

University of Groningen

Inductive types in constructive languages

Bruin, Peter Johan de

IMPORTANT NOTE: You are advised to consult the publisher's version (publisher's PDF) if you wish to cite from it. Please check the document version below.

Document Version

Publisher's PDF, also known as Version of record

Publication date:

1995

[Link to publication in University of Groningen/UMCG research database](#)

Citation for published version (APA):

Bruin, P. J. D. (1995). *Inductive types in constructive languages*. s.n.

Copyright

Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

The publication may also be distributed here under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license. More information can be found on the University of Groningen website: <https://www.rug.nl/library/open-access/self-archiving-pure/taverne-amendment>.

Take-down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): <http://www.rug.nl/research/portal>. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.

Chapter 8

Existence of inductively defined sets

We characterized inductive types in chapter 5 by means of polynomial functors; now we shall show that for a polynomial functor F , in set theory, an initial F -algebra and a final F -coalgebra indeed exist. The axioms of set theory are listed in section A.1. We outline two alternative proofs. Manes [54, p. 74] lists a number of works that present a rigorous construction of (an equivalent of) initial F -algebras for polynomial functors F .

The first proof, in section 8.1, is the standard construction of an initial F -algebra, by taking the transfinite limit $F^{(u)}. \emptyset$ for some ordinal number u .

Section 8.2 gives a more elementary proof, after an idea of Kerkhoff [45], which works in type theory too, and which dualizes yielding final co-algebras.

In section 8.3, we add equations to an initial F -algebra. For adding equations to a final F -coalgebra, we refer to section 7.4.

8.1 Using transfinite ordinal induction

Given a polynomial F , we are going to define a transfinite sequence of sets so that its limit gives an initial F -algebra.

For κ (*kappa*) a cardinal number, we define

$$Y \subseteq_{\kappa} X := Y \subseteq X \wedge \text{card } Y \leq \kappa .$$

A functor $F: \mathbf{SET} \rightarrow \mathbf{SET}$ is *bounded* iff it has some *rank*. It has rank κ (or is κ -based) iff for all $X: \mathbf{Set}$,

$$F.X = \bigcup (Y: \subseteq_{\kappa} X :: F.Y) .$$

Note that bounded functors are monotonic: they preserve (\subseteq).

Theorem 8.1 1. Any polynomial functor F is bounded.

2. For any bounded functor F there exists an initial F -algebra $(T; \tau)$. Actually, τ can be the identity, so that $F.T = T$.

Proof 1. Let $F.X = \Sigma(x: A :: X^{Bx})$. We calculate a rank κ of F .

$$\begin{aligned}
& \Sigma(x: A :: X^{Bx}) \subseteq \bigcup(Y: \subseteq_{\kappa} X :: \Sigma(x: A :: Y^{Bx})) \\
\Leftrightarrow & \Sigma(x: A :: X^{Bx}) \subseteq \Sigma(x: A :: \bigcup(Y: \subseteq_{\kappa} X :: Y^{Bx})) \\
\Leftarrow & \forall x: A :: X^{Bx} \subseteq \bigcup(Y: \subseteq_{\kappa} X :: Y^{Bx}) \\
\Leftarrow & \forall x: A; u: X^{Bx} :: \exists Y: \subseteq_{\kappa} X :: u \in Y^{Bx} \\
\Leftarrow & \forall x: A; u: X^{Bx}; Y := \{i: Bx :: u_i\} :: Y \subseteq_{\kappa} X \wedge u \in Y^{Bx} \\
\Leftarrow & \forall x: A; u: X^{Bx} :: \text{card}\{i: Bx :: u_i\} \leq \kappa \wedge \text{True} \\
\Leftarrow & \kappa = \max(x: A :: \text{card } Bx)
\end{aligned}$$

2. Let the rank of F be κ . For some ordinal u , we define a transfinite sequence of sets $T: \mathbf{Set}^{S^u}$ by ordinal recursion:

$$\begin{aligned}
T_0 & := \emptyset \\
T_{S_n} & := F.T_n \\
T_v & := \bigcup(w: < v :: T_w) \quad \text{for limit ordinals } v
\end{aligned}$$

Now, T_u is the limit of the whole sequence, and if it satisfies $F.T_u \subseteq T_u$ then $(T_u; !)$ is an F -algebra. So we try:

$$\begin{aligned}
& F.T_u \subseteq T_u \\
\Leftrightarrow & \bigcup(Y: \subseteq_{\kappa} T_u :: F.Y) \subseteq T_u \\
\Leftrightarrow & \forall(Y: \subseteq_{\kappa} T_u :: F.Y \subseteq T_u)
\end{aligned}$$

To prove this condition, assume $Y: \subseteq_{\kappa} T_u$. Requiring that u is a limit ordinal (requirement 1), we have that for all $y: \in Y$ there is some $v_y < u$ with $y \in T_{v_y}$. Hence $Y \subseteq T_{\max v}$, and:

$$F.Y \subseteq F.T_{\max v} = T_{S(\max v)}. \quad (8.1)$$

Now note that if $\kappa < u$ (2) then:

$$\text{card}(\text{Dom } v) = \text{card } Y \leq \kappa < u.$$

So if u is a regular cardinal (3) (see section A.4), then:

$$\max v \leq \sum v < u,$$

hence $s(\max v) < u$ as u is a limit. Combined with (8.1) and monotony of T , we obtain our present goal, $F.Y \subseteq T_u$.

So we are done if we find a u that is a limit ordinal (requirement 1), that is bigger than κ (2), and that is a regular cardinal (3). Taking $u := \max\langle \kappa^+, \omega \rangle$ satisfies all this. (κ^+ is the least regular cardinal greater than κ , see section A.4.) \blacksquare

This proof does not dualize to final coalgebras, because T_0 would then have to be a set of all sets, which does not exist. Indeed, final coalgebras of the form $(U; !)$ generally do not exist in ZFC. But within Aczel's set theory with anti-foundation (section A.7), it is possible to build such algebras; see [4].

8.2 Kerkhoff's proof

An alternative construction is the following one, somewhat simplified from Kerkhoff [45]. It needs no ordinal recursion but only natural numbers and powersets. Furthermore, it can be dualized to model co-inductive types, and it can be formalized within our extended type theory as well.

We want to construct an initial algebra $(T; \Sigma(x: A :: T^{Bx}) \rightarrow T)$. Elements of T are built from some $a: A$ and a tuple $s: T^{Ba}$ of sub-elements, so we think of them as trees, where each node has a label $x: A$ and a tuple of subtrees indexed over Bx . The idea is to represent such a tree by its set of nodes, where each node is characterized by its label together with the sequence of indices from $\bigcup(x: A :: Bx)$ that leads to the node.

Theorem 8.2 (Kerkhoff) *For polynomial F , there exists an initial F -algebra.*

Proof. As in paragraph 2.9.2, let $X^* := \Sigma(n: \mathbb{N} :: X^n)$ be the type of finite sequences, so that $\langle \rangle: X^*$, and $\langle x \rangle \# l: X^*$ for $x: X, l: X^*$. We'll define S to be the type of arbitrary sets of node representations, τ to be the operator that combines a tuple of such sets into a new one with a single root node, so that $(S; \tau)$ is an F -algebra, and then define T to be the subalgebra of S generated by τ .

Let $F.X = \Sigma(x: A :: X^{Bx})$.

$$\begin{aligned} S &:= \mathcal{P}(\bigcup(A; B)^* \times A) \\ \tau.(a: A; s: S^{Ba}): S &:= \{(\langle \rangle, a)\} \cup \{y: Ba; (l, x): \in s_y :: (\langle y \rangle \# l, x)\} \\ T &:= \bigcap(X: \subseteq S \mid \tau[F.X] \subseteq X) \end{aligned}$$

Now, theorem 4.3 says that $(T; \tau)$ is initial, provided that τ is injective. To prove this, assume:

$$\tau.(a; s) = \tau.(a'; s')$$

First, as we have $(\langle \rangle, a) \in \tau.(a; s)$, and as it is not possible that $(\langle \rangle, a) = (\langle y' \rangle \# l', x')$ for some $y': Ba; (l', x'): \in s'_{y'}$, it follows that $(\langle \rangle, a) = (\langle \rangle, a')$ so $a = a'$.

Secondly, we prove $s_y \subseteq s'_y$ for arbitrary $y: Ba$.

$$\begin{aligned} &(l, x) \in s_y \\ \Rightarrow &(\langle y \rangle \# l, x) \in \tau.(a; s) && \{\text{def. } \tau\} \\ \Leftrightarrow &(\langle y \rangle \# l, x) \in \tau.(a; s') && \{\text{assumption}\} \\ \Leftrightarrow &\exists y': Ba; (l'; x'): \in s'_{y'} :: (\langle y \rangle \# l, x) = (\langle y' \rangle \# l', x') && \{\text{def. } \tau\} \\ \Leftrightarrow &\exists y': Ba; (l'; x'): \in s'_{y'} :: y = y' \wedge l = l' \wedge x = x' \\ \Leftrightarrow &(l, x) \in s'_y \end{aligned}$$

By symmetry we have $s'_y \subseteq s_y$, so that $s = s'$. ■

The difference with Kerkhoff is that he constructed the “free” F -algebra over a set C , which is the initial $(F + \mathsf{K}C)$ -algebra; he had

$$S := \mathcal{P}((C \cup A \cup \bigcup(A; B))^*)$$

$$\begin{aligned}
\tau.(a; s) &:= \{ \langle a \rangle \} \cup \{ y: Ba; l: \in s_y :: \langle a, y \rangle \# l \} \\
\eta.c &:= \{ \langle c \rangle \} \\
T &:= \bigcap (X: \subseteq S \mid \tau[F.X] \cup \eta[C] \subseteq X)
\end{aligned}$$

A dual construction (dual with respect to set inclusion) yields an F -coalgebra $(U; \delta)$. The proof that this coalgebra is final is very different, though.

Theorem 8.3 *For polynomial F , there exists a final F -coalgebra.*

Proof. Let S and τ be as above. Then define:

$$\begin{aligned}
U &:= \bigcup (X: \subseteq S \mid X \subseteq \tau[F.X]) \\
\delta.(t: U) &:= (a; s) \text{ where } (\langle \rangle, a) \in t, \\
&\quad s_y := \{ (l, x) \mid (\langle y \rangle \# l, x) \in t \}.
\end{aligned}$$

Note that $\delta: U \rightarrow F.U$ is the inverse of τ (on U , not on S). This gives an F -coalgebra $(U; \delta)$; we'll prove that it is final. Let $(V; \gamma)$ be another F -coalgebra; we have to construct a unique homomorphism $f: (V; \gamma) \rightarrow (U; \delta)$, so that $\delta \circ f = F.f \circ \gamma$, or:

$$\forall v: V :: f.v = \tau.(F.f.(\gamma.v)) \quad (8.2)$$

An inductive definition of f would yield only a partial function. Rather, we define the collection of subsets $f.v: S$ for $v: V$ by simultaneous induction as the least tuple of sets such that

$$\forall v: V :: \tau.(F.f.(\gamma.v)) \subseteq f.v.$$

That is, for $v: V$ and $(a; w) := \gamma.v$:

$$\begin{aligned}
& (\langle \rangle, a) \in f.v \\
y: Ba; (l, x): S \vdash (l, x) \in f.w_y & \Rightarrow (\langle y \rangle \# l, x) \in f.v
\end{aligned}$$

This has the form of a fixed point equation on the lattice $(S; \subseteq)^V$, so by Knaster-Tarski (theorem 3.6) we have indeed $f: V \rightarrow S$ and (8.2).

We first check the type of f :

$$\begin{aligned}
& f \in V \rightarrow U \\
\Leftrightarrow & f[V] \subseteq U \\
\Leftarrow & f[V] \subseteq \tau[F.f[V]] \quad \{ \text{definition } U \} \\
\Leftarrow & (F.f \circ \gamma)[V] \subseteq F.f[V] \quad \{ f = \tau \circ F.f \circ \gamma \} \\
\Leftrightarrow & F.f \circ \gamma \in V \rightarrow F.f[V] \\
\Leftarrow & F.f \in F.V \rightarrow F.f[V] \quad \{ \gamma: V \rightarrow F.V \} \\
\Leftarrow & f \in V \rightarrow f[V] \\
\Leftrightarrow & \text{True}
\end{aligned}$$

Thus, f is a homomorphism indeed. For uniqueness, suppose g is a homomorphism too. As $\tau.(F.g.(\gamma.v)) = g.v$ and f is minimal, we have $f.v \subseteq g.v$. But then $f.v = g.v$, because of the following lemma, and we are done.

Lemma. *If $u, u' \in U$ and $u \subseteq u'$, then $u = u'$.*

We prove for $l: \bigcup(A; B)^*$, $x: A$ the following, by induction on the length of the finite sequence l :

$$\forall u, u': U; u \subseteq u' :: (l, x) \in u' \Rightarrow (l, x) \in u \quad (8.3)$$

First we note that for any u, u' , by definition of U we have $u = \tau.(a; s)$ and $u' = \tau.(a'; s')$ for certain $a, a': A$, $s: U^{Ba}$, $s': U^{Ba'}$. Given $u \subseteq u'$ and the definition of τ , it follows then that $a = a'$ and $s_y \subseteq s'_y$ for all $y: Ba$.

We check (8.3) for the empty list: if $(\langle \rangle, x) \in u'$, then we have $x = a' = a$ so $(\langle \rangle, x) \in u$.

Then, assume (8.3) as induction hypothesis. If $(\langle y \rangle \# l, x) \in u'$, then we have $(l, x) \in s'_y$, so by hypothesis $(l, x) \in s_y$, hence $(\langle y \rangle \# l, x) \in u$. This completes the induction, the lemma, and the theorem. ■

8.3 Algebras with equations

In section 4.4, we introduced equations or laws. We show now that one can always add laws to an initial F -algebra in **TYPE** (and also in **SET**), when quotient-types are available. The dual theorem, that one can always add laws to a final coalgebra, was already shown in section 7.4 .

Theorem 8.4 *For any polynomial endofunctor F on **TYPE**, and law $E = (H; r)$, if **ALG** F has an initial object, then **ALG** $(F; E)$ has one as well.*

Proof. Let $F.X = \Sigma(x: A :: X^{Bx})$, and $(T; \tau)$ be initial in **ALG** F . We define a congruence relation $R: \mathcal{P}(T^2)$ as follows. It is inductively defined by the clauses:

$$h: H.T \vdash \quad r(T; \tau).h \in R \quad (8.4)$$

$$a: A; t, t': T^{Ba} \vdash \quad \forall y: Ba :: (t_y, t'_y) \in R \Rightarrow (\tau.(a; t), \tau.(a; t')) \in R \quad (8.5)$$

$$|\! =_T | \subseteq R$$

$$R^\cup \subseteq R$$

$$R \cdot R \subseteq R$$

The first two clauses may be written as $r(T; \tau) \in H.T \rightarrow R$ and $(\tau, \tau) \in F.R \rightarrow R$.

Using the quotient types of C.4.2, we take T' to be T modulo this congruence, and we define τ' so that $\lambda // \text{in}_R: (T \triangleright T // R)$ is a homomorphism, $\lambda // \text{in}_R: (T; \tau) \rightarrow (T'; \tau')$:

$$T' := T // R$$

$$\tau' := \{a: A; t: T^{Ba} :: ((a; (y :: // \text{in}_R t_y)), // \text{in}_R(\tau.(a; t))) \}$$

First we have to show that this τ' is really a function. So assume $a: A$, $t, t': T^{Ba}$, and $// \text{in}_R t_y = // \text{in}_R t'_y$ for $y: Ba$. As R is an equivalence relation, we have $(t_y, t'_y) \in R$ for all y , hence $(\tau.(a; t), \tau.(a; t')) \in R$ by the last clause of R , and $// \text{in}_R(\tau.(a; t)) = // \text{in}_R(\tau.(a; t'))$.

Secondly, F -algebra $(T'; \tau')$ should satisfy law E , that is, $r_0(T'; \tau') =_{H.T' \rightarrow T'} r_1(T'; \tau')$. As the $r_j: HU \rightarrow U$ are natural transformations, and as $\lambda // \text{in}_R$ is a homomorphism, we have

$$\lambda // \text{in}_R \circ r_j(T; \tau) = r_j(T'; \tau') \circ H.(\lambda // \text{in}_R) .$$

But by the first clause of R , we have

$$\lambda//\text{in}_R \circ r_0(T; \tau) = \lambda//\text{in}_R \circ r_1(T'; \tau') ,$$

so we are done if $H.(\lambda//\text{in}_R)$ is surjective, that is, has a right-inverse. Now note that $\lambda//\text{in}_R$ must have a right-inverse g by the axiom of choice, and then $H.g$ is a right-inverse of $H.(\lambda//\text{in}_R)$.

Thirdly, supposing that $(U; \psi)$ is another $(F; E)$ -algebra, we must provide a unique homomorphism $f: (T'; \tau') \rightarrow (U; \psi)$. For a function $f: T' \rightarrow U$ we have:

$$\begin{aligned} & f \text{ is a homomorphism} \\ \Leftrightarrow & f \circ \tau' = \psi \circ F.f \\ \Leftrightarrow & f \circ \tau' \circ F.(\lambda//\text{in}) = \psi \circ F.f \circ F.(\lambda//\text{in}) \quad \{\lambda//\text{in} \text{ is surjective}\} \\ \Leftrightarrow & f \circ \lambda//\text{in} \circ \tau = \psi \circ F.(f \circ \lambda//\text{in}) \quad \{\text{def. } \tau'\} \\ \Leftrightarrow & f \circ \lambda//\text{in} = ([U; \psi]) \quad \{\text{initiality } (T; \tau) \} \end{aligned}$$

So we can take

$$f := \lambda//\text{elim} ([U; \psi]) . ,$$

where we must prove that for $(x, x') \in R$, one has $([\psi]).x = ([\psi]).x'$. For this we need the minimality of R . So defining

$$S := \{ (x, x'): T^2 \mid ([\psi]).x = ([\psi]).x' \} ,$$

we prove that $R \subseteq S$ by checking that S satisfies the five clauses that define R . Relation S is clearly reflexive, symmetric, and transitive. To check (8.4), we have for $h: H.T$

$$([\psi]).(r_0(T; \tau).h) = ([\psi]).(r_1(T; \tau).h)$$

because $([\psi]) \circ r_j(T; \tau) = r_j(U; \psi) \circ H.([\psi])$ by naturality of r_j , and $r_0(U; \psi) = r_1(U; \psi)$ as $(U; \psi)$ satisfies law E .

To check (8.5), when $(t_y, t'_y) \in S$ for $y: Ba$, then we have $(\tau.(a; t), \tau.(a; t')) \in S$ because $([\psi])$ is a homomorphism, i.e. $([\psi]).(\tau.(a; t)) = \psi.(a; (y :: ([\psi]).t_y))$. ■