Inductive types in constructive languages

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Chapter 8

Existence of inductively defined sets

We characterized inductive types in chapter 5 by means of polynomial functors; now we shall show that for a polynomial functor F, in set theory, an initial F-algebra and a final F-coalgebra indeed exist. The axioms of set theory are listed in section A.1. We outline two alternative proofs. Manes [54, p. 74] lists a number of works that present a rigorous construction of (an equivalent of) initial F-algebras for polynomial functors F.

The first proof, in section 8.1, is the standard construction of an initial F-algebra, by taking the transfinite limit $F^{(u)}.\emptyset$ for some ordinal number u.

Section 8.2 gives a more elementary proof, after an idea of Kerkhoff [45], which works in type theory too, and which dualizes yielding final co-algebras.

In section 8.3, we add equations to an initial F-algebra. For adding equations to a final F-coalgebra, we refer to section 7.4.

8.1 Using transfinite ordinal induction

Given a polynomial F, we are going to define a transfinite sequence of sets so that its limit gives an initial F-algebra.

For κ (*kappa*) a cardinal number, we define

$$Y \subseteq_\kappa X \ := \ Y \subseteq X \wedge \mathsf{card}\, Y \leq \kappa \ .$$

A functor $F\colon \mathbf{SET} \to \mathbf{SET}$ is *bounded* iff it has some *rank*. It has rank κ (or is κ*-based*) iff for all $X\colon \mathbf{Set}$,

$$F.X = \bigcup(Y\colon \subseteq_\kappa X :: F.Y) \ .$$

Note that bounded functors are monotonic: they preserve $(\subseteq)$.

Theorem 8.1 1. *Any polynomial functor F is bounded.*

2. *For any bounded functor F there exists an initial F-algebra $(T;\tau)$. Actually, τ can be the identity, so that $F.T = T$.*

Proof 1. Let $F.X = \Sigma(x\colon A :: X^{Bx})$. We calculate a rank κ of F.

$$\begin{array}{cc}
 & \Sigma(x\colon A :: X^{Bx}) \subseteq \bigcup(Y\colon \subseteq_\kappa X :: \Sigma(x\colon A :: Y^{Bx})) \\
\Leftrightarrow & \Sigma(x\colon A :: X^{Bx}) \subseteq \Sigma(x\colon A :: \bigcup(Y\colon \subseteq_\kappa X :: Y^{Bx})) \\
\Leftarrow & \forall x\colon A ::\ X^{Bx} \subseteq \bigcup(Y\colon \subseteq_\kappa X :: Y^{Bx}) \\
\Leftrightarrow & \forall x\colon A;\ u\colon X^{Bx} :: \exists Y\colon \subseteq_\kappa X ::\ u \in Y^{Bx} \\
\Leftarrow & \forall x\colon A;\ u\colon X^{Bx};\ Y := \{i\colon Bx :: u_i\} ::\ Y \subseteq_\kappa X \ \wedge\ u \in Y^{Bx} \\
\Leftrightarrow & \forall x\colon A;\ u\colon X^{Bx} ::\ \mathsf{card}\{i\colon Bx :: u_i\} \leq \kappa \ \wedge\ \mathsf{True} \\
\Leftarrow & \kappa = \mathsf{max}(x\colon A :: \mathsf{card}\, Bx)
\end{array}$$

2. Let the rank of F be κ. For some ordinal u, we define a transfinite sequence of sets $T\colon \mathbf{Set}^{\mathsf{S}\,u}$ by ordinal recursion:

$$\begin{array}{rcll}
T_0 & := & \emptyset & \\
T_{\mathsf{s}\,n} & := & F.T_n & \\
T_v & := & \bigcup(w\colon < v :: T_w) & \text{for limit ordinals } v
\end{array}$$

Now, T_u is the limit of the whole sequence, and if it satisfies $F.T_u \subseteq T_u$ then $(T_u; \mathsf{I})$ is an F-algebra. So we try:

$$\begin{array}{cc}
 & F.T_u \subseteq T_u \\
\Leftrightarrow & \bigcup(Y\colon \subseteq_\kappa T_u :: F.Y) \subseteq T_u \\
\Leftrightarrow & \forall(Y\colon \subseteq_\kappa T_u :: F.Y \subseteq T_u)
\end{array}$$

To prove this condition, assume $Y\colon \subseteq_\kappa T_u$. Requiring that u is a limit ordinal (requirement 1), we have that for all $y\colon\in Y$ there is some $v_y < u$ with $y \in T_{v_y}$. Hence $Y \subseteq T_{\mathsf{max}\, v}$, and:

$$F.Y \subseteq F.T_{\mathsf{max}\, v} = T_{\mathsf{s}(\mathsf{max}\, v)}\ . \tag{8.1}$$

Now note that if $\kappa < u$ (2) then:

$$\mathsf{card}(\mathsf{Dom}\, v) = \mathsf{card}\, Y \leq \kappa < u\ .$$

So if u is a regular cardinal (3) (see section A.4), then:

$$\mathsf{max}\, v \leq \sum v < u\ ,$$

hence $\mathsf{s}(\mathsf{max}\, v) < u$ as u is a limit. Combined with (8.1) and monotony of T, we obtain our present goal, $F.Y \subseteq T_u$.

So we are done if we find a u that is a limit ordinal (requirement 1), that is bigger than κ (2), and that is a regular cardinal (3). Taking $u := \mathsf{max}\langle \kappa^+, \omega\rangle$ satisfies all this. (κ^+ is the least regular cardinal greater than κ, see section A.4.) ■

This proof does not dualize to final coalgebras, because T_0 would then have to be a set of all sets, which does not exist. Indeed, final coalgebras of the form $(U; \mathsf{I})$ generally do not exist in ZFC. But within Aczel's set theory with anti-foundation (section A.7), it is possible to build such algebras; see [4].

8.2 Kerkhoff's proof

An alternative construction is the following one, somewhat simplified from Kerkhoff [45]. It needs no ordinal recursion but only natural numbers and powersets. Furthermore, it can be dualized to model co-inductive types, and it can be formalized within our extended type theory as well.

We want to construct an initial algebra $(T\colon \mathbf{Set};\ \tau\colon \Sigma(x\colon A :: T^{Bx}) \to T)$. Elements of T are built from some $a\colon A$ and a tuple $s\colon T^{Ba}$ of sub-elements, so we think of them as trees, where each node has a label $x\colon A$ and a tuple of subtrees indexed over Bx. The idea is to represent such a tree by its set of nodes, where each node is characterized by its label together with the sequence of indices from $\bigcup(x\colon A :: Bx)$ that leads to the node.

Theorem 8.2 *(Kerkhoff) For polynomial F, there exists an initial F-algebra.*

Proof. As in paragraph 2.9.2, let $X^* := \Sigma(n\colon \mathbb{N} :: X^n)$ be the type of finite sequences, so that $\langle\rangle\colon X^*$, and $\langle x\rangle \mathbin{+\!\!+} l : X^*$ for $x\colon X$, $l\colon X^*$. We'll define S to be the type of arbitrary sets of node representations, τ to be the operator that combines a tuple of such sets into a new one with a single root node, so that $(S;\tau)$ is an F-algebra, and then define T to be the subalgebra of S generated by τ.

Let $F.X = \Sigma(x\colon A :: X^{Bx})$.

$$
\begin{array}{rcl}
S & := & \mathcal{P}(\,\bigcup(A;B)^* \times A\,) \\
\tau.(a\colon A; s\colon S^{Ba})\colon S & := & \{\,(\langle\rangle, a)\,\} \,\cup\, \{y\colon Ba;\ (l,x)\colon\in s_y :: (\langle y\rangle \mathbin{+\!\!+} l, x)\,\} \\
T & := & \bigcap(X\colon\subseteq S \mid\colon \tau[F.X] \subseteq X)
\end{array}
$$

Now, theorem 4.3 says that $(T;\tau)$ is initial, provided that τ is injective. To prove this, assume:

$$\tau.(a;s) = \tau.(a';s')$$

First, as we have $(\langle\rangle, a) \in \tau.(a;s)$, and as it is not possible that $(\langle\rangle, a) = (\langle y'\rangle \mathbin{+\!\!+} l', x')$ for some $y'\colon Ba$; $(l',x')\colon\in s'_{y'}$, it follows that $(\langle\rangle, a) = (\langle\rangle, a')$ so $a = a'$.

Secondly, we prove $s_y \subseteq s'_y$ for arbitrary $y\colon Ba$.

$$
\begin{array}{lcl}
 & (l,x) \in s_y & \\
\Rightarrow & (\langle y\rangle \mathbin{+\!\!+} l, x) \in \tau.(a;s) & \{\text{def. } \tau\} \\
\Leftrightarrow & (\langle y\rangle \mathbin{+\!\!+} l, x) \in \tau.(a;s') & \{\text{assumption}\} \\
\Leftrightarrow & \exists y'\colon Ba;\ (l';x')\colon\in s'_{y'} :: (\langle y\rangle \mathbin{+\!\!+} l, x) = (\langle y'\rangle \mathbin{+\!\!+} l', x') & \{\text{def. } \tau\} \\
\Leftrightarrow & \exists y'\colon Ba;\ (l';x')\colon\in s'_{y'} :: y = y' \wedge l = l' \wedge x = x' & \\
\Leftrightarrow & (l,x) \in s'_y &
\end{array}
$$

By symmetry we have $s'_y \subseteq s_y$, so that $s = s'$. ■

The difference with Kerkhoff is that he constructed the "free" F-algebra over a set C, which is the initial $(F + \mathsf{K}\,C)$-algebra; he had

$$S \;:=\; \mathcal{P}(\,(C \cup A \cup \bigcup(A;B))^*\,)$$

$$\begin{array}{rcl} \tau.(a;s) & := & \{\langle a\rangle\} \cup \{y{:}\,Ba;\ l{:}\in s_y :: \langle a,y\rangle +\!\!+\, l\} \\ \eta.c & := & \{\langle c\rangle\} \\ T & := & \bigcap(X{:}\subseteq S \mid: \tau[F.X] \cup \eta[C] \subseteq X) \end{array}$$

A dual construction (dual with respect to set inclusion) yields an F-coalgebra $(U;\delta)$. The proof that this coalgebra is final is very different, though.

Theorem 8.3 *For polynomial F, there exists a final F-coalgebra.*

Proof. Let S and τ be as above. Then define:

$$\begin{array}{rcl} U & := & \bigcup(X{:}\subseteq S \mid: X \subseteq \tau[F.X]) \\ \delta.(t{:}\,U) & := & (a;s)\ \underline{\text{where}}\ \ (\langle\rangle, a) \in t\ , \\ & & \qquad\qquad\qquad s_y := \{(l,x) \mid: (\langle y\rangle +\!\!+\, l, x) \in t\}\ . \end{array}$$

Note that $\delta{:}\,U \to F.U$ is the inverse of τ (on U, not on S). This gives an F-coalgebra $(U;\delta)$; we'll prove that it is final. Let $(V;\gamma)$ be another F-coalgebra; we have to construct a unique homomorphism $f{:}\,(V;\gamma) \to (U;\delta)$, so that $\delta \circ f = F.f \circ \gamma$, or:

$$\forall v{:}\,V :: f.v \ = \ \tau.(F.f.(\gamma.v)) \tag{8.2}$$

An inductive definition of f would yield only a partial function. Rather, we define the collection of subsets $f.v{:}\,S$ for $v{:}\,V$ by simultaneous induction as the least tuple of sets such that

$$\forall v{:}\,V :: \tau.(F.f.(\gamma.v)) \subseteq f.v\ .$$

That is, for $v{:}\,V$ and $(a;w) := \gamma.v$:

$$\begin{array}{rrcl} & & & (\langle\rangle, a) \in f.v \\ y{:}\,Ba;\ (l,x){:}\,S\ \vdash & (l,x) \in f.w_y & \Rightarrow & (\langle y\rangle +\!\!+\, l, x) \in f.v \end{array}$$

This has the form of a fixed point equation on the lattice $(S;\subseteq)^V$, so by Knaster-Tarski (theorem 3.6) we have indeed $f{:}\,V \to S$ and (8.2).

We first check the type of f:

$$\begin{array}{cll} & f \in V \to U & \\ \Leftrightarrow & f[V] \subseteq U & \\ \Leftarrow & f[V] \subseteq \tau[F.f[V]] & \{\text{definition } U\} \\ \Leftarrow & (F.f \circ \gamma)[V] \subseteq F.f[V] & \{f = \tau \circ F.f \circ \gamma\} \\ \Leftrightarrow & F.f \circ \gamma \in V \to F.f[V] & \\ \Leftarrow & F.f \in F.V \to F.f[V] & \{\gamma{:}\,V \to F.V\} \\ \Leftarrow & f \in V \to f[V] & \\ \Leftrightarrow & \mathsf{True} & \end{array}$$

Thus, f is a homomorphism indeed. For uniqueness, suppose g is a homomorphism too. As $\tau.(F.g.(\gamma.v)) = g.v$ and f is minimal, we have $f.v \subseteq g.v$. But then $f.v = g.v$, because of the following lemma, and we are done.

Lemma. *If $u, u' \in U$ and $u \subseteq u'$, then $u = u'$.*
We prove for $l\colon \bigcup(A; B)^*$, $x\colon A$ the following, by induction on the length of the finite sequence l:

$$\forall u, u'\colon U;\ u \subseteq u' :: (l, x) \in u' \Rightarrow (l, x) \in u \tag{8.3}$$

First we note that for any u, u', by definition of U we have $u = \tau.(a; s)$ and $u' = \tau.(a'; s')$ for certain $a, a'\colon A$, $s\colon U^{Ba}$, $s'\colon U^{Ba'}$. Given $u \subseteq u'$ and the definition of τ, it follows then that $a = a'$ and $s_y \subseteq s'_y$ for all $y\colon Ba$.

We check (8.3) for the empty list: if $(\langle\rangle, x) \in u'$, then we have $x = a' = a$ so $(\langle\rangle, x) \in u$.

Then, assume (8.3) as induction hypothesis. If $(\langle y\rangle \mathbin{+\!\!+} l, x) \in u'$, then we have $(l, x) \in s'_y$, so by hypothesis $(l, x) \in s_y$, hence $(\langle y\rangle \mathbin{+\!\!+} l, x) \in u$. This completes the induction, the lemma, and the theorem. ∎

8.3 Algebras with equations

In section 4.4, we introduced equations or laws. We show now that one can always add laws to an initial F-algebra in **TYPE** (and also in **SET**), when quotient-types are available. The dual theorem, that one can always add laws to a final coalgebra, was already shown in section 7.4 .

Theorem 8.4 *For any polynomial endofunctor F on* **TYPE**, *and law $E = (H; r)$, if* **ALG** *F has an initial object, then* **ALG**$(F; E)$ *has one as well.*

Proof. Let $F.X = \Sigma(x\colon A :: X^{Bx})$, and $(T; \tau)$ be initial in **ALG** F. We define a congruence relation $R\colon \mathcal{P}(T^2)$ as follows. It is inductively defined by the clauses:

$$\begin{array}{rrcll}
 & h\colon H.T \ \vdash & & r(T;\tau).h \in R & (8.4)\\
a\colon A;\ t, t'\colon T^{Ba} \ \vdash & \forall y\colon Ba :: (t_y, t'_y) \in R & \Rightarrow & (\tau.(a;t), \tau.(a;t')) \in R & (8.5)\\
 & |{=_T}| & \subseteq & R & \\
 & R^{\cup} & \subseteq & R & \\
 & R \cdot R & \subseteq & R &
\end{array}$$

The first two clauses may be written as $r(T; \tau) \in H.T \to R$ and $(\tau, \tau) \in F.R \to R$.

Using the quotient types of C.4.2, we take T' to be T modulo this congruence, and we define τ' so that $/\!/\_\mathsf{in}_R\colon (T \triangleright T/\!/R)$ is a homomorphism, $\lambda/\!/\_\mathsf{in}_R\colon (T; \tau) \to (T'; \tau')$:

$$\begin{array}{rcl}
T' & := & T/\!/R\\
\tau' & := & \{a\colon A;\ t\colon T^{Ba} :: ((a; (y :: /\!/_\mathsf{in}_R\, t_y)), /\!/_\mathsf{in}_R(\tau.(a;t)))\}
\end{array}$$

First we have to show that this τ' is really a function. So assume $a\colon A$, $t, t'\colon T^{Ba}$, and $/\!/\_\mathsf{in}_R\, t_y = /\!/\_\mathsf{in}_R\, t'_y$ for $y\colon Ba$. As R is an equivalence relation, we have $(t_y, t'_y) \in R$ for all y, hence $(\tau.(a;t), \tau.(a;t')) \in R$ by the last clause of R, and $/\!/\_\mathsf{in}_R(\tau.(a;t)) = /\!/\_\mathsf{in}_R(\tau.(a;t'))$.

Secondly, F-algebra $(T'; \tau')$ should satisfy law E, that is, $r_0(T'; \tau') =_{H.T' \to T'} r_1(T'; \tau')$. As the $r_j\colon HU \dot{\to} U$ are natural transformations, and as $\lambda/\!/\_\mathsf{in}_R$ is a homomorphism, we have

$$\lambda/\!/\_\mathsf{in}_R \circ r_j(T; \tau) \;=\; r_j(T'; \tau') \circ H.(\lambda/\!/\_\mathsf{in}_R)\ .$$

But by the first clause of R, we have

$$\lambda//\_\mathsf{in}_R \circ r_0(T;\tau) \;=\; \lambda//\_\mathsf{in}_R \circ r_1(T';\tau') \;,$$

so we are done if $H.(\lambda//\_\mathsf{in}_R)$ is surjective, that is, has a right-inverse. Now note that $\lambda//\_\mathsf{in}_R$ must have a right-inverse g by the axiom of choice, and then $H.g$ is a right-inverse of $H.(\lambda//\_\mathsf{in}_R)$.

Thirdly, supposing that $(U;\psi)$ is another $(F;E)$-algebra, we must provide a unique homomorphism $f\colon(T';\tau') \to (U;\psi)$. For a function $f\colon T' \to U$ we have:

$$\begin{array}{cll}
 & f \text{ is a homomorphism} & \\
\Leftrightarrow & f \circ \tau' = \psi \circ F.f & \\
\Leftrightarrow & f \circ \tau' \circ F.(\lambda\,//_\mathsf{in}) = \psi \circ F.f \circ F.(\lambda\,//_\mathsf{in}) & \{\lambda\,//_\mathsf{in} \text{ is surjective}\} \\
\Leftrightarrow & f \circ \lambda\,//_\mathsf{in} \circ \tau = \psi \circ F.(f \circ \lambda\,//_\mathsf{in}) & \{\text{def. } \tau'\} \\
\Leftrightarrow & f \circ \lambda\,//_\mathsf{in} = (\![U;\psi]\!) & \{\text{initiality } (T;\tau)\ \}
\end{array}$$

So we can take

$$f \;:=\; \lambda\,//\_\mathsf{elim}\,(\![U;\psi]\!). \;,$$

where we must prove that for $(x,x')\colon\in R$, one has $(\![\psi]\!).x = (\![\psi]\!).x'$. For this we need the minimality of R. So defining

$$S \;:=\; \{\,(x,x')\colon T^2 \;|\colon (\![\psi]\!).x = (\![\psi]\!).x'\}\;,$$

we prove that $R \subseteq S$ by checking that S satisfies the five clauses that define R. Relation S is clearly reflexive, symmetric, and transitive. To check (8.4), we have for $h\colon H.T$

$$(\![\psi]\!).(r_0(T;\tau).h) \;=\; (\![\psi]\!).(r_1(T;\tau).h)$$

because $(\![\psi]\!) \circ r_j(T;\tau) = r_j(U;\psi) \circ H.(\![\psi]\!)$ by naturality of r_j, and $r_0(U;\psi) = r_1(U;\psi)$ as $(U;\psi)$ satisfies law E.
To check (8.5), when $(t_y, t'_y) \in S$ for $y\colon Ba$, then we have $(\tau.(a;t), \tau.(a;t')) \in S$ because $(\![\psi]\!)$ is a homomorphism, i.e. $(\![\psi]\!).(\tau.(a;t)) = \psi.(a;(y :: (\![\psi]\!).t_y))$. ■