only be closed provided we introduce more fields than the gauge fields associated to each of the generators of the algebra. To be precise, the $\mathcal{N} = 2$ super-Poincaré algebra requires besides the usual gauge fields the introduction of an extra Abelian gauge field. In the non-relativistic case, one would expect that, similarly, extra fields are needed to close the algebra. We have performed the four-dimensional gauging procedure and verified that it is not enough to introduce a single Abelian vector field in the non-relativistic case. More fields are needed and that is what makes the four-dimensional case more complicated. In the conclusions

we will comment on this issue.

This chapter is organized as follows. As a warming-up exercise, we will first review in section 2 the gauging, leading to Newton-Cartan gravity, and subsequent gauge fixing, leading to Galilean gravity, in the bosonic case. In section 3 we present the 3D $\mathcal{N} = 2$ super-Bargmann algebra. In section 4 we perform the gauging of this algebra, following the procedure outlined for the bosonic case in chapter four and reviewed in section 2. We explicitly perform the gauge fixing that brings us to the frame of a Galilean observer in section 5 and show how the Newton-Cartan supergravity theory reduces to a Galilean supergravity theory in terms of a Newton potential and its supersymmetric partner. We present our conclusions in section 6.

6.2 Newton-Cartan and Galilean gravity

In this section, we recall shortly how the Newton-Cartan theory is obtained by gauging the Bargmann algebra, and how subsequently Galilean gravity can be obtained by partial gauge fixing.

6.2.1 Newton-Cartan gravity

Our starting point is the Bargmann algebra (4.46), but now specifically for three dimensions. In this case the algebra simplifies a bit. Namely, in two spatial dimensions there is only one spatial rotation. As such rotations will commute and form an Abelian subalgebra, i.e. $[J_{ab}, J_{cd}] = 0$. In table 1 we have indicated the symmetries, gauge fields, local parameters and curvatures that we associated to each of the generators.

symmetry	generators	gauge field	parameters	curvatures
time translations	H	τ_μ	$\zeta(x^\nu)$	$R_{\mu\nu}(H)$
space translations	P^a	e_μ^a	$\zeta^a(x^\nu)$	$R_{\mu\nu}^a(P)$
boosts	G^a	ω_μ^a	$\lambda^a(x^\nu)$	$R_{\mu\nu}^a(G)$
spatial rotations	J^{ab}	ω_μ^{ab}	$\lambda^{ab}(x^\nu)$	$R_{\mu\nu}^{ab}(J)$
central charge transf.	Z	m_μ	$\sigma(x^\nu)$	$R_{\mu\nu}(Z)$

Table 6.1: This table indicates the generators of the Bargmann algebra and the gauge fields, local parameters and curvatures that are associated to each of these generators.

According to the Bargmann algebra the gauge fields transform under spatial rotations, boosts and central charge transformations as described by eqn.(4.48). We will not consider temporal and spatial translations because later these will effectively be removed by the

second and third equation of the constraints (4.60). The curvatures which transform covariantly under the transformations (4.48) are then given by eqns.(4.49)-(4.53).

We then proceed by imposing the second and third equation of the constraints (4.60)

$$R_{\mu\nu}{}^a(P) = 0, \quad R_{\mu\nu}(Z) = 0. \quad (6.11)$$

These are the conventional constraints. On top of this, we impose the additional constraints

$$R_{\mu\nu}(H) = 0, \quad R_{\mu\nu}{}^{ab}(J) = 0. \quad (6.12)$$

The first equation defines the foliation of a Newtonian spacetime. The second one is needed to obtain Newton gravity in flat space. The constraints (6.11), together with the first constraint of (6.12) can then be used to convert the H - and P^a -transformations, with parameters $\zeta(x^\nu)$ and $\zeta^a(x^\nu)$, of the algebra into general coordinate transformations, with parameters $\xi^\lambda(x^\nu)$. The gauge fields τ_μ and $e_\mu{}^a$ can now be interpreted as the temporal and spatial dreibeins. Their projective inverses, τ^μ and $e^\mu{}_a$, are defined by the equations (4.55)-4.57. Using these projective inverses one can use the conventional constraints (6.11) to solve for the spin-connections fields $\omega_\mu{}^{ab}(x^\nu)$ and $\omega_\mu{}^a(x^\nu)$ in terms of $\tau_\mu, e_\mu{}^a$ and m_μ . These solutions are given by eqns.(4.63) and (4.66), which we repeat here for convenience:

$$\omega_\mu{}^{ab}(x^\nu) = 2e^{\rho[a}\partial_{[\rho}e_{\mu]}{}^{b]} + e_\mu{}^c e^{\rho a} e^{\nu b} \partial_{[\rho}e_{\nu]}{}^c - \tau_\mu e^{\rho a} e^{\nu b} \partial_{[\rho}m_{\nu]} , \quad (6.13)$$

$$\omega_\mu{}^a(x^\nu) = e^{\nu a} \partial_{[\mu}m_{\nu]} + e_\mu{}^b e^{\nu a} \tau^\rho \partial_{[\nu}e_{\rho]}{}^b + \tau^\nu \partial_{[\mu}e_{\nu]}{}^a + \tau_\mu \tau^\nu e^{\rho a} \partial_{[\nu}m_{\rho]} . \quad (6.14)$$

At this point, the only non-zero curvature left is the one corresponding to the boost transformations. Plugging the previous constraints into the Bianchi identities one finds that the *only* non-zero components of the boost curvature are given by⁴

$$R_{0(a}{}^{b)}(G) \neq 0. \quad (6.15)$$

The dynamical vacuum equation defining Newton-Cartan gravity is given by the trace of the above expression, plus its boost transformation (which vanishes automatically by the second constraint of eqn.(6.12)):

$$R_{0a}{}^a(G) = 0, \quad R_{0a}{}^{ab}(J) = 0. \quad (6.16)$$

These equations of motion are invariant under general coordinate transformations, local boosts, local spatial rotations and local central charge transformations, with parameters $\xi^\lambda(x^\mu), \lambda^a(x^\mu), \lambda^{ab}(x^\mu)$ and $\sigma(x^\mu)$, respectively.

6.2.2 Galilean gravity

To obtain Galilean gravity, described in terms of a Newton potential $\Phi(x^\mu)$, we perform a partial gauge fixing of the Newton-Cartan theory which we will now describe. First, we

⁴Note that the flat zero-component, i.e. the contraction of a curved index μ with τ^μ , is indicated as 0.

solve the constraints (6.12) by imposing the gauge fixing conditions

$$\tau_\mu(x^\nu) = \delta_\mu^0, \quad \omega_\mu^{ab}(x^\nu) = 0. \quad (6.17)$$

This fixes the local time translations and spatial rotations to constant transformations:

$$\xi^0(x^\nu) = \xi^0, \quad \lambda^{ab}(x^\nu) = \lambda^{ab}. \quad (6.18)$$

No compensating transformations are induced by these gauge fixings. Next, we gauge fix the spatial dependence of the spatial translations by imposing the gauge fixing condition

$$e_i^a(x^\nu) = \delta_i^a. \quad (6.19)$$

Requiring $\delta e_i^a = 0$ leads to the condition

$$\xi^a(x^\nu) = \xi^a(t) - \lambda_{ai}x^i. \quad (6.20)$$

The solution (6.20) for the spatial dependence of the spatial translation parameters expresses the fact that, after imposing the gauge fixing condition (6.19), the i index should be treated as an a index and therefore only feels the constant spatial rotations. Note that after imposing the gauge fixing (6.19) space is flat and we do not distinguish anymore between the i and a indices and upper and down indices.

At this stage the independent temporal and spatial dreibein components and their projective inverses are given by⁵

$$\begin{aligned} \tau_\mu(x^\nu) &= \delta_\mu^0, & e_\mu^a(x^\nu) &= (-\tau^a(x^\nu), \delta_i^a), \\ \tau^\mu(x^\nu) &= (1, \tau^a(x^\nu)), & e^\mu_a(x^\nu) &= (0, \delta^i_a), \end{aligned} \quad (6.21)$$

where the $\tau^a(x^\nu)$ are the only non-constant dreibein components left. The only other independent gauge field left is the central charge gauge field $m_\mu(x^\nu)$. Taking into account the compensating gauge transformation given in (6.20) we find that the remaining fields $\tau^a(x^\nu)$, $m_0(x^\nu)$ and $m_i(x^\nu)$ transform as follows:

$$\begin{aligned} \delta\tau^a(x^\nu) &= \lambda^a_b \tau^b(x^\nu) - \lambda^c_d x^d \partial_c \tau^a(x^\nu) + \xi^0 \partial_0 \tau^a(x^\nu) + \xi^j(t) \partial_j \tau^a(x^\nu) - \\ &\quad - \dot{\xi}^a(t) - \lambda^a(x^\nu), \end{aligned} \quad (6.22)$$

$$\begin{aligned} \delta m_i(x^\nu) &= \xi^0 \partial_0 m_i(x^\nu) + \xi^j(t) \partial_j m_i(x^\nu) + \lambda_i^j m_j(x^\nu) - \lambda^j_k x^k \partial_j m_i(x^\nu) + \\ &\quad + \lambda_i(x^\nu) + \partial_i \sigma(x^\nu), \end{aligned} \quad (6.23)$$

$$\begin{aligned} \delta m_0(x^\nu) &= \xi^0 \partial_0 m_0(x^\nu) + \dot{\xi}^i(t) m_i(x^\nu) + \xi^i(t) \partial_i m_0(x^\nu) - \lambda^i_j x^j \partial_i m_0(x^\nu) - \\ &\quad - \lambda^a(x^\nu) \tau_a(x^\nu) + \dot{\sigma}(x^\nu). \end{aligned} \quad (6.24)$$

⁵Remember that $\tau^i = \tau^a \delta_a^i$ and that we do not distinguish between τ^i and τ^a anymore.

The three fields $\tau^a(x^\nu)$, $m_i(x^\nu)$ and $m_0(x^\nu)$ are not independent. Since the gauge field $\omega_\mu^{ab}(x^\nu)$ which we gauge fixed to zero, see eq. (6.17), is dependent we need to investigate its consequences. It turns out that the spatial part of these conditions does not lead to restrictions on the above fields but the time component does. Using the other gauge fixing conditions as well, we find that the gauge fixing condition $\omega_0^{ab}(x^\nu) = 0$ leads to the following restriction:

$$\partial_{[i}\tau_{j]}(x^\nu) + \partial_{[i}m_{j]}(x^\nu) = 0. \quad (6.25)$$

This implies that, locally, one can write⁶

$$\tau_i(x^\nu) + m_i(x^\nu) = \partial_i m(x^\nu). \quad (6.26)$$

Without loss of generality, we can thus eliminate $m_i(x^\nu)$ for $\tau_i(x^\nu)$ and $m(x^\nu)$, which is what we will do in the following. The transformation rule for $m(x^\nu)$ can be found from $\delta\tau_i(x^\nu)$ and $\delta m_i(x^\nu)$:

$$\delta m(x^\nu) = \xi^0 \partial_0 m(x^\nu) - \xi^k(t)x^k + \xi^j(t)\partial_j m(x^\nu) - \lambda^j_k x^k \partial_j m(x^\nu) + \sigma(x^\nu) + Y(t), \quad (6.27)$$

where $Y(t)$ is an arbitrary time-dependent shift. At this point we are left with three independent fields $\tau^i(x^\nu)$, $m_0(x^\nu)$ and $m(x^\nu)$ whose transformation laws are given by (6.22), (6.24), (6.27), respectively.

From the transformation rule (6.27), we see that the central charge transformation acts as a Stückelberg shift on the field $m(x^\nu)$. We can thus partially fix the central charge transformations by imposing

$$m(x^\nu) = 0. \quad (6.28)$$

This fixes the central charge transformations according to

$$\sigma(x^\mu) = \sigma(t) + \dot{\xi}^a(t)x_a, \quad (6.29)$$

where it is understood that we also fix $Y(t) = -\sigma(t)$ in (6.27). After this gauge fixing the transformation rules of the two independent fields $\tau^i(x^\nu)$ and $m_0(x^\nu)$ are given by:

$$\begin{aligned} \delta\tau^i(x^\nu) &= \lambda^i_j \tau^j(x^\nu) - \lambda^j_k x^k \partial_j \tau^i(x^\nu) + \xi^0 \partial_0 \tau^i(x^\nu) + \xi^j(t)\partial_j \tau^i(x^\nu) - \dot{\xi}^i(t) - \lambda^i(x^\nu), \\ \delta m_0(x^\nu) &= \xi^0 \partial_0 m_0(x^\nu) - \dot{\xi}^i(t)\tau_i(x^\nu) + \xi^i(t)\partial_i m_0(x^\nu) + \ddot{\xi}^k(t)x^k \\ &\quad - \lambda^i_j x^j \partial_i m_0(x^\nu) - \lambda^i(x^\nu)\tau_i(x^\nu) + \dot{\sigma}(t). \end{aligned} \quad (6.30)$$

We note that the local boost transformations, with local parameters $\lambda^i(x^\nu)$, end up as a Stückelberg symmetry. This Stückelberg symmetry can be fixed by imposing the final gauge condition

$$\tau^a(x^\nu) = 0. \quad (6.31)$$

⁶Note that we freely lower and raise the i or a index on τ^i here and in the following. So, τ_i no longer refers to the i -components of τ_μ at this point.

This leads to the following compensating transformations:

$$\lambda^i(x^\nu) = -\dot{\xi}^i(t). \quad (6.32)$$

The only independent field left now is

$$m_0(x^\nu) \equiv \Phi(x^\nu), \quad (6.33)$$

which in a minute we will identify as the Newton potential. Using the gauge condition (6.31) and taking into account the compensating transformations (6.32) we find that the transformation rule of this field is given by

$$\delta\Phi(x^\nu) = \xi^0\partial_0\Phi(x^\nu) + \xi^i(t)\partial_i\Phi(x^\nu) + \ddot{\xi}^k(t)x^k - \lambda^i_j x^j \partial_i\Phi(x^\nu) + \dot{\sigma}(t). \quad (6.34)$$

The fact that we identify the field $m_0(x^\nu)$ with the Newton potential $\Phi(x^\nu)$ is justified by looking at the equations of motion. In terms of $\Phi(x^\nu)$ the expressions for the only non-zero dependent boost spin-connection field is given by

$$\omega_0^a(x^\nu) = -\partial^a\Phi(x^\nu). \quad (6.35)$$

If we now plug this expression for the boost spin-connection components into the equation of motion (6.16) we find the expected Poisson equation for the Newton potential:

$$\Delta\Phi = \partial_a\partial^a\Phi = 0. \quad (6.36)$$

This equation is invariant under the acceleration extended Galilei symmetries (6.34).

The transformations (6.34) form a closed algebra on $\Phi(x^\nu)$. One finds the following non-zero commutators:

$$\begin{aligned} [\delta_{\xi^0}, \delta_{\xi^i(t)}] \Phi(x^\nu) &= \delta_{\xi^i(t)} \left(-\xi^0 \dot{\xi}^i(t) \right) \Phi(x^\nu), \\ [\delta_{\xi^0}, \delta_{\sigma(t)}] \Phi(x^\nu) &= \delta_{\sigma(t)} \left(-\xi^0 \dot{\sigma}(t) \right) \Phi(x^\nu), \\ [\delta_{\xi_1^i(t)}, \delta_{\xi_2^j(t)}] \Phi(x^\nu) &= \delta_{\sigma(t)} \left(\dot{\xi}_1^j(t) \xi_2^j(t) - \dot{\xi}_2^j(t) \xi_1^j(t) \right) \Phi(x^\nu), \\ [\delta_{\xi^i(t)}, \delta_{\lambda^{jk}}] \Phi(x^\nu) &= \delta_{\xi^i(t)} \left(\lambda^i_j \dot{\xi}^j(t) \right) \Phi(x^\nu), \end{aligned} \quad (6.37)$$

where we have indicated the parameters of the transformations on the right-hand-side in the brackets. Note that in calculating the commutator on $\Phi(x^\nu)$ we do not vary the explicit x^a that occurs in this transformation rule. This x^a -dependence follows from solving a parameter, see eq. (6.29), and we do not vary the parameters of the transformations when calculating commutators.

This finishes our review of the bosonic case. For the convenience of the reader we have summarized all gauge conditions and resulting compensating transformations in Table 2.

gauge condition/restriction	compensating transformation
$\tau_\mu(x^\nu) = \delta_\mu^0$	$\xi^0(x^\nu) = \xi^0$
$\omega_\mu^{ab}(x^\nu) = 0$	$\lambda^{ab}(x^\nu) = \lambda^{ab}$
$e_i^a(x^\nu) = \delta_i^a$	$\xi^a(x^\nu) = \xi^a(t) - \lambda_{ai}x^i$
$\tau_i(x^\nu) + m_i(x^\nu) = \partial_i m(x^\nu)$	–
$m(x^\nu) = 0$	$\sigma(x^\nu) = \sigma(t) + \dot{\xi}^a(t)x_a$
$\tau^a(x^\nu) = 0$	$\lambda^i(x^\nu) = -\dot{\xi}^i(t)$
$m_0(x^\nu) = \Phi(x^\nu)$	$\omega_0^a(x^\nu) = -\partial^a \Phi(x^\nu)$

Table 6.2: This table indicates the gauge fixing conditions and corresponding compensating transformations that lead to Galilean gravity. We have also included the restrictions that follow from the fact that the spin-connection field ω_μ^{ab} is dependent. At the bottom of the table we have summarized the expressions of the non-zero remaining gauge fields in terms on the Newton potential $\Phi(x^\nu)$.

6.3 The 3D $\mathcal{N} = 2$ Super-Bargmann Algebra

A supersymmetric extension of the Bargmann algebra can be obtained by contracting the super-Poincaré algebra with a central extension, similar to how the Bargmann algebra can be obtained from a trivially extended Poincaré algebra. It turns out that in order to obtain a true supersymmetric extension of the Bargmann algebra in which the anti-commutator of two supersymmetry generators gives both a time and a space translation we need at least two supersymmetries [81]. In this chapter we will consider the minimal case, i.e. $\mathcal{N} = 2$ supersymmetry.

Our starting point is therefore the 3D $\mathcal{N} = 2$ super-Poincaré algebra with central extension \mathcal{Z} , whose non-zero commutation relations are given by

$$\begin{aligned}
[M_{BC}, P_A] &= -2\eta_{A[B}P_{C]}, & [M_{CD}, M_{EF}] &= 4\eta_{[C[E}M_{F]D]}, \\
[M_{AB}, Q_\alpha] &= -\frac{1}{2}[\gamma_{AB}]_\alpha{}^\beta Q_\beta, \\
\{Q_\alpha^i, Q_\beta^j\} &= -[\gamma^A \gamma^0]_{\alpha\beta} P_A \delta^{ij} + \epsilon_{\alpha\beta} \epsilon^{ij} \mathcal{Z}.
\end{aligned} \tag{6.38}$$

The indices $A, B, \dots = \underline{0}, 1, 2$ are flat Lorentz indices, $\alpha = 1, 2$ are 3D spinor indices and $i = 1, 2$ count the number of supercharges. We have collected the 4 supercharges into two 2-component Majorana spinors Q_α^i .⁷

⁷We use a Majorana representation for the γ -matrices, in which the charge conjugation matrix C is given by $C = \gamma^0$. For notational convenience we will write γ^0 instead of $\gamma^{\underline{0}}$.

Following [101], we define the linear combinations

$$Q_\alpha^\pm \equiv Q_\alpha^1 \pm \epsilon_{\alpha\beta} Q_\beta^2 \quad (6.39)$$

and apply the following rescaling, with a real parameter ω , of the generators and the central extension:

$$\begin{aligned} Q_\alpha^- &\rightarrow \sqrt{\omega} Q_\alpha^-, & Q_\alpha^+ &\rightarrow \frac{1}{\sqrt{\omega}} Q_\alpha^+, & (6.40) \\ \mathcal{Z} &\rightarrow -\omega Z + \frac{1}{\omega} H, & P_0 &\rightarrow \omega Z + \frac{1}{\omega} H, & M_{a0} &\rightarrow \omega G_a. \end{aligned}$$

We furthermore rename $M_{ab} = J_{ab}$.

The non-relativistic contraction of the algebra (6.38) is now defined by taking the limit $\omega \rightarrow \infty$. This leads to the following 3D $\mathcal{N} = 2$ super-Bargmann algebra:

$$\begin{aligned} [J_{ab}, P_c] &= -2\delta_{c[a} P_{b]}, & [J_{ab}, G_c] &= -2\delta_{c[a} G_{b]}, \\ [G_a, H] &= -P_a, & [G_a, P_b] &= -\delta_{ab} Z, \\ [J_{ab}, Q^\pm] &= -\frac{1}{2} \gamma_{ab} Q^\pm, & [G_a, Q^+] &= -\frac{1}{2} \gamma_{a0} Q^-, & (6.41) \\ \{Q_\alpha^+, Q_\beta^+\} &= 2\delta_{\alpha\beta} H, & \{Q_\alpha^+, Q_\beta^-\} &= -[\gamma^{a0}]_{\alpha\beta} P_a, \\ \{Q_\alpha^-, Q_\beta^-\} &= 2\delta_{\alpha\beta} Z. \end{aligned}$$

The bosonic part of the above algebra is the Bargmann algebra, involving the Hamiltonian H , the spatial translations P_a , the spatial rotations J_{ab} , the Galilean boosts G_a and the central charge Z . Note that the bosonic Bargmann generators and the central charge, together with the fermionic Q^- generators form the following $\mathcal{N} = 1$ subalgebra [99]:

$$\begin{aligned} [J_{ab}, P_c] &= -2\delta_{c[a} P_{b]}, & [J_{ab}, G_c] &= -2\delta_{c[a} G_{b]}, \\ [G_a, H] &= -P_a, & [G_a, P_b] &= -\delta_{ab} Z, & (6.42) \\ [J_{ab}, Q^-] &= -\frac{1}{2} \gamma_{ab} Q^-, & \{Q_\alpha^-, Q_\beta^-\} &= 2\delta_{\alpha\beta} Z. \end{aligned}$$

The same does not apply if we include the Q^+ generators instead of the Q^- generators. This is due to the $[G, Q]$ commutator, see (6.41), in which the Q^+ and Q^- generators occur asymmetrically. The $\mathcal{N} = 1$ sub-algebra (6.42) is not a true supersymmetry algebra in the sense that the anti-commutator of two Q^- supersymmetries does not give a time and space translation but a central charge transformation. Although the $\mathcal{N} = 2$ supersymmetry algebra (6.41) is a true supersymmetry algebra the converse is not true: not every $\mathcal{N} = 2$ super-algebra is necessarily a true supersymmetry algebra. Finally, we note that the above 3D $\mathcal{N} = 2$ super-Bargmann algebra can be embedded, via a null reduction, into a $\mathcal{N} = 1$ super-Poincaré algebra [102].

6.4 3D $\mathcal{N}=2$ Newton-Cartan Supergravity

In this section we apply a gauging procedure to the $\mathcal{N} = 2$ super-Bargmann algebra (6.41) thereby extending the bosonic discussion of section 2 to the supersymmetric case. As a first step in this gauging procedure we associate a gauge field to each of the symmetries of the $\mathcal{N} = 2$ super-Bargmann algebra and we promote the constant parameters describing these transformations to arbitrary functions of the spacetime coordinates $\{x^\mu\}$, see table 3.

symmetry	generators	gauge field	parameters	curvatures
time translations	H	τ_μ	$\zeta(x^\nu)$	$\hat{R}_{\mu\nu}(H)$
space translations	P^a	e_μ^a	$\zeta^a(x^\nu)$	$\hat{R}_{\mu\nu}^a(P)$
boosts	G^a	ω_μ^a	$\lambda^a(x^\nu)$	$\hat{R}_{\mu\nu}^a(G)$
spatial rotations	J^{ab}	ω_μ^{ab}	$\lambda^{ab}(x^\nu)$	$\hat{R}_{\mu\nu}^{ab}(J)$
central charge transf.	Z	m_μ	$\sigma(x^\nu)$	$\hat{R}_{\mu\nu}(Z)$
two supersymmetries	Q_α^\pm	$\psi_{\mu\pm}$	$\epsilon_\pm(x^\nu)$	$\hat{\psi}_{\mu\nu\pm}$

Table 6.3: This table indicates the generators of the $\mathcal{N} = 2$ super-Bargmann algebra and the gauge fields, local parameters and super-covariant curvatures that are associated to each of these generators. The fermionic generators are indicated below the double horizontal line.

The corresponding gauge-invariant curvatures, see table 3, are given by:

$$\begin{aligned}
\hat{R}_{\mu\nu}(H) &= 2\partial_{[\mu}\tau_{\nu]} - \frac{1}{2}\bar{\psi}_{[\mu}\gamma^0\psi_{\nu]+}, \\
\hat{R}_{\mu\nu}^a(P) &= 2\partial_{[\mu}e_{\nu]}^a - 2\omega_{[\mu}^{ab}e_{\nu]b} - 2\omega_{[\mu}^a\tau_{\nu]} - \bar{\psi}_{[\mu}\gamma^a\psi_{\nu]-}, \\
\hat{R}_{\mu\nu}^a(G) &= 2\partial_{[\mu}\omega_{\nu]}^a - 2\omega_{[\mu}^{ab}\omega_{\nu]b}, \\
\hat{R}_{\mu\nu}^{ab}(J) &= 2\partial_{[\mu}\omega_{\nu]}^{ab}, \\
\hat{R}_{\mu\nu}(Z) &= 2\partial_{[\mu}m_{\nu]} - 2\omega_{[\mu}^ae_{\nu]a} - \bar{\psi}_{[\mu}\gamma^0\psi_{\nu]-}, \\
\hat{\psi}_{\mu\nu+} &= 2\partial_{[\mu}\psi_{\nu]+} - \frac{1}{2}\omega_{[\mu}^{ab}\gamma_{ab}\psi_{\nu]+}, \\
\hat{\psi}_{\mu\nu-} &= 2\partial_{[\mu}\psi_{\nu]-} - \frac{1}{2}\omega_{[\mu}^{ab}\gamma_{ab}\psi_{\nu]-} + \omega_{[\mu}^a\gamma_{a0}\psi_{\nu]+}.
\end{aligned} \tag{6.43}$$

According to the $\mathcal{N} = 2$ super-Bargmann algebra (6.41) the gauge fields given in table 3

transform under spatial rotations, boosts and central charge transformations as follows:

$$\begin{aligned}
\delta\tau_\mu &= 0, \\
\delta e_\mu^a &= \lambda^a{}_b e_\mu^b + \lambda^a \tau_\mu, \\
\delta\omega_\mu^{ab} &= \partial_\mu \lambda^{ab} + 2\lambda^{c[a} \omega_\mu^{b]c}, \\
\delta\omega_\mu^a &= \partial_\mu \lambda^a - \lambda^b \omega_\mu^a{}_b + \lambda^{ab} \omega_{\mu b}, \\
\delta m_\mu &= \partial_\mu \sigma + \lambda^a e_{\mu a}, \\
\delta\psi_{\mu+} &= \frac{1}{4} \lambda^{ab} \gamma_{ab} \psi_{\mu+}, & \delta\psi_{\mu-} &= \frac{1}{4} \lambda^{ab} \gamma_{ab} \psi_{\mu-} - \frac{1}{2} \lambda^a \gamma_{a0} \psi_{\mu+}.
\end{aligned} \tag{6.44}$$

We will discuss the other transformations of the $\mathcal{N} = 2$ super-Bargmann algebra below.

The next step in the gauging procedure is to impose a set of constraints on the curvatures. We first impose the following set of conventional constraints:

$$\hat{R}_{\mu\nu}{}^a(P) = 0, \quad \hat{R}_{\mu\nu}(Z) = 0. \tag{6.45}$$

These conventional constraints can be used to solve for the spin connections in terms of the other gauge fields as follows:⁸

$$\begin{aligned}
\omega_\mu^{ab} &= 2e^{\rho[a} \left(\partial_{[\rho} e_{\mu]}^{b]} - \frac{1}{2} \bar{\psi}_{[\rho} \gamma^b \psi_{\mu]-} \right) + e_\mu^c e^{\rho a} e^{\nu b} \left(\partial_{[\rho} e_{\nu]}^c - \frac{1}{2} \bar{\psi}_{[\rho} \gamma^c \psi_{\nu]-} \right) \\
&\quad - \tau_\mu e^{\rho a} e^{\nu b} \left(\partial_{[\rho} m_{\nu]} - \frac{1}{2} \bar{\psi}_{[\rho} \gamma^0 \psi_{\nu]-} \right),
\end{aligned} \tag{6.46}$$

$$\begin{aligned}
\omega_\mu^a &= e^{\nu a} \left(\partial_{[\mu} m_{\nu]} - \frac{1}{2} \bar{\psi}_{[\mu} \gamma^0 \psi_{\nu]-} \right) + e_\mu^b e^{\nu a} \tau^\rho \left(\partial_{[\nu} e_{\rho]}^b - \frac{1}{2} \bar{\psi}_{[\nu} \gamma^b \psi_{\rho]-} \right) \\
&\quad + \tau^\nu \left(\partial_{[\mu} e_{\nu]}^a - \frac{1}{2} \bar{\psi}_{[\mu} \gamma^a \psi_{\nu]-} \right) + \tau_\mu \tau^\nu e^{\rho a} \left(\partial_{[\nu} m_{\rho]} - \frac{1}{2} \bar{\psi}_{[\nu} \gamma^0 \psi_{\rho]-} \right).
\end{aligned} \tag{6.47}$$

On top of this we impose the following additional constraints:

$$\hat{R}_{\mu\nu}(H) = 0, \quad \hat{\psi}_{\mu\nu+} = 0, \quad \hat{R}_{\mu\nu}{}^{ab}(J) = 0. \tag{6.48}$$

The first constraint defines a foliation of Newtonian spacetime. As we will see below the second constraint follows by supersymmetry from the first constraint and, similarly, the third constraint follows from the second one. This third constraint defines flat space Newton-Cartan supergravity. Note that, unlike in the bosonic case, this constraint is enforced upon us by supersymmetry, whereas in the purely bosonic theory this constraint was optional. The constraints (6.45), together with the first constraint of (6.48) can be used to convert the time and space translations into general coordinate transformations, with parameter $\xi^\mu(x^\nu)$.

The supersymmetry variation of the conventional constraints does not lead to new constraints as they are used to determine the supersymmetry transformation rules of the now dependent gauge

⁸The projective inverses τ^μ and $e^\mu{}_a$ of τ_μ and e_μ^a are defined in eqns.(4.55)-4.57.

fields (6.46) and (6.47). We find the following rules for these gauge fields:⁹

$$\begin{aligned}
\delta_Q \omega_\mu^{ab} &= -\frac{1}{2} \bar{\epsilon}_+ \gamma^{[a} \hat{\psi}^{b]}_{\mu-} + \frac{1}{4} e_{\mu c} \bar{\epsilon}_+ \gamma^c \hat{\psi}^{ab} - \frac{1}{2} \tau_\mu \bar{\epsilon}_- \gamma^0 \hat{\psi}^{ab} - \\
&\quad - \frac{1}{2} \bar{\epsilon}_- \gamma^{[a} \hat{\psi}^{b]}_{\mu+} + \frac{1}{4} e_{\mu c} \bar{\epsilon}_- \gamma^c \hat{\psi}^{ab} +, \\
\delta_Q \omega_\mu^a &= \frac{1}{2} \bar{\epsilon}_- \gamma^0 \hat{\psi}_\mu^a - + \frac{1}{2} \tau_\mu \bar{\epsilon}_- \gamma^0 \hat{\psi}_0^a - + \frac{1}{4} e_{\mu b} \bar{\epsilon}_+ \gamma^b \hat{\psi}^a_{0-} + \frac{1}{4} \bar{\epsilon}_+ \gamma^a \hat{\psi}_{\mu 0-} \\
&\quad + \frac{1}{4} e_{\mu b} \bar{\epsilon}_- \gamma^b \hat{\psi}^a_{0+} + \frac{1}{4} \bar{\epsilon}_- \gamma^a \hat{\psi}_{\mu 0+}.
\end{aligned} \tag{6.49}$$

In contrast, we must investigate the supersymmetry variations of the non-conventional constraints (6.48). In order to do this, we must first determine the supersymmetry rules of the independent gauge fields. According to the super-Bargmann algebra (6.41) the supersymmetry transformations of the independent gauge fields are given by

$$\begin{aligned}
\delta_Q \tau_\mu &= \frac{1}{2} \bar{\epsilon}_+ \gamma^0 \psi_{\mu+}, \\
\delta_Q e_\mu^a &= \frac{1}{2} \bar{\epsilon}_+ \gamma^a \psi_{\mu-} + \frac{1}{2} \bar{\epsilon}_- \gamma^a \psi_{\mu+}, \\
\delta_Q m_\mu &= \bar{\epsilon}_- \gamma^0 \psi_{\mu-}, \\
\delta_Q \psi_{\mu+} &= D_\mu \epsilon_+, \\
\delta_Q \psi_{\mu-} &= D_\mu \epsilon_- + \frac{1}{2} \omega_\mu^a \gamma_{a0} \epsilon_+,
\end{aligned} \tag{6.50}$$

where ω_μ^a is the dependent boost gauge field. The covariant derivative D_μ is only covariantized with respect to spatial rotations. When acting on the parameters ϵ_\pm , it is given by

$$D_\mu \epsilon_\pm = \partial_\mu \epsilon_\pm - \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \epsilon_\pm \tag{6.51}$$

in terms of the dependent connection field ω_μ^{ab} .

At this point we have obtained the supersymmetry rules of *all* gauge fields, both the dependent as well as the independent ones. We find that with these supersymmetry transformations the supersymmetry algebra closes on-shell. To be precise, the commutator of two supersymmetry transformations closes and is given by the following soft algebra:

$$\begin{aligned}
[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] &= \delta_{\text{g.c.t.}}(\xi^\mu) + \delta_{J_{ab}}(\lambda^a{}_b) + \delta_{G_a}(\lambda^a) + \delta_{Q_+}(\epsilon_+) + \\
&\quad + \delta_{Q_-}(\epsilon_-) + \delta_Z(\sigma),
\end{aligned} \tag{6.52}$$

provided the following equations hold:

$$\gamma^\mu \tau^\nu \hat{\psi}_{\mu\nu-} = 0, \quad e^\mu{}_a e^\nu{}_b \hat{\psi}_{\mu\nu-} = 0. \tag{6.53}$$

The first equation can be seen as an equation of motion, the second one does not contain any time derivatives and should be viewed as a fermionic constraint. Here g.c.t. denotes a general coordinate

⁹Recall that $\hat{\psi}_{ab} = e_a{}^\mu e_b{}^\nu \hat{\psi}_{\mu\nu}$.

transformation and the field-dependent parameters are given by

$$\begin{aligned}
\xi^\mu &= \frac{1}{2}(\bar{\epsilon}_{2+}\gamma^0\epsilon_{1+})\tau^\mu + \frac{1}{2}(\bar{\epsilon}_{2+}\gamma^a\epsilon_{1-} + \bar{\epsilon}_{2-}\gamma^a\epsilon_{1+})e^\mu{}_a, \\
\lambda^a{}_b &= -\xi^\mu\omega_\mu{}^a{}_b, \\
\lambda^a &= -\xi^\mu\omega_\mu{}^a, \\
\epsilon_\pm &= -\xi^\mu\psi_{\mu\pm}, \\
\sigma &= -\xi^\mu m_\mu + (\bar{\epsilon}_{2-}\gamma^0\epsilon_{1-}).
\end{aligned} \tag{6.54}$$

We are now in a position to investigate the supersymmetry variations of the three constraints (6.48) and of the equation of motion/constraint (6.53). One may verify that under supersymmetry the first constraint in (6.48) transforms to the second one and that the supersymmetry variation of the second constraint leads to the third one. This third constraint does not lead to new constraints because the supersymmetry variation of $\omega_\mu{}^{ab}$ vanishes on-shell, see eq. (6.49). Substituting the constraints into the super-Bianchi identities, it follows that the only non-zero bosonic curvature we are left with is the boost curvature $\hat{R}_{\mu\nu}{}^a(G)$ and we find that only the following components are non-vanishing:

$$\tau^\mu e^\nu{}_{(a}\hat{R}_{\mu\nu}{}^{b)}(G) \equiv \hat{R}_{\underline{0}(a}{}^{b)}(G) \neq 0. \tag{6.55}$$

Using this it follows that the supersymmetry variation of the second constraint in (6.53) does not lead to a new constraint. On the other hand, the supersymmetry variation of the fermionic equation of motion, i.e. the first constraint in (6.53), leads to the bosonic equation of motion

$$\hat{R}_{\underline{0}a}{}^a(G) = 0. \tag{6.56}$$

To finish the consistency check of the gauging procedure we should check whether the supersymmetry variation of the bosonic equation of motion (6.56) does not lead to new constraints and/or equations of motion. Instead of doing this we shall show in the next section that after gauge fixing all constraints can be solved leading to a consistent system with a closed algebra. This finishes our construction of the 3D $\mathcal{N} = 2$ Newton-Cartan supergravity theory.

6.5 3D Galilean Supergravity

In this section we will perform a partial gauge fixing of the bosonic and fermionic symmetries to derive the Newton-Cartan supergravity theory from the Galilean observer point of view. We will define a supersymmetric Galilean observer as one for which only a supersymmetric extension of the acceleration extended Galilei symmetries are retained. Due to the constant time translations, this implies in particular that only half of the supersymmetries will be gauged, see below. We closely follow the analysis given in section 2 for the bosonic case. First, we solve the constraints (6.48) by imposing the gauge fixing conditions

$$\tau_\mu(x^\nu) = \delta_\mu^0, \quad \omega_\mu{}^{ab}(x^\nu) = 0, \quad \psi_{\mu+}(x^\nu) = 0. \tag{6.57}$$

This fixes the local time translations, spatial rotations and ϵ_+ transformations to constant transformations:

$$\xi^0(x^\nu) = \xi^0, \quad \lambda^{ab}(x^\nu) = \lambda^{ab}, \quad \epsilon_+(x^\nu) = \epsilon_+. \tag{6.58}$$

No compensating transformations are induced by these gauge fixings. We now partially gauge fix the spatial translations by imposing the gauge choice

$$e_i^a(x^\nu) = \delta_i^a. \quad (6.59)$$

This gauge choice implies that we may use from now on the expressions (6.21) for the temporal and spatial dreibein components and their projective inverses. We will derive the required compensating transformation below. First, using the above gauge choices and the fact that the purely spatial components $\hat{R}_{ij}^a(G)$ of the curvatures of boost transformations and the purely spatial components $\hat{\psi}_{ij-}$ of the curvature of ϵ_- transformations are zero, for their expressions see eq. (6.43), we derive that

$$\partial_{[i}\omega_{j]}^a = 0, \quad \partial_{[i}\psi_{j]-} = 0. \quad (6.60)$$

The first equation we solve locally by writing

$$\omega_i^a = \partial_i \omega^a, \quad (6.61)$$

where ω^a is a dependent field since ω_i^a is dependent. This also explains why we have not added a purely time-dependent piece to the r.h.s. of the above solution. We next partially gauge fix the ϵ_- transformations by imposing the gauge choice

$$\psi_{i-}(x^\nu) = 0. \quad (6.62)$$

This fixes the ϵ_- transformations according to

$$\epsilon_-(x^\nu) = \epsilon_-(t) - \frac{1}{2}\omega^a \gamma_{a0} \epsilon_+. \quad (6.63)$$

Given the gauge choice (6.62) the spatial translations are now fixed without the need for any fermionic compensating transformation. Indeed, from the total variation of the gauge fixing condition (6.59) we find:

$$\xi^i(x^\nu) = \xi^i(t) - \lambda^i_j x^j. \quad (6.64)$$

At this point, we are left with the remaining fields τ^a , m_i , m_0 and ψ_{0-} . These fields are not independent since the gauge field ω_μ^{ab} which we gauge fixed to zero is dependent, see eq. (6.46). Like in the bosonic case, only the time component $\omega_0^{ab} = 0$ leads to a restriction:¹⁰

$$\partial_{[i}(\tau_{j]} + m_{j])}(x^\nu) = 0. \quad (6.65)$$

As in the bosonic case, this implies that we can write locally:

$$\tau_i(x^\nu) + m_i(x^\nu) = \partial_i m(x^\nu). \quad (6.66)$$

Without loss of generality we will use this equation to eliminate m_i in terms of the other two fields. The variation of m is determined by writing the variation of $\tau_i + m_i$ as a ∂_i -derivative. This is trivial for most of the terms, except for the ϵ_+ term. Before addressing this issue below, it is convenient to write down the total variation of $\partial_i m$ instead of m . From eq. (6.66) we find

$$\delta \partial_i m = \xi^0 \partial_0 \partial_i m + \xi^j(t) \partial_j \partial_i m + \lambda_i^j \partial_j m - \lambda^m_n x^n \partial_m \partial_i m + \partial_i \sigma(x^\nu) - \xi^i(t) - \frac{1}{2} \bar{\epsilon}_+ \gamma_i \psi_{0-}. \quad (6.67)$$

¹⁰Recall that $\tau_i = \tau^a \delta_{ia}$. Note also that under supersymmetry the variation of this constraint gives $\bar{\epsilon}_+ \gamma_{[i} \partial_{j]} \psi_{0-} = 0$ which is equivalent to the fermionic equation of motion (which after gauge fixing takes the form (6.80)). Therefore, this constraint is consistent with supersymmetry.

Note that the terms proportional to the local boost parameters $\lambda^i(x^\nu)$ have cancelled out. We may now partially gauge fix the central charge transformations by putting

$$m(x^\nu) = 0. \quad (6.68)$$

We thus obtain

$$\partial_i \sigma(x^\nu) = \dot{\xi}^i(t) + \frac{1}{2} \bar{\epsilon}_+ \gamma_i \psi_{0-}(x^\nu), \quad (6.69)$$

which is sufficient to calculate the transformation rule of $\partial_i m_0$. After this gauge fixing, taking into account all the compensating transformations, see table 4 below, and the restriction (6.66) with $m = 0$ substituted, we find the following transformation rules for the remaining independent fields:

$$\begin{aligned} \delta \tau_i &= \xi^0 \partial_0 \tau_i + \xi^j(t) \partial_j \tau_i - \dot{\xi}^i(t) + \lambda_{ij} \tau^j - \lambda^k_l x^l \partial_k \tau_i - \lambda_i(x^\nu) - \frac{1}{2} \bar{\epsilon}_+ \gamma_i \psi_{0-}, \\ \delta \partial_i m_0 &= \xi^0 \partial_0 \partial_i m_0 + \xi^j(t) \partial_j \partial_i m_0 + \ddot{\xi}^i(t) - \dot{\xi}^j(t) \partial_i \tau_j + \lambda_i{}^j \partial_j m_0 - \lambda^m_n x^n \partial_m \partial_i m_0 - \\ &\quad - \partial_i (\lambda^j(x^\nu) \tau_j) + \bar{\epsilon}_-(t) \gamma^0 \partial_i \psi_{0-} + \frac{1}{2} \partial_i (\omega^a \bar{\epsilon}_+ \gamma_a \psi_{0-}) + \frac{1}{2} \bar{\epsilon}_+ \gamma_i \dot{\psi}_{0-}, \\ \delta \psi_{0-} &= \xi^0 \partial_0 \psi_{0-} + \xi^i(t) \partial_i \psi_{0-} - \lambda^i_j x^j \partial_i \psi_{0-} + \frac{1}{4} \lambda^{ab} \gamma_{ab} \psi_{0-} \\ &\quad + \dot{\epsilon}_-(t) + \frac{1}{2} (\omega_0^a - \dot{\omega}^a) \gamma_{a0} \epsilon_+. \end{aligned} \quad (6.70)$$

Note that ω_0^a and ω^a depend on the fields τ_i, m_0 . Using expression (6.47) for the dependent boost gauge field $\omega_\mu{}^a$ one can calculate that

$$\omega_i{}^a \equiv \partial_i \omega^a = -\partial_i \tau^a \quad \rightarrow \quad \omega^a = -\tau^a, \quad (6.71)$$

$$\omega_0^a = -\dot{\tau}^a - \partial_a \left(m_0 - \frac{1}{2} \tau^i \tau^i \right). \quad (6.72)$$

As a final step we now fix the local boost transformations by imposing

$$\tau^i(x^\nu) = 0, \quad (6.73)$$

which leads to the following compensating transformations:

$$\lambda^i(x^\nu) = -\dot{\xi}^i(t) - \frac{1}{2} \bar{\epsilon}_+ \gamma_i \psi_{0-}(x^\nu). \quad (6.74)$$

One now finds that

$$\omega^a = 0, \quad \omega_0^a = -\partial^a m_0 \equiv -\partial^a \Phi, \quad (6.75)$$

where Φ is the Newton potential. In terms of the ‘Newton force’ Φ_i and its supersymmetric partner Ψ defined by

$$\Phi_i = \partial_i \Phi, \quad \Psi = \psi_{0-}, \quad (6.76)$$

one thus obtains the following transformation rules:

$$\delta \Phi_i = \xi^0 \partial_0 \Phi_i + \xi^j(t) \partial_j \Phi_i + \ddot{\xi}^i(t) + \lambda_i{}^j \partial_j \Phi - \lambda^m_n x^n \partial_m \Phi_i + \bar{\epsilon}_-(t) \gamma^0 \partial_i \Psi + \frac{1}{2} \bar{\epsilon}_+ \gamma_i \dot{\Psi}, \quad (6.77)$$

$$\delta \Psi = \xi^0 \partial_0 \Psi + \xi^i(t) \partial_i \Psi - \lambda^i_j x^j \partial_i \Psi + \frac{1}{4} \lambda^{ab} \gamma_{ab} \Psi + \dot{\epsilon}_-(t) - \frac{1}{2} \Phi^i \gamma_{i0} \epsilon_+. \quad (6.78)$$

Note that the central charge transformations only act on the Newton potential, not on the Newton force. Determining the transformation rule of the Newton potential Φ is non-trivial, due to the

fact that the last term of (6.77) cannot be manifestly written as a ∂_i -derivative. The above transformation rules are consistent with the integrability condition

$$\partial_{[i}\Phi_{j]}(x^\nu) = 0, \quad (6.79)$$

by virtue of the fermionic equations of motion (6.53) which, after gauge fixing, take on the form

$$\gamma^i \partial_i \Psi(x^\nu) = 0 \quad \Leftrightarrow \quad \partial_{[i}\gamma_{j]} \Psi(x^\nu) = 0. \quad (6.80)$$

Under supersymmetry these fermionic equations of motion lead to the following bosonic equation of motion:

$$\partial^i \Phi_i(x^\nu) = 0. \quad (6.81)$$

The same bosonic equation of motion also follows from eq. (6.56) after gauge fixing. In order to obtain transformation rules for the Newton potential Φ and its fermionic superpartner, we need to solve the fermionic equations of motion/constraint (6.80). The second form of this constraint makes it clear that the equations of motion are solved by a spinor χ , that obeys:

$$\gamma_i \Psi = \partial_i \chi. \quad (6.82)$$

Note that this only determines χ up to a purely time-dependent shift. From (6.82), it follows that χ obeys the constraint:

$$\gamma^1 \partial_1 \chi = \gamma^2 \partial_2 \chi. \quad (6.83)$$

Ψ can thus be expressed in terms of χ in a number of equivalent ways:

$$\Psi = \gamma^1 \partial_1 \chi = \gamma^2 \partial_2 \chi = \frac{1}{2} \gamma^i \partial_i \chi. \quad (6.84)$$

It is now possible to determine the transformation rule of Φ by rewriting $\delta\Phi_i$ as a ∂_i -derivative:

$$\delta\Phi_i = \partial_i(\delta\Phi). \quad (6.85)$$

The resulting transformation rule for the Newton potential is

$$\delta\Phi = \xi^0 \partial_0 \Phi + \xi^i(t) \partial_i \Phi + \ddot{\xi}^i(t) x^i - \lambda^m{}_n x^n \partial_m \Phi + \frac{1}{2} \bar{\epsilon}_-(t) \gamma^{0i} \partial_i \chi + \frac{1}{2} \bar{\epsilon}_+ \dot{\chi} + \sigma(t). \quad (6.86)$$

Note that we have allowed for an arbitrary time-dependent shift $\sigma(t)$ in the transformation rule, whose origin stems from the fact that $\Phi_i = \partial_i \Phi$ only determines Φ up to an arbitrary time-dependent shift. In order to determine the transformation rule of χ , we try to rewrite $\gamma_i \delta\Psi$ as a ∂_i -derivative:¹¹

$$\gamma_i \delta\Psi = \partial_i(\delta\chi). \quad (6.87)$$

Most of the terms in $\gamma_i \delta\Psi$ can be straightforwardly written as a ∂_i -derivative. Only for the ϵ_+ transformation, the argument is a bit subtle. We thus focus on the terms in $\gamma_i \delta\Psi$, given by

$$-\frac{1}{2} \gamma_i \Phi^j \gamma_{j0} \epsilon_+ = -\frac{1}{2} \gamma_i \partial^j \Phi \gamma_{j0} \epsilon_+ = -\frac{1}{2} \partial^j \Phi \gamma_{ij0} \epsilon_+ - \frac{1}{2} \partial_i \Phi \gamma_0 \epsilon_+. \quad (6.88)$$

¹¹Note that even though $\Psi = \frac{1}{2} \gamma^i \partial_i \chi$, the correct transformation rule of χ cannot be found by writing $\delta\Psi$ as $\frac{1}{2} \gamma^i \partial_i$ of an expression. In particular, one would miss the term involving the dual Newton potential Ξ in the transformation rule of χ . This is due to the fact that $\Psi = \frac{1}{2} \gamma^i \partial_i \chi$ is a consequence of the defining equations $\gamma_i \Psi = \partial_i \chi$, but is not equivalent to it.

The last term is already in the desired form. To rewrite the first term in the proper form, we note that the Newton potential Φ can be dualized to a ‘dual Newton potential’ Ξ via

$$\partial_i \Phi = \varepsilon_{ij} \partial^j \Xi, \quad \partial_i \Xi = -\varepsilon_{ij} \partial^j \Phi. \quad (6.89)$$

Using the convention that $\gamma_{ij0} = \epsilon_{0ij} = \epsilon_{ij}$, we then get

$$-\frac{1}{2} \gamma_i \Phi^j \gamma_{j0} \epsilon_+ = \frac{1}{2} \partial_i \Xi \epsilon_+ - \frac{1}{2} \partial_i \Phi \gamma_0 \epsilon_+. \quad (6.90)$$

One thus obtains the following transformation rule for χ , which includes the dual Newton potential Ξ :

$$\delta \chi = \xi^0 \partial_0 \chi + \xi^i(t) \partial_i \chi - \lambda^m{}_n x^n \partial_m \chi + \frac{1}{4} \lambda^{mn} \gamma_{mn} \chi + x^i \gamma_i \dot{\epsilon}_-(t) + \frac{1}{2} \Xi \epsilon_+ - \frac{1}{2} \Phi \gamma_0 \epsilon_+ + \eta(t). \quad (6.91)$$

Note that we have again allowed for a purely time-dependent shift $\eta(t)$, whose origin lies in the fact that (6.82) only determines χ up to a purely time-dependent shift.

Now, in order to calculate the algebra on Φ , χ , we also need the transformation rule of the dual potential Ξ . This rule is determined by dualizing the transformation rule of Φ :

$$\partial_i(\delta \Xi) = -\varepsilon_{ij} \partial^j(\delta \Phi). \quad (6.92)$$

By repeatedly using (6.82) and (6.89), we get:

$$\delta \Xi = \xi^0 \partial_0 \Xi + \xi^i(t) \partial_i \Xi + \ddot{\xi}^i(t) \varepsilon_{ij} x^j - \lambda^m{}_n x^n \partial_m \Xi + \frac{1}{2} \bar{\epsilon}_-(t) \gamma^i \partial_i \chi - \frac{1}{2} \bar{\epsilon}_+ \gamma_0 \dot{\chi} + \tau(t), \quad (6.93)$$

where we again allowed for a purely time-dependent shift $\tau(t)$. The algebra then closes on Φ and χ , using (6.82), (6.83), (6.89). One finds the following non-zero commutators between the fermionic symmetries:

$$\begin{aligned} [\delta_{\epsilon_{1-}(t)}, \delta_{\epsilon_{2-}(t)}] &= \delta_{\sigma(t)} \left(\frac{d}{dt} (\bar{\epsilon}_{2-}(t) \gamma^0 \epsilon_{1-}(t)) \right), \\ [\delta_{\epsilon_{1+}}, \delta_{\epsilon_{2+}}] &= \delta_{\xi^0} \left(\frac{1}{2} (\bar{\epsilon}_{2+} \gamma^0 \epsilon_{1+}) \right), \\ [\delta_{\epsilon_+}, \delta_{\epsilon_-(t)}] &= \delta_{\xi^i(t)} \left(\frac{1}{2} (\bar{\epsilon}_-(t) \gamma^i \epsilon_+) \right), \\ [\delta_{\eta(t)}, \delta_{\epsilon_+}] &= \delta_{\sigma(t)} \left(\frac{1}{2} (\bar{\epsilon}_+ \dot{\eta}(t)) \right). \end{aligned} \quad (6.94)$$

The non-zero commutators between the bosonic and fermionic symmetries are given by:

$$\begin{aligned} [\delta_{\xi^i(t)}, \delta_{\epsilon_+}] &= \delta_{\epsilon_-(t)} \left(\frac{1}{2} \dot{\xi}^i(t) \gamma_0 \epsilon_+ \right), & [\delta_{\lambda^{ij}}, \delta_{\epsilon_+}] &= \delta_{\epsilon_+} \left(-\frac{1}{4} \lambda^{ij} \gamma_{ij} \epsilon_+ \right), \\ [\delta_{\xi^0}, \delta_{\epsilon_-(t)}] &= \delta_{\epsilon_-(t)} (-\xi^0 \dot{\epsilon}_-(t)), & [\delta_{\xi^i(t)}, \delta_{\epsilon_-(t)}] &= \delta_{\eta(t)} (-\xi^i(t) \gamma_i \dot{\epsilon}_-(t)), \\ [\delta_{\lambda^{ij}}, \delta_{\epsilon_-(t)}] &= \delta_{\epsilon_-(t)} \left(-\frac{1}{4} \lambda^{ij} \gamma_{ij} \epsilon_-(t) \right), & [\delta_{\sigma(t)}, \delta_{\epsilon_+}] &= \delta_{\eta(t)} \left(\frac{1}{2} (\sigma(t) \gamma^0 \epsilon_+) \right), \\ [\delta_{\xi^0}, \delta_{\eta(t)}] &= \delta_{\eta(t)} (-\xi^0 \dot{\eta}(t)), & [\delta_{\lambda^{ij}}, \delta_{\eta(t)}] &= \delta_{\eta(t)} \left(-\frac{1}{4} \lambda^{ij} \gamma_{ij} \eta(t) \right). \end{aligned} \quad (6.95)$$

The bosonic commutators are not changed with respect to the purely bosonic case and are given by (6.37).

It is interesting to comment on the appearance of holomorphic functions in the above description. In a basis in which

$$\gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (6.96)$$

the constraint (6.83) on χ reduces to the Cauchy-Riemann equations for a holomorphic function $\chi_2 + i\chi_1$, where the indices 1, 2 refer to spinor indices. Interestingly, the appearance of the dual potential implies that a holomorphic function, given by $\Phi + i\Xi$, also emerges in the bosonic sector. Indeed, the definition of (6.89) corresponds to the Cauchy-Riemann equations for this function. Both the real and imaginary parts of this holomorphic function then satisfy the two-dimensional Laplace equation. This finishes our discussion of the $\mathcal{N} = 2$ Galilean supergravity theory. Like in the bosonic case, see the end of section 2, we have summarized all gauge fixing conditions and resulting compensating transformations in table 4.

gauge condition/restriction	compensating transformation
$\tau_\mu(x^\nu) = \delta_\mu^0$	$\xi^0(x^\nu) = \xi^0$
$\omega_\mu^{ab}(x^\nu) = 0$	$\lambda^a(x^\nu) = \lambda^{ab}$
$\psi_{\mu+}(x^\nu) = 0$	$\epsilon_+(x^\nu) = \epsilon_+$
$e_i^a(x^\nu) = \delta_i^a$	$\xi^i(x^\nu) = \xi^i(t) - \lambda^i_j x^j$
$\psi_{i-}(x^\nu) = 0$	$\epsilon_-(x^\nu) = \epsilon_-(t) - \frac{1}{2}\omega^a(x^\nu)\gamma_{a0}\epsilon_+$
$\tau_i(x^\nu) + m_i(x^\nu) = \partial_i m(x^\nu)$	–
$m(x^\nu) = 0$	$\partial_i \sigma(x^\nu) = \dot{\xi}^i(t) + \frac{1}{2}\bar{\epsilon}_+\gamma_i\psi_{0-}(x^\nu)$
$\tau^a(x^\nu) = 0$	$\lambda^i(x^\nu) = -\dot{\xi}^i(t) - \frac{1}{2}\bar{\epsilon}_+\gamma_i\psi_{0-}(x^\nu)$
$m_0(x^\nu) = \Phi(x^\nu), \quad \omega_0^a(x^\nu) = -\partial^a\Phi(x^\nu)$	$\psi_{0-}(x^\nu) = \Psi(x^\nu)$

Table 6.4: This table indicates the gauge fixing conditions and corresponding compensating transformations that lead to 3D Galilean supergravity. We have also included the restrictions that follow from the fact that the spin-connection field ω_μ^{ab} is dependent. At the bottom of the table we have summarized the expressions of the non-zero remaining gauge fields in terms of the Newton potential $\Phi(x^\nu)$ and its supersymmetric partner $\chi(x^\nu)$, which is related to $\Psi(x^\nu)$ via (6.82).

6.6 Discussion

In this chapter we constructed a supersymmetric extension of three-dimensional Newton-Cartan gravity by gauging the $\mathcal{N} = 2$ supersymmetric Bargmann algebra. An, at first sight, unusual feature we encountered is that only half of the $\mathcal{N} = 2$ supersymmetry is realized locally, the other half manifests itself as a fermionic Stückelberg symmetry. After fixing the Stückelberg symmetry the second supersymmetry is realized only as a global supersymmetry. A similar feature occurs in the bosonic case where the time reparametrizations occur as a Stückelberg symmetry that after

fixing leaves us with constant time translations only.

We have discussed a full gauging, corresponding to ‘Newton-Cartan supergravity’ and a medium gauging, obtained by partial gauge fixing, corresponding to ‘Galilean supergravity’. In the latter formulation, we have been able to realize the supersymmetry algebra on a multiplet containing the Newton potential, as well as its dual. The Newton potential and its dual correspond to the real and imaginary parts of a holomorphic function. This holomorphic structure is reminiscent of the three-dimensional relativistic case [100], as well as of branes with two transverse directions such as cosmic strings and D7-branes [103, 104]. It would be interesting to see how these features can be generalized to higher dimensions.

The reason that in this chapter we restricted ourselves to three-dimensional Newton-Cartan supergravity is that it is non-trivial to find the additional fields, beyond the gauge fields associated to the supersymmetric Bargmann algebra, that are needed to realize the supersymmetry algebra. This is different from the relativistic case where an off-shell counting of the field degrees of freedom restricts the possible choices. One way to make progress here is to better understand the representation theory of the super-Bargmann algebra thereby mimicking the relativistic case. Another useful approach could be to extend the work of [95] and approach the issue from a five-dimensional point of view. We note that the reduction of a 5D Poincaré multiplet to 4D gives an irreducible 4D $\mathcal{N} = 2$ Poincaré multiplet *plus* an $\mathcal{N} = 2$ vector multiplet. It is not clear that such a reducibility into two multiplets also occurs in the non-relativistic case. This might indicate that more fields, namely those of the vector multiplet, are needed to close the supersymmetry algebra in the non-relativistic case. It is clear that more work needs to be done to come at a full grasp of the possible Newton-Cartan supergravities in arbitrary dimensions. Hopefully this chapter, starting with the three-dimensional case, will help to better understand the higher-dimensional cases.

Chapter 7

Conclusions, Developments and Outlook

7.1 Summary of this thesis

In this thesis we have investigated non-relativistic theories of gravity in the formalism of Newton-Cartan theory. This formalism is developed as a gauge theory of the corresponding spacetime symmetries. In this theory the algebra is gauged, associating to every generator a gauge field and corresponding curvature. Curvature constraints are then imposed to remove the local spacetime translations, such that the algebra is deformed. In addition these constraints make the gauge fields belonging to the rotations and boosts dependent. Equations of motion can then be defined in terms of the remaining independent gauge fields. This procedure is analogous to the relativistic case, in which a gauging of the Poincaré algebra (which also contains spacetime translations) leads to the theory of General Relativity. The gauging procedure allows one to construct also theories exhibiting Newton-Cartan geometry but with extended non-relativistic symmetries. Two explicit extensions were considered: non-relativistic strings and a three-dimensional theory of Newton-Cartan Supergravity.

In chapter four we have seen how the Newton-Cartan theory can be obtained by applying a gauging procedure to the Bargmann algebra. This algebra is a centrally-extended Galilei algebra, where the central extension corresponds to particle number conservation. Besides the central extension the algebra also contains Galilei boosts, spatial rotations and spacetime translations. Upon gauging the Bargmann algebra, a non-relativistic but general-covariant theory of gravity is found. Such a theory was already constructed by Elie Cartan a few years after the development of General Relativity and is now known as Newton-Cartan theory. The gauging procedure however sheds new light on Newton-Cartan theory. First of all, the theory contains different possible constraints called the Trautman and Ehlers conditions, which are needed if one wishes to reproduce Newtonian gravity. In the gauge formulation on the other hand these constraints, plus the flat space condition, simply correspond to the vanishing of the curvature of spatial rotations. This curvature constraint is optional and one could stay more general, but the Einstein equations will then make a particular projection of this rotational curvature to vanish. Second, the theory contains metric-compatibility conditions which fix the connection up to a closed but otherwise arbitrary two-form.

In the gauge theory this two-form is the exterior derivative of the central extension gauge field. Third, the gauging procedure shows that in order to eliminate the local time translations one has to put the corresponding curvature to zero. However, this is not a conventional constraint (i.e. it cannot be solved for the spin connections), but a differential condition on the temporal vielbein which turns it into a Stückelberg field. Physically this constraint leads to Newton's absolute time. Finally, the gauge theory shows another important role of the central extension: Without it one is not able to make both the rotational and boost spin connections dependent fields. This also explains why an attempt to apply the gauging procedure to the Galilean conformal algebra does not work; this algebra does not allow for a central mass extension and as such does not contain the Bargmann algebra as subalgebra. To mimick a conformal tensor calculus for Newton-Cartan theory, one should instead turn to the Schrödinger algebra. The independent gauge fields of the gauged Bargmann theory are the temporal and spatial vielbeine and the central gauge field:

$$\{\tau_\mu, e_\mu^a, m_\mu\}. \quad (7.1)$$

In chapter five it is shown that the general-covariant action for a non-relativistic point particle needs a coupling to a vector field. This vector field has a particular transformation under boosts. Without this coupling the action would not be invariant under local boosts. Considering its transformation properties, this vector field turns out to be the central gauge field of the Bargmann algebra. As such this particle action is expressed in terms of the background fields (7.1).

The gauging procedure we just described paves the way to other non-relativistic theories of gravity. The Bargmann algebra is associated to the symmetries of point particles. In view of non-relativistic holography one could now also apply the gauging procedure to symmetry algebras associated to non-relativistic strings. The reason why this theory is interesting is because it has been pointed out in [92] that the non-relativistic limit of the AdS/CFT-correspondence involves a Newton-Cartan theory of strings. Such a theory exhibits stringy Newton-Hooke symmetries, which can be regarded as the non-relativistic limit of strings on an Anti-deSitter background. As for point particles, these stringy algebras can be obtained by a contraction of their relativistic counterparts. The algebra contraction is such that the longitudinal space keeps its relativistic symmetries, while the space transverse to the world-sheet becomes non-relativistic. This is different from the usual Newtonian limit of General Relativity, which is independent of the particular object one is looking at. The reason is that the usual limit only involves the time coordinate, while the algebra contraction involves on top of that one extra spatial coordinate. This extra rescaling, involving a spatial direction, is suggested by holography, where the radial direction of the Anti-deSitter background is the energy scale of the dual conformal field theory. As such one expects this radial coordinate to be rescaled in the contraction in the same way as the time coordinate, giving an AdS_2 space longitudinal to the string.

Another striking difference between the non-relativistic particle and the string is that the corresponding algebra of the latter does not involve a *central* extension anymore. Instead, the stringy extension consists of *two* generators $Z_{a'}$ and $Z_{a'b'}$, where $Z_{a'}$ is the stringy counterpart of the central element Z of the Bargmann algebra. Besides a gauge field $m_\mu^{a'}$ another gauge field $m_\mu^{a'b'}$ is obtained. However, it is shown that this extra gauge field drops out of the gravitational and geodesic equations. The independent gauge fields one is left with are just the stringy extensions

of the fields (7.1):

$$\{\tau_\mu{}^{a'}, e_\mu{}^a, m_\mu{}^{a'}\}. \quad (7.2)$$

The Newton potential is replaced by the trace of a tensor potential where the components $\Phi_{a'b'}$ form a 2×2 matrix. The procedure outlined here can easily be extended to non-relativistic p -branes.

Besides stringy modifications of Newton-Cartan gravity we also considered supersymmetric extensions. These are interesting in their own right, but also in light of holography. The simplest algebra to start from is the $\mathcal{N} = 1$ super-Bargmann algebra. However, a contraction procedure on this algebra leads to a trivial kind of supersymmetry, i.e. one in which two supertransformations do not give a space or time translation anymore. Instead, the anti-commutator of two supercharges gives merely a central charge, turning supersymmetry effectively into an internal $U(1)$ symmetry. In order to obtain non-trivial supersymmetry one has to go at least to the supersymmetric $\mathcal{N} = 2$ Bargmann algebra. This algebra consists of the usual Bargmann algebra, augmented by two supercharges. The gauging of this algebra in three spacetime dimensions was considered in chapter six. An important conclusion from this construction is that due to the appearance of absolute time, not only the temporal vielbein but also half of the gravitini are Stückelberg fields. This leaves one with only one dynamical gravitino, namely $\psi_{\mu-}$. The independent gauge fields of the theory are given by

$$\{\tau_\mu, e_\mu{}^a, m_\mu, \psi_{\mu-}, \psi_{\mu+}\}. \quad (7.3)$$

where now both τ_μ and $\psi_{\mu+}$ can be completely gauge-fixed. It is to be expected that one encounters this “decreasing of the amount of supersymmetry” in the construction of other (i.e. $\mathcal{N} > 2, D > 3$) Newton-Cartan Supergravity theories because the foliation of spacetime by an absolute time is a main characteristic of non-relativistic theories. An important difference with the purely bosonic theory is that now the vanishing of the rotational curvature is not optional anymore. Instead, this constraint is enforced upon us by supersymmetry. Physically it means that $D = 3$ Newtonian Supergravity without matter couplings only exists in flat space. Another feature of the supersymmetric theory is that in order to write down the transformation rules in terms of the Newton potential and its superpartner, one needs to introduce a field dual to the Newton potential.

This ends our conclusions.

7.2 Developments

Since the finishing of the papers used for this thesis a lot of additional research has been done concerning Newton-Cartan theory. Here we will briefly look at these developments.

First of all, the gauging procedure has been applied to other algebras besides the ones in this thesis. One of these algebras is the Schrödinger algebra [113], which was already briefly mentioned in the summary. This algebra has the Bargmann algebra as subalgebra, but on top of that contains one dilational and one special conformal generator. The dilational gauge field appears in the curvature of the temporal vielbein, introducing via the usual curvature constraints temporal torsion in the affine connection. This gives rise to torsional Newton-Cartan geometry, which plays an important role in Lifshitz holography [114]. The gauging of the Schrödinger superalgebra leads to Schrödinger Supergravity [116]. In this construction the so-called superconformal tensor

calculus is derived for theories of Newton-Cartan Supergravity. The purpose of this method is to describe matter couplings, using superconformal symmetries as a guideline. In the relativistic case one uses the superconformal algebra, which has the super-Poincaré algebra as a subalgebra. The non-relativistic analog of this method leads naturally to the Schrödinger superalgebra instead of the Galilean conformal superalgebra. The Newton-Cartan Supergravity theory has also been extended to include Newton-Hooke symmetries [115]. Besides non-relativistic algebras the gauging procedure also has been applied to ultra-relativistic algebras [117], also known as Carroll algebras. An important difference with the gauging of the Bargmann algebra is that one has to add an extra field by hand to solve for the spin connections. These Carrollian theories are interesting in the application of flat space holography, because in $2 + 1$ dimensional asymptotically flat space the asymptotic symmetries at infinity can be considered to be Carrollian. Finally, in this thesis we only considered algebra (i.e. Inönü-Wigner) contractions. Because the gauge fields are in the adjoint representation of the algebra, the contraction of the algebra suggests a contraction on the gauge fields. This contraction, which can be interpreted as a non-relativistic limit of the field theory, is derived in [118]. It is then used to derive the *off*-shell formulation of the three-dimensional Newton-Cartan Supergravity theory discussed in chapter six of this thesis.

Newton-Cartan geometry can also be used for a holographical description of an effective field theory of quantum Hall states [119]. However, there are still open questions concerning (non-relativistic) holography. The AdS/CFT correspondence in the first example by Maldacena is a duality between a strongly coupled and weakly coupled theory. The correspondence was made explicit for a type *IIB* Supergravity theory on an $AdS_5 \times S_5$ background and an $\mathcal{N} = 4$, $SU(N)$ super-conformal Yang-Mills theory. The setup involves a stack of *D3*-branes, which are solutions of the Supergravity theory. The couplings which are mapped are then constructed out of the different parameters of the two theories. The conjecture consists of claiming that the correspondence holds not only for the Supergravity theory ($\alpha' \rightarrow 0$) but for the *full* string theory, which is its *UV*-completion. An important guideline in the correspondence is the matching of symmetries; the isometry group of AdS_5 is generated by the algebra $\mathfrak{so}(4, 2)$, which also generates the conformal algebra in four dimensions of the super Yang-Mills theory. The isometries of the five-sphere S_5 are generated by $\mathfrak{so}(6)$, which is isomorphic to the R-symmetry algebra $\mathfrak{su}(4)$ of the superconformal theory. In the non-relativistic setting one also uses the symmetries as a guideline, but without an explicit embedding in string theory a relation between the couplings is not known. Because the duality involves strong versus weak couplings, an explicit proof of the conjecture is very hard since perturbation theory breaks down at strong coupling. In more general settings the gravitational theory should be embedded in a string theory, but this embedding is only known for a few examples. It is therefore desirable to develop precision tests in which one can further strengthen the correspondence and its extrapolations. One example of such a test consists of non-perturbatively acquired partition functions of the field theory at the boundary. These results can then be compared with a holographic calculation in the gravitational theory. These calculations require the background of the field theory to be curved. The off-shell formulation of the Newton-Cartan supergravity as described in chapter six of this thesis provides a tool for obtaining such non-relativistic field theories on curved backgrounds. SUSY-preserving background solutions of this off-shell formulation were studied in [120].

Kaluza-Klein reductions of Newton-Cartan gravity have been considered in [121], resulting in Galilean electromagnetism plus a scalar field. These two extra fields source the spatial components of the Ricci tensor, giving an explicit example of Newton-Cartan geometry without the full rotational curvature being zero. This gives an interesting extension of the usual Newton-Cartan theory where only the rest mass density sources the temporal Ricci components.

This finishes our update of recent developments. Finally we consider possible future research.

7.3 Outlook

There are various ways of continuing research of Newton-Cartan gravity and its stringy and supersymmetric extensions. For holographic applications it would be interesting to consider the supersymmetric extension of the *stringy* Newton-Cartan theory. From the discussion of [92] one expects such a theory to be dual to a Galilean superconformal theory. Another interesting question is whether it is possible to construct Newton-Cartan Supergravity theories without the flat space constraint. As such one could construct supersymmetric field theories on less trivial backgrounds, similar to [120]. However, this constraint follows from the vanishing of the temporal vielbein curvature, which expresses the foliation of spacetime by an absolute time. This is a defining feature of any non-relativistic theory, and it is not clear if and how one can circumvent this constraint without introducing matter couplings as in [121].

The $\mathcal{N} = 2$ theory in three spacetime dimensions also sheds some light on the construction of the $\mathcal{N} = 2$ theory in four dimensions. The graviton multiplet of the relativistic theory consists of the graviton, two gravitini and one vector. This theory cannot be obtained by a gauging of the corresponding algebra because the vector is not a gauge field of the SUSY-algebra. Non-relativistically the same problem holds. With trial and error we tried to write down the transformation rules for a multiplet consisting of (7.1) plus two gravitini and a vector field. With this natural Ansatz it was found that whereas the superalgebra closes on the bosonic fields, one or more extra fields are needed in the supermultiplet for the closure of the superalgebra on the gravitini. An interesting open question is the explicit construction of this supermultiplet. One possible way to do this would be by a null-reduction of the relativistic $\mathcal{N} = 2$ theory in *five* dimensions. Another way to construct this theory would be by a contraction procedure as proposed in [118]. A third method would be to linearize the four-dimensional theory and to derive the supercurrent [122] (the supersymmetric analog of the conserved energy-momentum tensor of General Relativity); the fields which are missing from our Ansatz and their transformations should then arise as a consistency requirement.

Appendices

Appendix A

Notation and conventions

A.1 Notation concerning indices, (A)dS and nomenclature

Our notation concerning indices, (Anti)-de Sitter space and nomenclature are as follows. We denote the number of spacetime dimensions by D . A positive cosmological constant $\Lambda > 0$ describes a deSitter space, whereas $\Lambda < 0$ describes an anti deSitter space. A few times we will explicitly write spinor indices as α, β, \dots . Flat target-space indices are given by $A = \{a', a\}$, where $\{a'\}$ is longitudinal and $\{a\}$ is transverse, e.g.

$$\zeta^A = \{\zeta^{a'}, \zeta^a\}. \quad (\text{A.1})$$

For a particle we write $\{a' = \underline{0}\}$ and $\{a = 1, \dots, D-1\}$, whereas for a string we write $\{a' = \underline{0}, \underline{1}\}$ and $\{a = 2 \dots D-1\}$. Curved target-space indices are given by $\mu = \{\alpha, i\}$, where $\{\alpha\}$ is longitudinal (unless we explicitly use it as a spinor index, see e.g. the derivation of the Fierz identity (A.11)) and $\{i\}$ is transverse, e.g.

$$\xi^\mu = \{\xi^\alpha, \xi^i\}. \quad (\text{A.2})$$

Turning curved into flat indices is done using the (inverse) vielbeins τ^μ and e^μ_a , as in the following example:

$$\begin{aligned} \hat{F}_{\underline{0}a} &= \tau^\mu e^\nu_a \hat{F}_{\mu\nu}, \\ \hat{F}_{ab} &= e^\mu_a e^\nu_b \hat{F}_{\mu\nu}. \end{aligned} \quad (\text{A.3})$$

Infinitesimal general coordinate transformations $x^\rho \rightarrow x^\rho + \xi^\rho$ on (dual) vectors are written as

$$\begin{aligned} \delta V^\mu &= \xi^\rho \partial_\rho V^\mu - V^\rho \partial_\rho \xi^\mu, \\ \delta \omega_\mu &= \xi^\rho \partial_\rho \omega_\mu + \omega_\rho \partial_\mu \xi^\rho, \end{aligned} \quad (\text{A.4})$$

where the partial derivatives can be replaced by covariant ones when torsion is not present. These expressions are naturally extended to more general tensors.

For a particle we write $\{\alpha = 0\}$ and $\{i = 1, \dots, D-1\}$, and for a string we write $\{\alpha = 0, 1\}$ and $\{i = 2, \dots, D-1\}$. For temporal components of generators of Lie algebras we will not use underlined indices, e.g. the temporal component of P_A will just be written as P_0 . For notational convenience we will do the same for gamma matrices, i.e. the zero-component of γ^A will just be

written as γ^0 instead of $\gamma^{\bar{0}}$. We indicate world-sheet indices with $\{\bar{\alpha}, \bar{\beta}, \dots\}$, and the world-sheet coordinates as $\{\sigma^{\bar{\alpha}}\}$. Finally, for timelike embedding coordinates $\{x^0\}$ we will sometimes write $\{x^0\} = \{ct\}$, or $\{x^0\} = \{t\}$ if the speed of light c is explicitly taken to be $c = 1$. This embedding coordinate should not be confused with the evolution parameter τ .

Because confusion can arise about nomenclature, we stress that the *non-relativistic limit* restricts the (transverse) speed of a particle, string or brane to be small with respect to the speed of light c , while the *Newtonian limit* on top of that restricts the gravitational field to be weak and static. The word “classical” is only used as “not quantum”.

A.2 Supersymmetry conventions

Our supersymmetry conventions for $D = 3$ follow [108], in which we choose $\epsilon = \eta = +1$. The Clifford algebra is given by

$$\{\gamma_A, \gamma_B\} = 2\eta_{AB}, \quad (\text{A.5})$$

Also,

$$\gamma_{A\dots B} \equiv \gamma_{[A\dots B]}, \quad (\text{A.6})$$

where we always (anti)symmetrize with total weight one, e.g.

$$\gamma_{ABC} = \frac{1}{3!}(\gamma_A\gamma_B\gamma_C + \dots). \quad (\text{A.7})$$

The charge conjugation matrix C , which obeys $C^T = -C$ and $C^\dagger = C^{-1}$, is chosen as

$$C = \gamma^0. \quad (\text{A.8})$$

We then have the identities $\gamma_A^\dagger = \gamma^0 \gamma_A \gamma^0$, $\gamma_{AB}^\dagger = \gamma^0 \gamma_{AB} \gamma^0$ etc.

For $D = 3$ one can choose Majorana spinors, which we will do. Being in an odd number of spacetime dimensions we can not define a chirality operator. Dirac conjugation is defined by

$$\bar{\psi} = i\psi^\dagger \gamma^0, \quad (\text{A.9})$$

giving $(\bar{\psi})^\dagger = i\gamma^0 \psi$. Then $(\bar{\psi} \gamma^A \lambda)^\dagger = -\bar{\lambda} \gamma^A \psi$, $(\bar{\psi} \gamma^{AB} \lambda)^\dagger = -\bar{\lambda} \gamma^{AB} \psi$, etc.

The following set of four matrices forms a complete basis for all 2×2 matrices:

$$\{\gamma\} = \{1, \gamma^A\}, \quad A = \{0, 1, 2\}. \quad (\text{A.10})$$

Given this set one can easily check the three-dimensional Fierz identity¹

$$\psi \bar{\lambda} = -\frac{1}{2}(\bar{\lambda} \psi) - \frac{1}{2}(\bar{\lambda} \gamma^0 \psi) \gamma_0 - \frac{1}{2}(\bar{\lambda} \gamma^a \psi) \gamma_a. \quad (\text{A.11})$$

These identities are crucial in checking the closure of the SUSY-algebra on the fermionic fields. The reason is that in applying the SUSY commutators on a fermionic field (e.g. the gravitino in Supergravity theories or the electron in supersymmetric QED) the free spinor index is not on the fermionic field itself but on one of the fermionic SUSY parameters ϵ . As such the on-shell closure of the SUSY algebra is not manifestly clear. With the Fierz identity (A.11) this free spinor index can be put on the fermionic field to make the closure of the algebra manifest.

¹In components this bi-spinor is $\psi_\alpha \lambda^\beta$. The trace of $\psi \bar{\lambda}$ is given by $\psi_\alpha \lambda^\alpha = -\lambda^\alpha \psi_\alpha = -\bar{\lambda} \psi$, giving a minus-sign.

Appendix B

Basic gauge theory

Symmetries in physics are described by groups G . The symmetries which are important in this thesis are Lie groups [109, 110], which describe continuous symmetries. The elements $g \in G$ of such groups are generated by a Lie algebra \mathfrak{g} . These Lie algebras are linear vector spaces, which make them convenient to analyze the group. If we write the elements of \mathfrak{g} as T_A , where $A = \{1 \dots N\}$ for some N , then $\mathfrak{g} = \text{span}\{T_A\}$ and a general group element g is written as

$$\begin{aligned} g &= e^{\theta^A T_A} \\ &= 1 + \theta^A T_A + \frac{1}{2} \theta^A \theta^B T_A T_B + \dots \end{aligned} \tag{B.1}$$

The parameters $\{\theta^A\}$ can be real or complex, depending on the particular algebra. The characteristic feature of groups is their multiplication structure; if $g_1 \in G$ and $g_2 \in G$, then $g_1 g_2 \in G$. This group multiplication structure is encoded completely in the underlying Lie algebra via the Lie bracket¹

$$[T_A, T_B] = f_{AB}^C T_C. \tag{B.2}$$

The structure constants $\{f_{AB}^C\}$ of the algebra \mathfrak{g} are manifestly antisymmetric in $\{AB\}$.

In a gauge theory a global symmetry on a set of fields $\{\phi\}$ is promoted to a local symmetry, which introduces gauge fields B_μ^A on which the Lie algebra \mathfrak{g} is realized. Usually these gauge fields come from the kinetic terms of the fields $\{\phi\}$. These kinetic terms are not invariant under the local transformations and therefore need compensation. If the fields $\{\phi\}$ transform as

$$\delta_\epsilon \phi = \epsilon^A T_A \phi, \tag{B.3}$$

where ϵ^A can be a bosonic or a fermionic transformation parameter, one can replace the ordinary derivative $\partial_\mu \phi$ in the kinetic terms by the covariant derivative

$$D_\mu \phi = \partial_\mu \phi - B_\mu^A T_A \phi, \tag{B.4}$$

which per construction transforms in the same way as the field itself:

$$\delta_\epsilon D_\mu \phi = \epsilon^A T_A D_\mu \phi. \tag{B.5}$$

¹Note that the algebra does not completely fix the group; a familiar example is the fact that the groups $SO(3)$ and $SU(2)$ are generated by the same Lie algebra.

In general we define objects to be covariant when they transform under all the transformations without a derivative on the transformation parameter ϵ^A . The transformation of the gauge fields then reads

$$\delta_\epsilon B_\mu^A = \partial_\mu \epsilon^A + \epsilon^B B_\mu^C f_{BC}^A, \quad (\text{B.6})$$

where a summation over all $\{BC\}$ is understood, such that the transformation (B.5) holds, and

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] B_\mu^A = \delta(\epsilon_3 = \epsilon_2^B \epsilon_1^C f_{BC}^D) B_\mu^A, \quad (\text{B.7})$$

i.e. the algebra closes on the gauge fields. This allows one to construct the corresponding field strength $R_{\mu\nu}^A$,

$$R_{\mu\nu}^A = 2\partial_{[\mu} B_{\nu]}^A + B_\mu^B B_\nu^C f_{BC}^A, \quad (\text{B.8})$$

which transforms in a covariant way:

$$\delta_\epsilon R_{\mu\nu}^A = \epsilon^B R_{\mu\nu}^C f_{BC}^A. \quad (\text{B.9})$$

Now, because gauge fields B_μ^A carry both a spacetime index $\{\mu\}$ and an internal index $\{A\}$, they transform under general coordinate transformations and the gauge transformations. With the explicit expressions given above one can check that the following relation holds:

$$\delta_{gct}(\xi^\lambda) B_\mu^A + \xi^\lambda R_{\mu\lambda}^A - \sum_{\{C\}} \delta(\xi^\lambda B_\lambda^C) B_\mu^A = 0. \quad (\text{B.10})$$

It is important to note that the gauge parameters in this relation are constructed out of the gauge fields B_μ^A and the parameter ξ^λ of the general coordinate transformation. The simplest example of this relation is provided by a $U(1)$ gauge theory, in which all the structure coefficients f_{BC}^A are zero. The gauge field A_μ with corresponding gauge parameter Λ transforms as

$$\delta_{gct}(\xi^\lambda) A_\mu = \xi^\lambda \partial_\lambda A_\mu + \partial_\mu \xi^\lambda A_\lambda, \quad \delta_\Lambda A_\mu = \partial_\mu \Lambda. \quad (\text{B.11})$$

The field strength (B.8) is written as $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$, and one can then check that

$$\delta_{gct}(\xi^\lambda) A_\mu + \xi^\lambda F_{\mu\lambda} - \delta_\Lambda(\xi^\lambda A_\lambda) A_\mu = 0. \quad (\text{B.12})$$

This implies that when one imposes the curvature constraint $F_{\mu\nu} = 0$ (making the gauge field pure gauge) a gauge transformation with field dependent gauge parameter $\Lambda = \xi^\lambda A_\lambda$ can be interpreted as a general coordinate transformation or vice versa. These field dependent gauge transformations do not obey the original $U(1)$ algebra anymore. Namely,

$$\begin{aligned} [\delta(\Lambda_1), \delta(\Lambda_2)] A_\mu &= [\delta(\xi_1^\lambda A_\lambda), \delta(\xi_2^\lambda A_\lambda)] A_\mu \\ &= \partial_\mu \left(\xi_2^\lambda \partial_\lambda (\xi_1^\rho A_\rho) - [1 \leftrightarrow 2] \right) \\ &\neq 0 \end{aligned} \quad (\text{B.13})$$

in general. In applying this gauging procedure to theories of gravity, the identity (B.10) is used to remove the local spacetime translations from the independent fields. As is clear from the $U(1)$ example above, this will in general deform the original algebra. In Supergravity theories the same happens for the $\{Q, Q\}$ -commutator, which in general will give a general coordinate transformation plus other transformations in the algebra, all with field-dependent parameters. Such an algebra with field-dependent structure constants is called a soft algebra.

Appendix C

Bianchi identities

Here the Bianchi identities of the Bargmann theory and stringy Newton-Cartan theory will be given.

For the Bargmann theory the Bianchi-identities read

$$D_{[\lambda}R_{\mu\nu]}(H) = 0, \quad (\text{C.1})$$

$$D_{[\lambda}R_{\mu\nu]}^a(P) = -R_{[\lambda\mu}{}^{ab}(J)e_{\nu]}{}^b - R_{[\lambda\mu}{}^a(G)\tau_{\nu]} + R_{[\lambda\mu}(H)\omega_{\nu]}^a, \quad (\text{C.2})$$

$$D_{[\lambda}R_{\mu\nu]}{}^{ab}(J) = 0, \quad (\text{C.3})$$

$$D_{[\lambda}R_{\mu\nu]}^a(G) = -R_{[\lambda\mu}{}^{ab}(J)\omega_{\nu]}{}^b, \quad (\text{C.4})$$

$$D_{[\lambda}R_{\mu\nu]}(Z) = R_{[\lambda\mu}{}^a(P)\omega_{\nu]}^a - R_{[\lambda\mu}{}^a(G)e_{\nu]}{}^a. \quad (\text{C.5})$$

The curvatures (6.43) of the stringy Newton-Cartan theory satisfy the Bianchi identities

$$D_{[\rho}R_{\mu\nu]}{}^{a'}(H) = -R_{[\rho\mu}{}^{a'b'}(M)\tau_{\nu]}{}^{b'},$$

$$D_{[\rho}R_{\mu\nu]}^a(P) = -R_{[\rho\mu}{}^{ab}(J)e_{\nu]}{}^b - R_{[\rho\mu}{}^{aa'}(G)\tau_{\nu]}{}^{a'},$$

$$D_{[\rho}R_{\mu\nu]}{}^{a'b'}(M) = 0,$$

$$D_{[\rho}R_{\mu\nu]}{}^{aa'}(G) = -R_{[\rho\mu}{}^{a'b'}(M)\omega_{\nu]}{}^{ab'} - R_{[\rho\mu}{}^{ab}(J)\omega_{\nu]}{}^{ba'}, \quad (\text{C.6})$$

$$D_{[\rho}R_{\mu\nu]}{}^{ab}(J) = 0,$$

$$D_{[\rho}R_{\mu\nu]}{}^{a'}(Z) = -R_{[\rho\mu}{}^{a'b'}(M)m_{\nu]}{}^{b'} + R_{[\rho\mu}{}^a(P)\omega_{\nu]}{}^{aa'} - R_{[\rho\mu}{}^{aa'}(G)e_{\nu]}{}^a,$$

$$- R_{[\rho\mu}{}^{a'}(H)m_{\nu]}{}^{a'b'} + R_{[\rho\mu}{}^{a'b'}(Z)\tau_{\nu]}{}^{b'},$$

$$D_{[\rho}R_{\mu\nu]}{}^{a'b'}(Z) = R_{[\rho\mu}{}^{c'[a'}(M)m_{\nu]}{}^{b']c'} + R_{[\rho\mu}{}^{aa'}(G)\omega_{\nu]}{}^{ab'} - R_{[\rho\mu}{}^{ab'}(G)\omega_{\nu]}{}^{aa'}.$$

Appendix D

Newton-Cartan geodesics

D.1 Point particle geodesic

Here we give some details about the derivation of the geodesic equations (5.17) and (5.75). We start with the point particle case. For that purpose we write the Lagrangian (5.14) as

$$\begin{aligned} L &= \frac{m}{2} N^{-1} \dot{x}^\mu \dot{x}^\nu \left(h_{\mu\nu} - 2m_\mu \tau_\nu \right) \\ &\equiv \frac{m}{2} N^{-1} \dot{x}^\mu \dot{x}^\nu H_{\mu\nu}, \end{aligned} \quad (\text{F.1})$$

where we defined

$$H_{\mu\nu} \equiv h_{\mu\nu} - 2m_{(\mu} \tau_{\nu)}, \quad N \equiv \tau_\mu \dot{x}^\mu. \quad (\text{F.2})$$

Varying the Lagrangian (F.1) with respect to $\{x^\lambda\}$ and using the metric compatibility condition $\partial_{[\mu} \tau_{\nu]} = 0$ gives

$$\begin{aligned} -Nm^{-1} \frac{\delta L}{\delta x^\lambda} &= \left(N^{-2} \dot{N} \tau_\lambda H_{\mu\nu} - \frac{1}{2} N^{-1} \tau_\lambda \partial_\rho H_{\mu\nu} \dot{x}^\rho - \frac{1}{2} \partial_\lambda H_{\mu\nu} + \partial_\nu H_{\mu\lambda} \right) \dot{x}^\mu \dot{x}^\nu \\ &\quad - N^{-1} \tau_\lambda H_{\mu\nu} \dot{x}^\mu \ddot{x}^\nu - N^{-1} \dot{N} H_{\mu\lambda} \dot{x}^\mu + H_{\mu\lambda} \ddot{x}^\mu = 0. \end{aligned} \quad (\text{F.3})$$

First we contract this equation with $h^{\lambda\sigma}$. This gives

$$h^{\lambda\sigma} \left(\partial_\nu H_{\mu\lambda} - \frac{1}{2} \partial_\lambda H_{\mu\nu} \right) \dot{x}^\mu \dot{x}^\nu + h^{\lambda\sigma} H_{\mu\lambda} \ddot{x}^\mu - N^{-1} \dot{N} h^{\lambda\sigma} H_{\mu\lambda} \dot{x}^\mu = 0. \quad (\text{F.4})$$

Using now the Newton-Cartan metric relations (4.24), $\partial_{[\mu} \tau_{\nu]} = 0$ and

$$\dot{N} = \tau_\mu \ddot{x}^\mu + \partial_\mu \tau_\nu \dot{x}^\mu \dot{x}^\nu, \quad (\text{F.5})$$

some manipulation shows that (F.4) gives the geodesic equation (5.17),

$$\ddot{x}^\mu + \Gamma_{\nu\rho}^\mu \dot{x}^\nu \dot{x}^\rho = \frac{\dot{N}}{N} \dot{x}^\mu, \quad (\text{F.6})$$

with the connection given by (4.70). Second one can contract (F.3) with τ^λ . The resulting expression contains, among others, terms proportional to \ddot{x}^μ . If one uses the geodesic equation (F.6) to rewrite these in terms of \dot{x}^μ one can finally show that this τ^λ -contraction of (F.3) is trivially satisfied.

D.2 String geodesic

The calculation concerning the string Lagrangian (5.69) leading to the stringy geodesic equation (5.75) can be done in a similar way. We first write

$$H_{\mu\nu} = h_{\mu\nu} - 2m_{(\mu}{}^a \tau_{\nu)}{}^a, \quad (\text{F.7})$$

such that (5.69) becomes

$$L = -\frac{T}{2} \sqrt{-\det(\tau)} \tau^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\alpha}} x^\mu \partial_{\bar{\beta}} x^\nu H_{\mu\nu}. \quad (\text{F.8})$$

We next use the relations

$$\begin{aligned} \delta \sqrt{-\det(\tau)} &= \frac{1}{2} \sqrt{-\det(\tau)} \tau^{\bar{\alpha}\bar{\beta}} \delta \tau_{\bar{\alpha}\bar{\beta}}, \\ \delta \tau^{\bar{\alpha}\bar{\beta}} &= -\tau^{\bar{\alpha}\bar{\gamma}} \tau^{\bar{\beta}\bar{\epsilon}} \delta \tau_{\bar{\gamma}\bar{\epsilon}}, \\ \delta \tau_{\bar{\alpha}\bar{\beta}} &= 2\partial_{\bar{\alpha}} x^\mu \partial_{\bar{\beta}} \delta x^\lambda \tau_{\mu\lambda} + \partial_{\bar{\alpha}} x^\mu \partial_{\bar{\beta}} x^\nu \partial_\lambda \tau_{\mu\nu} \delta x^\lambda, \\ \partial_{\bar{\alpha}} \left(\sqrt{-\det(\tau)} \tau^{\bar{\alpha}\bar{\beta}} \partial_{\bar{\beta}} x^\mu \right) &= \sqrt{-\det(\tau)} \tau^{\bar{\alpha}\bar{\beta}} \nabla_{\bar{\alpha}} \partial_{\bar{\beta}} x^\mu, \\ \partial_\rho \tau_{\mu\nu} + \partial_\mu \tau_{\rho\nu} - \partial_\nu \tau_{\rho\mu} &= \Gamma_{\mu\rho}^\lambda \tau_{\lambda\nu}, \end{aligned} \quad (\text{F.9})$$

where the last identity follows from the metric compatibility condition $\nabla_\rho \tau_{\mu\nu} = 0$. Varying (F.8) with respect to $\{x^\lambda\}$ now gives the geodesic equation (5.75),

$$\tau^{\bar{\alpha}\bar{\beta}} \left(\nabla_{\bar{\alpha}} \partial_{\bar{\beta}} x^\rho + \partial_{\bar{\alpha}} x^\mu \partial_{\bar{\beta}} x^\nu \Gamma_{\mu\nu}^\rho \right) = 0, \quad (\text{F.10})$$

with the connection $\Gamma_{\mu\nu}^\rho$ given by (5.65). This connection is equivalent to the connection (5.64) given by the vielbein postulates.

Appendix E

Some properties of AdS_2

The isometries of AdS_2 are described by the group $SO(2, 1)$, which is generated by the algebra

$$\begin{aligned} [H_a, H_b] &= R^{-2}M_{ab}, & [M_{bc}, H_a] &= -2\eta_{a[b}H_{c]}, \\ [M_{cd}, M_{ef}] &= 4\eta_{[c[e}M_{f]d]}. \end{aligned} \quad (G.1)$$

If we define $A = \{0, 1, 2\}$, $M_{2a} = RH_a$ where R is the radius of curvature, and $\eta_{AB} = \text{diag}(-1, +1, -1)$, the $\mathfrak{so}(2, 1)$ algebra is manifest:

$$[M_{CD}, M_{EF}] = 4\eta_{[C[E}M_{F]D]}. \quad (G.2)$$

We define the AdS_2 space via the embedding coordinates $\{y^A\}$ as

$$\eta_{AB}y^Ay^B = -(y^0)^2 + (y^1)^2 - (y^2)^2 = -R^2. \quad (G.3)$$

In terms of the AdS_2 coordinates $x^\alpha = \{t, z\}$ we choose

$$y^0 = \sqrt{z^2 + R^2} \sin\left(\frac{t}{R}\right), \quad y^1 = z, \quad y^2 = \sqrt{z^2 + R^2} \cos\left(\frac{t}{R}\right), \quad (G.4)$$

such that

$$t = R \tan^{-1}\left(\frac{y^0}{y^2}\right), \quad z = y^1. \quad (G.5)$$

The induced metric on the AdS_2 space is then

$$ds^2 = \eta_{AB}dy^A dy^B = -\left(1 + \frac{z^2}{R^2}\right)dt^2 + \left(1 + \frac{z^2}{R^2}\right)^{-1}dz^2, \quad (G.6)$$

with nonzero Christoffel components

$$\Gamma_{tt}^z = z\left(\frac{z^2 + R^2}{R^4}\right), \quad \Gamma_{zz}^z = \frac{-z}{z^2 + R^2}, \quad \Gamma_{zt}^t = \frac{z}{z^2 + R^2}. \quad (G.7)$$

The $SO(2, 1)$ group acts linearly on the embedding coordinates y^A via $y'^A = \Lambda^A_B y^B$. For the AdS_2 coordinates $x^\alpha = \{t, z\}$ this implies via (G.5) the non-linear realization

$$\begin{aligned} t' &= R \tan^{-1}\left(\frac{\Lambda^0_0 \sin\left(\frac{t}{R}\right) + \Lambda^0_1(z^2 + R^2)^{-1/2} + \Lambda^0_2 \cos\left(\frac{t}{R}\right)}{\Lambda^2_0 \sin\left(\frac{t}{R}\right) + \Lambda^2_1(z^2 + R^2)^{-1/2} + \Lambda^2_2 \cos\left(\frac{t}{R}\right)}\right), \\ z' &= \Lambda^1_0 \sqrt{z^2 + R^2} \sin\left(\frac{t}{R}\right) + \Lambda^1_1 z + \Lambda^1_2 \sqrt{z^2 + R^2} \cos\left(\frac{t}{R}\right). \end{aligned} \quad (G.8)$$

An H transformation is then performed via $-R\Lambda^2_0$ and $R\Lambda^2_1$, and an M transformation is performed via Λ^1_0 . From (G.8) it is clear that the identity transformation $t' = t$ and $z' = z$ is given by $\Lambda^0_0 = \Lambda^1_1 = \Lambda^2_2 = 1$ and the other Λ 's being zero. One can deduce the infinitesimal transformations δt and δz from the three Killing vectors of $\mathfrak{so}(2,1)$,

$$\xi_{\{AB\}} = -2Y_{[A}\partial_{B]}, \quad (\text{G.9})$$

which have components $\xi^C_{\{AB\}} = -2Y_{[A}\delta^C_{B]}$. One then has the infinitesimal transformation

$$\delta Y^C = \lambda^{AB}\xi^C_{\{AB\}} = \lambda^C_B Y^B, \quad (\text{G.10})$$

with λ^A_B as the infinitesimal components of an $SO(2,1)$ transformation such that an M -transformation is written as $M = \lambda^{AB}M_{AB}$. The Killing vectors become¹

$$\xi_{\{01\}} = \frac{zR \cos(\frac{t}{R})}{\sqrt{z^2 + R^2}} \partial_t + \sqrt{z^2 + R^2} \sin(\frac{t}{R}) \partial_z, \quad (\text{G.11})$$

$$\xi_{\{02\}} = -R\partial_t, \quad (\text{G.12})$$

$$\xi_{\{12\}} = \frac{Rz \sin(\frac{t}{R})}{\sqrt{z^2 + R^2}} \partial_t - \sqrt{z^2 + R^2} \cos(\frac{t}{R}) \partial_z, \quad (\text{G.13})$$

where $\xi_{\{12\}}$ and $\xi_{\{02\}}$ generate H -transformations, and $\xi_{\{01\}}$ generates the M transformation. One can check that these vectors indeed form an $\mathfrak{so}(2,1)$ algebra. Then

$$\begin{aligned} \delta_H t &= -R\lambda^2_0 - \lambda^2_1 \frac{Rz \sin(\frac{t}{R})}{\sqrt{z^2 + R^2}}, & \delta_H z &= \lambda^2_1 \sqrt{z^2 + R^2} \cos(\frac{t}{R}), \\ \delta_M t &= \lambda^1_0 \frac{zR \cos(\frac{t}{R})}{\sqrt{z^2 + R^2}}, & \delta_M z &= \lambda^1_0 \sqrt{z^2 + R^2} \sin(\frac{t}{R}). \end{aligned} \quad (\text{G.14})$$

¹Notice that $\xi_{\{02\}}$ describes the fact that the AdS_2 metric is static. We could rescale the time coordinate t with $-R$ to get $\xi_{\{02\}} = \partial_t$.

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Nederlandse Samenvatting

Tijdens het afronden van dit proefschrift was het honderd jaar geleden dat Einstein zijn algemene relativiteitstheorie publiceerde, een theorie die ons begrip van zwaartekracht en de natuurkunde in het geheel drastisch veranderde. De theorie werd door Einstein niet zozeer ontwikkeld uit empirische noodzaak, maar vooral uit theoretische overwegingen. Einstein meende namelijk dat de essentie van het zogenaamde equivalentieprincipe (zware massa is trage massa) niet voldoende was doorgrond. Deze overtuiging, plus de wens om zwaartekracht in zijn relativistische raamwerk te gieten, was voor hem een motivatie om de zwaartekrachtstheorie van Newton als onvolledig te beschouwen. Newton beschreef twee eeuwen voor Einsteins geboorte in zijn *Principia* zwaartekracht als een instantane aantrekkingskracht tussen massa's. Deze beschrijving was empirisch succesvol omdat ze onder andere de banen van de planeten correct beschreef, maar een onderliggend mechanisme bleef onduidelijk. Einstein herformuleerde zwaartekracht als de kromming van ruimtetijd en liet zien dat zwaartekracht niet instantaan werkt, maar in het vacuum zich voortplant met een eindige snelheid: de lichtsnelheid. Deze meetkundige beschrijving wordt gesuggereerd door het equivalentieprincipe. Het principe stelt namelijk dat lokaal in de ruimtetijd, zwaartekracht en versnelling dezelfde effecten hebben, net zoals lokaal de aarde vlak lijkt. Een belangrijke eigenschap van Einsteins theorie is *algemene covariantie*, het idee dat *alle* waarnemers dezelfde vergelijkingen gebruiken. Dit verschilt van Newtons vergelijkingen, waarvan de vorm alleen geldt voor een beperkte groep waarnemers. Wanneer een waarnemer bijvoorbeeld gaat roteren dan zal deze waarnemer inertiaalkrachten waarnemen, die eerst niet aanwezig waren in Newtons vergelijkingen. Wiskundig verschijnen deze inertiaalkrachten omdat Newtons vergelijkingen slechts tensoren zijn onder een beperkte groep van transformaties. Einstein meende aanvankelijk dat algemene covariantie een definiërende eigenschap van zijn theorie was, maar werd daarop al gauw gecorrigeerd. Onder andere Kretschmann suggereerde dat het ook mogelijk zou moeten zijn om theorieën zoals die van Newton algemeen-covariant te formuleren. Een paar jaar later bleek dat Kretschmann gelijk had.

Naast zwaartekracht was Einstein ook de persoon die met zijn verklaring van het foto-elektrisch effect Plancks kwantumhypothese niet alleen als een wiskundige truuk zag, maar als een natuurkundig principe. Daarmee was Einstein, tien jaar voordat hij zijn algemene relativiteitstheorie publiceerde, ook één van de grondleggers van de kwantummechanica. Het blijkt echter erg moeilijk te zijn om de algemene relativiteitstheorie en de kwantummechanica in één overkoepelend raamwerk onder te brengen. De verschillende pogingen om dit probleem op te lossen, bijvoorbeeld luskwantumzwaartekracht en snaartheorie, worden beide in meerdere aspecten nog steeds niet goed begrepen. Omdat kwantumzwaartekrachtseffecten empirisch erg moeilijk zijn te detecteren, moet men het al decennialang van gedachtenexperimenten hebben. In deze 'experimenten' wor-

den vaak zwarte gaten gebruikt, de simpelste zwaartekrachtsfenomenen die de natuur kent. Op dezelfde manier als dat het waterstofatoom door de kwantummechanica moest kunnen worden beschreven, verwachten we dat elementaire eigenschappen van zwarte gaten door een theorie van kwantumzwaartekracht voorspeld worden. De ontdekking van Jacob Bekenstein, dat je een zwart gat een entropie kunt toedichten die evenredig is met de *oppervlakte* van de waarnemershorizon, was een belangrijke leidraad. Een theorie van kwantumzwaartekracht moet een microscopische oorsprong van deze entropie beschrijven. Bekensteins resultaat impliceert ook een holografische opvatting van een zwart gat in D dimensies, waarbij de vrijheidsgraden in $D - 1$ dimensies kunnen worden beschreven. Dit idee werd gegeneraliseerd door Leonard Susskind en Gerard 't Hooft. Een belangrijke doorbraak kwam halverwege de jaren negentig, toen Juan Maldacena het idee van holografie expliciet maakte voor specifieke theorie: snaartheorie. Uit zijn analyse bleek dat een bepaald type supersnaartheorie, namelijk type *IIB* in een zogenaamde $AdS_5 \times S_5$ ruimtetijd, dual is aan een vierdimensionale kwantumveldentheorie *zonder* zwaartekracht, namelijk een $\mathcal{N} = 4$, $SU(N)$ superconforme Yang-Mills theorie. Dit dual-zijn betekent in de praktijk dat je elke eigenschap van de ene theorie uniek kunt relateren aan een eigenschap van de andere theorie. Een simpele (klassieke) analogie is de relatie tussen LC-ketens (een elektrische schakeling met een spanningsbron, een spoel en een condensator) en harmonische oscillatoren. Dit zijn twee totaal verschillende systemen. Toch bestaat er een 'dualiteit' tussen de stroom I door de schakeling en de positie x van de oscillator, en tussen $L \times C$ (de zelfinductie van de spoel maal de capaciteit van de condensator) en $\frac{m}{k}$ (de massa van de oscillator gedeeld door de veerconstante). De reden is dat de desbetreffende differentiaalvergelijkingen dezelfde vorm hebben, hoewel de onderliggende fysica heel anders is. De dualiteit die Maldacena vond, is vele malen ingewikkelder, maar het idee is vergelijkbaar. De dualiteit is met name bijzonder omdat het een kwantumveldentheorie *zonder* zwaartekracht relateert aan een theorie *met* zwaartekracht in één extra ruimtelijke dimensie, en omdat de bijbehorende koppelingen invers gerelateerd zijn. Deze koppelingen van beide theorieën behelzen combinaties van de dimensieloze parameters.

In Einsteins theorie wordt de koppeling door het correspondentieprincipe gegeven door een combinatie van de lichtsnelheid en Newtons constante. Wanneer je zwaartekracht bij lage energieën bekijkt, betekent dit op papier dat de zwaartekrachtsvelden statisch en zwak zijn (geen zelfinteractie) en/of de objecten in deze velden langzaam bewegen (lage kinetische energie). We weten via het correspondentieprincipe dat Einsteins algemene relativiteitstheorie dan weer moet overgaan in Newtons zwaartekrachtstheorie. De combinatie van lage (oftewel niet-relativistische) snelheden en zwakke, tijdsafhankelijke velden noemen we het Newtoniaanse regime. Elk tekstboek over algemene relativiteit behandelt deze limiet omdat het Newtons theorie moet kunnen reproduceren en omdat het de koppeling van Einsteins theorie vastlegt. Maar in verreweg de meeste gevallen gebeurt deze analyse op het niveau van de bewegingsvergelijkingen. Newtons theorie kan echter ook in een differentiaalmeetkundige vorm worden beschreven. Hierin wordt Newtoniaanse zwaartekracht beschreven als kromming van een Newtonse notie van ruimtetijd. De belangrijkste eigenschap van deze ruimtetijd is het bestaan van een absolute tijd, zoals Newton deze in zijn *Principia* beschreef. De theorie is algemeen-covariant en vormde een bevestiging van Kretschmanns kritiek op Einstein. Deze formulering van Newtons theorie werd voor het eerst afgeleid door Elie Cartan, slechts een paar jaar na Einsteins publicatie van zijn algemene relativiteitstheorie, en kennen we nu onder de naam *Newton-Cartan theorie*. Dit formalisme vormt het onderwerp van dit

proefschrift.

De algemene relativiteitstheorie beschrijft zwaartekracht dus als een meetkundig fenomeen. Voor zover we weten zijn er naast de zwaartekracht nog drie andere fundamentele interacties. Deze worden beschreven met een zogenaamde ijktheorie die we *Yang-Mills theorie* noemen, naar de twee ontdekkers ervan. Dit zijn theorieën die interacties beschrijven, waarin continue en interne symmetrieën de belangrijkste leidraad zijn. Deze continue symmetrieën worden beschreven met Lie-algebra's. Kortgezegd worden er in ijktheorieën interacties verkregen door symmetrieën uit te breiden. De theorie van een vrij elektron bijvoorbeeld heeft een interne globale symmetrie: wanneer je op elke plek en elk tijdstip het elektronveld met dezelfde hoek roteert, blijft de dynamica hetzelfde. Wanneer je deze hoek laat afhangen van de ruimte- en tijdcoördinaten, dat wil zeggen lokaal maakt, dan breek je deze symmetrie. Wanneer je vervolgens *eis*t dat deze lokale symmetrie toch aanwezig is, oftewel de symmetrie *ijkt*, dan moet je een extra veld introduceren dat koppelt aan het elektronveld. Deze koppeling beschrijft een interactie. Wanneer dit veld ook nog een eigen dynamica krijgt, vorm je een theorie waarin elektronen via dit extra veld (het zogenaamde *ijkveld*) met elkaar wisselwerken.

Deze manier van interacties beschrijven klinkt nogal anders dan Einsteins meetkundige aanpak, maar er blijkt een diepe connectie te zijn tussen beide beschrijvingen. Een ijktheorie kun je namelijk ook meetkundig interpreteren, en algemene covariantie kun je opvatten als een ijksymmetrie. In plaats van enkel Newtons inertiaalwaarnemers mag je in Einsteins theorie immers *elke* waarnemer kiezen die je wilt. Veertig jaar nadat Einstein zijn algemene relativiteitstheorie publiceerde, werd dan ook aangetoond dat je via een ijkprocedure de algemene relativiteitstheorie kunt afleiden. Deze ijkprocedure moet worden toegepast op de (globale) symmetrieën van de speciale relativiteitstheorie, en de bijbehorende ijkvelden zijn te relateren aan de meetkundige objecten die Einstein gebruikte. Zo krijg je een algebraïsche beschrijving van de algemene relativiteitstheorie. Dat maakt de theorie abstracter, maar ook toegankelijker voor eventuele uitbreidingen naar andere symmetrieën zoals supersymmetrie.

Een logische vraag is dan of ook de Newton-Cartan formulering met een ijkprocedure op niet-relativistische symmetrieën kan worden afgeleid. Deze vraag is het startpunt van dit proefschrift. Allereerst wordt er naar puntdeeltjes gekeken. Omdat de Lagrangiaan van een puntdeeltje naar een totale afgeleide transformeert onder zogenaamde Galileï boosts, wordt de Galileï algebra uitgebreid met een extra generator. Deze generator beschrijft de massa van de deeltjes in kwestie, en de behouden lading drukt behoud van deeltjes uit. Zo'n uitbreiding wordt een (centrale) extensie genoemd, omdat deze extensie commuteert met alle andere elementen van de algebra. Normaliter duiken deze extensies op wanneer je een klassieke veldentheorie de regels van de kwantummechanica wilt opleggen, maar het voorbeeld van het niet-relativistische puntdeeltje en de ijkprocedure van de bijbehorende algebra in dit proefschrift laat de relevantie van dit soort extensies in het klassieke geval zien. In de ijkprocedure levert de extensie één extra ijkveld op. De tijdscomponent van dit veld blijkt de Newtonse potentiaal te zijn en de koppeling aan het puntdeeltje heeft dezelfde vorm als de koppeling aan de vectorpotentiaal van het elektromagnetische veld. De massa van het deeltje speelt hierbij dan de rol van lading. Alleen met behulp van het centrale ijkveld kun je de meetkundige structuur van het Newton-Cartan formalisme volledig in termen van ijkvelden

opschrijven. Bovendien maakt het de algemeen-covariante actie van het puntdeeltje invariant onder lokale Galileï boosts. Ook wordt aangetoond dat enkele restricties die in de oorspronkelijke Newton-Cartan formulering werden opgelegd om Newtonse zwaartekracht te reproduceren, allemaal zijn te herleiden tot één enkele restrictie in de algebraïsche formulering. Deze ene restrictie, die optioneel is, zegt dat de veldsterkte van rotaties nul is. Fysisch impliceert de restrictie dat de ruimte vlak is, zoals in Newtons oorspronkelijke formulering van zijn mechanica.

De ijkprocedure is vervolgens ook uit te breiden naar de symmetrieën van een niet-relativistische snaar. Analoot aan het puntdeeltje suggereert de Lagrangiaan van de niet-relativistische snaar een extensie van de algebra. Deze bestaat uit twee generatoren en de ijking levert daarom twee ijkvelden op. Eén ijkveld komt niet in de vergelijkingen van het zwaartekrachtveld voor, terwijl het andere ijkveld de rol speelt van een snaarachtige uitbreiding van de Newtonse potentiaal. De resulterende Newtoniaanse zwaartekrachtstheorie blijkt anders te zijn dan die van een wereld waarin puntdeeltjes zich in de ruimtetijd bevinden. De ruimtetijd die parallel ligt aan het wereldoppervlak van de snaar blijft relativistisch, terwijl de ruimte die transversaal op dit wereldoppervlak ligt niet relativistisch is. Dit is een belangrijk verschil met de algemene relativiteitstheorie, waar de zwaartekrachtvergelijkingen niet afhangen van het feit of er deeltjes, snaren of branen door de ruimtetijd bewegen. In feite wordt in de gebruikelijke niet-relativistische limiet een beperking gelegd op *alle* ruimtelijke snelheden, terwijl in de ijking alleen de transversale snelheden worden beperkt. Zo bekeken definieert deze constructie een nieuwe manier om de Newtoniaanse limiet van de algemene relativiteitstheorie te nemen. Ook wordt uitgelegd hoe een kosmologische constante geïntroduceerd kan worden, zowel voor de snaar als voor het puntdeeltje.

De analyse in dit proefschrift maakt ook de constructie van niet-relativistische theorieën van supergravitatie toegankelijker. Supersymmetrie is een symmetrie die voortkomt uit de vraag hoeveel symmetrie er mogelijk is in het standaardmodel zonder de theorie triviaal te maken. Meer symmetrie betekent namelijk meer behouden grootheden, en voorbij een bepaalde hoeveelheid symmetrie zijn interacties niet meer mogelijk. Deze beperking kan worden omzeild door te pomen dat de behouden ladingen niet bosonisch, maar fermionisch zijn. Hierbij worden de generatoren die de symmetrieën van de speciale relativiteitstheorie beschrijven, aangevuld met fermionische generatoren. Deze fermionische generatoren noemen we *superladingen*. De algebra dicteert vervolgens dat elk bosonisch veld minstens één superpartner heeft en vice versa. De relatie tussen de super- en de ruimtetijd-transformaties is dat twee supertransformaties een ruimtetijd-translatie genereren. Eind jaren zeventig voerden Freedman, Ferrara en Van Nieuwenhuizen een ijkprocedure uit op de superalgebra, wat leidde tot een supersymmetrische uitbreiding van de algemene relativiteitstheorie. Deze procedure wordt in dit proefschrift toegepast op de niet-relativistische superalgebra in *drie* dimensies. De algemene relativiteitstheorie in drie dimensies bevat geen zwaartekrachtsgolven en zwaartekracht manifesteert zich alleen lokaal. De Newtonse limiet van de driedimensionale theorie stelt dan ook dat massa's onderling geen zwaartekracht ondergaan. Dat betekent echter niet dat er in drie dimensies geen Newtonse zwaartekrachtstheorie bestaat, maar enkel dat deze niet verkregen kunnen worden uit de gebruikelijke limietprocedure van de algemene relativiteitstheorie.

Voor niet-relativistische supersymmetrie blijken er tenminste twee verschillende superladingen nodig te zijn, omdat met één enkele superlading de karakteristieke relatie tussen supertransfor-

maties en ruimtetijd-translaties verdwijnt. De ijkprocedure kan vervolgens rechtstreeks worden toegepast en geeft een driedimensionale Newtonse theorie van supergravitatie. Deze constructie geeft een aantal inzichten die (waarschijnlijk) algemeen zijn. Ten eerste blijkt dat de helft van de gravitino's met een ijktransformatie kunnen worden verwijderd. Deze vermindering van de effectieve hoeveelheid supersymmetrie zal waarschijnlijk ook aanwezig zijn voor theorieën in meerdere dimensies en/of met meerdere superladingen. Ten tweede dwingt supersymmetrie je om de veldsterkte van de rotaties op nul te zetten. De oorzaak is het bestaan van een absolute tijd, die zo karakteristiek is voor de niet-relativistische ruimtetijd. De bijbehorende veldsterkte moet op nul worden gezet om algemene covariantie te verkrijgen, en supersymmetrie dicteert vervolgens dat ook de veldsterkte van één van de gravitino's en de rotaties verdwijnt. Ten derde blijkt dat als je de transformaties expliciet wilt opschrijven in termen van de Newtonse potentiaal en de bijbehorende superpartner, er een duale Newtonse potentiaal geïntroduceerd moet worden. De volgende stap is om een vier-dimensionale Newtonse theorie van superzwaartekracht op te schrijven, maar pogingen hiertoe zijn mislukt. De meest natuurlijke ansatz voor het multiplet, namelijk de onafhankelijke velden uit de bosonische theorie plus twee gravitino's en één vector, blijkt niet voldoende te zijn om de bijbehorende algebra te laten sluiten op de gravitino's. Wellicht dat een recent ontwikkelde limietprocedure [118] toegepast op de relativistische theorie meer inzicht biedt.

Deze in dit proefschrift ontwikkelde (supersymmetrische) uitbreidingen van Newton-Cartan theorie hebben interessante toepassingen voor het holografische principe. De niet-relativistische limiet van snaartheorie op een Anti-deSitter achtergrond zal een snaarachtige extensie van de gebruikelijke Newton-Cartan theorie zijn met zogenaamde snaarachtige Newton-Hooke symmetrieën. Deze hypothese wordt ondersteund door naar de onderliggende symmetrieën te kijken. De bosonische vorm van deze theorie is geconstrueerd in hoofdstuk vijf van dit proefschrift. De supersymmetrische uitbreiding van drie-dimensionale Newton-Cartan theorie zoals besproken in hoofdstuk zes van dit proefschrift biedt inzicht in hoe deze snaarachtige Newton-Cartan theorie supersymmetrisch gemaakt kan worden. Hiermee kan de niet-relativistische limiet van dit specifieke holografische geval beter begrepen worden. Ook kunnen zogenaamde localisatie-technieken toegepast worden op de theorie van driedimensionale Newton-Cartan supergravitatie. Een off-shell formulering van deze theorie is recentelijk gevonden in [118], waarmee vervolgens een veldentheorie verkregen kan worden op een achtergrond die (deels) supersymmetrie behoudt. Op deze symmetrische achtergronden kunnen partitiefuncties worden berekend, die vergeleken kunnen worden met de holografische berekening. Dit biedt mogelijk een interessante manier om niet-relativistische holografie een steviger fundament te geven.

Los van alle holografische toepassingen blijft de theorie van Newton-Cartan op zichzelf een interessant onderwerp. Het dwingt je om na te denken over fundamentele eigenschappen van (uitbreidingen van) de algemene relativiteitstheorie, zoals algemene covariantie, supersymmetrie en de niet-relativistische (en Newtonse) limiet. Of, om het in goed Nederlands te zeggen: *back to the basics*. Het feit dat een eeuw na de publicatie van zowel de algemene relativiteitstheorie als de Newton-Cartan formulering hierover nog zoveel valt te ontdekken, toont des te meer aan hoe subtiel Einsteins theorie is.

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