Symmetries in string theory

Boonstra, Harm Jan Hugo

Chapter 5

Duality

In this chapter we study a different kind of symmetry in string theory which is known under the name duality symmetry. Duality transformations act on background space-time fields in which the string moves, whereas the gauge transformations discussed in chapters 2 and 3 act on the world-sheet fields. In section 5.1 we present the low-energy effective action for the background fields and discuss a simple compactification procedure that gives an effective action in lower dimensional space-time. We discuss global symmetries that appear upon reduction to lower dimensions. Discrete subgroups of these symmetry groups are believed or known to be symmetries of the full string theory, called dualities. In section 5.2 we describe, from the sigma model point of view, a subclass of possible dualities, the so-called target-space duality or T-duality. Section 5.3 is about strong/weak coupling dualities in string theory.

5.1 Introduction

We first continue our discussion of strings moving in background fields started in section 2.3. Let us repeat here the action of the nonlinear sigma model which describes the bosonic string moving in a background determined by a space-time metric, antisymmetric tensor and dilaton:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}\{g_{\mu\nu}(X)h^{ab}\partial_a X^\mu\partial_b X^\nu + B_{\mu\nu}(X)\varepsilon^{ab}\partial_a X^\mu\partial_b X^\nu + \alpha' R^{(2)}\Phi(X)\}\,, \qquad (5.1)$$

This action does not turn into a free field action by going to the conformal gauge $h_{ab} = e^\phi\eta_{ab}$. Therefore, its quantization is not straightforward and one must resort to perturbation theory in the parameter α' which plays the role of the loop-counting parameter of the nonlinear sigma model viewed as a two-dimensional quantum field theory. As mentioned in section 2.3, demanding Weyl invariance of this quantum field theory results in a number of differential equations for the background fields. The Weyl anomaly is the trace of the energy-momentum tensor and is given in reference [54] (see also [84]),

$$< T^a{}_a >= \beta^\Phi \sqrt{-h} R^{(2)} + \beta^g_{\mu\nu} \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu + \beta^B_{\mu\nu} \sqrt{-h} \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu \,, \tag{5.2}$$

where β^Φ , β^g and β^B are the beta-functionals associated to the coupling functions $\Phi(X)$, $g_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$. They are given by

$$\begin{aligned} \beta^\Phi &\propto D - 26 + 3\alpha' \left(R + 4\nabla_\mu \Phi \nabla^\mu \Phi - 4\nabla^2 \Phi + \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) + O(\alpha'^2) \,, \\ \beta^g_{\mu\nu} &\propto R_{\mu\nu} - 2\nabla_\mu \nabla_\nu \Phi + \tfrac{9}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + O(\alpha') \,, \\ \beta^B_{\mu\nu} &\propto \nabla_\rho H^\rho{}_{\mu\nu} - 2H^\rho{}_{\mu\nu} \nabla_\rho \Phi + O(\alpha') \,. \end{aligned} \tag{5.3}$$

The field strength $H_{\mu\nu\rho}$ of the antisymmetric tensor field is defined by

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} \,, \tag{5.4}$$

using antisymmetrization conventions $A_{[\mu_1 \ldots \mu_n]} \equiv \frac{1}{n!} \sum_P (-1)^{|P|} A_{\mu_{P_1} \ldots \mu_{P_n}}$ where the sum is over all $n!$ permutations $\{P_1, \ldots P_n\}$ of $\{1, \ldots n\}$. $R_{\mu\nu}$ is the space-time Ricci tensor and $R = R_\mu{}^\mu$ is the space-time scalar curvature, not to be confused with the world-sheet scalar curvature $R^{(2)}$. The expressions (5.3) arise from one-loop calculations. They receive corrections of higher orders in α', but these will not concern us here.

From (5.2) it is clear that cancellation of the Weyl anomaly requires

$$\beta^\Phi = \beta^g = \beta^B = 0 \,. \tag{5.5}$$

Note that for vanishing dilaton and antisymmetric tensor fields the equation $\beta^g_{\mu\nu} = 0$ becomes Einstein's equation in D-dimensional empty space-time if we ignore also the α' corrections. Nonzero Φ and $B_{\mu\nu}$ provide sources for Einstein's equation.

The equations (5.5) can be viewed as equations of motion of the action

$$S = \int d^D X \sqrt{-g} e^{-2\Phi} \left(-\frac{D-26}{3} - R + 4\partial_\mu \Phi \partial^\mu \Phi - \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) \,. \tag{5.6}$$

This is the low-energy effective action. Only the massless fields from the string spectrum are represented in this action. Note that $B_{\mu\nu}$ is a gauge field; the action is invariant under the gauge transformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu} \Lambda_{\nu]}$. From now on we assume that we are in the critical dimension, i.e. that the first term in (5.6) vanishes. Thus the low-energy effective action for the bosonic string is the action of 26-dimensional gravity coupled to a scalar field and an antisymmetric tensor field.

For other string theories, one can follow the same steps to arrive at a low-energy effective action. A general procedure is to consider a string moving in a background with fields corresponding to the massless states in the string spectrum. The dynamics of the string in this background is described by a nonlinear sigma model, and the condition for conformal invariance at the quantum level supplies the equations of motion for the background fields. From these equations one may construct the low-energy effective action[1].

[1] Another method that has been used to derive the low-energy effective action is to calculate scattering amplitudes of the massless string modes within string perturbation theory and construct the field theory action for the background fields which reproduces these amplitudes.

The bosonic massless spectrum of the heterotic string contains, besides the metric, antisymmetric tensor and dilaton, gauge bosons of $E_8 \times E_8$ or $SO(32)$. The bosonic part of the low-energy effective action of the heterotic string is, to lowest order in α',

$$S = \int d^{10}x\sqrt{-g}e^{-2\Phi}\left(-R + 4\partial_\mu\Phi\partial^\mu\Phi - \tfrac{3}{4}H_{\mu\nu\rho}H^{\mu\nu\rho}\right), \tag{5.7}$$

which is, apart from the dimension, identical to (5.6). When fermion fields are included, the low-energy effective action to lowest order in α' is the $D = 10$ $N = 1$ supergravity action [58]. First order α' corrections involve Yang-Mills fields

$$S^{(\alpha')} \sim \alpha' \int d^{10}x\sqrt{-g}e^{-2\Phi}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \tag{5.8}$$

as well as Chern-Simons terms in the antisymmetric tensor field strength. Including fermions, one obtains supersymmetric Yang-Mills coupled to $N = 1$ supergravity [59, 25, 60].

It should be kept in mind that we use only lowest order approximations to the full effective actions. The full effective action is approximated by an expansion in two independent parameters. One of them is α' which is the loop-counting parameter from the sigma model point of view, as mentioned before. It defines a length scale $\sqrt{\alpha'}$ which is assumed to be small, probably not too far from the Planck length, and the background fields are assumed to vary slowly in space-time relative to this scale. The other parameter is the string loop parameter which is the expectation value of $e^{2\Phi}$. It counts the number of loops in a string scattering process, i.e. the genus of the world-sheet. This can be understood by noting that in the nonlinear sigma model the constant mode of the dilaton field multiplies the Euler characteristic

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma d^2\sigma\sqrt{h}R^{(2)} = 2 - 2g, \tag{5.9}$$

where g is the genus of the world-sheet Σ. Thus we see that in the (Euclidean) path integral genus g amplitudes are weighted by a factor $e^{(2g-2)\Phi} = (e^{2\Phi})^{g-1}$. Since each handle corresponds to two string 'vertices', the string coupling constant should be identified with e^{Φ}. As the dilaton exponential in (5.6), (5.7) and (5.8) indicates, these effective actions correspond to tree level string theory, i.e. to the classical zero-loop scatterings of massless string modes.

Higher order kinetic terms for the gravitational and Yang-Mills fields appear either as α' corrections or as string loop corrections or as both. For the gravitational field the first few terms in the expansion schematically look like

$$e^{-2\Phi}\left(R + \alpha' R^2 + (\alpha')^3R^4 + e^{2\Phi}(\alpha')^3R^4 + ...\right). \tag{5.10}$$

In particular, this implies that string theory predicts corrections with terms of higher order in derivatives to standard gravity or supergravity. Further information, in particular on aspects of supersymmetry concerning higher order corrections may be found in [183] and references therein.

5.1.1 Compactification and dimensional reduction

In order to eventually make contact with the world as we know it, it is necessary to 'compactify' the superstring theory from ten to four dimensions. The six internal dimensions should be hidden from our view, so these dimensions should form a compact space of tiny proportions, presumably of order the Planck length [124]. This means that we have to look for a solution of the background field equations (5.5) with the property that the ten-dimensional space-time M is a product of four-dimensional space-time M^4 and a six-dimensional compact internal manifold K, $M = M^4 \times K$. From now on we use hatted indices $\hat{\mu}, \hat{\nu}, ...$ for ten-dimensional space-time coordinates, unhatted $\mu, \nu, ...$ for four-dimensional space-time coordinates and $m, n, ...$ for six-dimensional internal coordinates, and we write $x^{\hat{\mu}} = (x^\mu, y^m)$. Moreover, ten-dimensional fields will be hatted and four-dimensional fields unhatted in our notation.

The 10d fields can be reinterpreted as 4d fields as follows. In general, a 10d spin-s field will decompose into several fields with spins $\leq s$ from the 4d point of view. This takes place according to the decomposition of the representations of the appropriate Lorentz groups, i.e. of $SO(1,9)$ representations into $SO(1,3) \times SO(6)$ representations. For example, the 10d metric $\hat{g}_{\hat{\mu}\hat{\nu}}$, which transforms in the 55-dimensional representation of $SO(1,9)$ decomposes under $SO(1,3) \times SO(6)$ as $\mathbf{55} = \mathbf{10} \times \mathbf{1} + \mathbf{4} \times \mathbf{6} + \mathbf{1} \times \mathbf{21}$, corresponding to the components $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu m}$ and $\hat{g}_{mn}$, respectively. Thus in four dimensions these fields correspond to a metric, 6 vector fields and 21 scalar fields, respectively.

In general, however, the fields depend on all ten coordinates x^μ and y^m. This means that the 10d fields give rise to an infinite number of 4d fields, essentially because in four dimensions the y-dependences are reinterpreted as internal degrees of freedom. However, only a finite number of these fields correspond to massless particles in four dimensions, the precise number depending on the topology of K, see e.g. [104]. All the other fields correspond to massive particles, presumably with masses of order M_{Planck}. Since we are (at first) interested in the low-energy effective theory, these massive particles may be ignored as an approximation.

We only consider the simplest way of compactification, namely toroidal compactification. Then the compact internal manifold is assumed to be a six-dimensional torus, $K = T^6 = S^1 \times ... \times S^1$. From the phenomenological point of view this may not be the most interesting way of compactifying but, as we will see, it gives rise to a number of interesting symmetry structures some of which are also expected (or known) to be present in more general compactifications. Let us just mention that other, more realistic compactification procedures, for example the ones introduced in [56], may produce four-dimensional models with $N = 1$ supersymmetry, chiral fermions, gauge groups which contain the standard model group $SU(3) \times SU(2) \times U(1)$, etc.. For a review of various compactification methods and corresponding descriptions of the internal degrees of freedom by conformal field theories, see [174].

In fact, we will consider only a special case of toroidal compactification; we require that all fields be independent of the compact coordinates y^m. This is called dimensional reduction. Such a y-independent field configuration should be a solution of the string equations of motion, and for this to be consistent, K must be flat (the metric on K is independent of the coordinates on K). A torus is flat, thus dimensional reduction is consistent with torus compactification. We now discuss how a four-dimensional action is obtained by dimensional reduction. Explicit formulas for dimensional reduction have been given in [171, 62, 147].

First of all, it should be noted that the gauge symmetries of the 10d action are necessarily broken because we require that the fields remain y-independent after a gauge transformation. Let us illustrate this with a vector field $\hat{A}_{\hat{\mu}}$. Under a 10d local coordinate transformation $\delta x^{\hat{\mu}} = -\xi^{\hat{\mu}}(x^{\hat{\nu}})$, it transforms according to

$$\delta \hat{A}_{\hat{\mu}} = \xi^{\hat{\nu}} \partial_{\hat{\nu}} \hat{A}_{\hat{\mu}} + \partial_{\hat{\mu}} \xi^{\hat{\nu}} \hat{A}_{\hat{\nu}} = \xi^{\nu} \partial_{\nu} \hat{A}_{\hat{\mu}} + \partial_{\hat{\mu}} \xi^{\nu} \hat{A}_{\nu} + \partial_{\hat{\mu}} \xi^{n} \hat{A}_{n} \tag{5.11}$$

where we used that $\partial_n \hat{A}_{\hat{\mu}} = 0$. Requiring $\partial_m \delta \hat{A}_{\hat{\mu}} = 0$ implies

$$\partial_m \xi^{\mu} = 0 \quad \text{and} \quad \partial_m \partial_{\hat{\mu}} \xi^{n} = 0\,, \tag{5.12}$$

which has the solution

$$\xi^{\mu} = \xi^{\mu}(x^{\nu}) \quad \text{and} \quad \xi^{m} = a^{m}{}_{n} y^{n} + \xi^{m}(x^{\nu})\,, \tag{5.13}$$

where $a^{m}{}_{n}$ are constants. Thus the remaining transformations originating from 10d general coordinate invariance are

$$\begin{aligned} \delta x^{\mu} &= -\xi^{\mu}(x^{\nu}) && \text{4d general coordinate invariance}\,, \\ \delta y^{m} &= -\xi^{m}(x^{\nu}) && \mathrm{U}(1)^6 \text{ 4d gauge invariance}\,, \\ \delta y^{m} &= -a^{m}{}_{n} y^{n} && \mathrm{GL}(6) \text{ global invariance}\,, \end{aligned} \tag{5.14}$$

where it should be noted that the second line really defines 4d gauge transformations, since the gauge parameters are arbitrary functions of the 4d coordinates. Below we will see how these transformations act on the 4d fields. A similar analysis of the 10d gauge transformations $\delta \hat{B}_{\hat{\mu}\hat{\nu}} = \partial_{[\hat{\mu}} \Lambda_{\hat{\nu}]}$ shows that the remaining 4d transformations, consistent with dimensional reduction, are

$$\begin{aligned} \delta \hat{B}_{\mu\nu} &= \partial_{[\mu} \lambda_{\nu]}\,, \\ \delta \hat{B}_{\mu n} &= \partial_{\mu} \lambda_{n} + b_{\mu n}\,, \\ \delta \hat{B}_{mn} &= b_{mn}\,, \end{aligned} \tag{5.15}$$

where λ_{ν} and λ_n are functions of x^{μ} only and $b_{\mu n}$ and b_{mn} are constants. The first equation shows the usual gauge invariance for a 4d antisymmetric tensor field. The second equation shows that there are six $U(1)$ gauge invariances and some constant shifts. The last equation expresses invariance under constant shifts of the 4d scalars $\hat{B}_{mn}$. Note that we have not yet defined the 4d fields. The fields in the above equations are components of 10d fields and the 4d fields should be defined in terms of them. However, they should be defined in a sensible way such that they transform in the usual way under 4d (gauge) transformations.

It is convenient to use the vielbein formulation in which the metric is written as $\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}{}^{\hat{\alpha}} \hat{e}_{\hat{\nu}}{}^{\hat{\beta}} \eta_{\hat{\alpha}\hat{\beta}}$, where $\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}}$ is the ten-dimensional vielbein and $\hat{\alpha}, \hat{\beta}, ...$ denote flat ten-dimensional indices. The metric is invariant under local Lorentz transformations $\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}} \to \Lambda(x)_{\hat{\beta}}{}^{\hat{\alpha}} \hat{e}_{\hat{\mu}}{}^{\hat{\beta}}$. This allows us to choose the gauge

$$\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}} = \begin{pmatrix} e_\mu{}^\alpha & A_\mu^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix} , \tag{5.16}$$

where $\alpha = 0, 1, 2, 3$ and $a = 4, 5, ...9$ are flat indices. This parametrization has been chosen such that

- $e_\mu{}^\alpha$ is the 4d vierbein, and $g_{\mu\nu} = e_\mu{}^\alpha e_\nu{}^\beta \eta_{\alpha\beta}$ is the 4d metric

- $e_m{}^a$ produces 4d scalars $G_{mn} = e_m{}^a e_n{}^b \delta_{ab}$

- A_μ^m are six 4d abelian vector fields.

This can be checked using (5.14). For example, under the residual 10d general coordinate transformations (5.14) with parameters $\xi^m(x^\nu)$, we have

$$\delta A_\mu^m = \partial_\mu \xi^m , \tag{5.17}$$

which shows that A_μ^m are the gauge fields belonging to the residual $U(1)^6$.

Other four-dimensional fields can also be defined conveniently using vielbeins. We give one example. Define the four-dimensional vector fields corresponding to the gauge transformation (5.15) by

$$B_{\mu m} = e_\mu{}^\alpha e_m{}^a \hat{e}_\alpha{}^{\hat{\mu}} \hat{e}_a{}^{\hat{\nu}} \hat{B}_{\hat{\mu}\hat{\nu}} , \tag{5.18}$$

which is equivalent to saying that $B_{\alpha a} = \hat{B}_{\alpha a}$. This ensures that B_μ^m transforms as a 4d vector under 10d general coordinate transformations. In particular, it guarantees that B_μ^m is invariant under the $U(1)$ transformations in (5.14). Using (the inverse of) (5.16), we find

$$B_{\mu m} = \hat{B}_{\mu m} - A_\mu^n B_{mn} , \tag{5.19}$$

where we defined $B_{mn} = \hat{B}_{mn}$, or equivalently, $B_{ab} = \hat{B}_{ab}$.

The vector fields A_μ^m are usually called Kaluza Klein vector fields, since Kaluza and Klein put forward the idea that gravity in higher dimensions gives rise to gravity plus gauge symmetry in lower dimensions. The fields $B_{\mu m}$ are usually called winding vector fields since they can be shown to couple to the states corresponding to strings that wind around the periodic y^m direction.

Let us write down the low-energy effective action (5.7) of the heterotic string, dimensionally reduced to four dimensions[2] [147],

$$\begin{aligned} S = \int d^4x \sqrt{-g} e^{-2\Phi} [& -R_g + 4\partial_\mu \Phi \partial^\mu \Phi - \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \\ & -\tfrac{1}{4} \mathcal{F}^T_{\mu\nu} LML \mathcal{F}^{\mu\nu} + \tfrac{1}{8} \mathrm{Tr}(\partial_\mu ML \partial^\mu ML)] . \end{aligned} \tag{5.20}$$

The various fields in this action are given by the definitions above and

$$\mathcal{F}_{\mu\nu}(\mathcal{A}) = \begin{pmatrix} F^m_{\mu\nu}(A) \\ F_{\mu\nu\, m}(B) \end{pmatrix} = \begin{pmatrix} \partial_\mu A^m_\nu - \partial_\nu A^m_\mu \\ \partial_\mu B_{\nu m} - \partial_\nu B_{\mu m} \end{pmatrix} ,$$

[2] We take $\int d^6 y = 1$ for the volume of the six-dimensional torus.

$$
\begin{aligned}
H_{\mu\nu\rho} &= \partial_{[\mu}B_{\nu\rho]} + \tfrac{1}{2}\mathcal{A}^T_{[\mu}L\mathcal{F}_{\nu\rho]}\,, \\
B_{\mu\nu} &= \hat{B}_{\mu\nu} + \tfrac{1}{2}A^m_\mu B_{\nu m} - \tfrac{1}{2}A^m_\nu B_{\mu m} - B_{mn}A^m_\mu A^n_\nu\,, \\
\Phi &= \hat{\Phi} - \tfrac{1}{4}\log\det G_{mn}\,.
\end{aligned} \tag{5.21}
$$

Furthermore, we defined the 12×12 matrix

$$
L = \begin{pmatrix} 0 & I_6 \\ I_6 & 0 \end{pmatrix}\,, \tag{5.22}
$$

where I_6 is the 6×6 identity matrix. Transformations that preserve L, $\Omega^T L \Omega = L$, form the group $O(6,6)$. The last term in (5.20) represents the kinetic terms of all scalars G_{mn} and B_{mn}, rearranged in the matrix

$$
M = \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G - BG^{-1}B \end{pmatrix}\,, \tag{5.23}
$$

where we use the matrix notation $(G)_{mn} = G_{mn}$ and $(B)_{mn} = B_{mn}$. This scalar matrix is in fact an $O(6,6)$ matrix, since $M^T L M = L$. It is also symmetric, $M^T = M$. Together, these facts imply that M parametrizes the coset $\frac{O(6,6)}{O(6)\times O(6)}$. The scalar fields G_{mn} and B_{mn} characterize the torus of compactification, and the coset they span is also called the moduli space of toroidal compactifications.

The way in which the terms in the action (5.20) have been arranged shows that it is $O(6,6)$ symmetric. For $\Omega \in O(6,6)$, the following transformations are symmetries of the action:

$$
M \to \Omega M \Omega^T\,, \qquad \mathcal{F}_{\mu\nu} \to \Omega \mathcal{F}_{\mu\nu}\,. \tag{5.24}
$$

We also see that Kaluza Klein and winding vector fields are transformed into each other under (5.24).

If fermions are included, (5.20) is the action of $D = 4$ $N = 4$ supergravity coupled to six abelian vector fields[3]. Toroidal compactification does not break any supersymmetries and therefore we get four (Majorana) supersymmetry charges in $D = 4$ from one (Majorana-Weyl) supersymmetry charge in $D = 10$. The non-compact global $O(6,6)$ symmetry of (5.20) is one example of a generic phenomenon in extended supergravity theories. In such theories there is a non-compact global symmetry group G with a maximal compact subgroup H. The scalar fields of the theory are associated with the coset G/H. This implies, in particular, that their number is $\dim(G) - \dim(H)$. In our case $G = O(6,6)$ and $H = O(6)\times O(6)$ and therefore $\dim(G) - \dim(H) = 66 - 2.15 = 36$ which corresponds to the number of scalars G_{mn} (21) and B_{mn} (15). The action (5.20) contains yet another scalar, the dilaton. We will see in section 3 of this chapter that it also belongs to a coset associated to an additional non-compact global symmetry.

The presence of an $O(6,6)$ symmetry of the dimensionally reduced action can partly be ascribed to residual 10d gauge transformations. A $GL(6)$ subgroup of $O(6,6)$, acting on indices $m, n, ...$, arises from the 10d coordinate transformations $\delta y^m = -a^m{}_n y^n$. Also, the shifts $B_{mn} \to B_{mn} + b_{mn}$, accompanied by $B_{\mu m} \to B_{\mu m} - b_{mn}A^n_\mu$, are relics of

[3]The supergravity multiplet itself contains the other six vector fields.

gauge transformations of the 10d antisymmetric tensor field, see equations (5.15) and (5.19). Together, they correspond to the subgroup

$$\begin{pmatrix} a & 0 \\ -(a^T)^{-1}b & (a^T)^{-1} \end{pmatrix} \subset O(6,6)\,, \tag{5.25}$$

where $a \in GL(6)$ and b is an antisymmetric 6×6 matrix. Whereas this subgroup acts linearly (in the sense of no quadratic or higher nonlinearities) on the scalars, the other $O(6,6)$ transformations do not.

The $O(6,6)$ transformations which do not correspond to 10d gauge transformations can be used as solution generating transformations[4]. It was shown by Sen [179] that these transformations form an $\frac{O(6)\times O(6)}{O(6)}$ coset, with transformation matrices

$$\Omega = \frac{1}{2}\begin{pmatrix} R+S & R-S \\ R-S & R+S \end{pmatrix}\,, \tag{5.26}$$

with $R, S \in O(6)$. The diagonal subgroup with $R = S$ is divided out since it is part of the $GL(6)$ transformations in (5.25).

This analysis can be generalized to include the vector fields (5.8) of the heterotic string. It is known that the requirement of anomaly cancellation restricts the gauge group to either $E_8 \times E_8$ or $SO(32)$ [103]. Toroidal compactification generically gives masses to all charged vector fields so that only the vector fields associated to the Cartan subalgebra remain massless. This gives 16 abelian vector fields in both cases. These vector fields can be grouped together with the 12 abelian vector fields coming from the metric and the antisymmetric tensor field to form an $O(6,22)$ vector multiplet. The scalars (including those that come from the additional 10d vector fields) now form a coset $\frac{O(6,22)}{O(6)\times O(22)}$. The dimensionally reduced action then has a global $O(6,22)$ symmetry.

Another straightforward generalization is to dimensionally reduce with respect to the d coordinates of a torus T^d. The resulting $10-d$ dimensional effective action will then have a global symmetry group $O(d,d)$, or $O(d,d+16)$[5] if additional vector fields are present.

5.2 T-duality in toroidal compactifications

It is well-known that whereas the low-energy effective action of the heterotic string has an $O(6,22)$ global symmetry, this can not be true for the full string theory. It turns out that only a discrete subgroup may survive as a symmetry of the full string theory. The reason for this is that in the full string theory the transformations $\Omega \in O(6,22)$ must

[4]These transformations can be used to obtain possibly new solutions starting from a known solution. All solutions must have (in this case) six isometries as they must be compatible with dimensional reduction.

[5]Larger symmetry groups, containing $O(d,d+16)$ as a subgroup, arise for $d \geq 6$, i.e. for $(D \leq 4)$-dimensional reduced actions, see e.g. [118] and references therein.

preserve a 28-dimensional even self-dual lattice of charges, with an $O(6,22)$ metric

$$L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix} . \tag{5.27}$$

This lattice should be thought of as the lattice of allowed [151] charges of string states with respect to the 28 massless abelian vector fields that appear upon toroidal compactification of the heterotic string to four dimensions. The lattice is spanned by vectors $(a_1, ...a_6, b_1, ...b_6, \vec{k})$ where a_i and b_i are integers and the vectors $\vec{k}$ belong to the root lattice of $E_8 \times E_8$ or $SO(32)$. The integers a_i and b_i can be interpreted as Kaluza Klein momenta (momenta along the internal directions) and winding numbers (number of times a closed string wraps around a compact direction), respectively. The transformations that preserve the lattice form a discrete group $O(6,22;\mathbb{Z})$. This is expected to be a symmetry of the full string theory, called target-space duality or T-duality in short. In fact, T-duality is known to hold in each order of string perturbation theory. Moreover, T-duality transformations have been shown to correspond to discrete gauge symmetries in the space of string backgrounds [71]. We refer to [98] for an extensive review.

Before, we called the coset $\frac{O(6,22)}{O(6)\times O(22)}$ the moduli space of toroidal compactification of the heteroric string to four dimensions. We now see, however, that the moduli space of *inequivalent* toroidal compactifications is this coset modulo $O(6,22;\mathbb{Z})$ T-duality transformations.

Next, we describe T-duality at the level of the sigma model, and we will see how it relates equivalent conformal field theories underlying the toroidal compactifications. Again we refer to review articles [4, 98] for additional information and references.

It was shown by Buscher [53] that given a sigma model in a background (g, B, Φ)[6] with an abelian isometry, there exists an equivalent sigma model with background fields $(\tilde{g}, \tilde{B}, \tilde{\Phi})$ related to the former by a duality transformation. If we choose an adapted coordinate system $(x^0, x^\mu) = (y, x^\mu)$, where the isometry acts by translations of y, i.e. the background fields are independent of y, the backgrounds of two equivalent sigma models are related by

$$\begin{aligned}
&\tilde{g}_{00} = 1/g_{00} \,, \qquad \tilde{g}_{0\mu} = -B_{0\mu}/g_{00} \,, \\
&\tilde{g}_{\mu\nu} = g_{\mu\nu} - (g_{0\mu}g_{0\nu} - B_{0\mu}B_{0\nu})/g_{00} \,, \\
&\tilde{B}_{0\mu} = -g_{0\mu}/G_{00} \,, \\
&\tilde{B}_{\mu\nu} = B_{\mu\nu} - (g_{0\mu}B_{0\nu} - g_{0\nu}B_{0\mu})/g_{00} \,, \\
&\tilde{\Phi} = \Phi - \tfrac{1}{2}\log|g_{00}| \,.
\end{aligned} \tag{5.28}$$

Note that these equations describe a $\mathbb{Z}_2$ transformation. For string theory this means that two string backgrounds (conformally invariant sigma models) with an abelian isometry in the y direction and related by (5.28) yield the same string dynamics.

[6] In the following discussion all fields considered are conventional sigma model background fields, and we omit hats.

As a simple example, let us take a diagonal metric g, and $B = 0$. If the coordinate $x^0 = y$ is compactified on a circle, and g and Φ are independent of y, the above transformations give $\tilde{g}_{00} = 1/g_{00}$ (and $\tilde{\Phi} = \Phi - \frac{1}{2}\log|g_{00}|$)[7]. This corresponds to an inversion of the radius of the circle, $R \to \frac{1}{R}$, or $R \to \frac{\alpha'}{2R}$ if we restore the unit of length. This suggests that string geometry near the Planck scale (more precisely, the string scale) may be quite different from ordinary geometry.

5.2.1 Canonical approach to T-duality

The Buscher duality transformation relates two equivalent conformal field theories. One would expect that such an equivalence should be obtainable in the Hamiltonian formalism as a canonical transformation. This is indeed the case as has been shown in [99, 3]. In this subsection, we generalize the canonical approach to arbitrary $O(d,d;\mathbb{Z})$ T-duality transformations.

We start with the nonlinear sigma model with Lagrangian

$$\mathcal{L} = \tfrac{1}{2} g_{\hat{\mu}\hat{\nu}}(X)\partial_a X^{\hat{\mu}} \partial^a X^{\hat{\nu}} + \tfrac{1}{2}\varepsilon^{ab} B_{\hat{\mu}\hat{\nu}}(X)\partial_a X^{\hat{\mu}} \partial_b X^{\hat{\nu}} \,, \tag{5.29}$$

where we chose the conformal gauge and a convenient normalization. The results will be for the bosonic string, but can also be applied to the (bosonic sector of the) heterotic string. We let $\hat{\mu}, \hat{\nu}, ... \in \{0, 1, ...\bar{D}-1\}$, $\mu, \nu, ... \in \{0, 1, ...D-1\}$ and $m, n, ... \in \{D, D+1, ...\bar{D}-1\}$, and we assume that the background fields $g_{\hat{\mu}\hat{\nu}}$ and $B_{\hat{\mu}\hat{\nu}}$ are independent of the coordinates X^m. Thus we have $d \equiv \bar{D} - D$ abelian isometries whose orbits we assume to be compact. This means we are considering toroidal compactification of a $\bar{D}$-dimensional sigma model background on a d-dimensional torus. For the periodicities of the compact coordinates we may take without loss of generality[8]

$$X^m \simeq X^m + 2\pi \,. \tag{5.30}$$

Closed strings[9] can wrap around the circles,

$$X^m(\sigma + 2\pi) = X^m(\sigma) + 2\pi b^m \,, \tag{5.31}$$

where the integers b^m are the winding numbers. We will demonstrate how the T-duality group $O(d,d;\mathbb{Z})$ emerges from the action of canonical transformations. We follow the same steps as in [3, 4] where a single abelian isometry is considered.

The momentum canonically conjugate to $X^{\hat{\mu}}$ is

$$P_{\hat{\mu}} \equiv \frac{\partial \mathcal{L}}{\partial \dot{X}^{\hat{\mu}}} = g_{\hat{\mu}\hat{\nu}} \dot{X}^{\hat{\nu}} + B_{\hat{\mu}\hat{\nu}} X'^{\hat{\nu}} \,, \tag{5.32}$$

[7] The shift of the dilaton ensures that the sigma model remains conformally invariant, at least to one-loop order in α' [53].

[8] We assume the information on the radii of the torus to be contained in the metric.

[9] We restrict ourselves to closed strings here. For open strings, T-duality also has interesting consequences. It interchanges Neumann and Dirichlet boundary conditions, and leads to the emergence of extended objects called D-branes. For a review, see [158].

and the Hamiltonian is given by

$$\begin{aligned}\mathcal{H} &\equiv P_{\hat\mu}\dot X^{\hat\mu} - \mathcal{L} = \tfrac{1}{2}g_{\hat\mu\hat\nu}\dot X^{\hat\mu}\dot X^{\hat\nu} + \tfrac{1}{2}g_{\hat\mu\hat\nu}X'^{\hat\mu}X'^{\hat\nu} \\ &= \tfrac{1}{2}g^{\hat\mu\hat\nu}(P_{\hat\mu} - B_{\hat\mu\hat\rho}X'^{\hat\rho})(P_{\hat\nu} - B_{\hat\nu\hat\sigma}X'^{\hat\sigma}) + \tfrac{1}{2}g_{\hat\mu\hat\nu}X'^{\hat\mu}X'^{\hat\nu}\,.\end{aligned} \tag{5.33}$$

Let us now rewrite $\mathcal{L}$ as follows

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}g_{mn}(X)\left(\dot X^m\dot X^n - X'^m X'^n\right) + B_{mn}\dot X^m X'^n \\ &\quad +(\dot X^m + X'^m)J_{m-} + (\dot X^m - X'^m)J_{m+} + V\,,\end{aligned} \tag{5.34}$$

where

$$\begin{aligned}&J_{m-} = \tfrac{1}{2}g_{m\mu}(\dot X^\mu - X'^\mu) + \tfrac{1}{2}B_{m\mu}(X'^\mu - \dot X^\mu) = \tfrac{1}{2}(g_{m\mu} - B_{m\mu})\partial_- X^\mu\,,\\ &J_{m+} = \tfrac{1}{2}g_{m\mu}(\dot X^\mu + X'^\mu) + \tfrac{1}{2}B_{m\mu}(X'^\mu + \dot X^\mu) = \tfrac{1}{2}(g_{m\mu} + B_{m\mu})\partial_+ X^\mu\,,\\ &V = \tfrac{1}{2}g_{\mu\nu}(\dot X^\mu\dot X^\nu - X'^\mu X'^\nu) + B_{\mu\nu}\dot X^\mu X'^\nu = \tfrac{1}{2}(g_{\mu\nu} - B_{\mu\nu})\partial_+ X^\mu\partial_- X^\nu\,.\end{aligned} \tag{5.35}$$

To obtain a dual sigma model using canonical transformations, we need to apply the Legendre transformation only to the internal coordinates $(X^m, \dot X^m)$. Then the momenta are[10]

$$P_m = G_{mn}\dot X^n + B_{mn}X'^n + J_{m-} + J_{m+}\,, \tag{5.36}$$

and the 'Hamiltonian' corresponding to this incomplete Legendre transformation is

$$\begin{aligned}\mathcal{H} &= P_m\dot X^m - \mathcal{L} = \tfrac{1}{2}G^{mn}P_mP_n - G^{mn}B_{nk}P_mX'^k + \tfrac{1}{2}(G - BG^{-1}B)_{mn}X'^mX'^n \\ &\quad +G^{mn}B_{mk}(J_{n+} + J_{n-})X'^k - G^{mn}(J_{n+} + J_{n-})P_m \\ &\quad +X'^m(J_{m+} - J_{m-}) + \tfrac{1}{2}G^{mn}(J_{m+} + J_{m-})(J_{n+} + J_{n-}) - V\,,\end{aligned} \tag{5.37}$$

where we eliminated $\dot X^m$ using (5.36). We may now perform canonical transformations using a generating function that depends on X^m and the new coordinates $\tilde X^m$. (A nice explanation of generating functions for canonical transformations can be found in [167].) We take as the generating function

$$F(X, \tilde X) = \int_0^{2\pi} d\sigma\left(a_{mn}X^m\tilde X'^n + \tfrac{1}{2}b_{mn}\tilde X^m\tilde X'^n\right)\,, \tag{5.38}$$

where a is a non-singular $d \times d$ matrix and b is an antisymmetric $d \times d$ matrix (the symmetric part of b would contribute a total derivative to the integrand in (5.38)). The momenta are given by

$$\begin{aligned}P_m &= \frac{\delta F}{\delta X^m} = a_{mn}\tilde X'^n\,,\\ \tilde P_m &= -\frac{\delta F}{\delta \tilde X^m} = a_{nm}X'^n + b_{nm}\tilde X'^n\,.\end{aligned} \tag{5.39}$$

The zero modes of the momenta P_m are integers. This follows from requiring single-valuedness of the wave functions associated to string states under $X^m \to X^m + 2\pi$.

[10] To stick with our notation in the previous section, we denote the 'internal' components of the metric by G_{mn}.

Moreover, the zero mode of X'^m is the winding number b^m. In order that the zero modes remain integer under duality transformations, we should restrict the coefficients a_{mn} and b_{mn} to integers. Thus we observe also here (and in fact for the same reason that the string spectrum is required to be duality-invariant) that the T-duality group is a discrete group.

To obtain the new Hamiltonian $\tilde{\mathcal{H}}$ we first substitute P_m of (5.39) in (5.37) to obtain an expression in terms of X' and $\tilde{X}'$, and then use the second line in (5.39) to eliminate X'. Note that because $\mathcal{H}$ has no explicit X^m dependence this leads to a local expression. The result is, after some simplifications,

$$\begin{aligned}\tilde{\mathcal{H}} &= \tfrac{1}{2}\tilde{X}'a^TG^{-1}a\tilde{X}' - \tilde{X}'a^TG^{-1}B(a^T)^{-1}(\tilde{P}+b\tilde{X}') \\ &\quad +\tfrac{1}{2}(\tilde{P}-\tilde{X}'b)a^{-1}(G-BG^{-1}B)(a^T)^{-1}(\tilde{P}+b\tilde{X}') \\ &\quad +(J_+ + J_-)G^{-1}B(a^T)^{-1}(\tilde{P}+b\tilde{X}') - (J_+ + J_-)G^{-1}a\tilde{X}' \\ &\quad +(J_+ - J_-)(a^T)^{-1}(\tilde{P}+b\tilde{X}') + \tfrac{1}{2}(J_+ + J_-)G^{-1}(J_+ + J_-) - V\,. \end{aligned} \tag{5.40}$$

We suppress all indices by using matrix notation. In order to get the new sigma model Lagrangian we perform the inverse Legendre transformation from $(\tilde{X},\tilde{P})$ to $(\tilde{X},\dot{\tilde{X}} = \frac{\partial\tilde{\mathcal{H}}}{\partial\tilde{P}})$. We get, after some rearrangements,

$$\begin{aligned}\tilde{\mathcal{L}} &= \tfrac{1}{2}\dot{\tilde{X}}a^TWa\dot{\tilde{X}} - \tfrac{1}{2}\tilde{X}'a^TWa\tilde{X}' - \dot{\tilde{X}}(a^TWBG^{-1}a+b)\tilde{X}' \\ &\quad -J_+(Wa+WBG^{-1}a)(\dot{\tilde{X}}-\tilde{X}') + J_-(Wa-WBG^{-1}a)(\dot{\tilde{X}}+\tilde{X}') \\ &\quad -2J_+(W+WBG^{-1})J_- + V\,. \end{aligned} \tag{5.41}$$

For notational convenience we defined the matrix

$$W = (G-BG^{-1}B)^{-1}\,, \tag{5.42}$$

and we used $G^{-1}BW = WBG^{-1}$.

This is the dual sigma model Lagrangian and we can read off the new background fields by comparing with (5.34). Thus we see that

$$\begin{aligned}\tilde{G}_{mn} &= (a^TWa)_{mn} = (a^T(G-BG^{-1}B)^{-1}a)_{mn}\,, \\ \tilde{B}_{mn} &= -(a^TWBG^{-1}a)_{mn} - b_{mn}\,. \end{aligned} \tag{5.43}$$

Furthermore,

$$\begin{aligned}\tilde{J}_+ &= -(a^TW - a^TG^{-1}BW)J_+\,, \\ \tilde{J}_- &= \ (a^TW + a^TG^{-1}BW)J_-\,, \end{aligned} \tag{5.44}$$

from which we derive, using the definitions of $J_\pm$ given in (5.35),

$$\begin{aligned}\tilde{g}_{m\mu} &= (a^TG^{-1}BW)_m{}^n g_{n\mu} - (a^TW)_m{}^n B_{n\mu}\,, \\ \tilde{B}_{m\mu} &= -(a^TW)_m{}^n g_{n\mu} + (a^TG^{-1}BW)_m{}^n B_{n\mu}\,. \end{aligned} \tag{5.45}$$

Finally, we get from $\tilde{V} = V - 2J_+(W + WBG^{-1})J_-$,

$$\begin{aligned}
\tilde{g}_{\mu\nu} &= g_{\mu\nu} - g_{m\mu}W^{mn}g_{n\nu} \\
&\quad +2g_{m(\mu}(WBG^{-1})^{mn}B_{n\nu)} + B_{m\mu}W^{mn}B_{n\nu}\,, \\
\tilde{B}_{\mu\nu} &= B_{\mu\nu} - 2g_{m[\mu}W^{mn}B_{n\nu]} \\
&\quad +g_{m\mu}(WBG^{-1})^{mn}g_{n\nu} - B_{m\mu}(WBG^{-1})^{mn}B_{n\nu}\,.
\end{aligned} \tag{5.46}$$

The transformations (5.43), (5.45) and (5.46) generalize the $\mathbb{Z}_2$ duality rules of equation (5.28).

We note that for $a_{mn} = \delta_{mn}$ and $b_{mn} = 0$, the transformation in (5.43) can be written as $E \to 1/E$, where $E_{mn} \equiv (G+B)_{mn}$. This generalizes the $R \to 1/R$ circle duality to a d-torus.

We have not yet obtained all $O(d,d;\mathbb{Z})$ transformations, because transformations connected to the identity (in the continuous group $O(d,d;\mathbb{R})$, that is) cannot be obtained using the generating function F in (5.38). To obtain the latter transformations, we have to use a generating function that depends on the pair $(X^m, \tilde{P}_m)$. Using

$$F(X,\tilde{P}) = \int_0^{2\pi} d\sigma \left(a_m{}^n X^m \tilde{P}_n + \tfrac{1}{2} b_{mn} X^m X'^n \right)\,, \tag{5.47}$$

together with

$$\begin{aligned}
P_m &= \frac{\delta F}{\delta X^m} = a_m{}^n \tilde{P}_n + b_{mn} X'^n\,, \\
\tilde{X}^m &= \frac{\delta F}{\delta \tilde{P}_m} = X^n a_n{}^m\,,
\end{aligned} \tag{5.48}$$

we obtain all transformations of the form (5.25), i.e. general linear transformations acting on the indices $m, n, \ldots$ and constant shifts of B_{mn}, where now we have integer transformation parameters because of the periodicities of the internal coordinates.

We can also apply the Legendre transformation to a single coordinate X^m only, and use a generating function

$$F^m(X,\tilde{X}) = \int_0^{2\pi} d\sigma X^m \tilde{X}'^m\,, \tag{5.49}$$

with no summation over m understood. It generates the $R \to 1/R$ duality for the single compact coordinate X^m. This completes our identification of all $O(d,d;\mathbb{Z})$ transformations with canonical transformations. Further details on different classes of $O(d,d;\mathbb{Z})$ transformations may be found in [98].

One thing to note is that a transformation rule for the dilaton like the one that is part of the Buscher rules does not follow from this canonical approach. This comes as no surprise because we started with a world-sheet action in the conformal gauge and without the dilaton. There is however more than one way to see that the dilaton has to transform in order to maintain conformal invariance of the sigma model. One way is to look at the vanishing of the β-functions (5.5), or equivalently to look at the nilpotency condition $Q^2_{BRST} = 0$. In order for these to be satisfied, the dilaton has to transform as

$$\tilde{\Phi} = \Phi - \tfrac{1}{2} \log |\det G_{mn}| \,. \tag{5.50}$$

As argued in [4], one of the advantages of the canonical approach is that the dual fields can be obtained in a covariant way for dualities with respect to arbitrary abelian isometries. This makes it easier to obtain global information on the dual manifold. The implementation of the canonical transformation in the path integral is described in [94, 4].

5.3 Strong/weak coupling dualities

In this final section we describe another type of duality symmetry in string theory, namely dualities that act nonlinearly on the string coupling constant. We will be rather brief. The following text is essentially that of [41]. It is based on the papers [42, 19].

At the moment, the interest in strong/weak coupling dualities in field theory and string theory is rapidly growing. Such dualities map the weak coupling region of a theory into the strong coupling region of the same or another theory. Strong/weak coupling dualities may prove to be very important since they provide a way to go beyond perturbation theory. We discuss, at the level of low-energy effective actions, an early example of a strong/weak coupling duality in the context of string theory, namely string/fivebrane duality. This duality was first proposed by Strominger in [182] where a fivebrane soliton solution of the heterotic string effective theory was found. Further support for string/fivebrane duality has been presented in [74, 75].

Whereas a particle couples naturally to a one-form gauge field, a generic p-brane, an extended object that sweeps out a $(p+1)$-dimensional world-volume in space-time, couples to a $(p+1)$-form gauge field. For a string this is the well-known coupling to the two-form field in the sigma model, as in (5.1). In D space-time dimensions, a $(p+1)$-form gauge field is related by Poincaré duality to a $(D-p-3)$-form gauge field carrying the same degrees of freedom. This $(D-p-3)$-form gauge field couples naturally to a $(D-p-4)$-brane and thus we see that the dual of a p-brane is a $(D-p-4)$-brane. The duality transformation interchanges electric and magnetic components of the field strengths. Because of a generalized Dirac quantization argument [152], electric-magnetic dualities are accompanied by a strong/weak coupling interchange.

Apart from string/fivebrane duality in $D = 10$, some other strong/weak coupling dualities have been conjectured. The first of these was the Montonen-Olive particle/particle duality [150, 101] in $D = 4$ supersymmetric field theory. More recently, string/string duality in $D = 6$ has received a lot of attention, see [72, 118, 195, 181, 107]. Many other dualities between different string theories in several dimensions have now been proposed.

Besides strong/weak coupling dualities between different string (or p-brane) theories, some other relations that usually act within one particular string (or p-brane) theory are also known under the name dualities. Target-space duality or T-duality (see [98] for a review) is a symmetry of string theory that relates different background field configurations in which the string dynamics is exactly the same. It is discussed in the previous section. Target-space duality is known to hold in each order of string perturbation theory. On the other hand, S-duality (see [180] and references therein) is a strong/weak coupling duality and therefore can only be true nonperturbatively in the string coupling.

An interesting observation was made in references [176, 34]: S-duality and T-duality seem to get interchanged under string/fivebrane duality. The roles of S-moduli and some of the T-moduli are interchanged in going from the string to the fivebrane formulation. This is also consistent with an earlier observation [74] that string/fivebrane duality interchanges world-sheet and space-time loop expansion parameters.

We first discuss to what extent T and S-dualities are interchanged in the case of heterotic string/fivebrane duality. A more extensive exposition may be found in [42]. Consequently we turn our attention to solutions of the low-energy field theory, and describe how dyonic solutions can be obtained using electric/magnetic duality rotations that are symmetries of the equations of motion in a particular dimensionally reduced effective action [19].

5.3.1 $SL(2,\mathbb{R})$ symmetry of strings and fivebranes

Unfortunately, it is unknown how to quantize the fivebrane starting from the sigma model, and therefore we have no means to derive the fivebrane effective action. What is known however, is that since the fivebrane couples to a six-form, this effective action must involve a six-form gauge field. A natural candidate for the fivebrane effective action then comes to mind: $D = 10$ $N = 1$ supergravity in the dual formulation [59] which contains a six-form gauge field $A^{(6)}$ instead of the two-form gauge field B. This seems to be the best guess for the fivebrane effective action. Another problem is that it is unknown how heterotic fivebranes couple to vector fields. However, we will again adopt the point of view that the low-energy effective action is obtained by applying the standard dualization procedure to the two-form field of $D = 10$ $N = 1$ supergravity coupled to Yang-Mills fields A. Let us briefly explain this dualization procedure. First, the 3-form field strength $H = dB + A \wedge F$ is regarded as an independent unconstrained field in the usual $D = 10$ $N = 1$ action, thus one ignores its expression in terms of B and the Yang-Mills Chern-Simons term. Instead, the Bianchi identity of H is imposed by adding a Lagrange multiplier term to the Lagrangian:

$$\mathcal{L}(B, ...) \to \mathcal{L}(H, ...) + A^{(6)} \wedge (dH - F \wedge F) , \qquad (5.51)$$

where $A^{(6)}$ is the six-form Lagrange multiplier and F is the Yang-Mills field strength. Using the equation of motion for $A^{(6)}$, we recover the Bianchi identity for H and get back to the original formulation. However, if we eliminate H by its equation of motion, we obtain an action in terms of the six-form gauge potential $A^{(6)}$ instead of B. As we see from (5.51), the Chern-Simons term in the original formulation becomes a topological term in the dual formulation.

The heterotic string dimensionally reduced to $D = 4$ has an $SL(2,\mathbb{R})$ invariance of the equations of motion, though not of the action. This invariance is related to S-duality, which consists of transformations belonging to the discrete subgroup $SL(2,\mathbb{Z})$ believed to be a symmetry of the full string theory[11]. In order to see the $SL(2,\mathbb{R})$ S-duality invariance for the fivebrane, we dimensionally reduced (the bosonic part of) the dual action to four dimensions. This reduction was first done by Chamseddine in [59] where the emphasis was on properties of the resulting scalar potential, in view of possible supersymmetry breaking. In [176], the reduction of the bosonic part of the dual ten-dimensional action without vector fields was performed, and an $SL(2,\mathbb{R})$ symmetry of the *action* was recognized. We now describe the results of [42], in which we performed a dimensional reduction including vector fields and investigated the resulting symmetries in four dimensions.

Let us compare the dimensional reductions of the two-form (string) and six-form (fivebrane) formulations. In the table below we compare the four-dimensional fields[12] coming from the two-form and six-form gauge fields. We use $\mu, \nu, ...$ as $D = 4$ space-time indices and $m, n, ...$ as indices belonging to the six internal coordinates.

two-form version	six-form version
$B_{\mu\nu}$ dualized to λ_1	$\lambda_1 \equiv \varepsilon^{m_1 \ldots m_6} A^{(6)}_{m_1 \ldots m_6}$
B_{mn}	$A_{\mu\nu mn} \equiv \varepsilon_{mn}{}^{m_1 \ldots m_4} A^{(6)}_{\mu\nu m_1 \ldots m_4}$ dualized to B_{mn}
B^m_μ	$B^m_\mu \equiv \varepsilon^{m m_1 \ldots m_5} A^{(6)}_{\mu m_1 \ldots m_5}$

Table 4. *Four-dimensional fields arising from the ten-dimensional antisymmetric tensor fields in the two and six-form formulations.*

In the table it is indicated that the two-forms in four dimensions have been dualized to scalars. In the two-form formulation this makes the $SL(2,\mathbb{R})$ symmetry of the equations of motion manifest. The pseudo-scalar λ_1 is usually referred to as the axion. As could be expected from the dualizations in the table above compared to the dualization in $D = 10$, the net difference between the two resulting four-dimensional actions boils down to a dualization of the six winding vector fields B^m_μ.

Let us compare the symmetries of both actions. We will not write down the full actions here, they can be found for example in [180] (string action) and [42] (fivebrane action including Yang-Mills vector fields). The four-dimensional string action is also given in equation (5.20) where the two-form field is not dualized. Let us first assume that there are no vector fields in $D = 10$.

In the string action, the kinetic terms for the vector fields that come from the dimen-

[11] For references, see [180].

[12] More precise formulas for the four-dimensional fields that ensure correct $D = 4$ transformation properties can be found in [176]. See also the discussion on dimensional reduction in section 2.1.1.

sional reduction take the form[13] (schematically)

$$\frac{1}{\sqrt{-g}}\mathcal{L}_{\rm string} \sim \lambda_2 \mathcal{F}^T LML\mathcal{F} + \lambda_1 \mathcal{F}^T L\, {}^*\mathcal{F}\,, \tag{5.52}$$

where

$$\mathcal{F} = \begin{pmatrix} F^m(A) \\ F^m(B) \end{pmatrix} \tag{5.53}$$

is the multiplet of twelve vector fields, six fields A_μ^m arising from the 10d metric (Kaluza Klein vector fields), and six fields B_μ^m arising from the 10d antisymmetric tensor (winding vector fields). The Hodge star acts as ${}^*F_{\mu\nu} = \frac{1}{2\sqrt{-g}}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. As in equation (5.23), M is the symmetric $O(6,6)$ matrix that parametrizes the coset $\frac{O(6,6)}{O(6)\times O(6)}$ of scalar fields G_{mn} and B_{mn} coming from the 10d metric and 10d antisymmetric tensor fields, and the constant 12×12 matrix L is the $O(6,6)$ metric as given in (5.22). The scalars λ_2 and λ_1 parametrize the coset $SL(2,\mathbb{R})/U(1)$ and are usually combined into the axion/dilaton field $S = \lambda_1 + i\lambda_2$ where λ_1 is the axion and $\lambda_2 = e^{-\phi}$ is the inverse string coupling as ϕ is the dilaton. This action is invariant under $O(6,6)$ transformations that act on the vector and scalar fields as

$$\mathcal{F} \to \Omega\mathcal{F}\,, \qquad M \to \Omega M \Omega^T\,. \tag{5.54}$$

We see that Kaluza-Klein and winding electric charges are transformed into one another by T-duality. The axion and dilaton fields are invariant under these transformations. There is, however, an additional symmetry of the *equations of motion*. This is an $SL(2,\mathbb{R})$ invariance, related to S-duality, that acts on the S field as

$$S \to \frac{dS+c}{bS+a}\,, \qquad ad - bc = 1\,. \tag{5.55}$$

Alternatively, in terms of the symmetric $SL(2,\mathbb{R})$ matrix

$$\mathcal{M} = \frac{1}{\lambda_2}\begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & \lambda_1^2+\lambda_2^2 \end{pmatrix}\,, \tag{5.56}$$

the transformation law is

$$\mathcal{M} \to \omega\mathcal{M}\omega^T\,, \qquad \omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,. \tag{5.57}$$

Simultaneously, $SL(2,\mathbb{R})$ acts on the vector fields by duality rotations[14],

$$\begin{pmatrix} {}^*\mathcal{F} \\ \mathcal{G} \end{pmatrix} \to \omega \begin{pmatrix} {}^*\mathcal{F} \\ \mathcal{G} \end{pmatrix}\,, \tag{5.58}$$

where $\mathcal{G}$ is the equation of motion tensor $\mathcal{G}^a_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}L^a{}_b \frac{\partial\mathcal{L}}{\partial\mathcal{F}_b^{\mu\nu}}$. Indices a, b label the twelve vector fields. Bianchi identities and equations of motion are rotated into each

[13] We work in the Einstein frame, in which the Einstein-Hilbert action takes its canonical form.

[14] For a general treatment of duality rotations of vector fields, see [93]

other which implies that it is only an on-shell symmetry, not a symmetry of the action. The canonical (Einstein) metric is invariant under $O(6,6) \times SL(2,\mathbb{R})$. We see that S-duality transforms Kaluza-Klein electric (magnetic) charges and winding magnetic (electric) charges into each other.

In the $D = 4$ fivebrane action, the relevant terms are

$$\frac{1}{\sqrt{-g}}\mathcal{L}_{\text{fivebrane}} \sim G_{mn}\mathcal{F}^{m\,T}\mathcal{L}\mathcal{M}\mathcal{L}\mathcal{F}^n + B_{mn}\mathcal{F}^{m\,T}\mathcal{L}\,{}^*\mathcal{F}^n\,, \tag{5.59}$$

where the constant 2×2 matrix $\mathcal{L}$ is the $SL(2,\mathbb{R})$ invariant metric, $\omega^T\mathcal{L}\omega = \mathcal{L}$. We clearly see that the roles of T-moduli (G_{mn}, B_{mn}) and S-moduli (λ_2, λ_1) are interchanged as compared to the string case. This also means that now $SL(2,\mathbb{R})$ is a symmetry of the action, acting as

$$\mathcal{F} \to \omega\mathcal{F}\,, \qquad \mathcal{M} \to \omega\mathcal{M}\omega^T\,. \tag{5.60}$$

The $O(6,6)$ transformations now act through duality rotations and therefore constitute a symmetry of the equations of motion only. We see that now $SL(2,\mathbb{R})$ transforms Kaluza-Klein electric fields and winding electric fields into each other. Thus there are strong indications for an interchange of T and S-dualities [176].

Now we include the vector fields in the ten-dimensional effective actions. We assume that n additional abelian vector fields are present after dimensional reduction to $D = 4$. That makes the total number of vector fields $12 + n$. Then it is known for the string action that the $O(6,6)$ symmetry of the action can be extended to an $O(6,6+n)$ symmetry of the action. $SL(2,\mathbb{R})$ remains a symmetry of the equations of motion which now also involves duality rotations of the additional vector fields.

For the fivebrane action, however, things change, as was noted before in [176]. The $SL(2,\mathbb{R})$ invariance is no longer a symmetry of the action but remains a symmetry of the equations of motion only. This is because of its action on the vector fields:

$$\begin{pmatrix} F(A) \\ F(B) \end{pmatrix} \to \omega \begin{pmatrix} F(A) \\ F(B) \end{pmatrix}\,, \qquad \begin{pmatrix} {}^*F(V) \\ G(V) \end{pmatrix} \to \omega \begin{pmatrix} {}^*F(V) \\ G(V) \end{pmatrix}\,, \tag{5.61}$$

where $F(V)$ are the field strength tensors of the additional vector fields and $G(V) = -\frac{2}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial F(V)}$. The additional vector fields transform by duality rotations under $SL(2,\mathbb{R})$. Therefore, it can only be an equation of motion symmetry. Thus, $SL(2,\mathbb{R})$ no longer acts as a pure T-duality symmetry transforming electric Kaluza Klein, winding and gauge fields into each other. Rather, it acts on the gauge vector fields as electric-magnetic duality rotations. It can no longer be interpreted as a T-duality symmetry. The other symmetry, $O(6,6+n)$, remains an invariance of the equations of motion.

One may try to dualize some of the vector fields into their duals in order to obtain a manifestly $SL(2,\mathbb{R})$ invariant action. However, this is impossible since there always remain vector fields on which $SL(2,\mathbb{R})$ must be realized by duality rotations. What we conclude is that $SL(2,\mathbb{Z})$ cannot be an ordinary T-duality symmetry of the fivebrane since it involves electric-magnetic duality rotations. However, we see no reason to regard this as evidence against string/fivebrane duality.

Suggestions for making an $SL(2,\mathbb{R})$ invariant $D=4$ action including vector fields have been presented in [176]. The price to pay is that one has to give up manifest general covariance. More recently, a construction involving more auxiliary fields has been shown to yield an $SL(2,\mathbb{R}) \times O(6,22)$ manifestly general covariant action [154].

It has been shown in [72] that the interchange of T and S-dualities is more clearly realized in $D=6$ string/string duality. It seems that at least until more knowledge about quantization of fivebranes is available, a study of string/string duality will be more fruitful. However, $D=10$ string/fivebrane duality and $D=6$ string/string duality might be the same thing, the latter being a six-dimensional version of the former.

5.3.2 String and fivebrane solutions of low-energy effective theories

Classical solutions of the low-energy effective theories may provide further clues for duality. In the context of string/fivebrane duality, it is known that a solitonic fivebrane solves the heterotic string equations of motion [182]. Also, the dual fivebrane (six-form) theory admits a solitonic string [73] as a classical solution. For a recent review on string solitons, see [76].

It is known that upon reducing the heterotic string effective action to six dimensions, ignoring all vector fields, there is a $\mathbb{Z}_2$ symmetry of the equations of motion. This is the string/string duality transformation [72, 118, 195, 181, 107]

$$\phi' = -\phi\,, \qquad H' = e^{-2\phi}\, {}^*H\,, \tag{5.62}$$

in the Einstein frame. It is a duality rotation of the antisymmetric tensor field, where the Bianchi identity and the equation of motion are interchanged. This maps the fundamental string solution [64] into the solitonic string which is the ten-dimensional fivebrane solution with four spatial directions wrapped around T^4. When vector fields are included, the transformation (5.62) maps the heterotic string on T^4 to the type IIA string on K3 [118, 195, 181, 107].

Type II string effective actions dimensionally reduced to D dimensions have a larger non-compact global symmetry group[15] than type I actions. For example, in six dimensions the U-duality group is $SO(5,5)$ (see [184]). In particular, these transformations involve duality rotations of the two-form fields (of which there are five in $D=6$). It is then interesting to see whether we can use this to generate dyonic solutions of the equations of motion. This can indeed be done, and we will see that in the type II theory one can interpolate between the electrically charged fundamental string and the magnetically charged solitonic string. We will now show this using a simple $D=6$ model derived from the type IIB string. This was described in [19].

It is known that the field equations of $D=10$ type IIB supergravity[16] cannot be derived from a covariant action. This is due to the presence of the self-dual four-

[15] The corresponding conjectured discrete symmetry of the full string theory is named U-duality [118].

[16] We use formulas given in [24] in which also the relation between type IIA and type IIB fields in the presence of one isometry is given.

form field $\hat{D}$[17]. It is possible though to write down an action which reproduces the type IIB equations (with all fermionic fields put to zero) except for the self-duality condition $\hat{F}(\hat{D}) = {}^*\hat{F}(\hat{D})$. The equation of motion for $\hat{D}$ derived from this action has to be identical to the Bianchi identity, in order to be consistent with self-duality. Then putting $\hat{F}(\hat{D}) = {}^*\hat{F}(\hat{D})$ will yield the correct type IIB equations. The action we are looking for is given here in the Einstein frame in which the $SL(2,\mathbb{R})$ symmetry of the IIB theory is manifest:

$$\begin{aligned} S_{\rm IIB} \;=\; & \int d^{10}x\,\sqrt{-\hat{g}}\,\Big\{-\hat{R} + \tfrac{1}{4}{\rm Tr}\left(\partial\mathcal{M}\partial\mathcal{M}^{-1}\right) + \tfrac{3}{4}\hat{\mathcal{H}}^T\mathcal{L}\mathcal{M}\mathcal{L}\hat{\mathcal{H}} \\ & -\tfrac{5}{6}(\hat{F}(\hat{D}))^2 - \tfrac{1}{96}\frac{\varepsilon}{\sqrt{-\hat{g}}}\hat{D}\hat{\mathcal{H}}^T\mathcal{L}\hat{\mathcal{H}}\Big\}\,, \end{aligned} \tag{5.63}$$

where $\mathcal{H}^T = d\mathcal{B}^T = (H^{(1)}, H^{(2)}) = (dB^{(1)}, dB^{(2)})$ with $B^{(1)}$ and $B^{(2)}$ the Neveu-Schwarz Neveu-Schwarz (NSNS) and Ramond Ramond (RR) antisymmetric tensor fields, respectively. The scalar matrix $\mathcal{M}$ is again given by (5.56) but here the axion is replaced by the RR scalar $\hat{\ell}$: $\lambda = \lambda_1 + i\lambda_2 = \hat{\ell} + ie^{-\hat{\phi}}$. The field strength of the four-form field is given by

$$\hat{F}_{\hat{\lambda}\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}(\hat{D}) = \partial_{[\hat{\lambda}}\hat{D}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}]} + \tfrac{3}{4}\hat{\mathcal{B}}^T_{[\hat{\lambda}\hat{\mu}}\mathcal{L}\hat{\mathcal{H}}_{\hat{\nu}\hat{\rho}\hat{\sigma}]}\,. \tag{5.64}$$

The $SL(2,\mathbb{R})$ symmetry is easily recognized:

$$\mathcal{H} \to \omega\mathcal{H}\,, \qquad \mathcal{M} \to \omega\mathcal{M}\omega^T\,. \tag{5.65}$$

From now on we will refer to this symmetry as $SL(2,\mathbb{R})_{\rm IIB}$. Note that these transformations act nonlinearly on the string coupling constant. It was conjectured in [118] that an $SL(2,\mathbb{Z})$ subgroup is an exact symmetry of the full type IIB string. It involves a $\mathbb{Z}_2$ transformation that inverts the string coupling constant and interchanges NSNS and RR two-form fields.

We now use a simple ansatz for dimensional reduction to $D = 6$. Our goal is to extend the $\mathbb{Z}_2$ symmetry (5.62) to an $SL(2,\mathbb{R})$ symmetry and we will accomplish this by taking into account two additional scalar fields coming from the internal metric and the four-form field $\hat{D}$. Our ansatz for the ten-dimensional fields is

$$\begin{aligned} \hat{g}_{\mu\nu} &= e^{-G}g_{\mu\nu}\,, \qquad \hat{g}_{mn} = -\delta_{mn}e^{G}\,, \\ \hat{B}^{(i)}_{\mu\nu} &= B^{(i)}_{\mu\nu}\,, \qquad \hat{\ell} = \ell\,, \qquad \hat{\phi} = \phi\,, \\ \hat{D}_{\mu\nu\rho\sigma} &= D_{\mu\nu\rho\sigma}\,, \qquad \hat{D}_{mnpq} = D_{mnpq}\,, \end{aligned} \tag{5.66}$$

with all other components of ten-dimensional fields vanishing. Unhatted fields are six-dimensional. The four-form field $D_{\mu\nu\rho\sigma}$ is dual to a scalar in $D = 6$. This scalar is related by the self-duality constraint to $D \equiv \varepsilon^{mnpq}D_{mnpq}$. Thus we obtain one scalar from the nonzero components of $\hat{D}$ in the ansatz (5.66).

[17] We put hats again on ten-dimensional fields.

We obtain the following six-dimensional action:

$$S = \int d^6x\sqrt{-g}\Big\{-R + \frac{2\partial\lambda\partial\bar{\lambda}}{(\lambda-\bar{\lambda})^2} + \frac{2\partial\kappa\partial\bar{\kappa}}{(\kappa-\bar{\kappa})^2} + \kappa_2\mathcal{H}^T\mathcal{L}\mathcal{M}\mathcal{L}\mathcal{H} + \kappa_1\mathcal{H}^T\mathcal{L}\,{}^*\mathcal{H}\Big\}\,, \tag{5.67}$$

where

$$\kappa = \kappa_1 + i\kappa_2 = \tfrac{1}{8}D + \tfrac{3}{4}ie^{2G}\,. \tag{5.68}$$

Now we observe that besides $SL(2,\mathbb{R})_{IIB}$, there is another $SL(2,\mathbb{R})$ invariance, namely the one that acts on the $SL(2,\mathbb{R})/U(1)$ coset described by κ. This $SL(2,\mathbb{R})$ is different in character from $SL(2,\mathbb{R})_{IIB}$ because it acts through duality rotations on the two-form fields:

$$\kappa \to \frac{p\kappa+q}{r\kappa+s}\,,\qquad \mathcal{H}\to(r\kappa_1+s)\mathcal{H}+r\kappa_2\mathcal{M}\mathcal{L}\,{}^*\mathcal{H}\,. \tag{5.69}$$

This symmetry is very similar to S-duality in the $D=4$ heterotic string. We will refer to this symmetry group as $SL(2,\mathbb{R})_{EM}$ to distinguish it from $SL(2,\mathbb{R})_{IIB}$ that was already present in ten dimensions.

We may now use $SL(2,\mathbb{R})_{IIB}\times SL(2,\mathbb{R})_{EM}$ as solution generating transformations. To this end we apply the most general $SL(2,\mathbb{R})_{\rm IIB}\times SL(2,\mathbb{R})_{EM}$ transformation, with parameters a,b,c,d $(ad-bc=1)$ and p,q,r,s $(ps-qr=1)$, respectively, to the neutral fivebrane solution [74, 55], reinterpreted as a $D=6$ solitonic string,

$$\begin{aligned} ds^2 &= (dx^0)^2-(dx^1)^2-e^{2\phi}\delta_{ab}dx^a dx^b\,,\\ H^{(1)}_{abc} &= \tfrac{2}{3}\varepsilon_{abcd}\partial^d\phi\,, \end{aligned} \tag{5.70}$$

where $a,b,\ldots$ are indices for the four transverse coordinates. The result is, written in the string frame,

$$\begin{aligned} ds^2 &= A\left[(dx^0)^2-(dx^1)^2\right]-Ae^{2C}\delta_{ab}dx^a dx^b\,,\\ \begin{pmatrix}H^{(1)}\\ H^{(2)}\end{pmatrix} &= \begin{pmatrix}asH-\frac{3}{4}bre^{-2C}\,{}^*H\\ csH-\frac{3}{4}dre^{-2C}\,{}^*H\end{pmatrix}\,,\\ \lambda &= \ell+ie^{-\phi} = \frac{ac+bde^{-2C}}{a^2+b^2e^{-2C}}+i\frac{e^{-C}}{a^2+b^2e^{-2C}}\,,\\ \kappa &= \tfrac{1}{8}D+\tfrac{3}{4}ie^{2G} = \frac{qs+\frac{9}{16}pre^{-2C}}{s^2+\frac{9}{16}r^2e^{-2C}}+\tfrac{3}{4}i\frac{e^{-C}}{s^2+\frac{9}{16}r^2e^{-2C}}\,, \end{aligned} \tag{5.71}$$

where A and H_{abc} are functions of C,

$$\begin{aligned} A &= \sqrt{(a^2+b^2e^{-2C})(s^2+\tfrac{9}{16}r^2e^{-2C})}\,,\\ H_{abc} &= \tfrac{2}{3}\varepsilon_{abcd}\partial^d C\,, \end{aligned} \tag{5.72}$$

and C depends only on the transverse coordinates x^a and satisfies $\Box e^{2C}=0$.

A characteristic feature of the above dyonic solutions is that nonzero RR fields are needed in order for the solution to carry electric as well as magnetic charges with respect to the two-form fields. See [19] for more details and for a discussion on how these solutions may be interpreted in ten dimensions. Let us conclude by identifying a few well-known purely electric or magnetic solutions:

- We start from the ten-dimensional neutral fivebrane solution (5.70), corresponding to the identity transformation of $SL(2,\mathbb{R})_{IIB} \times SL(2,\mathbb{R})_{EM}$, $a = d = p = s = 1$, $b = c = q = r = 0$.
- Applying the non-diagonal $SL(2,\mathbb{R})_{IIB}$ transformation ($b = -c = 1$, $a = d = 0$) one obtains (reinterpreted in $D = 10$) a fivebrane solution with a non-trivial RR two-form field. The string coupling is mapped to its inverse.
- The $\mathbb{Z}_2$ string/string duality (5.62) corresponds to choosing $b = -c = -\frac{4}{3}q = \frac{3}{4}r = 1$ and $a = d = p = s = 0$. This yields the fundamental string solution [64] which has no RR fields.
- Applying the non-diagonal $SL(2,\mathbb{R})_{EM}$ transformation ($-\frac{4}{3}q = \frac{3}{4}r = 1$, $p = s = 0$), one obtains a solution with non-trivial RR two-form field. This is the $D = 6$ version of the other type IIB string solution which may also be regarded as a solution of the type I string in the context of a heterotic/type I duality, see [63] for recent discussions.

We have not investigated any further properties of the solutions (5.71). However, we know that they break half of the type II supersymmetries and therefore saturate a Bogomolnyi bound because it is known that the two string and fivebrane solutions just described do, and both $SL(2,\mathbb{R})$ transformations are consistent with the full set of type II supersymmetries.

It would be interesting to study the properties of the dyonic solutions described above in more detail. For example, the fact that both electrically and magnetically charged solutions exist, implies that only discrete subgroups of the $SL(2,\mathbb{R})$ transformations give rise to possible (dyonic) strings. In [175], electrically charged string solutions of the $D = 10$ type IIB theory have been considered in detail. Here the $SL(2,\mathbb{R})_{IIB}$ symmetry of the effective action is used together with a charge quantization rule (due to the existence of magnetic fivebrane solutions) to obtain an $SL(2,\mathbb{Z})_{IIB}$ multiplet of $D = 10$ string solutions with NSNS and/or RR charges. Other work on dyonic p-branes has appeared in [122], in which dyonic membranes are constructed. Membranes may carry both electric and magnetic charges in $D = 8$. A general discussion of p-brane solutions and their charges is given in [189].

At the moment of writing, the field of strong/weak coupling dualities is developing very fast. Let me conclude this chapter by mentioning one of the recent developments which also has some connection with the work presented in this subsection.

The solutions with nonzero RR charges in type II theories have always been something of a mystery. On the one hand, these solutions are singular which means that they probably need a sigma model source term. On the other hand, however, it is expected that the string sigma model only couples to solutions with NSNS charges, related to the fact that the perturbative string spectrum contains NSNS charged states but no states with RR charges. A solution to this problem is described in [156]. There it is shown that Dirichlet-branes (D-branes), extended objects to which the endpoints of open strings with mixed Dirichlet and Neumann boundary conditions are confined, carry a complete set of electric and magnetic RR charges. It is then argued that D-branes are intrinsic to type II string theory and are the RR sources required by string duality.