University of Groningen

Ten-dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields

Bergshoeff, E.; Roo, M. de; Wit, B. de

Published in:
Nuclear Physics B

DOI:
10.1016/0550-3213(82)90050-5

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date:
1982

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):
Bergshoeff, E., Roo, M. D., & Wit, B. D. (1982). Ten-dimensional Maxwell-Einstein supergravity, its currents, and the issue of its auxiliary fields. *Nuclear Physics B*, *195*(1). https://doi.org/10.1016/0550-3213(82)90050-5

Copyright
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment.

Take-down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Download date: 21-04-2025

Nuclear Physics B195 (1982) 97–136
© North-Holland Publishing Company

TEN-DIMENSIONAL MAXWELL-EINSTEIN SUPERGRAVITY, ITS CURRENTS, AND THE ISSUE OF ITS AUXILIARY FIELDS

E. BERGSHOEFF

Instituut-Lorentz, Nieuwsteeg 18, 2311 SB Leiden, The Netherlands

M. DE ROO

Instituut voor Theoretische Natuurkunde, P.O. Box 800, 9700 AV Groningen, The Netherlands

B. DE WIT and P. VAN NIEUWENHUIZEN[1]

NIKHEF-H, P.O. Box 41882, 1009 DB Amsterdam, The Netherlands

Received 1 September 1981

The $d = 10, N = 1$ Yang-Mills system is coupled to $d = 10, N = 1$ supergravity in a locally scale-invariant way. An analysis of the currents agrees with the Noether coupling results and reveals the existence of two ordinary axial and more low-dimension auxiliary fields. The coupling of the photon A_μ to antisymmetric tensors $A_{\mu\nu}$ is consistent because the Maxwell transformation $\delta A_\mu = \partial_\mu \Lambda$ is extended to $\delta A_{\mu\nu} = \kappa \Lambda F_{\mu\nu}$.

1. Introduction

A central problem in supergravity is to find sets of auxiliary fields which, when added to the physical fields, lead to a closed gauge algebra*. Only when one has a closed gauge algebra, has one an off-shell representation of the local symmetry group; without auxiliary fields the gauge algebra closes only on-shell, and one has a representation in terms of states, not of fields. Supergravity theories are labelled by a parameter N which runs from zero (Einstein gravity) to eight, and which counts the number of gravitini (spin $\frac{3}{2}$ fields) in the theory. No models with $N > 8$ are possible because they would contain fields with spin $J > 2$, whereas it seems that one cannot couple such fields in a consistent way to gravity [2]. Only for $N = 1$ ordinary [3] (sometimes called Poincaré) and $N = 1$ conformal [4] supergravity, for $N = 2$ ordinary and conformal supergravity [5], and recently for $N \leqslant 4$ conformal supergravity [6], have auxiliary fields been found.

[1] On leave from the Institute for Theoretical Physics, State University of New York at Stony Brook, NY 11794, USA.

* For a general introduction see ref. [1].

The issue of auxiliary fields has two separate aspects. First of all, one wants to find a set of fields with a closed gauge algebra; in other words, an off-shell representation of the local algebra. Secondly, one wants to construct actions for such a field representation, which are invariant under the gauge transformations of the algebra. This paper deals with the first aspect, but one should realize that a solution of the representation problem does not imply the existence of meaningful actions.

In this article we will study the auxiliary field problem by leaving $d=4$ dimensions and by considering supergravity models in as high a dimension as possible. Our motivation is that the higher in d one goes, the simpler the model becomes. Our methods will need the coupling of a supersymmetric matter system to supergravity. Since beyond $d=10$ no matter exists [7] (only an $N=1$ gauge action exists in $d=11$ [8]), whereas in $d=10$ only $N=1$ supersymmetric Yang-Mills matter exists [9] (for matter systems N counts the number of global supersymmetries), we are led to study the coupling to the $N=1, d=10$ gauge action. In $d=10$ the Dirac matrices are 32×32 and spinors can satisfy both a reality (Majorana) condition and a chirality (Weyl) condition. In this way every spinor in $d=10$ decomposes into four spinors in $d=4$. Therefore, the one gravitino of the $N=1, d=10$ model leads to four gravitini in $d=4$. Thus our work is expected to give information about the auxiliary fields of ordinary (non-conformal) $N=4$ supergravity. If $N=1, d=11$ or $N=2, d=10$ matter existed, we could have gone all the way, and dealt with the $N=8, d=4$ model by studying the $N=1, d=11$ model. In principle the $d=10, N=1$ model is equivalent to the $d=4, N=4$ model coupled to matter. Besides algebraic simplicity, the reason we go up in dimensions as much as possible is that we obtain a multiplet which is larger than the set of fields of $d=4, N=4$ conformal supergravity.

There is no royal road to auxiliary fields as yet. Rather, several logically independent approaches can give insight. We will consider two such approaches. First of all, in the coupling of matter systems to supergravity, both the action and the transformation rules of the gauge fields (modified due to the presence of matter) and the transformation rules of the matter fields contain certain combinations of fields, which can be replaced by auxiliary fields in such a way that an impressive simplification occurs. In this way, the $d=4$ Maxwell-Einstein supergravity system revealed the existence of an axial auxiliary field [10]. Similarly, we will find hints of two such axial auxiliary fields in $d=10$ dimensions. A second source of information about auxiliary fields is given by that multiplet of matter currents which contains the energy-momentum tensor $\theta_{\mu\nu}$ and the supersymmetry current J_μ. Associating a field with every current, one finds a multiplet of fields with a closed gauge algebra [11]. For example, in $d=4$ dimensions, the improved currents of the spin $(1,\frac{1}{2})$ supersymmetric Maxwell system thus yield the multiplet of conformal $N=1$ fields $(e_\mu{}^m, \psi_\mu, A_\mu)$, while a set of non-improved currents may lead to the minimal multiplet of ordinary $N=1$ supergravity fields $(e_\mu{}^m, \psi_\mu, A_\mu, S, P)$. Similarly, one has found the multiplet of $N\leqslant 4$ conformal supergravity from that multiplet of currents which contains the improved stress tensor [6]. More general multiplets of currents not containing $\theta_{\mu\nu}$

have recently been constructed [12], but we will consider here only the most interesting multiplet, viz., the one containing $\theta_{\mu\nu}$. Finally, it is possible to start from a supergravity theory with more local symmetries than ordinary supergravity. In practice this is always conformal supergravity, and after coupling superconformal matter to the fields of the superconformal gauge algebra one recovers ordinary supergravity with a closed gauge algebra by eliminating the extra symmetries by fixing certain fields [13, 14]. The multiplet of ordinary supergravity fields thus found is larger than that of conformal supergravity; it is, in fact, equivalent to a reducible conformal multiplet.

Part of our work is the $d = 10$ counterpart of the analysis of Howe and Lindström [15], who considered the multiplet of currents in $d = 5$ which is obtained from the $d = 5, N = 4$ globally supersymmetric Yang-Mills system. These authors found the surprising but somewhat disappointing result that the multiplet of currents leads to a multiplet of gauge fields which, when reduced to $d = 4$, coincides with the multiplet of $N = 4$ conformal supergravity and not of $N = 4$ ordinary supergravity. This result is somewhat puzzling since there are arguments [16] that in $d = 5$ no conformal supergravity exists. Perhaps the $d = 5$ model has more symmetries than ordinary $d = 5$ supergravity (although these extra symmetries are not of the conformal type), such that one finds a smaller irreducible multiplet for this larger symmetry group. We will find in $d = 10$ a larger multiplet of currents than they found in $d = 5$ and it is possible that our multiplet decomposes upon dimensional reduction into a number of irreducible multiplets, one of which is theirs. They find in $d = 5$ that all fields associated with currents have acceptable dimensions, whereas we will find in $d = 10$ that we get currents with many derivatives, and hence fields with disturbingly low dimensions. It seems probable that our multiplet is a Poincaré multiplet, since in $d = 10$ no simple superconformal algebra seems to exist [16], but we will find extremely interesting indications of a hidden superconformal symmetry in $d = 10$.

As soon as one considers the problem of how to couple the $N = 1, d = 10$ (A_μ, χ) Yang-Mills multiplet to the physical gauge fields $(e_\mu{}^m, \psi_\mu, A_{\mu\nu}, \lambda, \phi)$ of the $N = 1, d = 10$ gauge action, some interesting questions arise. For instance, will this coupling shed new light on the well-known problem of how to couple antisymmetric tensor fields in a consistent way to fields other than gravity [17]? As we shall see, a mechanism is found, which is made possible by the modification of the Maxwell law $\delta A_\mu = \partial_\mu \Lambda, \delta A_{\mu\nu} = 0$, into $\delta A_\mu = \partial_\mu \Lambda, \delta A_{\mu\nu} = \kappa\Lambda(\partial_\mu A_\nu - \partial_\nu A_\mu)$. This modification has been found before in a different context by Nicolai and Townsend [18]. A second exciting aspect is that of local scale invariance. In ordinary gravity, the Maxwell action is only Weyl invariant in $d = 4$, but in supergravity this action is modified by scalar fields ϕ (in a kind of Brans-Dicke way), and by defining Weyl transformations of ϕ appropriately, one can make the Maxwell action Weyl invariant in $d = 10$. The question arises whether one can extend this to a full invariance of the whole action, and to superconformal invariance.

The article is organized as follows. In sect. 2 we dimensionally reduce the

$N=1, d=11$ model to $d=10$, and truncate it to the $N=1, d=10$ gauge action. This has been done before by Chamseddine [19], but since our results disagree with his in the Noether coupling, we give full details. The four-fermion terms are found by requiring supercovariance of the fermion field equations. In sect. 3 we couple this gauge action to the $N=1, d=10$ Yang-Mills system. It is here that the modified Maxwell invariance is discussed. In sect. 4 we construct the gauge algebra. In sect. 5 we study the multiplet of currents, and find the corresponding superfield. In sect. 6, we compare the results of the matter coupling approach and the current multiplet approach. Here we also present our results on Weyl invariance in $d=10$. The conclusions are given in sect. 7, while a number of algebraic and differential identities in $d=10$ are derived in the appendix.

2. The $N=1, d=10$ supergravity theory from $d=11$ dimensions

2.1. THE ACTION

In $d=11$, the physical supergravity fields are the elfbein $E_\mu{}^m$, one gravitino $\Psi_\mu{}^a$ ($a=1,\ldots,32$), and an antisymmetric tensor $A_{\mu\nu\rho}$. The gravitino is a Majorana spinor

$$\overline{\Psi}_\mu = \Psi_\mu{}^{\mathrm{T}} C, \qquad C\Gamma_\mu C^{-1} = -\Gamma_\mu{}^{\mathrm{T}}, \qquad 1 \leqslant \mu \leqslant 11. \tag{2.1}$$

Since all fields are gauge fields, a dimensional argument [10] states that the action must be polynomial in all fields except $E_\mu{}^m$. Putting $\kappa=1$ always, the $d=11$ action reads

$$\begin{aligned}
\mathfrak{L}(d=11) &= -\tfrac{1}{2}ER(E,\Omega) - \tfrac{1}{2}E\overline{\Psi}_\mu \Gamma^{\mu\rho\sigma} D_\rho\left(\frac{\Omega+\hat{\Omega}}{2}\right)\Psi_\sigma - \tfrac{1}{48}EF^2_{\mu\nu\rho\sigma} \\
&\quad - \tfrac{1}{384}\sqrt{2}\,E\left(\overline{\Psi}_\mu \Gamma^{\mu\alpha\beta\gamma\delta\nu}\Psi_\nu + 12\overline{\Psi}^\alpha \Gamma^{\beta\gamma}\Psi^\delta\right)(F+\hat{F})_{\alpha\beta\gamma\delta} \\
&\quad - \frac{1}{36\times 96} i\sqrt{2}\,\varepsilon^{\mu_1\cdots\mu_{11}} F_{\mu_1\cdots\mu_4} F_{\mu_5\cdots\mu_8} A_{\mu_9\cdots\mu_{11}}, \\
F_{\mu\nu\rho\sigma} &= \partial_\mu A_{\nu\rho\sigma} + 23 \text{ terms}.
\end{aligned} \tag{2.2}$$

The hats denote supercovariantization, $\Gamma^{\mu\nu\rho} = \Gamma^{[\mu}\Gamma^\nu\Gamma^{\rho]}$ with strength one; $\hat{\Omega}_{\mu mn}$ is the usual supercovariantization of $\Omega_{\mu mn}(E)$, but $\Omega_{\mu mn}(E,\Psi)$ is the solution of the Ω field equation, which differs from $\hat{\Omega}$:

$$\Omega_{\mu mn} = \hat{\Omega}_{\mu mn} - \tfrac{1}{8}\overline{\Psi}^\alpha \Gamma_{\alpha\mu mn\beta}\Psi^\beta. \tag{2.3}$$

The transformation rules read

$$\delta E_\mu{}^m = \tfrac{1}{2}\bar{\varepsilon}\Gamma^m\Psi_\mu, \qquad \delta A_{\mu\nu\rho} = -\tfrac{1}{8}\sqrt{2}\,\bar{\varepsilon}\Gamma_{[\mu\nu}\Psi_{\rho]},$$

$$\delta\Psi_\mu = D_\mu(\hat{\Omega})\varepsilon + \tfrac{1}{288}\sqrt{2}\left(\Gamma_\mu{}^{\alpha\beta\gamma\delta} - 8\delta_\mu{}^\alpha\Gamma^{\beta\gamma\delta}\right)\varepsilon\hat{F}_{\alpha\beta\gamma\delta}. \tag{2.4}$$

The gravitino law is clearly supercovariant, which is a reflection of the fact that in $d=11$ one finds in the $[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]$ commutator only δ_Q terms with $\delta_Q(-\xi^\mu\Psi_\mu)$, where $\xi^\mu = \frac{1}{2}\bar{\varepsilon}_2\Gamma^\mu\varepsilon_1$. In $d=10$ we will find extra $\bar{\psi}\lambda\varepsilon$ terms in $\delta\psi_\mu$, and this leads to extra structure functions in the gauge algebra, just as in the $d=4$, $N\geqslant 3$ models (see for instance [1], sect. 6).

Dimensional reduction of the $N=1, d=11$ model leads to the $N=2, d=10$ model, which might yield the $N=1, d=10$ Maxwell-Einstein system if one finds the appropriate truncation. We will truncate further down, to the $N=1, d=10$ gauge action. This truncation is achieved by putting

$$E_{\hat{\mu}}{}^{\hat{m}} = \begin{pmatrix} E_\mu{}^m & 0 \\ 0 & E_{\dot{11}}{}^{11} \end{pmatrix}, \qquad A_{\mu\nu\rho} = 0,$$

$$\Psi_\mu{}^{\mathrm{R}} \equiv \tfrac{1}{2}(1-\Gamma_{11})\Psi_\mu = 0, \qquad \Psi_{\dot{11}}{}^{\mathrm{L}} \equiv \tfrac{1}{2}(1+\Gamma_{11})\Psi_{\dot{11}} = 0. \tag{2.5}$$

In (2.5) and from now on, 11-dimensional indices carry a hat, while a dot on top of an index indicates that this index is curved [19]. Hence, $\Gamma_{\dot{11}} = E_{\dot{11}}{}^{11}\Gamma_{11}$. From $\delta\Psi_\mu{}^{\mathrm{R}} = \partial_\mu\varepsilon^{\mathrm{R}} + \cdots$ it follows that for consistency also $\varepsilon^{\mathrm{R}} = 0$. Hence, $\delta E_{\dot{11}}{}^m = \frac{1}{2}\bar{\varepsilon}\Gamma^m\Psi_{\dot{11}} = 0$, while also $\delta E_\mu{}^{11} = \frac{1}{2}\bar{\varepsilon}\Gamma^{11}\Psi_\mu = \frac{1}{2}\bar{\varepsilon}\Psi_\mu = 0$. This means that no compensating local Lorentz rotation needs to be added to (2.4) to maintain the form of the vielbein in (2.5). The consistency of (2.5) follows further from

$$\hat{\Omega}_{\dot{11}mn} = \hat{\Omega}_{\mu 11 n} = F_{\alpha\beta\gamma\delta} = \hat{F}_{\alpha\beta\gamma\delta} = 0. \tag{2.6}$$

After dimensional reduction to $d=10$ and truncation as in (2.5), the kinetic terms are cast in canonical form by a suitable Weyl rescaling of the zehnbein and field redefinitions of the other fields. These follow easily from (2.2). The Einstein action has an extra factor $E_{\dot{11}}{}^{11}$ due to E, so that if we redefine $E_\mu{}^m = e_\mu{}^m\phi^\gamma$, with $\phi = E_{\dot{11}}{}^{11}$, the ϕ factors in the leading part cancel in d dimensions if $\gamma = -(d-2)^{-1}$ (in our case $d=10$):

$$E_{\hat{\mu}}{}^{\hat{m}} = \begin{pmatrix} \phi^{-(d-2)^{-1}} e_\mu{}^m & 0 \\ 0 & \phi \end{pmatrix}. \tag{2.7}$$

The ϕ-dependent terms in the Einstein action are only due to the torsion terms in Ω,

$$\Omega_{\mu mn}(E) = \omega_{\mu mn}(e) - \frac{1}{d-2}\left(e_{m\mu}e_n{}^\nu - e_{n\mu}e_m{}^\nu\right)\partial_\nu\phi/\phi, \tag{2.8}$$

and can be written down easily if one uses the well-known Palatini identity (always under the integral sign)

$$-\tfrac{1}{2}eR(\omega(e)+\tau)=-\tfrac{1}{2}eR(\omega(e))+\tfrac{1}{2}e\left(\tau^{mcn}\tau_{ncm}-\tau^{2}_{mmc}\right). \tag{2.9}$$

The contributions coming from

$$\Omega_{\dot{1}11n}=\phi^{1/(d-2)}e_n{}^{\mu}\partial_\mu\phi \tag{2.10}$$

yield a vanishing result since in terms of the zehnbein $E_\mu{}^m$ one finds

$$EE_n{}^{\mu}D_\mu(\Omega(E))\Omega_{\dot{1}11n}, \tag{2.11}$$

which is a total derivative. The final result is that one finds the canonical Einstein action plus a physical scalar,

$$-\tfrac{1}{2}ER(E,\Omega(E))=-\tfrac{1}{2}eR(e,\omega(e))-\tfrac{1}{2}e\left(\frac{d-1}{d-2}\right)\left(\frac{\partial_\mu\phi}{\phi}\right)^2. \tag{2.12}$$

That the scalar field ϕ has the correct sign for its kinetic term can be understood by noting that in the locally scale-invariant action

$$-e'R(e')=-e\phi^2R(e)+4e\frac{(d-1)}{(d-2)}(\partial_\mu\phi)^2,$$

the improvement scalar is a ghost, but that we use, rather, the reverse relation

$$-e\phi^2R(e)=-e'R(e')-4e\frac{(d-1)}{(d-2)}(\partial_\mu\phi)^2$$

with e on the r.h.s. expressed in terms of e' and ϕ.

Next we consider the Rarita-Schwinger action. In the leading part the factors of ϕ due to the rescaling of the zehnbein in (2.7) are removed by rescaling ψ_μ by a factor $\phi^{-[2(d-2)]^{-1}}=\phi^{-1/16}$. No $\partial_\mu\phi$ terms are introduced by the scaling because the gravitino is a Majorana spinor, while the $\partial_\mu\phi$ terms due to the spin connection cancel. There are also cross terms between ψ_μ and $\psi_{\dot{1}1}$ but no terms quadratic in $\psi_{\dot{1}1}$. One diagonalizes the spinor action by shifting $\psi_\mu\to\psi_\mu+\frac{1}{8}\Gamma_\mu\psi_{\dot{1}1}$, and normalizes the resulting part quadratic in $\psi_{\dot{1}1}$ by rescaling $\psi_{\dot{1}1}$ which does not introduce $\partial_\mu\phi$ terms for the same reasons as above. We do not shift $\psi_{\dot{1}1}\to\psi_{\dot{1}1}+A\Gamma\cdot\psi$, because we wish

to avoid a $\not{D}\varepsilon$ term in $\delta\psi_{\dot{1}1}$. In this way one arrives at the following result:

$$-\tfrac{1}{2}E\bar{\Psi}_\mu\Gamma^{\mu\rho\sigma}D_\rho(\Omega(E))\Psi_\sigma = -\tfrac{1}{2}e\bar{\psi}_\mu\Gamma^{\mu\rho\sigma}D_\rho(\omega(e))\psi_\sigma$$

$$-\tfrac{1}{2}e\bar{\lambda}\Gamma^\mu D_\mu(\omega(e))\lambda - \tfrac{3}{8}\sqrt{2}\,e\bar{\psi}_\mu\frac{\not{\partial}\phi}{\phi}\Gamma^\mu\lambda,$$

$$\Psi_\mu^{\mathrm{L}} = \phi^{-1/16}\left(\psi_\mu + \tfrac{1}{12}\sqrt{2}\,\Gamma_\mu\lambda\right), \qquad \Psi_{\dot{1}1}^{\mathrm{R}} = \tfrac{2}{3}\sqrt{2}\,\phi^{17/16}\lambda. \tag{2.13}$$

The term with $\not{\partial}\phi$ is due to (2.8) and the rescaling of $\psi_{\dot{1}1}$.

Let us now turn to the photon kinetic term. Defining

$$A_{\mu\nu} = 6A_{\mu\nu\dot{1}1}, \qquad F_{\mu\nu\rho} = \partial_{[\mu}A_{\nu\rho]}, \tag{2.14}$$

one finds easily its reduction. In addition to (2.6) one has

$$\hat{F}_{\alpha\beta\gamma\dot{1}1} = 3\hat{F}_{\alpha\beta\gamma} - \tfrac{7}{96}\sqrt{2}\,\phi^{3/4}\bar{\lambda}\Gamma_{\alpha\beta\gamma}\lambda,$$

$$\hat{F}_{\alpha\beta\gamma} = F_{\alpha\beta\gamma} - \tfrac{1}{4}\sqrt{2}\,\phi^{3/4}\bar{\psi}_{[\alpha}\Gamma_\beta\psi_{\gamma]} + \tfrac{1}{4}\phi^{3/4}\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma]}\lambda. \tag{2.15}$$

The coupling involving $F_{\mu\nu\rho}$ does not contain $F\lambda^2$, but only $F\psi^2$ and $F\psi\lambda$ terms. This we will discuss in subsect. 2.3.

The action, except the four-fermion couplings which will be given in (2.28), reads

$$\mathfrak{L}(N=1, d=10) = -\tfrac{1}{2}eR(e,\omega(e)) - \tfrac{1}{2}e\bar{\psi}_\mu\Gamma^{\mu\rho\sigma}D_\rho(\omega(e))\psi_\sigma - \tfrac{3}{4}e\phi^{-3/2}F^2_{\mu\nu\rho}$$

$$-\tfrac{1}{2}e\bar{\lambda}\Gamma^\mu D_\mu(\omega(e))\lambda - \tfrac{9}{16}e\left(\partial_\mu\phi/\phi\right)^2 - \tfrac{3}{8}\sqrt{2}\,e\bar{\psi}_\mu(\not{\partial}\phi/\phi)\Gamma^\mu\lambda$$

$$+\tfrac{1}{16}\sqrt{2}\,e\phi^{-3/4}F_{\alpha\beta\gamma}\left(\bar{\psi}_\mu\Gamma^{\mu\alpha\beta\gamma\nu}\psi_\nu + 6\bar{\psi}^\alpha\Gamma^\beta\psi^\gamma - \sqrt{2}\,\bar{\psi}_\mu\Gamma^{\alpha\beta\gamma}\Gamma^\mu\lambda\right). \tag{2.16}$$

Most of these results agree with [19], but instead of the Noether coupling $\bar{\psi}_\mu\not{\partial}\phi\Gamma^\mu\lambda$, a coupling $\bar{\psi}_\mu\partial^\mu\phi\lambda$ appears in [19].

2.2. THE TRANSFORMATION LAWS

The action has been obtained in canonical form on the basis $(e_\mu{}^m, \psi_\mu, \lambda, A_{\mu\nu}, \phi)$, but the transformation rules of $e_\mu{}^m$ and the $\delta\psi_\mu = \partial_\mu\varepsilon$ part are no longer canonical. We remedy this by redefining the supersymmetry parameter and by adding a

field-dependent local Lorentz rotation to the supersymmetry transformations:

$$\delta_Q(\eta, d=10) = \delta_Q(\varepsilon, d=11) + \delta_L\left(-\tfrac{1}{24}\sqrt{2}\,\bar{\eta}\Gamma^{mn}\lambda\right),$$

$$\eta = \varepsilon\phi^{1/16}. \tag{2.17}$$

In this way one finds

$$\delta e_\mu{}^m = \tfrac{1}{2}\bar{\eta}\Gamma^m\psi_\mu. \tag{2.18}$$

It is straightforward to find the transformation rule of the fields ϕ and $A_{\mu\nu}$. As a prelude to $\delta\psi_\mu$ and $\delta\lambda$, we reduce the supercovariant spin connections $\hat{\Omega}$. The result is

$$\hat{\Omega}_{\dot{1}11n} = \phi^{1/8}\hat{D}_n\phi, \qquad \hat{\Omega}_{\mu 11n} = \hat{\Omega}_{\dot{1}1mn} = 0,$$

$$\hat{\Omega}_{\mu mn} = \hat{\omega}_{\mu mn}(e,\psi) + \tfrac{1}{24}\sqrt{2}\,\bar{\psi}_\mu\Gamma_{mn}\lambda$$

$$-\tfrac{1}{288}\bar{\lambda}\Gamma_{\mu mn}\lambda - \tfrac{1}{8}\left(e_{m\mu}\hat{D}_n\phi - e_{n\mu}\hat{D}_m\phi\right)\phi^{-1}. \tag{2.19}$$

The presence of $\bar{\psi}_\mu\Gamma_{mn}\lambda$ terms in $\hat{\Omega}_{\mu mn}$ does not mean, of course, that an 11-dimensional supercovariant tensor would not be supercovariant in $d=10$. Rather, undoing the Lorentz rotation in (2.17) the $\partial_\mu\varepsilon$ terms from $\delta_L\hat{\omega}_{\mu mn}$ cancel those from $\delta\psi_\mu \to \partial_\mu\varepsilon$. Notice that this subtlety can only occur for quantities that are not covariant under local Lorentz transformations. This explains why $F_{\mu\nu\rho\dot{1}1}$ in (2.15) is supercovariant.

It is now obvious how to obtain $\delta\psi_\mu$ and $\delta\lambda$. We start from (2.4), and replace ε by η, and $(\Psi_\mu, \Psi_{\dot{1}1})$ by (ψ_μ, λ), and use (2.19). For $\delta\lambda$ we find a remarkable result: all $\lambda^2\varepsilon$ terms cancel. In $\delta\psi_\mu$ we do find $\lambda^2\varepsilon$ terms; we will explain these results in the next subsection.

The final result for the transformation rules reads:

$$\delta e_\mu{}^m = \tfrac{1}{2}\bar{\eta}\Gamma^m\psi_\mu, \qquad \delta\phi = -\tfrac{1}{3}\sqrt{2}\,\bar{\eta}\lambda\phi,$$

$$\delta A_{\mu\nu} = \tfrac{1}{4}\sqrt{2}\,\phi^{3/4}\left(\bar{\eta}\Gamma_\mu\psi_\nu - \bar{\eta}\Gamma_\nu\psi_\mu - \tfrac{1}{2}\sqrt{2}\,\bar{\eta}\Gamma_{\mu\nu}\lambda\right),$$

$$\delta\lambda = -\tfrac{3}{8}\sqrt{2}\,(\hat{\not{D}}\phi/\phi)\eta + \tfrac{1}{8}\phi^{-3/4}\Gamma^{\alpha\beta\gamma}\eta\hat{F}_{\alpha\beta\gamma},$$

$$\delta\psi_\mu = D_\mu(\hat{\omega}(e,\psi))\eta + \tfrac{1}{32}\sqrt{2}\,\phi^{-3/4}\left(\Gamma_\mu{}^{\alpha\beta\gamma} - 9\delta_\mu^\alpha\Gamma^{\beta\gamma}\right)\eta\hat{F}_{\alpha\beta\gamma}$$

$$-\frac{1}{16\times 32}\left(\Gamma_\mu{}^{\alpha\beta\gamma} - 5\delta_\mu^\alpha\Gamma^{\beta\gamma}\right)\eta\bar{\lambda}\Gamma_{\alpha\beta\gamma}\lambda$$

$$+\tfrac{1}{96}\sqrt{2}\left[\left(\bar{\psi}_\mu\Gamma_{mn}\lambda\right)\Gamma^{mn}\eta + \left(\bar{\lambda}\Gamma_{mn}\eta\right)\Gamma^{mn}\psi_\mu\right.$$

$$\left. +2\left(\bar{\psi}_\mu\lambda\right)\eta - 2\left(\bar{\lambda}\eta\right)\psi_\mu + 4\left(\bar{\psi}_\mu\Gamma_m\eta\right)\Gamma^m\lambda\right]. \tag{2.20}$$

2.3. THE FOUR-FERMION COUPLING

The action in (2.16) was complete up to four-fermion couplings $I^{(4)}$ (as in $d=4$, a dimensional argument [10] shows that there cannot be six or more fermion couplings). We find $I^{(4)}$ by requiring that the ψ_μ and λ field equations be supercovariant. Let us review this argument. Gauge invariance of the action implies $(\delta I/\delta\phi^j)(\delta\phi^j) \equiv 0$; hence, under a second gauge variation δ' one has

$$\left[\delta'(\delta I/\delta\phi^j)\right]\delta\phi^j = -(\delta I/\delta\phi^j)\left[\delta'(\delta\phi^j)\right]. \tag{2.21}$$

This equation makes it plausible that the variation δ' of a field equation must be a sum of field equations. Since bosonic field equations need two derivatives in the leading term, no $\partial\varepsilon$ can appear in the variation of a fermionic field equation. Therefore, spinor field equations should be supercovariant. Notice that this argument is only valid in the absence of auxiliary fields.

The λ field equation is obtained from (2.16). Using left-derivatives, it reads

$$\begin{aligned}\delta I/\delta\bar\lambda = &-\not{D}(\omega(e))\lambda - \tfrac{3}{8}\sqrt{2}\,\Gamma^\mu\Gamma^\nu\psi_\mu(\partial_\nu\phi/\phi)\\ &+\tfrac{1}{8}\Gamma^\mu\Gamma^{\alpha\beta\gamma}\psi_\mu F_{\alpha\beta\gamma}\phi^{-3/4} + \delta I^{(4)}/\delta\bar\lambda.\end{aligned} \tag{2.22}$$

Using (2.20), the $\partial\eta$ term from the variation of $\partial_\nu\phi$ and the $\partial\eta$ term from the variation $\delta\lambda \sim (\hat D_\nu\phi - \partial_\nu\phi)\eta$ are the same as those obtained by varying a particular $\bar\psi\psi\lambda$ term which we will write down in a moment. Similarly the variation of $F_{\alpha\beta\gamma}$ and $\delta\lambda \sim (\hat F - F)\eta$ determine other $\bar\psi\psi\lambda$ and $\bar\psi\psi\psi$ terms. Finally, we know how to covariantize $\omega(e)$. Thus we find at once

$$\begin{aligned}\delta I^{(4)}/\delta\bar\lambda = &-\tfrac{1}{4}\Gamma^\mu\Gamma^{mn}\lambda\left(\hat\omega_{\mu mn}(e,\psi) - \omega_{\mu mn}(e)\right) - \tfrac{1}{4}\Gamma^\mu\Gamma^\nu\psi_\mu\left(\bar\psi_\nu\lambda\right)\\ &+\tfrac{1}{32}\Gamma^\mu\Gamma^{\alpha\beta\gamma}\psi_\mu\left(\bar\psi_\alpha\Gamma_{\beta\gamma}\lambda\right) - \tfrac{1}{32}\sqrt{2}\,\Gamma^\mu\Gamma^{\alpha\beta\gamma}\psi_\mu\left(\bar\psi_\alpha\Gamma_\beta\psi_\gamma\right).\end{aligned} \tag{2.23}$$

In other words, the variations of $\delta(I - I^{(4)})/\delta\bar\lambda$ occur in pairs in such a way that we can by inspection add 3-fermion terms which covariantize $\delta I/\delta\bar\lambda$ and hence must coincide with $\delta I^{(4)}/\delta\bar\lambda$. As shown in the appendix, there cannot be λ^4 couplings, hence all terms in $\delta I^{(4)}/\delta\bar\lambda$ contain ψ's and can be found by covariantizing the λ field equation.

We must now integrate $\delta I^{(4)}/\delta\bar\lambda$. The terms without λ are simply integrated by multiplication with $\bar\lambda$, but the terms linear in λ must have come from terms in $I^{(4)}$ proportional to $(\bar\lambda\Gamma^{\alpha\beta\gamma}\lambda)$. Hence, by Fierz reordering terms such that they are of the form $(\bar\psi\psi)(\Gamma^{(n)}\lambda)$, the terms with $\Gamma^{(1)}$ and $\Gamma^{(5)}$ should cancel, after which one integrates by replacing $\Gamma^{\alpha\beta\gamma}\lambda$ by $\frac{1}{2}(\bar\lambda\Gamma^{\alpha\beta\gamma}\lambda)$. The $\psi^2\lambda$ terms from the $(\hat\omega - \omega)$ terms in (2.23) are essential to cancel the $\Gamma^{(1)}$ and $\Gamma^{(5)}$ terms, and one finds the $\psi^2\lambda^2$ and $\psi^3\lambda$ terms in $I^{(4)}$ in this way.

There are no $\lambda^3\psi$ couplings in the action. Indeed, the only candidates are $(\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda)(\bar{\psi}_\alpha\Gamma_{\beta\gamma}\lambda)$ and $(\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda)(\bar{\psi}_\mu\Gamma^\mu\Gamma^{\alpha\beta\gamma}\lambda)$, which vanish because of the identity (A.4) in the appendix. Therefore, the λ field equation does not contain $\lambda^2\psi_\mu$ contributions. Covariance of the field equation then implies that the supersymmetry variation of $\delta(I-I^{(4)})/\delta\bar{\lambda}$ does not contain $\lambda^2\partial_\mu\eta$ terms. Such terms could have come from $\delta\lambda\sim\lambda^2\eta$, and from a possible λF term in the field equations. One easily checks that these two sources lead to independent contributions, which must therefore both vanish. This proves, as promised, the absence of a $\delta\lambda\sim\lambda^2\eta$ variation, and of a $\lambda^2 F$ term in the action.

The ψ^4 couplings follow from the supercovariantization of the gravitino field equation. The gravitino field equation follows from (2.16) and reads

$$\delta I/\delta\bar{\psi}_\mu = -\Gamma^{\mu\rho\sigma}D_\rho(\omega(e))\psi_\sigma + \tfrac{1}{8}\sqrt{2}\,\phi^{-3/4}F_{\alpha\beta\gamma}\left(\Gamma^{\mu\alpha\beta\gamma\nu}\psi_\nu + 6g^{\mu\alpha}\Gamma^\beta\psi^\gamma\right)$$
$$-\tfrac{3}{8}\sqrt{2}\,(\partial\!\!\!/\phi/\phi)\Gamma^\mu\lambda - \tfrac{1}{8}\Gamma^{\alpha\beta\gamma}\Gamma^\mu\lambda F_{\alpha\beta\gamma}\phi^{-3/4} + \delta I^{(4)}/\delta\bar{\psi}_\mu. \qquad (2.24)$$

Since we are only interested in the remaining ψ^4 terms, we note that variation of $\delta I^{(4)}/\delta\bar{\psi}_\mu\sim\psi^3$ terms leads to $\psi^2\partial\eta$ terms. Hence we collect these variations only. They come from $\delta\psi\sim(\hat{F}-F)\eta$ and δF, and from $\delta\omega(e)$. Covariantization of the ψ_μ field equation thus requires

$$\delta I^{(4)}(\psi^4\text{ terms})/\delta\bar{\psi}_\mu = -\tfrac{1}{4}\Gamma^{\mu\rho\sigma}\Gamma^{mn}\psi_\sigma\left(\hat{\omega}_{\rho mn}(e,\psi) - \omega_{\rho mn}(e)\right)$$
$$-\tfrac{1}{16}\left(\Gamma^{\mu\alpha\beta\gamma\nu}\psi_\nu + 6g^{\mu\alpha}\Gamma^\beta\psi^\gamma\right)\left(\bar{\psi}_{[\alpha}\Gamma_\beta\psi_{\gamma]}\right). \qquad (2.25)$$

Integration of this equation is difficult, since in this case we cannot use the fact that only a particular combination of fields (like $\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda$) can occur in $I^{(4)}$. Rather, we will derive first $I^{(4)}$ from $d=11$ by dimensional reduction, and then see how we should have integrated (2.25). The ψ^4 terms are easily deduced from the action in $d=11$ since according to (2.13) one only needs to replace Ψ_μ by $\phi^{-1/16}\psi_\mu$. In the Noether coupling $\frac{1}{2}(F_{\alpha\beta\gamma\delta}+\hat{F}_{\alpha\beta\gamma\delta})$ goes over into $\frac{3}{2}(F_{\alpha\beta\gamma}+\hat{F}_{\alpha\beta\gamma})$, and these terms yield

$$I^{(4)}(\psi^4\text{ terms in Noether}) = -\tfrac{1}{64}e\bar{\psi}_\rho\Gamma_m\psi_n\left(\bar{\psi}_\alpha\Gamma^{\alpha\rho mn\beta}\psi_\beta + 6\bar{\psi}^{[\rho}\Gamma^m\psi^{n]}\right). \qquad (2.26)$$

The only other ψ^4 terms come from the Einstein action, for which we use (2.9), and from the Rarita-Schwinger action with spin connection $\frac{1}{2}(\omega+\hat{\omega})$. The square of spinor bilinears with $\Gamma^{(5)}$ cancels, and one finds

$$I^{(4)}(\psi^4\text{ terms in E + RS}) = \tfrac{1}{8}e\left(\bar{\psi}\cdot\Gamma\psi_m\right)^2 - \tfrac{1}{32}e\left(\bar{\psi}_\mu\Gamma_\nu\psi_\rho\right)^2 - \tfrac{1}{16}e\left(\bar{\psi}_\mu\Gamma_\nu\psi_\rho\right)\left(\bar{\psi}^\mu\Gamma^\rho\psi^\nu\right)$$
$$-\tfrac{1}{64}e\left(\bar{\psi}_\rho\Gamma_m\psi_n\right)\left(\bar{\psi}_\alpha\Gamma^{\alpha\rho mn\beta}\psi_\beta\right). \qquad (2.27)$$

One can show that the variation of (2.26), (2.27) leads to (2.25) by using (A.13).

The final result is that the four-fermion terms are given by

$$
\begin{aligned}
e^{-1}I^{(4)} = {} & \tfrac{1}{8}e\left(\bar{\psi}\cdot\Gamma\psi_m\right)^2 - \tfrac{1}{32}e\left(\bar{\psi}_\mu\Gamma_\nu\psi_\rho\right)^2 - \tfrac{1}{16}e\left(\bar{\psi}_\mu\Gamma_\nu\psi_\rho\right)\left(\bar{\psi}^\mu\Gamma^\rho\psi^\nu\right) \\
& - \tfrac{1}{64}\left(\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda\right)\left(\tfrac{1}{12}\bar{\psi}^\rho\Gamma_{\alpha\beta\gamma}\psi_\rho + \tfrac{1}{2}\bar{\psi}^\rho\Gamma_{\rho\alpha\beta}\psi_\gamma - \bar{\psi}_\alpha\Gamma_\beta\psi_\gamma\right) \\
& - \tfrac{1}{32}\sqrt{2}\left(\bar{\lambda}\Gamma^\mu\Gamma^{\alpha\beta\gamma}\psi_\mu\right)\left(\bar{\psi}_\alpha\Gamma_\beta\psi_\gamma\right) \\
& - \tfrac{1}{32}\left(\bar{\psi}_\rho\Gamma_m\psi_n\right)\left(\bar{\psi}_\alpha\Gamma^{\alpha\rho mn\beta}\psi_\beta\right) - \tfrac{3}{32}\left(\bar{\psi}_\rho\Gamma_m\psi_n\right)\left(\bar{\psi}^{[\rho}\Gamma^m\psi^{n]}\right). \qquad (2.28)
\end{aligned}
$$

The question now arises whether all these four-fermion terms are supercovariantizations; in other words, whether using $\omega(e,\psi,\lambda)$ in $\mathfrak{L}(\mathrm{E})$, $\frac{1}{2}(\omega+\hat{\omega})$ as spin connection in $\mathfrak{L}(\psi_\mu)$ and $\mathfrak{L}(\lambda)$, and further taking in both Noether couplings $\frac{1}{2}(F+\hat{F})$ and $\frac{1}{2}(\not{\partial}\phi+\hat{\not{D}}\phi)$, accounts for all terms in $I^{(4)}$. Of course, the ψ^4 terms are of this kind, since they were of this kind in $d=11$. The next simplest terms are the $\psi^3\lambda$ terms since they would only come from the Noether terms with $\frac{1}{2}(F+\hat{F})$. This Noether coupling gives

$$
\tfrac{1}{128}\sqrt{2}\left(\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma]}\lambda\right)\left(\bar{\psi}_\mu\Gamma^{\mu\alpha\beta\gamma\nu}\psi_\nu + 6\bar{\psi}^\alpha\Gamma^\beta\psi^\gamma\right) - \tfrac{1}{64}\sqrt{2}\left(\bar{\psi}_\alpha\Gamma_\beta\psi_\gamma\right)\left(\bar{\lambda}\Gamma^\mu\Gamma^{\alpha\beta\gamma}\psi_\mu\right), \qquad (2.29)
$$

and this does not agree with (2.28), as can be seen by again using (A.13). Hence the four-fermion terms cannot be rewritten in terms of the covariantizations mentioned above.

3. Maxwell-Einstein supergravity in $d=10$

In this section we couple the $d=10$ Maxwell system to the $N=1, d=10$ gauge action which we derived in sect. 2. Ten dimensions is thought to be the highest dimension where matter exists.

The $d=10, N=1$ globally supersymmetric Maxwell system reads

$$
\mathfrak{L}^{(0)} = -\tfrac{1}{4}F_{\mu\nu}^2 - \tfrac{1}{2}\bar{\chi}\not{\partial}\chi, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (3.1)
$$

The action is invariant under the following transformation rules:

$$
\delta A_\mu = \tfrac{1}{2}\bar{\varepsilon}\Gamma_\mu\chi, \qquad \delta\chi = -\tfrac{1}{4}\Gamma\cdot F\varepsilon, \qquad \Gamma\cdot F \equiv \Gamma^{mn}F_{mn}. \qquad (3.2)
$$

Although not necessary for the invariance of this free action*, we restrict χ to be chiral in order to have equal numbers of bosons and fermions. From (3.2) it follows that χ and ε must have the same chirality, whereas χ and λ have opposite chirality.

* In the free scalar multiplet in $d=4$ a similar situation exists: one may omit the scalar (or pseudoscalar) without losing invariance of the action.

Since there is a dimensionless scalar field ϕ present in the supergravity action (2.16) we expect non-polynomial behaviour in ϕ. All other fields can only appear polynomially. In fact, since ϕ came from the elfbein, we certainly expect non-polynomial couplings in ϕ. Rather than to find these terms order-by-order in κ, we will work directly to all orders in ϕ. Hence we start from

$$e^{-1}\mathcal{L}^{(0)} = -\tfrac{1}{4}F_{\mu\nu}^2 f^2(\phi) - \tfrac{1}{2}\bar{\chi}\Gamma^\mu D_\mu(\omega(e))\chi,$$

$$\delta A_\mu = \tfrac{1}{2}\bar{\varepsilon}\Gamma_\mu\chi g(\phi), \qquad \delta\chi = -\tfrac{1}{4}\Gamma\cdot F\varepsilon h(\phi). \tag{3.3}$$

A possible function $k(\phi)$ multiplying the Dirac action has been cancelled by rescaling χ. No $\partial_\mu k$ terms are created in this way, since χ is a Majorana spinor, so that $\bar{\chi}\gamma^\mu\chi = 0$.

To order κ^0 the variations cancel if

$$\mathcal{L}^N = -\tfrac{1}{4}e\kappa\bar{\psi}_\mu\Gamma\cdot F\Gamma^\mu\chi h, \qquad h = f^2 g. \tag{3.4}$$

If one requires that the supersymmetry commutator produces the correctly normalized translation on A_μ, one finds $gh = 1$. Therefore, we choose in (3.3) $g = f^{-1}$ and $h = f$. To order κ, there are now new variations proportional to $\partial_\mu f$ (since $f = 1$ in flat space, $\partial_\mu f$ is of order κ) coming from δA_μ and $\delta\chi$ in $\mathcal{L}^{(0)}$:

$$\delta\mathcal{L}^{(0)} = \tfrac{1}{4}e\bar{\chi}\Gamma\cdot F\not{\partial} f\varepsilon. \tag{3.5}$$

These terms cannot be cancelled by using an order κ variation in the order κ^0 part of the action, since they are not proportional to the A_μ or χ equation of motion. Thus, one needs a new term of order κ or κ^2 in the action. We will come back to these variations later, but first repeat the analysis of the $d = 4$ Maxwell-Einstein system in $d = 10$ [20, 1]. The variations we will consider below consist of three groups: the $\kappa F^2\psi\varepsilon$ terms, the $\kappa D_\mu\chi^2\psi\varepsilon$ terms, and many other, rather easy, terms.

We begin with the variations of the form $\kappa F^2\psi\varepsilon$. These come from varying the zehnbein in the Maxwell action and from $\delta\chi$ in the Noether action. The $\Gamma^{(1)}$ terms cancel as in $d = 4$, since the non-polynomial functions in front of the Maxwell and Noether actions are related. In [20] the vanishing of the terms $\kappa\bar{\varepsilon}\Gamma\psi F^2$ was proved by invoking a typical $d = 4$ relation (namely that $F_{\mu\alpha}\tilde{F}_{\nu\alpha}$ is proportional to $g_{\mu\nu}F\cdot\tilde{F}$), but this cancellation holds even without this property, and for that reason it also holds in $d = 10$. The $\Gamma^{(3)}$ terms cancel, as in $d = 4$, because terms like $F_{\mu\alpha}F_{\nu\alpha}\bar{\psi}_\rho\Gamma^{\mu\nu\rho}\varepsilon$ clearly vanish. The $\Gamma^{(5)}$ terms are new in $d = 10$. They only come from $\delta\chi$ in $\mathcal{L}^{(N)}$ and read

$$\delta\mathcal{L} = \tfrac{1}{16}e\kappa F_{\alpha\beta}F_{\rho\sigma}\bar{\psi}_\mu\Gamma^{\alpha\beta\mu\rho\sigma}\varepsilon f^2$$

$$= \tfrac{1}{8}e\kappa A_\beta F_{\rho\sigma}f^2\big((D_\alpha\bar{\varepsilon})\Gamma^{\alpha\beta\mu\rho\sigma}\psi_\mu + \bar{\varepsilon}\Gamma^{\alpha\beta\mu\rho\sigma}D_\alpha\psi_\mu\big) + O(\kappa^2). \tag{3.6}$$

The terms with $D_\alpha\bar{\varepsilon}$ can be cancelled by adding a term $\sim f^2\kappa^2\psi^2 A_\beta F_{\rho\sigma}$ to the action, but the term with $D_\alpha\psi_\mu$ is not proportional to the gravitino field equation R^μ. Using

$$R^\mu = \Gamma^{\mu\rho\sigma}D_\rho(\omega(e))\psi_\sigma, \qquad \Gamma\cdot R = 8\Gamma^{\rho\sigma}D_\rho\psi_\sigma,$$

$$\Gamma^\sigma(D_\sigma\psi_\alpha - D_\alpha\psi_\sigma) = R_\alpha - \tfrac{1}{8}\Gamma_\alpha\Gamma\cdot R, \tag{3.7}$$

one can write $\Gamma^{\alpha\beta\mu\rho\sigma}D_\alpha\psi_\mu$ as a gravitino field equation except for terms of the form $\bar{\varepsilon}\Gamma_{[\alpha}D_\beta\psi_{\gamma]}$. Unlike in $d=4$, $\bar{\varepsilon}\Gamma_{[\alpha}D_\beta\psi_{\gamma]}$ is not proportional to R_μ. However, the theory solves this problem in another manner: the variation of $F_{\alpha\beta\gamma}$ is also proportional to $\bar{\varepsilon}\Gamma_{[\alpha}D_\beta\psi_{\gamma]}$. In this way all order $\kappa F^2\psi\varepsilon$ variations are cancelled if one takes

$$\begin{aligned} e^{-1}\mathfrak{L} &= -\tfrac{1}{4}F_{\mu\nu}^2 f^2 - \tfrac{1}{2}\bar{\chi}\not{D}(\omega(e))\chi - \tfrac{1}{4}\kappa\bar{\psi}_\mu\Gamma\cdot F\Gamma^\mu\chi f \\ &\quad - \tfrac{1}{16}\kappa^2 A_\nu F_{\alpha\beta}\bar{\psi}_\mu\Gamma^{\mu\nu\alpha\beta\rho}\psi_\rho f^2 + \tfrac{3}{4}\sqrt{2}\,\kappa A_\mu F_{\nu\rho}F^{\mu\nu\rho}\phi^{-3/4}f^2, \\ \delta\bar{\psi}_\mu(\text{extra}) &= \tfrac{1}{32}\kappa A_{[\nu}F_{\alpha\beta]}\bar{\varepsilon}\left(\Gamma^{\nu\alpha\beta}{}_\mu + 9\delta_\mu^\alpha\Gamma^{\nu\beta}\right)f^2. \end{aligned} \tag{3.8}$$

It looks as if the Maxwell invariance $\delta A_\mu = \partial_\mu\Lambda$ is broken by (3.8), but things are more subtle and interesting as we shall see.

The next set of variations we consider are the $\kappa D_\mu\chi^2\psi\varepsilon$ terms. Their analysis resembles the $d=4$ analysis, so that we follow the same path [20, 1]. The reader who is not interested in details may skip the discussion till (3.15); however, it is interesting to see how in $d=10$ the same results as in $d=4$ emerge for different algebraic reasons. The contributions come from varying the vielbeins in the Dirac action, and from varying A_μ in $\mathfrak{L}^{(N)}$, and read

$$\begin{aligned} e^{-1}\delta\mathfrak{L} &= -\tfrac{1}{4}\kappa\bar{\varepsilon}\Gamma\cdot\psi\bar{\chi}\not{D}\chi - \tfrac{1}{8}\bar{\chi}\Gamma^\mu\Gamma^{mn}\chi\delta\omega_{\mu mn}(e) \\ &\quad + \tfrac{1}{4}\kappa(\bar{\varepsilon}\Gamma^\mu\psi_\nu)(\bar{\chi}\Gamma^\nu D_\mu\chi) - \tfrac{1}{4}\kappa\left(\bar{\psi}_\mu\Gamma^{\alpha\beta}\Gamma^\mu\chi\right)fD_\alpha\left(\bar{\varepsilon}\Gamma_\beta\chi f^{-1}\right). \end{aligned} \tag{3.9}$$

The first term gives a χ field equation. After adding ψ-torsion in the Dirac action in order to cancel $\partial\varepsilon$ terms in $\delta\omega_{\mu mn}$, we have

$$\begin{aligned} \delta\hat{\omega}_{\mu mn}(e,\psi) &= \tfrac{1}{4}\kappa\bar{\varepsilon}\left(\Gamma_n\psi_{\mu m} - \Gamma_m\psi_{\mu n} - \Gamma_\mu\psi_{mn}\right), \\ \psi_{\mu\nu} &\equiv D_\mu\psi_\nu - D_\nu\psi_\mu. \end{aligned} \tag{3.10}$$

As in $d=4$, only the totally antisymmetric part of $\delta\hat{\omega}_{\mu mn}$ contributes since $\bar{\chi}\Gamma^\mu\Gamma^{mn}\chi = \bar{\chi}\Gamma^{\mu mn}\chi$, but unlike in $d=4$, $\delta\hat{\omega}_{[\mu mn]}$ is not proportional to the gravitino field equation but rather, as we have discussed, to $\delta F_{\mu mn}$. Hence the $\delta\omega$ variations in (3.9)

are eliminated by adding a new term to the action proportional to $F_{\alpha\beta\gamma}$:

$$\mathcal{L}(\text{extra}) = -\tfrac{1}{16}\sqrt{2}\, e\kappa\bar{\chi}\Gamma^{\alpha\beta\gamma}\chi F_{\alpha\beta\gamma}\phi^{-3/4}. \tag{3.11}$$

We must now deal with the two last terms in (3.9). In the very last term we bring Γ_μ next to ψ_μ in order to recognize the gravitino field equations (3.7). Since $D_\alpha f^{-1}$ is of order κ, we ignore $D_\alpha f$ terms. Thus all f-dependence cancels and the last term in (3.9) yields

$$-\tfrac{1}{4}\kappa(\bar{\psi}\cdot\Gamma\Gamma^{\alpha\beta}\chi)D_\alpha(\bar{\varepsilon}\Gamma_\beta\chi) - \tfrac{1}{2}\kappa(\bar{\psi}^\beta\Gamma^\alpha\chi)D_\alpha(\bar{\varepsilon}\Gamma_\beta\chi) + \tfrac{1}{2}\kappa(\bar{\psi}^\mu\Gamma^\beta\chi)D_\mu(\bar{\varepsilon}\Gamma_\beta\chi). \tag{3.12}$$

Partially integrating the first two terms in (3.12), and using (3.7), one finds new $\mathcal{L}^{(4)}, \delta\chi, \delta\psi_\mu$, plus a remainder $\delta\mathcal{L} = \frac{1}{4}e\kappa(\bar{\psi}\cdot\overleftarrow{D}\Gamma_\beta\chi)(\bar{\varepsilon}\Gamma^\beta\chi)$ to which we will return in (3.14). In the last term in (3.12), the $D_\mu\bar{\varepsilon}$ contribution yields a new $\mathcal{L}^{(4)}$ while the $D_\mu\chi$ contribution must be combined, as in $d=4$, with the last but one term in (3.9). This is done as follows. Fierz reordering the $D_\mu\chi$ part of the last term in (3.12) yields terms proportional to $\bar{\chi}\Gamma^{(1)}D_\mu\chi$ and $\bar{\chi}\Gamma^{(3)}D_\mu\chi$ [terms with $\Gamma^{(5)}$ cancel due to (A.6)]. The $\Gamma^{(1)}$ term and the last but one term in (3.9) add up to

$$\begin{aligned} e^{-1}\delta\mathcal{L} &= \tfrac{1}{4}\kappa(\bar{\varepsilon}\Gamma_\mu\psi_\nu)\bar{\chi}(\Gamma^\nu D^\mu - \Gamma^\mu D^\nu)\chi \\ &= \tfrac{1}{8}\kappa\bar{\chi}\Gamma^{\nu\mu\lambda}\chi D_\lambda(\bar{\varepsilon}\Gamma_\mu\psi_\nu) + \not{\partial}\chi \text{ terms}, \end{aligned} \tag{3.13}$$

according to the differential identities derived in the appendix. Clearly the $D_\lambda\bar{\varepsilon}$ term yields a new $\mathcal{L}^{(4)}$, while $\bar{\varepsilon}\Gamma_{[\mu}D_\lambda\psi_{a]}$ is cancelled by adding a term to the action containing an $F_{\mu\lambda a}$, and the $\not{\partial}\chi$ terms are taken care of by new $\delta\chi$ variations.

All that is left of (3.9) at this stage is

$$e^{-1}\delta\mathcal{L} = \tfrac{1}{4}\kappa(\bar{\psi}\cdot\overleftarrow{D}\Gamma_\beta\chi)(\bar{\varepsilon}\Gamma^\beta\chi) + \tfrac{1}{48}\kappa(\bar{\varepsilon}\Gamma^{(3)}\psi^\mu)(\bar{\chi}\Gamma^{(3)}D_\mu\chi). \tag{3.14}$$

Partially integrating the first term, the term with $D_\mu\bar{\varepsilon}$ yields an $\mathcal{L}^{(4)}$ while the terms with $D_\mu\chi$ yield a result proportional to $(\bar{\varepsilon}O\psi_\mu + \bar{\psi}_\mu O\varepsilon)$ after Fierzing. Hence $O = \Gamma^{(3)}$, and this cancels the second term in (3.14). The final result is that the $\delta\mathcal{L} \sim \kappa D_\mu\chi^2\psi\varepsilon$ terms yield

$$\begin{aligned} \delta\chi &= -\tfrac{1}{4}\kappa(\bar{\varepsilon}\Gamma\cdot\psi)\chi + \tfrac{1}{4}\kappa\Gamma^\mu\Gamma_\beta\psi_\mu(\bar{\varepsilon}\Gamma^\beta\chi) - \tfrac{1}{4}\kappa(\bar{\psi}\cdot\Gamma\chi)\varepsilon + \tfrac{1}{4}\kappa(\bar{\varepsilon}\Gamma_\mu\psi_\nu)\Gamma^{\mu\nu}\chi, \\ \delta\psi_\mu &= -\frac{\kappa}{8\times 32}\left(\Gamma_\mu^{\alpha\beta\gamma} - 5\delta_\mu^{\ \alpha}\Gamma^{\beta\gamma}\right)\varepsilon\bar{\chi}\Gamma_{\alpha\beta\gamma}\chi, \\ e^{-1}\mathcal{L}^{(4)} &= \tfrac{1}{16}\sqrt{2}\,\kappa\bar{\chi}\Gamma^{\alpha\beta\gamma}\chi F_{\alpha\beta\gamma}\phi^{-3/4} + \tfrac{1}{8}\kappa^2(\bar{\psi}\cdot\Gamma\chi)^2 - \tfrac{1}{4}\kappa^2(\bar{\psi}_\mu\Gamma_\beta\chi)(\bar{\psi}^\mu\Gamma^\beta\chi) \\ &\quad + \tfrac{1}{16}\kappa^2\bar{\chi}\Gamma^{\alpha\beta\gamma}\chi(\bar{\psi}_\alpha\Gamma_\beta\psi_\gamma). \end{aligned} \tag{3.15}$$

The $\psi_\mu \chi \varepsilon$ variations in $\delta\chi$ can be shown to coincide with a supercovariantization of $F_{\mu\nu}$ in the lowest-order χ transformation (3.3).

At the order κ level, the variations left to cancel come from the following sources

(i) the term $\frac{1}{4} e \bar{\chi} F \varepsilon \partial\!\!\!/ f$ in (3.6);

(ii) $\delta\psi_\mu \sim F_{\rho\sigma\tau}\varepsilon$ in $\mathfrak{L}^{(N)} \sim \kappa \bar{\psi} \chi F_{\rho\sigma} f$;

(iii) the κ^0 variations in $\mathfrak{L} \sim \kappa A_\mu F_{\nu\rho} F_{\mu\nu\rho} f^2 \phi^{-3/4}$;

(iv) the κ^0 variations in $\mathfrak{L} \sim \kappa \bar{\chi} \chi F_{\rho\sigma\tau} \phi^{-3/4}$;

(v) the variation of f in $\mathfrak{L}^{(0)}$.

Let us first consider the variations of the form $\kappa \bar{\chi} F_{\mu\nu} F_{\alpha\beta\gamma} \varepsilon$. They come from (ii)–(iv). In order to cancel these contributions one must add to the action a term $\mathfrak{L} \sim \kappa \bar{\chi} \lambda F_{\mu\nu}$, which is the only remaining possible interaction in order κ. Actually, since $\lambda \sim \psi_{11}$, we expect such a term as the 11-dimensional counterpart of $\mathfrak{L}^{(N)}$. Thus we must also consider

(vi) the κ^0 variation in $\mathfrak{L} \sim \kappa \bar{\lambda} \chi F_{\rho\sigma} f$.

Modulo an $A_{\mu\nu}$ field equation the desired cancellation takes place for

$$\mathfrak{L}(\text{extra}) = -\tfrac{1}{8}\sqrt{2}\, e \kappa (\bar{\chi} \Gamma \cdot F \lambda) f. \tag{3.16}$$

The $A_{\mu\nu}$ field equation is cancelled by an extra variation

$$\delta A_{\mu\nu}(\text{extra}) = \tfrac{1}{2}\sqrt{2}\, \kappa \phi^{3/8} \bar{\varepsilon} \Gamma_{[\mu} \chi A_{\nu]}. \tag{3.17}$$

Now we will meet variations which determine how the arbitrary function f depends on ϕ. Consider the variation $\delta\mathfrak{L} = \frac{1}{4} e \bar{\chi} \Gamma \cdot F \partial\!\!\!/ f \varepsilon$ in (i). The only other source of such terms is the variation $\delta\lambda \sim \partial\!\!\!/ \phi / (\kappa\phi)$ in (3.16). Their sum must cancel:

$$e^{-1}\delta\mathfrak{L} = \tfrac{1}{4} \bar{\chi} \Gamma \cdot F \partial\!\!\!/ f \varepsilon + \tfrac{3}{32} \bar{\chi} \Gamma \cdot F (\partial\!\!\!/ \phi / \phi) \varepsilon = 0, \tag{3.18}$$

from which we find that

$$f(\phi) = \phi^{-3/8}. \tag{3.19}$$

At this stage there are two kinds of order κ variations left. The $\delta F_{\mu\nu\rho} \sim \partial \bar{\varepsilon} \lambda$ variations in (iv) plus the $\delta A_\mu \sim \bar{\varepsilon} \chi$ variations in (vi) yield a total of

$$\begin{aligned} e^{-1}\delta\mathfrak{L} = & -\tfrac{1}{64}\sqrt{2}\, \kappa \bar{\chi} \Gamma^{\alpha\beta\gamma} \chi \partial_\alpha \left(\bar{\varepsilon} \Gamma_{\beta\gamma} \lambda \phi^{3/4} \right) \phi^{-3/4} \\ & - \tfrac{1}{8}\sqrt{2}\, \kappa \bar{\chi} \Gamma^{\alpha\beta} \lambda \phi^{-3/8} \partial_\alpha \left(\bar{\varepsilon} \Gamma_\beta \chi \phi^{3/8} \right). \end{aligned} \tag{3.20}$$

Partially integrating the second term one finds $\delta\chi$ and $\delta\lambda$ corrections, plus a term with $[(\partial_\sigma \bar{\chi})\lambda - \bar{\chi} \partial_\sigma \lambda](\bar{\varepsilon} \Gamma^\sigma \chi)$ which clearly yields an $\mathfrak{L}^{(4)}$, a new $\delta\chi$ and a term $-2(\bar{\chi} \partial_\sigma \lambda)(\bar{\varepsilon} \Gamma^\sigma \chi)$. If one Fierz reorders this last term, it cancels the first term in

(3.20). Finally we consider the $\delta F_{\mu\nu\rho} \sim \partial\bar{\varepsilon}\lambda$ variations in (iii), yielding $\delta\mathfrak{L} \sim A_\mu F_{\nu\rho}\partial\bar{\varepsilon}\lambda$. They combine with $\delta\chi \sim \Gamma \cdot F\varepsilon$ in (3.16), but a term $\sim \kappa\bar{\varepsilon}\lambda(F^2_{\alpha\beta})f^2$ remains. This term nicely cancels the variation of f in $\mathfrak{L}^{(0)}$, which we already announced in (v).

Let us pause for a moment and ask ourselves which terms one still can expect from an analysis of $\delta\mathfrak{L}$ at order κ^2 and higher. As far as four-fermion couplings are concerned, all couplings with one or more gravitino must have been found since they show up at the order κ level when one uses $\delta\psi_\mu = \kappa^{-1}D_\mu\varepsilon$. Since no couplings with three λ's or χ's are possible according to the appendix, the only undetermined four-fermion coupling is proportional to $(\bar{\chi}\Gamma_{\alpha\beta\gamma}\chi)(\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda)$. A relatively easy way to fix its coefficient is to consider the $\delta\mathfrak{L} \sim \kappa^2\chi^2\lambda\partial\!\!\!/\phi\varepsilon$ variation, to which it contributes by means of $\delta\lambda$. Dimensionally, the couplings $\kappa^2 A_\mu F_{\nu\rho}(\bar{\lambda}\Gamma^{\mu\nu\rho}\lambda)$ and $\kappa^2(A_\mu F_{\nu\rho})^2$ are possible, but the former are absent and the latter are not. This will follow from the modified Maxwell invariance of the action. Hence, let us turn to a new subject temporarily, and try to come to grips with the issue of the Maxwell invariances.

The flat-space Maxwell system is invariant under $\delta A_\mu = \partial_\mu\Lambda$. In order κ, a coupling was found in (3.8),

$$\mathfrak{L} = \tfrac{3}{4}\sqrt{2}\,\kappa e A_\mu F_{\nu\rho}F^{\mu\nu\rho}\phi^{-3/4}, \tag{3.21}$$

which violates this invariance by an amount proportional to $\Lambda F_{\nu\rho}\partial_\mu F^{\mu\nu\rho}$. However, we can restore Maxwell gauge invariance by using once more the familiar Noether method. This leads to a Maxwell transformation rule of $A_{\mu\nu}$:

$$\delta_{\mathrm{M}} A_{\mu\nu} = \tfrac{1}{2}\sqrt{2}\,\kappa\Lambda F_{\mu\nu}. \tag{3.22}$$

We can now introduce a Maxwell-covariant $A_{\mu\nu}$ curl,

$$F'_{\mu\nu\rho} = F_{\mu\nu\rho} - \tfrac{1}{2}\sqrt{2}\,\kappa A_{[\mu}F_{\nu\rho]}. \tag{3.23}$$

Maxwell invariance now implies that the field A_μ can only occur through its field strength $F_{\mu\nu}$, or through the covariantization in $F'_{\mu\nu\rho}$. Of course, one now eagerly looks back at the action and transformations, and investigates whether the replacement of $F_{\mu\nu\rho}$ by $F'_{\mu\nu\rho}$ causes simplifications. Indeed all terms containing A_μ can be understood in this way. In the gauge action this replacement absorbs the $A_\mu F_{\nu\rho}F^{\mu\nu\rho}$ coupling, found above, while also the exact expression for the $(A_\mu F_{\nu\rho})^2$ coupling is predicted:

$$\mathfrak{L}(\text{extra}) = -\tfrac{3}{8}e\kappa^2 A_{[\mu}F_{\nu\rho]}(A^\mu F^{\nu\rho})\phi^{-3/2}. \tag{3.24}$$

The $\kappa^2 A_\mu F_{\nu\rho}\psi^2$ and $\kappa^2 A_\mu F_{\nu\rho}\psi\lambda$ couplings are explained in the same way. Furthermore, we understand why there is no $A_\mu F_{\nu\rho}\bar{\lambda}\Gamma^{\mu\nu\rho}\lambda$ coupling: there was no $F_{\mu\nu\rho}\bar{\lambda}\Gamma^{\mu\nu\rho}\lambda$ to begin with! It comes as no surprise that also the $\bar{\chi}\chi A_\mu F_{\nu\rho}$ terms are Maxwell-covariantizations of $\bar{\chi}\chi F_{\mu\nu\rho}$.

Spurred by this success we look for further simplifications. As suggested by the $d=4$ Maxwell-Einstein system [20] we rewrite the action as follows:

$$\begin{aligned}\mathfrak{L} = {} & \mathfrak{L}\left(N=1 \text{ gauge, but with } F'_{\mu\nu\rho}\right) - \tfrac{1}{4}e\phi^{-3/4}F_{\mu\nu}^2 - \tfrac{1}{2}e\bar{\chi}\not{D}(\hat{\omega})\chi \\ & - \tfrac{1}{8}\kappa e\phi^{-3/8}\bar{\chi}\Gamma^{\mu}\Gamma^{\rho\sigma}\left(F_{\rho\sigma} + \hat{F}_{\rho\sigma}\right)\left(\psi_{\mu} + \tfrac{1}{12}\sqrt{2}\,\Gamma_{\mu}\lambda\right) \\ & + \tfrac{1}{16}\sqrt{2}\,\kappa e\phi^{-3/4}\bar{\chi}\Gamma^{\alpha\beta\gamma}\chi\hat{F}'_{\alpha\beta\gamma} \\ & - \frac{1}{16\times 96}\sqrt{2}\,\kappa^2 e\bar{\chi}\Gamma_{\alpha\beta\gamma}\chi\bar{\psi}_{\mu}(4\Gamma^{\alpha\beta\gamma}\Gamma^{\mu} + 3\Gamma^{\mu}\Gamma^{\alpha\beta\gamma})\lambda \\ & - \tfrac{1}{512}\kappa^2 e\bar{\chi}\Gamma_{\alpha\beta\gamma}\chi\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda, \end{aligned} \tag{3.25}$$

where $\hat{\omega}$ contains only ψ-torsion. The supercovariant field strengths $\hat{F}_{\mu\nu}$ and $\hat{F}'_{\mu\nu\rho}$ are defined by

$$\hat{F}_{\mu\nu} = F_{\mu\nu} - \kappa\bar{\psi}_{[\mu}\Gamma_{\nu]}\chi\phi^{3/8}, \tag{3.26}$$

$$\hat{F}'_{\mu\nu\rho} = F'_{\mu\nu\rho} - \tfrac{1}{4}\kappa\phi^{3/4}\left(\sqrt{2}\,\bar{\psi}_{[\mu}\Gamma_{\nu}\psi_{\rho]} - \bar{\psi}_{[\mu}\Gamma_{\nu\rho]}\lambda\right). \tag{3.27}$$

Note that in the Noether coupling the combination $\psi_{\mu} + \frac{1}{12}\sqrt{2}\,\Gamma_{\mu}\lambda$ found in (2.13) reappears. We have, of course, replaced the matter curl $F_{\mu\nu}$ by $\frac{1}{2}(F+\hat{F})_{\mu\nu}$ because this is a standard substitution based on the requirement of supercovariant fermion field equations. The same argument led us to replace $F'_{\mu\nu\rho}$ by $\hat{F}'_{\mu\nu\rho}$. Notice that explicit $\psi^2\chi^2$ terms are now absent in the action. The transformation rules under which (3.25) is invariant will be discussed in sect. 4.

It is of interest to consider the supercovariant curl $\hat{F}'_{\mu\nu\rho}$ in more detail. It might appear that the covariantizations in (3.27) are not complete, since they coincide with those of the pure-gauge $F_{\mu\nu\rho}$ in (2.15). However, this is not the case. The supersymmetry transformation (3.17) leads to a term $(\partial_{[\mu}\bar{\varepsilon})\Gamma_{\nu}A_{\rho]}\phi^{3/8}$ in the transformation of (3.27), which is precisely cancelled by the variation of the AF modification defined in (3.23). Hence (3.27) is supercovariant and, because the A_{μ} dependence is that of $F'_{\mu\nu\rho}$, it is simultaneously covariant with respect to Maxwell transformations.

4. The gauge algebra

The gauge algebra of $N=1, d=10$ supergravity is expected to resemble that of $N=1, d=11$ supergravity since the $d=10$ theory is obtained by reduction and subsequent truncation from the $d=11$ theory. Similarly, if the Maxwell-Einstein system turns out to be a truncation of the $N=2$, $d=10$ theory (which is directly obtained by reducing the $N=1, d=11$ theory), the gauge algebra of this matter

coupled system ought to resemble its 11-dimensional cradle. We will first derive these $d=10$ algebras and then compare with the $d=11$ case.

The transformation rules of the pure gauge theory were obtained in subsect. 2.2 and given in (2.20). It is always easiest to evaluate the gauge algebra on bosonic fields since one does not need to Fierz rearrange, while the commutators close on bosonic fields. So we begin with the zehnbein. The $\delta\psi_\mu = D_\mu(\hat{\omega})\varepsilon$ terms give the same algebra as in $d=4$,

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_{gc}(\xi^\mu) + \delta_Q(-\xi^\mu\psi_\mu) + \delta_L(\xi^\mu\hat{\omega}_{\mu mn}), \tag{4.1}$$

while the $\delta\psi_\mu \sim \hat{F}\varepsilon$ and $\delta\psi_\mu \sim \bar{\lambda}\lambda\varepsilon$ terms must lead to extra local Lorentz rotations (they cannot give δ_Q terms since one needs a gravitino in $\delta_Q e_\mu{}^m$). The parameter of these extra Lorentz transformations is added to the one in (4.1):

$$\begin{aligned}\lambda_{12,mn} = {} & \xi^\mu\hat{\omega}_{\mu mn} + \left(\bar{\varepsilon}_2\Gamma_{mn}{}^{\alpha\beta\gamma}\varepsilon_1\right)\left(\tfrac{1}{32}\sqrt{2}\,\hat{F}_{\alpha\beta\gamma}\phi^{-3/4} - \frac{1}{16\times 32}\bar{\lambda}\Gamma_{\alpha\beta\gamma}\lambda\right) \\ & + \xi^\mu\left(\tfrac{2}{8}\sqrt{2}\,\hat{F}_{mn\mu}\phi^{-3/4} - \frac{5}{8\times 32}\bar{\lambda}\Gamma_{mn\mu}\lambda\right).\end{aligned} \tag{4.2}$$

Consider now the $\bar{\psi}\lambda\varepsilon$ terms in $\delta\dot{\psi}_\mu$. They should not lead to new Lorentz rotations, since in that case the parameters of these Lorentz rotations would not be supercovariant, whereas they should be according to the expected commutator $[\delta_L(\lambda_{mn}), \delta_Q(\varepsilon)] = -\delta_Q(\frac{1}{4}\lambda_{mn}\Gamma^{mn}\varepsilon)$. Thus, they should lead to new local supersymmetry transformations, and, indeed, the $\delta\psi_\mu \sim \bar{\psi}\lambda\varepsilon$ variations in $\delta e_\mu{}^m = \frac{1}{2}\bar{\varepsilon}\Gamma^m\psi_\mu$ can be cast in the form $\frac{1}{2}\bar{\varepsilon}'\Gamma^m\psi_\mu$. The total supersymmetry parameter reads

$$\varepsilon_{12} = -\xi^\mu\psi_\mu + \frac{1}{96\times 160}\sqrt{2}\left(\bar{\varepsilon}_2\Gamma^{(5)}\varepsilon_1\right)\Gamma^{(5)}\lambda - \tfrac{7}{64}\sqrt{2}\left(\bar{\varepsilon}_2\Gamma^\alpha\varepsilon_1\right)\Gamma_\alpha\lambda. \tag{4.3}$$

We turn to the Maxwell terms in the commutator. On $A_{\mu\nu}$ the commutator $[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]$ has terms with $D_\mu\varepsilon$ from $\delta\psi_\mu$. We expect to need such terms already for the general coordinate transformation $\delta_{gc}(\xi^\mu)$ on $A_{\mu\nu}$, but there is an excess of $D_\mu\varepsilon$ terms. These we interpret as Maxwell transformations

$$\delta_M^{(2)}A_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu. \tag{4.4}$$

(The superscript (2) distinguishes the Maxwell transformations of $A_{\mu\nu}$ from those of A_μ, henceforth denoted by $\delta_M^{(1)}A_\mu = \partial_\mu\Lambda$.) The $\hat{F}$ terms in $\delta\psi_\mu$ and $\delta\lambda$ contribute in the well-known way both to general coordinate transformations and to Maxwell transformations, and one finds a Maxwell parameter

$$\Lambda_{12,\mu} = -\tfrac{1}{2}\sqrt{2}\,\xi_\mu\phi^{3/4} - \xi^\nu A_{\nu\mu}. \tag{4.5}$$

Rather subtle Dirac algebra is required to show that the $\lambda^2\varepsilon$ terms in $[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]A_{\mu\nu}$ are indeed given by $\delta_Q(\varepsilon')$ with ε' given in (4.3). As a check we evaluate the supersymmetry commutator on ϕ, and find agreement. The complete gauge commutator for $N=1, d=10$ supergravity reads

$$\begin{aligned}[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] &= \delta_{gc}(\xi^\mu) + \delta_Q(\varepsilon_{12} \text{ in } (4.3)) \\ &\quad + \delta_L(\lambda_{12,mn} \text{ in } (4.2)) + \delta_M^{(2)}(\Lambda_{12,\mu} \text{ in } (4.5)). \end{aligned} \tag{4.6}$$

One sees that the Maxwell parameter agrees with the dimensionally reduced $d=11$ Maxwell parameter defined by $\delta A_{\mu\nu\rho} = \partial_\mu \Lambda_{\nu\rho}$ + cyclic terms, where

$$\Lambda_{12,\mu\nu}(d=11) = -\tfrac{1}{24}\sqrt{2}\,\bar{\varepsilon}_2 \Gamma_{\mu\nu}\varepsilon_1 - \xi^\sigma A_{\sigma\mu\nu}, \tag{4.7}$$

taking into account our rescaling of $A_{\mu\nu\dot{1}\dot{1}} = \frac{1}{6}A_{\mu\nu}$. Similarly, one can compare the $d=11$ Lorentz parameter

$$\lambda^{mn}(d=11) = \xi^\mu \hat{\Omega}_\mu{}^{mn} + \frac{1}{8\times 36}\sqrt{2}\,\bar{\varepsilon}_2(\Gamma^{mn\alpha\beta\gamma\delta} + 24 e^{m\alpha} e^{n\beta}\Gamma^{\gamma\delta})\varepsilon_1 \hat{F}_{\alpha\beta\gamma\delta}, \tag{4.8}$$

$$\begin{aligned}\hat{\Omega}_\mu{}^{mn} &= \hat{\omega}_\mu{}^{mn}(e,\psi) - \tfrac{1}{8}(e_\mu{}^m e^{n\nu} - e_\mu{}^n e^{m\nu})\hat{D}_\nu \phi/\phi \\ &\quad - \tfrac{1}{288}\bar{\lambda}\Gamma_\mu{}^{mn}\lambda + \tfrac{1}{24}\sqrt{2}\,\bar{\psi}_\mu \Gamma^{mn}\lambda, \end{aligned} \tag{4.9}$$

with the $\lambda^{mn}(d=10)$ in (4.2), but one should take into account that we have added a Lorentz rotation to the dimensionally reduced supersymmetry transformation in order to bring $\delta e_\mu{}^m$ in canonical form [see (2.17)]. For example, the $\hat{F}$ terms with $\Gamma^{(5)}$ do not only come from reducing $\lambda^{mn}(d=11)$, but also from $\delta_Q(\varepsilon_1)\delta_L(-\frac{1}{24}\sqrt{2}\,\bar{\varepsilon}_2\Gamma^{mn}\lambda)$. This also explains the origin of the λ-dependent supersymmetry structure functions, because acting with $\delta_L(-\frac{1}{24}\sqrt{2}\,\bar{\varepsilon}_1\Gamma^{mn}\lambda)$ on $\delta_Q(\eta_2)$ one finds a new δ_Q transformation with Lorentz-rotated η_2.

We now turn to the gauge algebra as it follows from the transformation rules in the presence of matter. In the gauge field laws the only modification is that $\hat{F}$ is everywhere replaced by $\hat{F}'$ given in (3.27). In addition, the following matter contributions are added:

$$\delta A_{\mu\nu}(\text{matter}) = \tfrac{1}{2}\kappa\phi^{3/8}\bar{\varepsilon}\Gamma_{[\mu}\chi A_{\nu]}, \tag{4.10}$$

$$\delta\lambda(\text{matter}) = \frac{1}{12\times 36}\sqrt{2}\,\kappa(\bar{\chi}\Gamma^{\alpha\beta\gamma}\chi)\Gamma_{\alpha\beta\gamma}\varepsilon, \tag{4.11}$$

$$\delta\psi_\mu(\text{matter}) = -\tfrac{1}{256}\kappa(\bar{\chi}\Gamma^{\alpha\beta\gamma}\chi)(\Gamma_{\mu\alpha\beta\gamma} - 5g_{\mu\alpha}\Gamma_{\beta\gamma})\varepsilon, \tag{4.12}$$

$$\delta A_\mu = \tfrac{1}{2}\phi^{3/8}\bar{\varepsilon}\Gamma_\mu\chi, \tag{4.13}$$

$$\delta\chi = -\tfrac{1}{4}\phi^{-3/8}\Gamma\cdot\hat{F}\varepsilon + \tfrac{1}{64}\sqrt{2}\,\kappa\big(3(\bar{\lambda}\chi)\varepsilon - \tfrac{3}{2}(\bar{\lambda}\Gamma^{\alpha\beta}\chi)\Gamma_{\alpha\beta}\varepsilon - \tfrac{1}{24}(\bar{\lambda}\Gamma^{\alpha\beta\gamma\delta}\chi)\Gamma_{\alpha\beta\gamma\delta}\varepsilon\big). \tag{4.14}$$

Evaluating the supersymmetry commutator on the zehnbein, one finds that only the Lorentz parameter receives a matter correction term, namely, $\hat{F}_{\alpha\beta\gamma}$ is replaced by $\hat{F}'_{\alpha\beta\gamma}$, while one also finds the replacement

$$\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda \to \bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda + 2\bar{\chi}\Gamma^{\alpha\beta\gamma}\chi. \tag{4.15}$$

This is the same combination of λ^2 and χ^2 terms as in $\delta\psi_\mu$, but in $\delta\lambda$ only a χ^2 but no λ^2 term is present, for reasons explained in subsect. 2.3. Hence one gets here a clear signal that there are at least two three-index auxiliary fields, assuming that not only the $\bar{\chi}\chi$ terms but also the $\bar{\lambda}\lambda$ terms come from auxiliary fields.

Evaluating the gauge commutator on A_μ, one expects to find a Maxwell transformation, and indeed, one finds the standard result with parameter $\Lambda = -\xi^\sigma A_\sigma$. One needs some juggling with duality and chirality to prove this, for example

$$(\bar{\varepsilon}_2\Gamma_{\mu\alpha\beta\gamma\delta}\varepsilon_1)(\bar{\lambda}\Gamma^{\alpha\beta\gamma\delta}\chi) = \tfrac{1}{5}(\bar{\varepsilon}_2\Gamma_{\alpha_1\cdots\alpha_5}\varepsilon_1)\bar{\lambda}(\Gamma_{\alpha_1\cdots\alpha_5}\Gamma_\mu - 5g_{\alpha_5\mu}\Gamma_{\alpha_1\cdots\alpha_4})\chi. \tag{4.16}$$

Using this relation, one has a good check on the $\bar{\lambda}\chi\varepsilon$ terms in $\delta\chi$.

The commutator of two local supersymmetry variations of $A_{\mu\nu}$ is interesting, because it will reveal whether the Maxwell transformation on $A_{\mu\nu}$ is modified by the presence of the (A_μ, χ) matter. We already have found this modification in (3.17), but even if we did not know the answer at this point, evaluation of the gauge algebra would tell us this modification. Consider first the variation $\delta\chi = -\frac{1}{4}\Gamma\cdot F\varepsilon$ in (4.10). It yields $\sqrt{2}\,\xi^\rho F_{\rho[\mu}A_{\nu]}$. There are also $F_{\mu\nu}A_\rho$ terms in $F'_{\mu\nu\rho}$, and since $A_{\mu\nu}$ is Lorentz inert, the $F_{\mu\nu\rho}$ are not needed for the Lorentz rotations. Their contribution is $\frac{3}{2}\sqrt{2}\,\xi^\rho A_{[\nu}F_{\mu\rho]}$, and the total result is

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)]\,A_{\mu\nu} = (\text{as for } e_\mu{}^m) + \tfrac{1}{2}\sqrt{2}\,\kappa\Lambda_{12}F_{\mu\nu}, \tag{4.17}$$

where $\Lambda_{12} = -\xi^\mu A_\mu$. Comparison with (3.22) shows that we indeed have recovered the modified Maxwell transformation rules. Of course there remains the independent Maxwell transformation $\delta A_{\mu\nu} = \partial_\mu\Lambda_\nu - \partial_\nu\Lambda_\mu$ in the commutator, with Λ_μ given in (4.5).

Summarizing, the commutator of two local supersymmetry transformations for the

Maxwell-Einstein system reads

$$
\begin{aligned}
\left[\delta_{\mathrm{Q}}(\varepsilon_1), \delta_{\mathrm{Q}}(\varepsilon_2)\right] &= \delta_{\mathrm{gc}}(\xi^\mu) + \delta_{\mathrm{Q}}\left(-\xi^\mu \psi_\mu\right) + \delta_{\mathrm{L}}(\xi \cdot \hat{\omega}^{mn}) \\
&+ \delta_{\mathrm{Q}}\left(\frac{1}{96 \times 160} \sqrt{2}\left(\bar{\varepsilon}_2 \Gamma^{(5)} \varepsilon_1\right) \Gamma^{(5)} \lambda - \tfrac{7}{32} \sqrt{2}\, \xi^\alpha \Gamma_\alpha \lambda\right) \\
&+ \delta_{\mathrm{M}}^{(1)}\left(-\xi^\mu A_\mu\right) + \delta_{\mathrm{M}}^{(2)}\left(-\tfrac{1}{2}\sqrt{2}\, \phi^{3/4} \xi_\mu - \xi^\nu A_{\nu\mu}\right) \\
&+ \delta_{\mathrm{L}}\Big\{\bar{\varepsilon}_2 \Gamma^{mn\alpha\beta\gamma} \varepsilon_1 \Big(\tfrac{1}{32}\sqrt{2}\, \phi^{-3/4} \hat{F}'_{\alpha\beta\gamma} - \frac{1}{16 \times 32}\left(\bar{\lambda} \Gamma_{\alpha\beta\gamma} \lambda + 2 \bar{\chi} \Gamma_{\alpha\beta\gamma} \chi\right)\Big) \\
&\quad + \xi^\mu \Big(\tfrac{9}{8}\sqrt{2}\, \phi^{-3/4} \hat{F}'_{mn\mu} - \frac{5}{8 \times 32}\left(\bar{\lambda} \Gamma_{mn\mu} \lambda + 2 \bar{\chi} \Gamma_{mn\mu} \chi\right)\Big)\Big\}, \qquad (4.18)
\end{aligned}
$$

where $\delta_{\mathrm{M}}^{(1)}(\Lambda) A_\mu = \partial_\mu \Lambda$ and $\delta_{\mathrm{M}}^{(1)} A_{\mu\nu} = \sqrt{\tfrac{1}{2}}\, \kappa \Lambda F_{\mu\nu}$. For completeness, we recall the definitions $\omega = \omega(e, \psi)$, $\xi^\alpha = \tfrac{1}{2} \bar{\varepsilon}_2 \Gamma^\alpha \varepsilon_1$; $\hat{F}'$ was defined in (3.27).

After this analysis of the [Q,Q] commutator we turn to the commutator of Maxwell transformations with supersymmetry. Both commutators vanish when the Maxwell and Einstein systems are decoupled, but one of them becomes non-vanishing in the presence of coupling:

$$
\begin{aligned}
&\left[\delta_{\mathrm{Q}}(\varepsilon), \delta_{\mathrm{M}}^{(1)}(\Lambda)\right] = \delta_{\mathrm{M}}^{(2)}\left(\tfrac{1}{2}\sqrt{2}\, \kappa \Lambda \phi^{3/8} \bar{\varepsilon} \Gamma_\mu \chi\right), \\
&\left[\delta_{\mathrm{Q}}(\varepsilon), \delta_{\mathrm{M}}^{(2)}(\Lambda_\mu)\right] = 0. \qquad (4.19)
\end{aligned}
$$

5. The multiplet of currents

From an arbitrary multiplet of global supersymmetry other multiplets can be constructed by considering bilinear combinations of the fields of the original multiplet, and their variations under supersymmetry. If the first multiplet corresponds to an on-shell representation of supersymmetry, the multiplets of bilinears are in general off-shell representations, since the product does not preserve the on-shell condition.

An example of such a multiplet of bilinear field combinations is the multiplet of currents (supercurrent) [11], which contains the energy-momentum tensor $\theta_{\mu\nu}$ and the supersymmetry current J_μ of the globally supersymmetric theory under consideration. It can be used to obtain information about the auxiliary fields of supergravity in the following way. If the globally supersymmetric theory is coupled to supergravity, the gauge fields of supergravity $h_{\mu\nu}$ and ψ_μ will couple to the Noether currents $\theta_{\mu\nu}$ and J_μ. This coupling of currents and fields forms the beginning of a globally supersymmetric invariant which defines a complete multiplet of fields conjugate to

the currents. Since the currents form an off-shell representation, so do the fields. Therefore, one obtains an off-shell multiplet of fields, which contains at least the gauge fields of supergravity, and of which the remaining fields may be interpreted as auxiliary fields.

In general it is advisable to consider massless matter systems in order to exclude on-shell central charges, which act on the matter fields and hence on the currents. In the present case there is anyhow no choice, since the only $d=10$ matter system is the massless Maxwell theory. In order to exclude off-shell central charges acting on the currents, one restricts one's attention to gauge-invariant currents. Fortunately, the multiplet we consider below, containing the stress tensor, is Maxwell invariant. In the absence of central charges, one has a supersymmetry algebra of the Clifford type, and this guarantees that one will obtain a finite representation. Notice that these arguments limit this approach to $d<11$, and in $d=4$ to $N\leqslant 4$.

The construction of the supercurrent and the corresponding field multiplet does not necessarily produce all auxiliary fields of Poincaré supergravity. In some cases the multiplet of currents generates only a submultiplet of supergravity. This is the case, for instance, when other gauge invariances besides those of Poincaré supergravity are present, as in conformally invariant theories.

In this section we shall use the $N=1$ supersymmetric on-shell Maxwell theory in 10 dimensions to generate a multiplet of currents corresponding to $N=1$ supergravity in 10 dimensions. To set the stage for this calculation, let us first consider the analogous situation in 4 dimensions. The Maxwell theory contains the photon A_μ, and a Majorana spinor χ. One starts from the energy-momentum tensor $\theta_{\mu\nu}$ and the supersymmetry current J_μ(A_μ and χ satisfy their field equations)

$$\theta_{\mu\nu}=4F_{\mu\alpha}F_{\nu\alpha}-F^2\delta_{\mu\nu}+\bar{\chi}(\gamma_\mu\partial_\nu+\gamma_\nu\partial_\mu)\chi,$$

$$J_\mu=\tfrac{1}{2}\sigma\cdot F\gamma_\mu\chi. \tag{5.1}$$

From the variation of J_μ one obtains

$$\delta J_\mu=-\tfrac{1}{8}\gamma^\lambda\varepsilon\theta_{\mu\lambda}-\tfrac{3}{8}i\gamma_5\left(\sigma_{\mu\lambda}\gamma_\rho+\tfrac{1}{3}\gamma_\rho\sigma_{\mu\lambda}\right)\partial_\lambda J_\rho^{(5)}\varepsilon, \tag{5.2}$$

where the axial vector current $J_\mu^{(5)}$ is given by

$$J_\mu^{(5)}=i\bar{\chi}\gamma_\mu\gamma_5\chi. \tag{5.3}$$

It is easy to see that the axial current transforms back into J_μ:

$$\delta J_\mu^{(5)}=-2i\bar{\varepsilon}\gamma_5 J_\mu. \tag{5.4}$$

Therefore, the multiplet of currents contains only three currents: $\theta_{\mu\nu}$, J_μ and $J_\mu^{(5)}$.

Notice that, due to four-dimensional identities, $\theta_{\mu\nu}$ and J_μ satisfy the conditions $\theta_{\mu\mu} = \gamma \cdot J = 0$, i.e. one automatically obtains the improved currents. The multiplet of fields contains $h_{\mu\nu}$ and ψ_μ, and an axial vector field A_μ^{aux} which couples to $J_\mu^{(5)}$. It is the multiplet of $N = 1$ conformal supergravity. This is, therefore, an example of a theory for which the multiplet of fields has additional gauge invariances. Finally, in 4 dimensions there is a second multiplet of bilinears. The combinations

$$\left(\bar{\chi}\chi, i\bar{\chi}\gamma_5\chi, \sigma \cdot F\chi, F \cdot F, iF \cdot \tilde{F}\right)$$

form a scalar multiplet, usually called the multiplet of anomalies.

Let us now go to higher dimensions. The simple superconformal group exists only in 4 dimensions, since simple superalgebras containing the conformal algebra cannot be constructed for $d \neq 4$ [16]. However, this does not guarantee that a multiplet of currents in $d \neq 4$ automatically generates all auxiliary fields of ordinary supergravity. This fact is illustrated by the recent work of Howe and Lindström [15]. They obtained the supercurrent of $N = 4$ Yang-Mills theory in 5 dimensions. This multiplet of currents has only $128 + 128$ bosonic and fermionic components, and on reduction to four dimensions it reproduces – up to field redefinitions – the multiplet of fields of $N = 4$ conformal supergravity [6]. Not all physical fields of $d = 5$ supergravity couple to the current multiplet containing $\theta_{\mu\nu}$. Likewise, we find the same situation in $d = 10$. We shall discuss this further in sect. 6.

To construct the supercurrent in 10 dimensions one must choose a ten-dimensional matter system. There is only one candidate*: the abelian version of $N = 1$ Yang-Mills, already considered in sect. 3. This will allow a comparison between the results obtained from the Noether coupling construction, and the current multiplet. The lagrangian density and transformation rules are as in four dimensions, and have been given in (3.1) and (3.2). The field equations are

$$\not{\partial}\chi = \partial_\mu F_{\mu\nu} = 0. \tag{5.5}$$

The construction of the supercurrent starts from the energy-momentum tensor $\theta_{\mu\nu}$ and the supersymmetry current J_μ. They have the same form as in four dimensions, and read (always using the field equations)

$$\theta_{\mu\nu} = 4F_{\mu\alpha}F_{\nu\alpha} - \delta_{\mu\nu}F^2 + \bar{\chi}\left(\Gamma_\mu\partial_\nu + \Gamma_\nu\partial_\mu\right)\chi, \tag{5.6}$$

$$J_\mu = \tfrac{1}{4}\Gamma \cdot F\Gamma_\mu\chi. \tag{5.7}$$

They are conserved, i.e. using the equations of motion (5.5) we find

$$\partial_\mu\theta_{\mu\nu} = \partial_\mu J_\mu = 0. \tag{5.8}$$

* The analogue of the $d = 4$ $(\frac{3}{2}, 1)$ multiplet, which can be obtained either directly or by reduction from $d = 11$, is not suitable for our purpose, since J_μ is not invariant under matter gravitino gauge transformations.

Note that in 10 dimensions $\theta_{\mu\nu}$ and J_μ are no longer traceless. They satisfy

$$\theta_{\mu\mu} = -6F^2, \qquad \Gamma_\mu J_\mu = \tfrac{3}{2}\Gamma \cdot F\chi. \tag{5.9}$$

Also it is now impossible to add local improvement terms such that $\theta_{\mu\nu}$ and J_μ are conserved and traceless.

Using (3.2), one now obtains the variations of (5.6) and (5.7),

$$\begin{aligned}
\delta\theta_{\mu\nu} &= \bar{\varepsilon}\left(\Gamma_{\mu\lambda}\partial_\lambda J_\nu + \Gamma_{\nu\lambda}\partial_\lambda J_\mu\right),\\
\delta J_\mu &= -\tfrac{1}{8}\Gamma_\nu \varepsilon\theta_{\mu\nu} - \tfrac{1}{8}\Gamma_{\mu\alpha\beta\gamma\delta}\varepsilon\partial_\alpha V_{\beta\gamma\delta}\\
&\quad + \tfrac{1}{96}\Gamma_{\mu\nu}\Gamma_{\alpha\beta\gamma}\varepsilon\partial_\nu X_{\alpha\beta\gamma},
\end{aligned} \tag{5.10}$$

where we have defined

$$V_{\alpha\beta\gamma} = A_{[\alpha}F_{\beta\gamma]} + \tfrac{1}{12}\bar{\chi}\Gamma_{\alpha\beta\gamma}\chi, \tag{5.11}$$

$$X_{\alpha\beta\gamma} = \bar{\chi}\Gamma_{\alpha\beta\gamma}\chi. \tag{5.12}$$

Let us give some details of the variation of J_μ. One starts with

$$\delta J_\mu = -\tfrac{1}{16}\Gamma_{\alpha\beta}\Gamma_\mu\Gamma_{\gamma\delta}\varepsilon F_{\alpha\beta}F_{\gamma\delta} + \tfrac{1}{4}\Gamma_{\alpha\beta}\Gamma_\mu\chi\bar{\varepsilon}\Gamma_\alpha\partial_\beta\chi.$$

The first term is written in terms of a $\Gamma^{(5)}$, a $\Gamma^{(3)}$ and a $\Gamma^{(1)}$. The $\Gamma^{(3)}$ vanishes due to the $(\alpha\beta \leftrightarrow \gamma\delta)$ pair-exchange symmetry. This fact is dimension-independent. The $\Gamma^{(1)}$ gives the usual term which we also saw in 4 dimensions, while the $\Gamma^{(5)}$ is included in the term with $V_{\alpha\beta\gamma}$. This term is, of course, absent in 4 dimensions. The χ^2 term requires a Fierz reordering. Using $\not{\partial}\chi = 0$ one replaces $\Gamma_{\alpha\beta}\Gamma_\mu$ by $-\Gamma_\beta\Gamma_\alpha\Gamma_\mu$ to obtain an expression proportional to $\bar{\chi}\overleftarrow{\partial}_\alpha\Gamma_\beta O\Gamma_\beta\Gamma_\alpha\Gamma_\mu\chi$. In 10 dimensions $O = \Gamma^{(1)}$ or $\Gamma^{(3)}$ because of (A.6). The remaining terms can be written completely in terms of $X_{\alpha\beta\gamma}$, and the χ^2 contribution to $\theta_{\mu\nu}$, using the differential identities of the appendix. In 4 dimensions the result is analogous, the two terms remaining in O are γ_μ and $\gamma_\mu\gamma_5$, and one obtains $\theta_{\mu\nu}$ as in 10 dimensions, plus the axial current $J_\mu^{(5)}$. So, in 4 dimensions the variation of J_μ gives only one new current due to the vanishing of an antisymmetrization over 5 indices.

We have chosen the particular combination $V_{\alpha\beta\gamma}$ for two reasons. Its supersymmetry variation gives only J_μ:

$$\delta V_{\alpha\beta\gamma} = \tfrac{1}{2}\bar{\varepsilon}\left(\Gamma_{[\alpha\beta}J_{\gamma]} - \tfrac{1}{9}\Gamma_{\alpha\beta\gamma}\Gamma\cdot J\right), \tag{5.13}$$

so that it does not lead to new current components. Secondly, our choice (5.11) is motivated by the result of sect. 3: we recognize that it is this particular combination

that couples to $F_{\alpha\beta\gamma}$, the field-strength tensor of the physical field $A_{\beta\gamma}$ of supergravity. Note that $V_{\alpha\beta\gamma}$ has gauge transformations $\delta V_{\alpha\beta\gamma} = \partial_{[\alpha}\Lambda_{\beta\gamma]}$. This is so because the term containing $V_{\alpha\beta\gamma}$ in (5.10) corresponds to a completely antisymmetric four-index tensor

$$W_{\alpha\beta\gamma\delta} = \tfrac{1}{2}F_{[\alpha\beta}F_{\gamma\delta]} + \tfrac{1}{12}\partial_{[\alpha}X_{\beta\gamma\delta]}, \tag{5.14}$$

which satisfies

$$\partial_{[\mu}W_{\alpha\beta\gamma\delta]} = 0. \tag{5.15}$$

$V_{\alpha\beta\gamma}$ is obtained by solving this constraint, and is defined only up to gauge transformations. On the other hand, the coupling $V^{\alpha\beta\gamma}F_{\alpha\beta\gamma}$ is not invariant under V gauge transformations. This suggests that there is originally a coupling $V^{\alpha\beta\gamma}t_{\alpha\beta\gamma}$ with $t_{\alpha\beta\gamma}$ conserved off-shell, while later t becomes F (which is only conserved on-shell). We will come back to this point in sect. 6.

It is therefore the current $X_{\alpha\beta\gamma}$ given in (5.12) which determines how the current multiplet will continue. At this point the calculation diverges from the one in 4 dimensions in a crucial way. The variation is

$$\delta X_{\alpha\beta\gamma} = \tfrac{1}{2}\bar{\varepsilon}\Gamma_{\mu\nu}\Gamma_{\alpha\beta\gamma}\chi_{\mu\nu}, \tag{5.16}$$

where

$$\chi_{\mu\nu} = F_{\mu\nu}\chi. \tag{5.17}$$

The new current contains, of course, the supersymmetry current J_μ. One can write $\chi_{\mu\nu}$ as

$$\chi_{\mu\nu} = \hat{\chi}_{\mu\nu} + \tfrac{1}{4}\Gamma_{[\mu}J_{\nu]} - \frac{7}{9\times 24}\Gamma_{\mu\nu}\Gamma\cdot J, \tag{5.18}$$

where $\hat{\chi}_{\mu\nu}$ satisfies $\Gamma_\mu\hat{\chi}_{\mu\nu} = 0$ so that J_μ cannot be constructed from $\hat{\chi}_{\mu\nu}$. Note that $\hat{\chi}_{\mu\nu}$ is not conserved. The differential condition on $\chi_{\mu\nu}$ is

$$\Gamma_{\mu\nu}\partial\!\!\!/\chi_{\mu\nu} = 0, \tag{5.19}$$

which is equivalent to $\partial_\mu J_\mu = 0$. Clearly, the appearance of the current $\hat{\chi}_{\mu\nu}$ is a crucial new aspect of $N = 1$ in 10 dimensions. In the calculation of Howe and Lindström [15] in 5 dimensions, there is only one current with the dimension of J_μ, namely J_μ itself.

Let us now vary $\chi_{\mu\nu}$, to verify that the multiplet of currents does not terminate at

this stage. We obtain

$$\delta\chi_{\mu\nu} = -\tfrac{1}{4}H_{\mu\nu,\alpha\beta}\Gamma^{\alpha\beta}\varepsilon + \tfrac{1}{32}\left(\theta_{\alpha[\mu} - \tfrac{7}{54}\delta_{\alpha[\mu}\theta_{\lambda\lambda}\right)\Gamma_{\nu]\alpha}\varepsilon$$

$$-\tfrac{1}{2}\partial_{[\mu}V_{\nu\alpha\beta]}\Gamma^{\alpha\beta}\varepsilon - \frac{1}{36\times 64}B_{\mu\nu,\alpha\beta\gamma\delta}\Gamma^{\alpha\beta\gamma\delta}\varepsilon$$

$$+\tfrac{1}{64}\Big[\partial_\lambda X_{\mu\nu\lambda} + \tfrac{1}{2}\partial_{[\mu}X_{\nu]\alpha\beta}\Gamma^{\alpha\beta} + \tfrac{3}{2}\partial_\alpha X_{\mu\nu\beta}\Gamma^{\alpha\beta}$$

$$+\partial^\alpha X_{\alpha\beta[\mu}\Gamma_{\nu]\beta} + \tfrac{1}{3}\Gamma_{\alpha\beta\gamma[\mu}\partial_{\nu]}X_{\alpha\beta\gamma} - \tfrac{1}{18}\partial_\lambda X_{\alpha\beta\gamma}\Gamma_{\mu\nu\lambda\alpha\beta\gamma}\Big]\varepsilon, \quad (5.20)$$

where we have had to introduce the new currents

$$H_{\mu\nu,\alpha\beta} = \tfrac{2}{3}\left(F_{\mu(\nu}F_{\alpha)\beta} - F_{\mu(\nu}F_{\beta)\alpha}\right) - \text{traces}, \quad (5.21)$$

$$B_{\mu\nu,\alpha\beta\gamma\delta} = \bar{\chi}\Gamma_{\mu\nu[\alpha\beta\gamma}\overleftrightarrow{\partial}_{\delta]}\chi - \bar{\chi}\Gamma_{\alpha\beta\gamma\delta[\mu}\overleftrightarrow{\partial}_{\nu]}\chi. \quad (5.22)$$

The traces in (5.21) have all been written in terms of $\theta_{\mu\nu}$. It is obvious by now that we are generating a very large multiplet, and it is an open question how many steps will be needed to complete the multiplet. It is not very illuminating to continue the construction of the supercurrent in the same way. At the present stage we have obtained the lowest dimensional components of the current multiplet. The corresponding fields will therefore be the highest dimensional ones. On dimensional grounds one argues that the fields that couple to $X_{\alpha\beta\gamma}$ and $V_{\alpha\beta\gamma}$ can occur quadratically in the ten-dimensional action, and might thus play the role of usual auxiliary fields. The fields that couple to higher dimensional currents – except the physical fields which appear with derivatives – have too low a dimension to occur in a conventional action, and we expect that they cannot be identified directly with particular contributions to the action and transformation rules of the Maxwell system coupled to supergravity. So at this stage we should have obtained that part of the supercurrent which is most directly related to the results of sect. 3, and we leave the completion of the multiplet of currents for later work.

It is possible to translate the results of this section into superfield language. The lowest dimensional current is $X_{\alpha\beta\gamma}$, and we therefore expect that the supercurrent can be written in terms of a superfield $\Phi_{\alpha\beta\gamma}(x,\theta)$*. (Since in the variation of the currents only products of $F_{\mu\nu}$ and χ appear, it seems natural to restrict the superfield to gauge-invariant components. This rules out $V_{\alpha\beta\gamma}$ as first component.) Since θ is a Majorana-Weyl spinor, the analysis of the expansion of the superfield in powers of θ is straightforward, at least for the low-θ sectors. One can write

$$\Phi_{\alpha\beta\gamma}(x,\theta) = X_{\alpha\beta\gamma} + \bar{\theta}\psi_{\alpha\beta\gamma} + \bar{\theta}\Gamma_{\mu\nu\rho}\theta Y_{\alpha\beta\gamma,\mu\nu\rho} + \cdots. \quad (5.23)$$

* This has also been noted by P. Howe, private communication.

Clearly the superfield has to be constrained to restrict it to the multiplet of currents. From (5.16) we see that the θ-sector should contain only a two-index spinor field $\chi_{\mu\nu}$, and not a three-index spinor $\psi_{\alpha\beta\gamma}$. Therefore,

$$\psi_{\alpha\beta\gamma} = \Gamma_{\mu\nu}\Gamma_{\alpha\beta\gamma}\chi_{\mu\nu}, \tag{5.24}$$

and solving $\chi_{\mu\nu}$ in terms of $\psi_{\alpha\beta\gamma}$, substitution of $\chi_{\mu\nu}(\psi)$ back into (5.24), leads to the constraint on the superfield (with $D = \partial/\partial\bar{\theta} + \not{\partial}\theta$):

$$(D\Phi)_{\mu\nu\rho} = \left(-\tfrac{1}{48}\Gamma_{\alpha\beta}\Gamma_{\mu\nu\rho}\Gamma_{\gamma} - \tfrac{5}{168}\Gamma_{\alpha}\Gamma_{\mu\nu\rho}\Gamma_{\beta\gamma} - \tfrac{1}{84}\Gamma_{\mu\nu\rho}\Gamma_{\alpha\beta\gamma}\right)(D\Phi)_{\alpha\beta\gamma}. \tag{5.25}$$

$\chi_{\mu\nu}$ must, of course, satisfy a (not necessarily independent) differential constraint to ensure that the conservation law of J_{μ} is satisfied. In terms of the superfield this condition reads

$$\left(\Gamma_{\alpha\beta}\partial_{\gamma} + \tfrac{1}{4}\not{\partial}\Gamma_{\alpha\beta\gamma}\right)(D\Phi)_{\alpha\beta\gamma} = 0. \tag{5.26}$$

One easily checks that this implies (5.19) for the θ-sector. To show that the constraints (5.25) and (5.26) are sufficient (at least for the low-θ sectors), we must now show that the θ^2 sector has indeed the field content that we have previously obtained by direct variation of the bilinear expressions. This is the case, but we shall spare the reader the tedious but straightforward algebra.

Finally, let us count the number of components which this part of the multiplet contains. In the unconstrained superfield $\Phi_{\alpha\beta\gamma}$, the numbers are 120, $120 \cdot 16$, and $120 \cdot 120$ for $X_{\alpha\beta\gamma}$, $\psi_{\alpha\beta\gamma}$ and $Y_{\alpha\beta\gamma,\mu\nu\lambda}$ respectively. The algebraic constraint (5.25) reduces $\psi_{\alpha\beta\gamma}$ to $\chi_{\mu\nu}$, which has only $45 \cdot 16$ components, of which a further 16 are eliminated by the conservation law (5.26). In the θ^2 sector we have $H_{\mu\nu,\lambda\rho}$, $V_{\mu\nu\lambda}$ and $\theta_{\mu\nu}$, and the six-index tensor $B_{\mu\nu,\alpha\beta\gamma\delta}$. For $H_{\mu\nu,\lambda\rho}$ we count 770 components, for $V_{\mu\nu\lambda}$ 84 (since 36 of its degrees of freedom can be gauged away) and for $\theta_{\mu\nu}$ 45 components. $B_{\mu\nu,\alpha\beta\gamma\delta}$ has 1050 degrees of freedom. Obviously, the multiplet easily exceeds the $128 + 128$ components of $N = 4$ conformal supergravity, but at this stage one cannot yet see how much remains after reduction to 4 dimensions.

6. Implications of the analysis of currents and the Noether coupling for auxiliary fields

In sects. 3 and 5 we have developed two tools, which in principle should lead to information about the auxiliary fields of $N = 1$ supergravity in 10 dimensions. On the one hand, we have constructed a coupling of matter fields to supergravity in 10 dimensions: the Maxwell-Einstein system. We recall, that the first indications of the presence of an auxiliary axial vector field in 4 dimensions also came from the supersymmetric Maxwell-Einstein theory. On the other hand, we have obtained

partial results on the multiplet of currents of the same supersymmetric Maxwell theory in 10 dimensions. Analysis of the supercurrent has led to knowledge of auxiliary fields in a number of cases. In this section we shall try to extract such information from both sources.

Let us start with the multiplet of currents. Our first task is obviously the construction of a conjugate multiplet of fields. Therefore we write the coupling

$$\tfrac{1}{4}\theta^{\mu\nu}s_{\mu\nu} - \bar{J}^{\mu}\phi_{\mu} + \tfrac{3}{4}\sqrt{2}\, V^{\alpha\beta\gamma}t_{\alpha\beta\gamma} + X^{\alpha\beta\gamma}u_{\alpha\beta\gamma} + \cdots . \tag{6.1}$$

The requirement that (6.1) is an invariant leads to transformation rules for the fields $s_{\mu\nu}, \phi_{\mu}, t_{\alpha\beta\gamma}, u_{\alpha\beta\gamma}, \ldots$, since the transformation rules of the corresponding currents are known. The expression (6.1), of course, goes on with products of the currents $\hat{\chi}_{\mu\nu}$, $H_{\mu\nu,\alpha\beta}$, $B_{\mu\nu,\alpha\beta\gamma\delta}$, and others, with their corresponding fields, but for our present purposes (6.1) will be sufficient. This implies, however, that the transformation rules of some of the fields will be incomplete.

We emphasize that at this point there is no reason to identify $s_{\mu\nu}$ and ϕ_{μ} with the physical fields $h_{\mu\nu}$ and ψ_{μ} of linearized $N=1$ supergravity. We have chosen the coefficients in (6.1) such that at a later stage the transformation rules will be those of the actual fields of supergravity.

Let us now summarize the transformation rules of the currents in (6.1). They are

$$\begin{aligned}
\delta\theta_{\mu\nu} &= 2\bar{\varepsilon}\Gamma_{(\mu\lambda}\partial_{\lambda}J_{\nu)},\\
\delta J_{\mu} &= -\tfrac{1}{8}\Gamma_{\nu}\varepsilon\theta_{\mu\nu} - \tfrac{1}{8}\Gamma_{\mu}{}^{\alpha\beta\gamma\delta}\varepsilon\partial_{\alpha}V_{\beta\gamma\delta} + \tfrac{1}{96}\Gamma_{\mu\nu}\partial_{\nu}X_{\alpha\beta\gamma}\Gamma^{\alpha\beta\gamma}\varepsilon,\\
\delta V_{\alpha\beta\gamma} &= \tfrac{1}{2}\bar{\varepsilon}\left(\Gamma_{[\alpha\beta}J_{\gamma]} - \tfrac{1}{9}\Gamma_{\alpha\beta\gamma}\Gamma\cdot J\right),\\
\delta X_{\alpha\beta\gamma} &= -12\bar{\varepsilon}\Gamma_{[\alpha}\hat{\chi}_{\beta\gamma]} + 3\bar{\varepsilon}\left(\Gamma_{[\alpha\beta}J_{\gamma]} - \tfrac{5}{54}\Gamma_{\alpha\beta\gamma}\Gamma\cdot J\right).
\end{aligned} \tag{6.2}$$

Note, that it is only through $X_{\alpha\beta\gamma}$ that the currents communicate with the remaining part of the current multiplet. Using (6.2) one easily determines the transformation rules of the fields by requiring that (6.1) is invariant. One obtains

$$\begin{aligned}
\delta s_{\mu\nu} &= \tfrac{1}{4}\bar{\varepsilon}\left(\Gamma_{\mu}\phi_{\nu} + \Gamma_{\nu}\phi_{\mu}\right),\\
\delta\phi_{\mu} &= -\tfrac{1}{2}\partial_{\lambda}s_{\rho\mu}\Gamma_{\lambda\rho}\varepsilon + \tfrac{1}{24}\sqrt{2}\left(\Gamma_{\mu}{}^{\alpha\beta\gamma} - 9\delta_{\mu}^{\alpha}\Gamma^{\beta\gamma}\right)t_{\alpha\beta\gamma}\varepsilon\\
&\quad + \tfrac{5}{18}\left(\Gamma_{\mu}\Gamma^{\alpha\beta\gamma} - \tfrac{54}{5}\delta_{\mu}^{\alpha}\Gamma^{\beta\gamma}\right)u_{\alpha\beta\gamma} + \text{more},\\
\delta t_{\alpha\beta\gamma} &= \tfrac{1}{12}\sqrt{2}\,\bar{\varepsilon}\Gamma^{\mu\nu}{}_{\alpha\beta\gamma}\partial_{\mu}\phi_{\nu} + \text{more},\\
\delta u_{\alpha\beta\gamma} &= \tfrac{1}{96}\bar{\varepsilon}\Gamma_{\alpha\beta\gamma}\Gamma^{\mu\nu}\partial_{\mu}\phi_{\nu} + \text{more}.
\end{aligned} \tag{6.3}$$

The transformation rules of ϕ_μ, $t_{\alpha\beta\gamma}$ and $u_{\alpha\beta\gamma}$ still contain terms which depend on the fields that couple to $\hat{\chi}_{\mu\nu}$, $H_{\mu\nu,\alpha\beta}$ and $B_{\mu\nu,\alpha\beta\gamma\delta}$. They have not been written explicitly. We remark that $\delta t_{\alpha\beta\gamma}$ satisfies $\partial^\alpha \delta t_{\alpha\beta\gamma} = 0$, in agreement with the transformation $\delta V_{\alpha\beta\gamma} = \partial_{[\alpha}\Lambda_{\beta\gamma]}$ discussed in sect. 5. One could attempt to identify $h_{\mu\nu}$ with $s_{\mu\nu}$, and ψ_μ with ϕ_μ, but the transformation rules (6.3) are not the linearized form of (2.20). Moreover, the physical fields ϕ, λ and $A_{\mu\nu}$ do not appear at all in (6.1) and (6.3). One can ask to which currents ϕ and λ might couple ($A_{\mu\nu}$ plays a somewhat different role, to which we come back later). The only currents of the right dimension that can be constructed out of the fields of the on-shell Maxwell system, and that can couple to ϕ and λ, are obviously $F_{\alpha\beta}F^{\alpha\beta}$ and $\Gamma \cdot F\chi$, respectively. But these currents are already contained in $\theta_{\mu\nu}$ and J_μ [see (5.9)].

Let us now use the information obtained from the Noether coupling of sect. 3. If one has a complete set of auxiliary fields, the trilinear Noether sector is separately invariant under linearized global transformations, when the matter fields are on-shell. Therefore, we expect that elimination of the auxiliary fields in (6.1) leads to the Noether couplings of sect. 3. If we look at the lagrangian (3.25) of the coupled system we notice that indeed λ couples (in lowest order) to $\Gamma \cdot J$, and that indeed ϕ (or rather $\ln\phi$) couples to the trace of $\theta_{\mu\nu}$. This coupling is

$$-\tfrac{1}{12}\sqrt{2}\,\bar{J}^\mu \Gamma_\mu \lambda - \tfrac{1}{32}(\ln\phi)\theta_{\mu\mu}. \tag{6.4}$$

The couplings of the fields $h_{\mu\nu}$, ψ_μ and $A_{\mu\nu}$ of linearized ten-dimensional supergravity can also be read off from (3.25). We obtain for the complete Noether coupling to order κ of all physical fields

$$\tfrac{1}{4}\theta^{\mu\nu}h_{\mu\nu} - \tfrac{1}{32}\theta_{\mu\mu}\ln\phi - \bar{J}^\mu\psi_\mu - \tfrac{1}{12}\sqrt{2}\,\bar{J}^\mu\Gamma_\mu\lambda + \tfrac{3}{4}\sqrt{2}\,V^{\alpha\beta\gamma}\partial_{[\alpha}A_{\beta\gamma]}. \tag{6.5}$$

On comparing this with (6.1), we see that it is tempting to make the identification

$$s_{\mu\nu} = h_{\mu\nu} - \tfrac{1}{8}\delta_{\mu\nu}\ln\phi, \tag{6.6a}$$

$$\phi_\mu = \psi_\mu + \tfrac{1}{12}\sqrt{2}\,\Gamma_\mu\lambda, \tag{6.6b}$$

$$t_{\alpha\beta\gamma} = \partial_{[\alpha}A_{\beta\gamma]} = F_{\alpha\beta\gamma}. \tag{6.6c}$$

At this stage it is interesting to look back to sect. 2, where we have discussed the reduction of supergravity from 11 to 10 dimensions. The combinations (6.6a) and (6.6b) both appear at an intermediate stage in sect. 2. The elfbein, when reduced to ten dimensions, had to be rescaled by a multiplicative factor $\phi^{-1/8}$, of which we see the lowest order contribution in (6.6a). Likewise the eleven-dimensional gravitino Ψ_μ (with a ten-dimensional index) had to be shifted as in (6.6b) to diagonalize the ten-dimensional action [see (2.13)]. So we conclude that the fields occurring in (6.1)

are the fields which are obtained directly from 11 dimensions, before further manipulations are performed. Even though the matter system has been derived in ten and does not even exist in eleven dimensions, its multiplet of currents seems to know about eleven-dimensional supergravity. The reason for this could well be that the coupled Maxwell-Einstein system is itself a truncation of the eleven-dimensional supergravity theory (if one sets only the $(\frac{3}{2},1)$ multiplet to zero but not the Maxwell multiplet in the reduction) so that by the result (6.6a, b), it is just asking us to be put in its most natural form: the one emerging from 11 dimensions.

Let us now discuss (6.6c). The identification of $t_{\alpha\beta\gamma}$ with $F_{\alpha\beta\gamma}$ suggests, that $t_{\alpha\beta\gamma}$ is an auxiliary field of supergravity in ten dimensions, which, by its equation of motion, becomes equal to $F_{\alpha\beta\gamma}$. However, we have seen that $t_{\alpha\beta\gamma}$ must be conserved, which seems to contradict this identification. A nice way around this problem, which ultimately makes contact again with the non-diagonal form of the action, is to introduce $A_{\mu\nu}$ as a Lagrange multiplier in the supergravity action. If we assume that $A_{\mu\nu}$, $t_{\alpha\beta\gamma}$ and the current $V^{\alpha\beta\gamma}$ occur in the Maxwell-Einstein action as

$$\tfrac{3}{4}t_{\alpha\beta\gamma}t^{\alpha\beta\gamma}+\tfrac{3}{2}A_{\mu\nu}\partial_\rho t^{\mu\nu\rho}+\tfrac{3}{4}\sqrt{2}\,V^{\alpha\beta\gamma}t_{\alpha\beta\gamma}, \tag{6.7}$$

then the result is in exact agreement with the action obtained in sect. 3. The equation of motion of $A_{\mu\nu}$ takes care of conservation of $t_{\mu\nu\rho}$. The equation of motion for $t_{\mu\nu\rho}$ leads to

$$t_{\alpha\beta\gamma}=\partial_{[\alpha}A_{\beta\gamma]}-\tfrac{1}{2}\sqrt{2}\,V_{\alpha\beta\gamma}. \tag{6.8}$$

If we now consider the explicit form of $V_{\alpha\beta\gamma}$ [see (5.11)] we see that the substitution of (6.8) into (6.7) gives us exactly all terms $F'_{\alpha\beta\gamma}$ in (3.25), including the proper covariantization corresponding to the extra Maxwell transformations of $A_{\mu\nu}$ induced by the matter coupling.

To see how (6.7) makes contact with the non-diagonal formulation of the supergravity action, let us write a trial lagrangian for the fields contained in (6.1):

$$\mathfrak{L}=-\frac{E}{2\kappa^2}R(E)-\tfrac{1}{2}E\bar{\phi}_\mu\Gamma^{\mu\nu\rho}D_\nu\phi_\rho+\tfrac{3}{4}Et_{\alpha\beta\gamma}t^{\alpha\beta\gamma}+\tfrac{3}{2}EA_{\mu\nu}\partial_\rho t^{\mu\nu\rho}. \tag{6.9}$$

Here the vielbein $E_\mu{}^m$ is related to the symmetric tensor $s_{\mu\nu}$ appearing in (6.1) by

$$\delta_\mu{}^m+\kappa s_\mu{}^m=\tfrac{1}{2}\left(E_\mu{}^m+E_m{}^\mu\right).$$

The lagrangian (6.9) is not invariant under the transformations (6.3). However, if one requires invariance, one easily derives that (6.9) must also include terms which involve a spinor field λ, and a scalar field ϕ. These terms take the form

$$\mathfrak{L}\sim-\tfrac{2}{3}\sqrt{2}\,E\bar{\lambda}\Gamma^{\mu\nu}D_\mu\phi_\nu-\frac{E}{2\kappa^2}(\ln\phi)R, \tag{6.10}$$

while the supersymmetry transformations of ϕ, λ and $A_{\mu\nu}$ which follow from (6.9), (6.10), are

$$\delta\lambda = \tfrac{1}{8}\Gamma^{\alpha\beta\gamma} t_{\alpha\beta\gamma}\varepsilon, \tag{6.11}$$

$$\delta(\ln\phi) = -\tfrac{1}{3}\sqrt{2}\,\kappa\bar{\varepsilon}\lambda, \tag{6.12}$$

$$\delta A_{\mu\nu} = \tfrac{1}{2}\sqrt{2}\,\bar{\varepsilon}\left(\Gamma_{[\mu}\phi_{\nu]} - \tfrac{1}{3}\sqrt{2}\,\Gamma_{\mu\nu}\lambda\right). \tag{6.13}$$

Note that the transformation rule of $A_{\mu\nu}$ is exactly the transformation rule as it was obtained from dimensional reduction, before making the shift (6.6b) which relates ϕ_μ to ψ_μ. Also, the results (6.11), (6.12) are precisely the correct rules for λ and ϕ, if indeed one sets $t_{\alpha\beta\gamma}$ equal to $F_{\alpha\beta\gamma}$ according to (6.8). Finally, the transformation of λ and ϕ_μ into $\partial\!\!\!/\phi/\phi$ can be determined by using the variation of $E_\mu{}^m$ into ψ_μ. The result agrees with the transformation rules which were obtained at an intermediate stage in sect. 2.

We have not yet discussed the field $u_{\alpha\beta\gamma}$, which in (6.1) couples to the current $X_{\alpha\beta\gamma}$. It does not play a role in the identification of the physical fields, so it seems reasonable to add a term to the action

$$\mathcal{L} \sim u^{\alpha\beta\gamma} u_{\alpha\beta\gamma}. \tag{6.14}$$

It is not quite possible to do this at the present stage, because the auxiliary fields which we have not incorporated, i.e. the ones coupling to $\hat{\chi}$, H and B, interfere with new contributions in $\delta\mathcal{L}$ due to (6.14) by producing variations with the same structure. So although the term (6.14) is likely to be present in the final action, one cannot yet unambiguously work out its consequences.

Let us summarize the procedure followed. Starting from the fields of the current multiplet $s_{\mu\nu}$ (or $E_\mu{}^m$), ϕ_μ, $t_{\mu\nu\rho}$, $u_{\mu\nu\rho}$, ..., we have identified $s_{\mu\nu}$ and ϕ_μ with the zehnbein and gravitino in the Maxwell-Einstein action before shifting. The field $t_{\mu\nu\rho}$ couples to the same source as $F_{\mu\nu\rho}$, but since $t_{\mu\nu\rho}$ is conserved off-shell we had to introduce a gauge field $A_{\mu\nu}$ as a Lagrange multiplier to take care of this constraint. A suitable quadratic action for these fields was given in (6.9) but requiring full invariance we discovered the lacking physical fields λ and ϕ. Thus this method yields an action and transformation rules which agree with the results of sects. 2 and 3, after the elimination of $t_{\mu\nu\rho}$ from (6.9) by its field equation. In principle one could continue this program by including the remaining fields of the current multiplet. However, one then would encounter the real problem, viz., how to construct an action for fields with low dimension. It might be that one would need other multiplets to achieve this. In fact we have seen already traces of such a multiplet, namely the Lagrange multipliers $A_{\mu\nu}$, λ, ϕ in (6.9) and (6.10).

As far as the field $t_{\mu\nu\rho}$ is concerned, the above analysis is complete. It is therefore interesting to rewrite the action and transformation rules for the complete Maxwell-Einstein system, on the basis of (6.9). One finds

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\mathrm{SG}} + \mathcal{L}_{\mathrm{matter}}, \\
\mathcal{L}_{\mathrm{SG}} &= -\tfrac{1}{2}eR(e,\omega(e)) - \tfrac{1}{2}e\bar{\psi}_\mu\Gamma^{\mu\rho\sigma}D_\rho(\omega(e))\psi_\sigma \\
&\quad -\tfrac{1}{2}e\bar{\lambda}\not{D}(\omega(e))\lambda - \tfrac{9}{16}e(\partial_\mu\phi/\phi)^2 + \tfrac{3}{4}et_{\mu\nu\rho}^2 - \tfrac{3}{2}e\phi^{-3/4}t^{\mu\nu\rho}\partial_\mu A_{\nu\rho} \\
&\quad +\tfrac{1}{16}e\sqrt{2}\,t_{\alpha\beta\gamma}\left(\bar{\psi}_\mu\Gamma^{\mu\alpha\beta\gamma\nu}\psi_\nu + 6\bar{\psi}^\alpha\Gamma^\beta\psi^\gamma - \sqrt{2}\,\bar{\psi}_\mu\Gamma^{\alpha\beta\gamma}\Gamma^\mu\lambda\right) \\
&\quad -\tfrac{3}{8}\sqrt{2}\,e\bar{\psi}_\mu(\not{\partial}\phi/\phi)\Gamma^\mu\lambda + \text{four-fermion terms}, \\
\mathcal{L}_{\mathrm{matter}} &= -\tfrac{1}{4}e\phi^{-3/4}F_{\mu\nu}^2 - \tfrac{1}{2}e\bar{\chi}\not{D}(\omega(e))\chi \\
&\quad -\tfrac{1}{4}e\kappa\phi^{-3/8}\bar{\chi}\Gamma^\mu\Gamma^{\rho\sigma}F_{\rho\sigma}\left(\psi_\mu + \tfrac{1}{12}\sqrt{2}\,\Gamma_\mu\lambda\right) \\
&\quad +\tfrac{1}{4}\sqrt{2}\,e\kappa t^{\mu\nu\rho}\left(3\phi^{-3/4}A_\mu F_{\nu\rho} + \tfrac{1}{4}\bar{\chi}\Gamma_{\mu\nu\rho}\chi\right) \\
&\quad + \text{four-fermion terms}.
\end{aligned} \tag{6.15}$$

Upon solving $t_{\mu\nu\rho}$ by its fields equation

$$\begin{aligned}
t_{\mu\nu\rho} &= \phi^{-3/4}F'_{\mu\nu\rho} - \tfrac{1}{24}\sqrt{2}\,\kappa\left(\bar{\psi}^\alpha\Gamma_{\alpha\mu\nu\rho\beta}\psi^\beta + 6\bar{\psi}_{[\mu}\Gamma_\nu\psi_{\rho]}\right. \\
&\quad \left. -\sqrt{2}\,\bar{\psi}_\alpha\Gamma_{\mu\nu\rho}\Gamma^\alpha\lambda\right) - \tfrac{1}{24}\sqrt{2}\,\kappa\bar{\chi}\Gamma_{\mu\nu\rho}\chi,
\end{aligned} \tag{6.16}$$

the action (6.15) reduces to (3.25). We have arranged matters such that the $\delta\psi \sim t\varepsilon$ equation [with t given by (6.16)] is free from explicit ϕ-factors.

We have argued that it is the absence of conformal supergravity in 10 dimensions which makes it attractive to look at Poincaré supergravity and matter coupling there. The same is true, of course, for 5 dimensions. Nevertheless one could claim that the results of Howe and Lindström constitute an indication for the presence of a higher symmetry. Therefore, it is of interest to consider the possible symmetries of the ten-dimensional model in more detail, and we will do this on the basis of (6.15).

We shall show that it is possible to define local Weyl weights for all fields in such a way that the coupling of the Maxwell system to the fields $(e_\mu{}^m, \psi_\mu, A_{\mu\nu}, \lambda, \phi)$ is locally scale (= Weyl) invariant. The $N=1$, $d=10$ gauge action itself is not Weyl invariant, nor are the transformation rules of its fields Weyl covariant. However, the transformation rules of the matter fields (A_μ, χ) are again Weyl covariant, whether or not the coupling has taken place. As we have seen, the currents of the $d=10$

Maxwell system only couple to a subset of the supergravity fields. Therefore one expects that extra local symmetries are present in the coupling of the Maxwell system to supergravity. The Weyl invariance announced above, indicates that these extra symmetries have conformal echos.

The situation is strongly reminiscent of the $d=4$ case. In $N=1$ the Maxwell system couples only to the superconformal fields $(e_\mu{}^m, \psi_\mu, A_\mu)$ and is locally superconformal (and hence Weyl) invariant, while the $N=1$ Poincaré gauge action has uniform global scale weight $2-d=-2$, as the reader may verify. The Poincaré action can be obtained by coupling $(e_\mu{}^m, \psi_\mu, A_\mu)$ to a compensating superconformal matter multiplet, but after fixing the conformal invariances the global Weyl invariance uses other weights than the original local Weyl weights of the matter multiplet. Even more striking is the analogy with the $N=2$ case [14]. The coupling of the $N=2$ Maxwell system with complex fields $(A, \Psi^i, V_\mu, B_{ij}=B_{ji})$ to the $N=2$ superconformal fields $(e_\mu{}^m, \psi_\mu{}^i, \chi^i, T_{mn}, D, V_{\mu j}^i, A_\mu)$ has the same features as we discussed for the $N=1$ case. In addition, there is an analogy between the $d=10$ field $t_{\mu\nu\rho}$ and the $d=4$ field $T_{mn}=-T_{nm}$. The field T_{mn} appears in the $N=2$ Poincaré gauge action as $T_{mn}^2+T_{mn}\partial_m B_n+(\partial_{[m}B_{n]})^2$, where B_m is the physical photon of $N=2$ Poincaré supergravity, while the gravitino law $\delta\psi_\mu$ contains a term $T_{mn}\varepsilon$. The same holds for $t_{\mu\nu\rho}$ in $d=10$, except that a term $(\partial_\mu A_{\nu\rho})^2$ is absent in (6.15) because $t_{\mu\nu\rho}$ has to be conserved off-shell.

If one takes these analogies with $d=4$ seriously, one might attempt to go back and arrive at a formulation in which the $N=1, d=10$ gauge action is a locally Weyl (superconformal?) coupling of a compensating matter multiplet to Weyl (superconformal) fields. The results of the analysis of currents indeed suggest that the Lagrange multipliers $A_{\mu\nu}$, λ, ϕ are the beginning of this matter multiplet, while the fields which couple directly to the currents, fall into the gauge multiplet.

Let us now demonstrate the Weyl invariance in the $d=10$ Maxwell-Einstein coupling. In ordinary gravity, the Maxwell action in curved space is only Weyl invariant in $d=4$. The reason is that the photon has always vanishing local Weyl weight (to exclude $\partial_\mu\Lambda$ terms in the variation of the action, where Λ is the Weyl parameter), while $\bar{g}g^{\mu\rho}g^{\nu\sigma}$ is Weyl invariant in $d=4$ only. *Supergravity improves this situation* because it dictates an extra factor $\phi^{-3/4}$ in front of the Maxwell action. By choosing the Weyl weight of the zehnbein $e_\mu{}^m$ to be -1 (which is the conventional normalization), the Weyl weight of ϕ must be $\frac{4}{3}(4-d)$ in order that the Maxwell action be Weyl invariant. The Dirac action for χ is locally scale invariant if one chooses the weight of χ equal to $\frac{1}{2}(d-1)$. This follows easily by first considering constant Λ, and then by noting that if Λ becomes local, $\partial_\mu\Lambda$ terms cancel since χ is a Majorana spinor [actually, for complex (Dirac) spinors, the same result holds since under a local scale transformation $\gamma^\mu D_\mu(\omega(e))\chi$ is free from $\partial_\mu\Lambda$ terms]. The Weyl weights of ψ_μ and λ follow from the Noether couplings and are $-\frac{1}{2}$ and $+\frac{1}{2}$, respectively. Let us now consider the $\bar{\chi}\chi F_{\mu\nu\rho}$ coupling in (3.25). One finds that $F_{\mu\nu\rho}$ has to transform homogeneously (without $\partial_\mu\Lambda$ terms!) with local weight

$2-d$. If Weyl invariance is to hold, we are therefore forced to use a formulation in which $F_{\mu\nu\rho}$ is replaced by an (auxiliary) field which transforms homogeneously under Weyl transformations. This strongly suggests to use $t_{\mu\nu\rho}$, and we continue our analysis based on (6.15). This choice is supported by the fact that the only term (as we shall see) in the coupling which breaks local scale invariance is $(A_\alpha F_{\beta\gamma})^2$, while this term is absent in the first-order formulation in (6.15) where $t_{\mu\nu\rho}$ is an independent field. All couplings in (6.15) are now Weyl invariant. In particular the covariantization of $\phi^{-3/4}F_{\mu\nu\rho}$ has the same weight as $t_{\mu\nu\rho}$, while also the four-fermion terms $\chi^2\lambda^2$ and $\chi^2\psi\lambda$ are Weyl invariant, despite the explicit factors κ^{-2} in front of them. This implies that they can be removed by rescaling ψ and λ.

Having fixed the Weyl weights to obtain a Weyl invariant *coupling*

$$w(e_\mu{}^m) = -1, \qquad w(\psi_\mu) = -\tfrac{1}{2}, \qquad w(t_{\mu\nu\rho}) = -2, \qquad w(\lambda) = +\tfrac{1}{2},$$

$$w(\phi) = \tfrac{4}{3}(4-d), \qquad w(A_\mu) = 0, \qquad w(\chi) = \tfrac{1}{2}(d-1),$$

one easily verifies that the terms in the pure gauge action have a uniform non-vanishing weight, namely $2-d$, except that the Lagrange multiplier term with $\partial_\mu A_{\nu\rho}$ has vanishing local Weyl weight $w(A_{\mu\nu}) = 0$ (if one assigns a global weight $2-d$ to $A_{\mu\nu}$, all terms in the gauge action have weight $2-d$). Again it is necessary to use $t_{\mu\nu\rho}$ as an independent field instead of $F_{\mu\nu\rho}$. Let us now consider the transformation rules. The factors of ϕ are precisely such that the laws δA_μ and $\delta\chi$ are covariant if one defines the Weyl weight of ε to be $-\frac{1}{2}$. This is in agreement with the $\{Q, Q\} \sim P$ commutator, since from $\delta\psi_\mu = \partial_\mu\varepsilon + \cdots$ and $\delta e_\mu{}^m = \frac{1}{2}\bar{\varepsilon}\gamma^m\psi_\mu$ one finds that ε and ψ_μ both have constant weights $-\frac{1}{2}$. *All* transformation rules are covariant under constant scale transformations, but only δA_μ and $\delta\chi$ are also locally Weyl covariant (the readers may check that the same surprising (?) features hold in $d=4$).

7. Conclusions

Our conclusions are that there are at least two axial vector auxiliary fields in ten-dimensional supergravity. We can also conclude that it will be difficult if not impossible to write down an action for this theory which includes the complete set of auxiliary fields. In our work this point appears most clearly in the fact that the extent of the multiplet of currents implies the existence of fields with too low a dimension to appear in a conventional action.

Part of the off-shell structure of $d=10$ supergravity is related to the multiplet of currents, but we have also found indications of a multiplet of Lagrange multipliers. The multiplet of currents can be written as a superfield $\Phi_{\alpha\beta\gamma}(x,\theta)$, which is subject to the two constraints (5.25), (5.26). Nilsson [21] has found that the fields of on-shell $d=10$ supergravity can be expressed in terms of a scalar superfield Φ. We note that the current multiplet $\Phi_{\alpha\beta\gamma}$ cannot be written as $\bar{D}\Gamma_{\alpha\beta\gamma}DK$ with K a scalar current superfield, although $\bar{D}\Gamma_{\alpha\beta\gamma}DK$ is a solution of both constraints. The reason is that

the θ^3 sector of K equals $(\bar{\theta}\Gamma^{\mu\nu\rho}\theta)(\bar{\theta}\Gamma_\mu\chi_{\nu\rho})$ with $\Gamma^\nu\chi_{\nu\rho}=0$, whereas $\Phi_{\alpha\beta\gamma}(\theta=0)$ varies into a $\chi_{\mu\nu}$ which is not traceless. Therefore the precise relation between field and current superfield remains an interesting problem.

We obtained the $N=1, d=10$ gauge action by dimensional reduction and truncation from the $N=1, d=11$ action in (2.16) and (2.28), and its transformation laws in (2.20). The absence of $\delta\lambda\sim\lambda^2\varepsilon$ variations and of $F_{\mu\nu\rho}\bar{\lambda}\lambda$ couplings was shown to be due to typical $d=10$ Majorana-Weyl identities, such as (A.3). The four-fermion couplings were obtained by requiring supercovariance of the fermionic field equations, but, as in $d=5$ [22], they are not obtained if one uses $\frac{1}{2}(\omega+\hat{\omega})$ as spin connection and replaces F by $\frac{1}{2}(F+\hat{F})$ and $\not{\partial}\phi$ by $\frac{1}{2}(\not{\partial}\phi+\hat{\not{D}}\phi)$ in the Noether couplings. We coupled this system to $N=1, d=10$ Yang-Mills theory, the action being obtained in (3.25), while the transformation laws were given in (4.10)–(4.14). The gauge algebra of the coupled system, given in (4.18) and (4.19) revealed the interesting fact that due to the coupling the Maxwell transformations no longer commute with supersymmetry. Thus, after coupling, one ends up with a bigger irreducible gauge group than before, similar to what happens when one couples chiral electrodynamics [23] to supergravity [24]. It might be possible that this intertwining of symmetries can be undone by introducing compensating fields [25].

One of our results which is of interest also outside supergravity, concerns the coupling of antisymmetric tensor fields to fields other than gravitation [17] in a consistent way. We have found in the coupling of $A_{\mu\nu}$ and A_μ that when A_μ transforms into $\partial_\mu\Lambda$, one must simultaneously rotate $A_{\mu\nu}$ into $\Lambda F_{\mu\nu}$. This rather unexpected mechanism might also be helpful in other cases where one couples several gauge fields to each other. In fact, this particular example was first found in the $N=4, d=4$ model [18].

Another unexpected result is that the coupling of the ten-dimensional Maxwell system to supergravity is Weyl invariant. The Maxwell action in ordinary gravity is Weyl invariant only in $d=4$, but in $d=10$ supergravity the scalar field ϕ (the 11-11 component of the elfbein) restores Weyl invariance to the Maxwell action, and in fact to the full coupling if one has introduced the auxiliary field $t_{\alpha\beta\gamma}$. Whether the coupling is invariant under a bigger group remains to be investigated.

The search for the auxiliary fields of extended supergravity has been successful so far for $N=1$ and $N=2$. In this paper, we have obtained some information about the elusive auxiliary fields of $N=4$ supergravity. To do this we have applied the "standard" methods of four-dimensional supergravity to $N=1$ in 10 dimensions: matter coupling and supercurrent analysis. One should admit that in 4 dimensions it was realized only *a posteriori* how close the results from matter coupling had come to the minimal set of auxiliary fields of $N=1$ supergravity. This is one of our reasons for presenting our results, which are somewhat inconclusive, in such detail. We have had the feeling throughout this work that ten-dimensional supergravity was giving us hints which we perhaps didn't appreciate sufficiently well. We leave it up to the careful reader to do better.

For three of us (E.B., B.d.W., P.v.N.) this work is part of the research program of the "Stichting voor Fundamenteel Onderzoek der Materie" (F.O.M.).

Appendix

DIRAC ALGEBRA IN TEN DIMENSIONS

Because in ten dimensions our spinors are both chiral ($\Gamma_{11}\lambda = \pm\lambda$) and Majorana spinors ($\bar{\lambda} = \lambda^{\mathrm{T}}C$), more identities exist than in eleven dimensions. In particular, the duality and differential identities derived below are interesting. We use the same charge conjugation matrix as in $d = 11$ ($C\Gamma_\mu C^{-1} = -\Gamma_\mu^{\mathrm{T}}$ for $\mu = 1,\ldots,10$) because the other C in $d = 10$ (with $C\Gamma_\mu C^{-1} = +\Gamma_\mu^{\mathrm{T}}$) does not admit Majorana spinors (see last reference in [16]).

Fierz rearrangements

Bilinears $\bar{\psi}\Gamma_{\alpha_1\cdots\alpha_n}\chi$ vanish when n is either even or odd, depending on whether ψ and χ have the same chirality or not. Since for $n > 5$ the product of n gamma matrices can be written as an ε-tensor times a product of $10 - n$ gamma matrices, times Γ_{11} (the $d = 10$ equivalent of γ_5 in $d = 4$), one need only retain terms with $n < 5$ in the Fierz rearrangement formula. A complete set of 32×32 matrices O_I satisfying $\mathrm{tr}\, O_I O_J = 32\delta_{IJ}$ is given by $O_I = \{I, \Gamma_\alpha, i\Gamma_{\alpha\beta}, i\Gamma_{\alpha\beta\gamma}, \Gamma_{\alpha\beta\gamma\delta}, \ldots, i\Gamma^{(10)}\}$, where $\Gamma^{(10)}$ is the product of 10 gamma matrices and will be denoted by Γ_{11}

$$i\Gamma^{(10)} \equiv \Gamma_{11}, \qquad \Gamma_{11}\Gamma_{11} = +1, \qquad \Gamma_{11} \text{ hermitian}. \tag{A.1}$$

The Fierz rearrangement formula reads

$$(\bar{\psi}M\chi)(\bar{\lambda}N\phi) = -\tfrac{1}{32}\sum_{n=0}^{5} C_n(\bar{\psi}\Gamma^{(n)}\phi)(\bar{\lambda}N\Gamma^{(n)}M\chi),$$

$$C_0 = 2, \qquad C_1 = 2, \qquad C_2 = -1, \qquad C_3 = -\tfrac{1}{3}, \qquad C_4 = \tfrac{1}{12}, \qquad C_5 = \tfrac{1}{120}. \tag{A.2}$$

The normalizations are such that in $\Gamma^{(n)}$ the sum over $\alpha_1 \cdots \alpha_n$ is unrestricted. For example, $\Gamma^{(0)}$ and $i\Gamma^{(10)}$ combine and yield $C_0 = 2$, but $\Gamma^{(5)}$ does not combine and then yields $1/5! = \frac{1}{120}$.

Symmetries

$\bar{\lambda}\Gamma_{\alpha_1\cdots\alpha_n}\chi$ is symmetric in λ and χ for $n = 0,3,4,7,8$ and antisymmetric for $n = 1,2,5,6,9,10$. If λ and χ have the same chirality, n is odd, otherwise n is even. Hence, for a Majorana-Weyl spinor λ, the only non-vanishing bilinear in $d = 10$ is the axial current $\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda$. If one Fierzes $(\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda)\bar{\lambda}$, one finds

$$(\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda)\bar{\lambda} = \tfrac{1}{2}(\bar{\lambda}\Gamma^{\tau[\alpha\beta}\lambda)\bar{\lambda}\Gamma_\tau\Gamma^{\gamma]}. \tag{A.3}$$

Contracting with $\Gamma_{\alpha\beta}$, one finds $-\frac{1}{2}$ the original expression. Hence the identity [21]

$$(\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda)\Gamma_{\alpha\beta}\lambda = 0. \tag{A.4}$$

A simple counting argument proves the existence of an identity of the form (A.3): a trilinear in λ has $\binom{16}{3} = 35 \times 16$ components. Since $I_3 = (\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda)\bar{\lambda}$ would have 120×16 components, while $I_2 = (\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda)\bar{\lambda}\Gamma_\alpha$ would have 45×16 and $I_1 = (\bar{\lambda}\Gamma^{\alpha\beta\gamma}\lambda)\bar{\lambda}\Gamma_{\alpha\beta}$ would have 10×16 components, one expects that I_3 can be expressed in terms of I_2 but that the contraction I_1 of I_2 vanishes. This yields indeed (A.3). In the proof one uses

$$\Gamma_\alpha\Gamma^{(n)}\Gamma^\alpha = (-1)^n(10-2n)\Gamma^{(n)}. \tag{A.5}$$

This equation contains several $d = 10$ analogues of the $d = 4$ relation $\gamma^m\sigma_{ab}\gamma_m = 0$, namely

$$\Gamma^{(5)}\Gamma^{\beta_1\cdots\beta_5}\Gamma^{(5)} = \Gamma^{(5)}\Gamma^\beta\Gamma^{(5)} = \Gamma_\alpha\Gamma^{(5)}\Gamma^\alpha = 0. \tag{A.6}$$

This identity can be generalized to $\Gamma^{(5)}\Gamma^{(2n+1)}\Gamma^{(5)} = 0$ and $\Gamma^{(2n+1)}\Gamma^{(5)}\Gamma^{(2n+1)} = 0$. The symbol $\Gamma^{(n)}$ denotes a product of n gamma matrices completely antisymmetrized.

Three-ψ_μ identities

By Fierz reordering $(\bar{\psi}_\mu\Gamma^{(1)}\psi_\nu)\psi_\rho$ and $(\bar{\psi}_\mu\Gamma^{(5)}\psi_\nu)\psi_\rho$ and antisymmetrizing in $(\mu\nu\rho)$ one finds only terms with $\Gamma^{(1)}$ and $\Gamma^{(5)}$ matrices. Using (A.6) one derives

$$\begin{aligned} \left(\bar{\psi}_{[\mu}\Gamma^\alpha\psi_\nu\right)\Gamma^\alpha\psi_{\rho]} &= 0, \\ \left(\bar{\psi}_{[\mu}\Gamma^{(5)}\psi_\nu\right)\Gamma^{(5)}\psi_{\rho]} &= 0. \end{aligned} \tag{A.7}$$

These relations are the counterpart of the Yang-Mills relation $\varepsilon_{abc}(\bar{\lambda}^a\gamma^\mu\lambda^b)(\gamma_\mu\lambda^c) = 0$ in $d = 4$. The first relation in (A.7) holds in $d = 4$ too, and is used there to show that in first-order formalism the gravitino equation reads $\Gamma^{\mu\rho\sigma}D_\rho(\omega)\psi_\sigma = 0$.

Duality

Consider $(\bar{\psi}_\mu\Gamma^{(5)}\psi_\nu)(\Gamma^{(5)}\psi_\rho)$. Writing the first $\Gamma^{(5)}$ as a 10-dimensional ε-tensor times the complementary $\Gamma^{(5)}$ times $-i\Gamma_{11}$, the Γ_{11} is absorbed by the chiral spinor, while the ε-tensor migrates to the second $\Gamma^{(5)}$ and inverts also this $\Gamma^{(5)}$ into its complementary $\Gamma^{(5)}$ times $-i\Gamma_{11}$. The net result is that one can strengthen (A.7) to

$$\left(\bar{\psi}_\mu\Gamma^{(5)}\psi_\nu\right)\left(\Gamma^{(5)}\psi_\rho\right) = 0 \qquad \text{not antisymmetric in } \mu\nu\rho. \tag{A.8}$$

One can extend the manipulations to more complicated ones. For example,

$$\left(\bar{\psi}_\mu \Gamma^{(k)}\Gamma^{(5)}\Gamma^{(l)}\psi_\nu\right)\Gamma^{(5)}\Gamma^{(m)}\psi_\rho = 0, \qquad \text{if } l+m=\text{even},$$

$$\left(\bar{\psi}_\mu \Gamma^{(k)}\Gamma^{(5)}\Gamma^{(l)}\psi_\nu\right)\Gamma^{(5)}\Gamma^{(n)}\lambda = 0, \qquad \text{if } l+n=\text{odd}, \tag{A.9}$$

where λ and ψ_μ have opposite chiralities.

Suppose one has a tensor

$$T^{\mu\nu} = \left(\bar{\psi}\Gamma^{\alpha\beta\gamma\delta\mu}\phi\right)\left(\bar{\xi}\Gamma_{\alpha\beta\gamma\delta}{}^{\nu}\chi\right), \tag{A.10}$$

with chiral spinors. Using

$$\Gamma^{\alpha\beta\gamma\delta\mu} = \tfrac{1}{120} i\varepsilon^{\alpha\beta\gamma\delta\mu\lambda_1\cdots\lambda_5}\Gamma_{\lambda_1\cdots\lambda_5}\Gamma_{11}, \tag{A.11}$$

and idem for $\Gamma_{\alpha\beta\gamma\delta\nu}$, one finds, after contracting the two ε-symbols, a relation of the form

$$T^{\mu\nu} = \pm\left(\tfrac{1}{5}g^{\mu\nu}T_\lambda{}^\lambda - T^{\nu\mu}\right). \tag{A.12}$$

The sign depends on the chirality of the spinors. Because of (A.8), the trace vanishes if the chirality of ϕ and χ is equal.

A peculiar identity

We are now able to prove the identity, which we needed in (2.33) when we discussed the four-fermion couplings in the $N=1$ gauge action. The identity reads

$$\tfrac{1}{16}\left(\Gamma^{\tau\alpha\beta\gamma\nu}\psi_\nu\right)\left(\bar{\psi}_\alpha\Gamma_\beta\psi_\gamma\right) - \tfrac{1}{16}\left(\bar{\psi}_\mu\Gamma^{\mu\tau\beta\gamma\nu}\psi_\nu\right)\left(\Gamma_\beta\psi_\gamma\right) + \tfrac{1}{2}g^{\beta[\tau}\Gamma^{\alpha\nu\gamma]}\psi_\nu\left(\bar{\psi}_\alpha\Gamma_\beta\psi_\gamma\right) = 0. \tag{A.13}$$

First we observe that due to (A.7) the first term is proportional to the last one

$$\Gamma^{\tau\alpha\beta\gamma\nu}\psi_\nu\left(\bar{\psi}_\alpha\Gamma_\beta\psi_\gamma\right) = -4\Gamma^{[\tau\alpha\gamma}g^{\nu]\beta}\psi_\nu\left(\bar{\psi}_\alpha\Gamma_\beta\psi_\gamma\right). \tag{A.14}$$

However, also the second term in (A.13) is proportional to the last one. To see this, write $\Gamma^{\mu\tau\beta\gamma\nu}$ as $\Gamma^\beta\Gamma^{\mu\tau\gamma\nu}$. No $\Gamma^{(3)}$ terms are needed due to the antisymmetry in $(\mu\nu)$. If one then Fierz reorders into the form $\bar{\psi}_\mu\Gamma^\rho O\Gamma_\beta\psi_\gamma$, only the terms with $O\sim\Gamma^\alpha$ remain, since the $\Gamma^{(3)}$ terms vanish due to the antisymmetry in $(\mu\gamma)$, while the $\Gamma^{(5)}$ cancel with (A.6). In this way one finds a total coefficient $-\frac{1}{4}-\frac{1}{4}+\frac{1}{2}=0$.

The identity (A.13) can be obtained by dimensional reduction of a similar identity in $d=11$, derived by Cremmer, Julia and Scherk [8]. In $d=11$, the Noether coupling

reads

$$I^{(4)}(\text{Noether}) = -\tfrac{1}{128}e\big(\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\big)\big(\bar{\psi}_\mu\Gamma^{\mu\alpha\beta\gamma\delta\nu}\psi_\nu + 12\bar{\psi}^\alpha\Gamma^{\beta\gamma}\psi^\delta\big), \qquad \text{(A.15)}$$

while $I^4(\mathrm{E}+\mathrm{RS})$ is, of course, the same as in $d=10$, namely given by (2.27). Supercovariance of the gravitino field equation requires that

$$\delta I^{(4)}/\delta\bar{\psi}_\mu = -\tfrac{1}{32}e\big(\Gamma^{\mu\alpha\beta\gamma\delta\nu}\psi_\nu + 12g^{\mu\alpha}\Gamma^{\beta\gamma}\psi^\delta\big)\big(\bar{\psi}_{[\alpha}\Gamma_{\beta\gamma}\psi_{\delta]}\big)$$
$$-\tfrac{1}{4}e\Gamma^{\mu\rho\sigma}\Gamma^{mn}\psi_\sigma\big(\hat{\omega}_{\rho mn} - \omega_{\rho mn}(e)\big). \qquad \text{(A.16)}$$

Equating both results for $\delta I^{(4)}/\delta\bar{\psi}_\mu$ leads to the identity of [8] (after contraction with ψ_δ and using that $\bar{\psi}_\alpha g^{\beta[\alpha}\Gamma^{\delta\mu\nu]}\psi_\delta = 0$)

$$-\tfrac{1}{64}\big(\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu\big)\big(\bar{\psi}_\alpha\Gamma_{\beta\gamma}\psi_\delta\big) + \tfrac{1}{64}\big(\Gamma_{\beta\gamma}\psi_\nu\big)\big(\bar{\psi}_\alpha\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\delta\big)$$
$$+\tfrac{1}{32}\big(\Gamma^{\mu\nu\alpha\beta\gamma\delta}\psi_\nu\big)\big(\bar{\psi}_\alpha\Gamma_\beta\psi_\delta\big) - \tfrac{1}{32}\big(\Gamma_\beta\psi_\nu\big)\big(\bar{\psi}_\alpha\Gamma^{\mu\nu\alpha\beta\delta}\psi_\delta\big)$$
$$-\tfrac{1}{2}g^{\beta[\mu}\Gamma^{\alpha\nu\delta]}\psi_\nu\big(\bar{\psi}_\alpha\Gamma_\beta\psi_\delta\big) = 0. \qquad \text{(A.17)}$$

The $\Gamma^{(6)}$ terms come from the Noether coupling in $d=11$, while the rest of the terms is independent of d. In $d=10$ the $\Gamma^{(6)}$ terms with factor $\frac{1}{64}$ become $\Gamma^{(5)}$ terms with factors $\frac{1}{32}$ (because β or γ in $\bar{\psi}_\alpha\Gamma_{\beta\gamma}\psi_\delta$ must take on the value 11), and the d-independent terms add to the d-dependent terms to yield the deceptively simple $d=10$ identity in (A.13).

Differential identities

Using the observation that $2\Gamma_{[\alpha}\partial_{\mu]} = \frac{1}{2}[\Gamma_{\alpha\mu}, \not{\partial}]$, one can prove that $\bar{\chi}\Gamma^{(n)}_{[\alpha_1\cdots\alpha_n}\overleftrightarrow{\partial}_{\mu]}\chi^m$ with $\chi^m = \overleftrightarrow{\partial}_{\mu_1}\cdots\overleftrightarrow{\partial}_{\mu_m}\chi$ is a total derivative, *provided one is on-shell* ($\not{\partial}\chi = 0$). Chirality restricts n to be odd, while we take $m=$ odd for $n=3,7$ and $m=$ even for $n=1,5,9$ since otherwise $\bar{\chi}\Gamma^{(n)}\overleftrightarrow{\partial}_\mu\chi^m$ vanishes.

Consider first the case that $n=1$, hence $m=$ odd. One has (see above)

$$2\bar{\chi}\Gamma_{[\alpha}\overleftrightarrow{\partial}_{\mu_1]}\chi^{m-1} = -\tfrac{1}{2}\bar{\chi}\big(\Gamma_{\alpha\mu_1}\overleftarrow{\not{\partial}} + \overrightarrow{\not{\partial}}\Gamma_{\alpha\mu_1}\big)\chi^{m-1}$$
$$= -(\partial_\lambda\bar{\chi})\Gamma_{\alpha\mu_1}\Gamma_\lambda\chi^{m-1} = \partial_\lambda\big(\bar{\chi}\Gamma_{\lambda\mu_1\alpha}\chi^{m-1}\big). \qquad \text{(A.18)}$$

Hence we end up with a tensor with $n=3$, and in the same way we derive ($k=m-2=$ odd)

$$4\bar{\chi}\Gamma^{(3)}_{[\alpha_1\alpha_2\alpha_3}\overleftrightarrow{\partial}_{\mu]}\chi^k = \partial_\lambda\big(\bar{\chi}\Gamma^{(5)}_{\lambda\mu\alpha_1\alpha_2\alpha_3}\chi^k\big). \qquad \text{(A.19)}$$

For $n=5$ we recover an $n=7$ tensor ($l=k-1=$ even)

$$6\bar{\chi}\Gamma^{(5)}_{[\alpha_1\cdots\alpha_5}\overleftrightarrow{\partial}_{\mu]}\chi^l=\partial_\lambda\left(\bar{\chi}\Gamma^{(7)}_{\lambda\mu\alpha_1\cdots\alpha_5}\chi^l\right) \tag{A.20}$$

and duality brings us back to the case $n=3$:

$$\Gamma^{(7)}_{\lambda\mu\alpha_1\cdots\alpha_5}=-\tfrac{1}{6}i\varepsilon_{\lambda\mu\alpha_1\cdots\alpha_5\beta_1\cdots\beta_3}\Gamma^{(3)}_{\beta_1\cdots\beta_3}\Gamma_{11}. \tag{A.21}$$

One may anticipate that the axial current plays an important role in descriptions based on differential forms since they are totally antisymmetric by definition.

References

[1] P. van Nieuwenhuizen, Phys. Reports 68 (1981) 189, sects. (1.9), (1.11), (6.2), (6.5)
[2] F.A. Berends, J.W. van Holten, B. de Wit and P. van Nieuwenhuizen, J. Phys. A13 (1980) 1643
[3] S. Ferrara and P. van Nieuwenhuizen, Phys. Lett. 74B (1978) 333;
K.S. Stelle and P.C. West, Phys. Lett. 74B (1978) 330
[4] M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, Phys. Rev. D17 (1978) 3179
[5] E.S. Fradkin and M.A. Vasiliev, Nuovo Cim. Lett. 25 (1979) 79; Phys. Lett. 85B (1979) 47;
B. de Wit and J.W. van Holten, Nucl. Phys. B155 (1979) 530
[6] E. Bergshoeff, M. de Roo and B. de Wit, Nucl. Phys. B182 (1981) 173
[7] W. Nahm, Nucl. Phys. B135 (1978) 149
[8] E. Cremmer, B. Julia and J. Scherk, Phys. Lett. 76B (1978) 409
[9] L. Brink, J. Scherk and J.H. Schwarz, Nucl. Phys. B121 (1977) 77;
F. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. B122 (1977) 253
[10] S. Ferrara, F. Gliozzi, J. Scherk and P. van Nieuwenhuizen, Nucl. Phys. B117 (1976) 333
[11] S. Ferrara and B. Zumino, Nucl. Phys. B87 (1975) 207;
V. Ogievetsky and E. Sokatchev, Nucl. Phys. B124 (1977) 309
[12] P. Howe, K.S. Stelle and P.K. Townsend, Nucl. Phys. B192 (1981) 332
[13] M. Kaku and P.K. Townsend, Phys. Lett. 76B (1978) 54;
A. Das, M. Kaku and P.K. Townsend, Phys. Rev. Lett. 40 (1978) 1215
[14] B. de Wit, J.W. van Holten and A. van Proeyen, Nucl. Phys. B184 (1981) 77;
B. de Wit, Conformal invariance in gravity and supergravity, Lectures at 18th Winter School of Theoretical Physics, Karpacz, February 19–March 4, 1981, preprint NIKHEF-H/81-20, Proc. to be published; Lectures at Spring School on Supergravity, Trieste, 1981
[15] P. Howe and U. Lindström, Phys. Lett. 103B (1981) 422
[16] J.H. Schwarz, Nucl. Phys. B185 (1981) 221;
B. S. DeWitt and P. van Nieuwenhuizen, ITP-SB-81-16;
P. van Nieuwenhuizen, Lectures at Spring School on Supergravity, Trieste, 1981
[17] W. Siegel, Phys. Lett. 93B (1980) 170;
E. Sezgin and P. van Nieuwenhuizen, Phys. Rev. D22 (1980) 301;
M. Duff and P. van Nieuwenhuizen, Phys. Lett. 94B (1980) 197;
D.Z. Freedman and P.K. Townsend, Nucl. Phys. B177 (1981) 282
[18] H. Nicolai and P.K. Townsend, Phys. Lett. 98B (1981) 257
[19] A.H. Chamseddine, Nucl. Phys. B185 (1981) 403
[20] S. Ferrara, J. Scherk and P. van Nieuwenhuizen, Phys. Rev. Lett. 37 (1976) 1035;
D.Z. Freedman and J.H. Schwarz, Phys. Rev. D15 (1977) 1007
[21] B.E.W. Nilsson, Nucl. Phys. B188 (1981) 176
[22] E. Cremmer, *in* Superspace and supergravity, ed. S.W. Hawking and M. Roček (Cambridge University Press, 1981)
[23] D.Z. Freedman, Phys. Rev. D15 (1977) 1173
[24] B. de Wit and P. van Nieuwenhuizen, Nucl. Phys. B139 (1978) 216
[25] K.S. Stelle and P.C. West, Nucl. Phys. B145 (1978) 175